**LNCS Transactions on Computational Science manuscript No.**
(will be inserted by the editor)

# Collision probability in an in-line machines model

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Received: date / Accepted: date

**Abstract** This paper presents a simple model of the manufacturing line which focuses on the performance of collision probability, and a method of application to the manufacture of Flat Panel Displays (FPDs) and semiconductors. We derive an approximate formula of the collision probability. When the processing time follows a normal distribution, we also did simulations to evaluate the exact probabilities and confirm that our approximation approach yields reasonable results compared to the simulated results. Moreover, we simplify our approximate formula of the collision probability. Concretely speaking, we derive a closed form formula when the processing time follows an exponential distribution. Finally, we present an optimization problem with the collision probability and show a method to solve it.

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**Keywords** In-line machines model · Tact time · Collision probability · Approximation · Closed form · Optimization

## 1 Introduction

The manufacturing process of Flat Panel Displays (FPDs) or semiconductors starts from cleaning, followed by such operations as film deposition, resist coating, and exposure (see [10], [11]). In the process, all given jobs are required to be completed as soon as possible. In order to do this, to date, various efforts to upgrade each machine have been made. As a result, this manufacturing system, consisting of a number of sophisticated machines, is too complicated and, in turn, creates the problem of how to increase efficiency. To solve this problem, we present a method which is not intended to upgrade each machine but to improve the efficiency of the whole manufacturing system.

To do this, we consider a variant on the stochastic flow shop model detailed in [8] (see Fig. 1). Concretely speaking, $m$ machines $(M_1, M_2, \ldots, M_m)$ are connected in a line, where each job is fed from an entrance, and conveyed to an exit after $m$ machines complete their operations. After each job is first processed by $M_1$, it is processed by $m-1$ machines in the order of $M_2, M_3, \ldots, M_m$. After each job is completed at a machine, it is automatically conveyed to the next machine. There is no intermediate buffer between successive machines. Moreover, we assume that (1) the inter-arrival time of jobs at the first machine $M_1$ is constant, and (2) the processing time follows a continuous probability distribution at each machine.

In the above model, dedicated processing equipment for FPDs or semiconductors is regarded as a machine, and a glass substrate that is the FPD or semiconductor material is regarded as a job. In addition, the above assumption (1) comes from the "tact time" constraint. Tact time is a Japanese-English word, which derives from the German word "takt", and was originally coined as part of the Toyota Production System (also known as the Just-In-Time System), but is now widely used in manufacturing practice. Next, assumption (2) reflects the character of the actual dedicated processing equipment. That is, because the actual processing time is uncertain and may vary according to conditions at that time due to solution foaming, chemicals, heat treating, etc., the processing time is treated as a random variable. These are the main reasons for assuming the above model.

In this paper, the phenomenon where a job is sent to a machine which is processing the previous job is called a "collision". Since the actual material used for the production of FPDs or semiconductors is expensive and fragile, manufacturers sustain big losses when collisions occur. Therefore, we consider the collision probability an important evaluation factor in this paper.

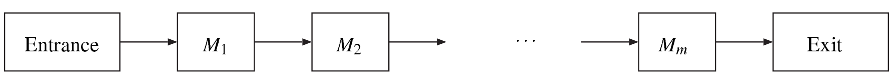


**Fig. 1** In-line $m$ machines model.

In a flow shop model, this collision-like phenomenon is often called "blocking", where the following rule is assumed: even if a machine $M_j$ completes a process, the machine $M_j$ keeps the completed job if the next machine $M_{j+1}$ is still processing. Then, the completed job is conveyed to the machine $M_{j+1}$ when it becomes empty. According to the above rule, the purpose is often to minimize an objective function based on the makespan. If the processing time is deterministic, there are many study results on blocking (see extensive survey in [2]). If the processing time is stochastic, study results are somewhat limited in comparison with their deterministic counterparts. For example, see [8] and [7], where the purpose is to minimize the expected makespan. On the other hand, in queueing theory, depending on the rule for processing blockings (blocked calls cleared, blocked calls delayed, etc.), previous work mainly focused on performance measures in the steady state. More specifically, major measures such as the distribution of the number of jobs, the mean number of jobs, and the mean waiting time on the queueing model exist. In addition, there are other measures; we can find, in fact, that a wide range of literature in the field of queuing theory has been investigated, for example in [3], [4] and [5]. In contrast, in this paper, given the number of jobs to be processed in the prescribed tact time span, we focus on a new measure, which is the probability that there will be at least one collision, called the collision probability. In a comparatively new manufacturing system such as one manufacturing FPDs, the evaluation item (i.e., collision probability) is the new focus of observation. Then we derive an approximation formula for this probability.

To evaluate the exact probabilities and confirm that our approximation approach yields reasonable results compared to the simulated results, we also carry out a number of simulations. At that time, we assume that the processing time follows a normal distribution in consideration of the actual situation.

Another contribution of this paper is to simplify the proposed approximation formula; in short, we show a closed form for the proposed formula of the collision probability when the processing time follows an exponential distribution.

Finally, we parameterize the collision probability and consider an optimization problem, which minimizes the tact time under the constraint that the collision probability is less than or equal to a given value. This reflects an actual problem in the field of manufacture. This approach is quite unique to the best of the authors' knowledge and has high application potential for the manufacture of FPDs or semiconductors.

The remainder of this paper is organized as follows. In Sect. 2, we describe a formal model of the production line. In Sect. 3, we derive an approximation formula of the collision probability. In Sect. 4, based on the assumption that the processing time follows a normal distribution, we show numerical results for the above approximation formula, as well as computer simulation results, confirming that the two types of results are almost the same. In Sect. 5, we derive a closed form for the proposed formula of the collision probability when the processing time follows an exponential distribution. In Sect. 6, we present an optimization problem with the collision probability and show a method to solve it. Finally, Sect. 7 sets forth the conclusions of this paper.

## 2 Model of in-line machines

We describe a formal model of the production line. For this, the following notations will be used:

- $M_1, M_2, \dots, M_m$: $m$ different machines in the line.
- $J_1, J_2, \dots, J_n$: $n$ jobs to be processed.
- $T_i^{(j)}$ $(> 0)$: Processing time of job $J_i$ on machine $M_j$.
- $T_{\text{tact}}$ $(> 0)$: Tact time, i.e., the time difference between the start time instants of $J_i$ and $J_{i+1}$ for all $1 \le i \le n-1$ at the entrance to the line.

The production model is illustrated in Fig. 1. With the same time interval $T_{\text{tact}}$, jobs are successively fed to the line from the entrance. Every job is first processed on machine $M_1$. It is then automatically transported to the next machine $M_2$ after it has been finished on $M_1$. It is assumed, for simplicity, that the transportation time between machines is nil. As soon as $M_2$ receives the job, it starts processing. In this manner, every job is processed on the machines in the order of $M_1, M_2, \dots, M_m$, and then sent to the exit. Moreover, we assume that the processing time $T_i^{(j)}$ on $M_j$ is a random variable that follows a continuous probability distribution, and all $T_i^{(j)}$ $(1 \le i \le n, 1 \le j \le m)$ are independent of each other.

In the above model, a collision occurs if the next job arrives at $M_j$ while $M_j$ is still processing the current job. We obtain the following lemma on the collision condition between jobs.

**Lemma 1** *Suppose that $T_i^{(j)} = t_i^{(j)}$ for all $1 \le i \le n$ and $1 \le j \le m$. For $n$ $(\ge 2)$ jobs, there is no collision in the above production line of $m$ machines if and only if*

$$\sum_{j=1}^{l} t_i^{(j)} \le T_{\text{tact}} + \sum_{j=1}^{l-1} t_{i+1}^{(j)}$$

*holds for all $1 \le i \le n-1$ and $1 \le l \le m$.*

*Proof* Easily proved by double induction on $n$ and $m$, so we omit it. □

## 3 Approximation of collision probability

In this section, we derive an approximation formula of the collision probability. By Lemma 1, the probability that there is no collision is given by

$$\Pr\left(\sum_{j=1}^{l} T_i^{(j)} \le T_{\text{tact}} + \sum_{j=1}^{l-1} T_{i+1}^{(j)} : 1 \le i \le n-1, 1 \le l \le m\right). \tag{1}$$

Unfortunately, it does not seem that Eq. (1) can be simplified further. Therefore, we try to approximate Eq. (1) by considering only two consecutive jobs.

The reason why we pay attention to two consecutive jobs is as follows: even if we consider $n$ jobs, a collision is the phenomenon which occurs between only two consecutive jobs. Therefore, we first pay attention to only two consecutive jobs, and then we derive the no-collision probability between them. After that, considering $n$ jobs, as the number of pairs of two consecutive jobs is $n-1$ ($J_1$ and $J_2$, $J_2$ and $J_3,\dots,J_{n-1}$ and $J_n$), we approximate the no-collision probability over all $n$ jobs using the $(n-1)$-th power of the above derived probability of two consecutive jobs.

For this, we introduce the following event $E_i$ for values of $i$ from 1 to $n-1$.

$E_i$ : Event that, under the assumption that there are only two consecutive jobs $J_i$ and $J_{i+1}$, there is no collision between them on $m$ machines.

Then, the probability of event $E_i$ occurring is given by

$$\Pr(E_i) = \Pr\left(\sum_{j=1}^{l} T_i^{(j)} \leq T_{\mathrm{tact}} + \sum_{j=1}^{l-1} T_{i+1}^{(j)} : 1 \leq l \leq m\right). \tag{2}$$

We introduce the following random variables:

$$X_l = \sum_{j=1}^{l} T_i^{(j)} - \sum_{j=1}^{l-1} T_{i+1}^{(j)} - T_{\mathrm{tact}} \text{ for all } 1 \leq l \leq m.$$

Then, Eq. (2) is given as follows:

$$\Pr\left(X_l \leq 0 : 1 \leq l \leq m\right) = \iint \dots \int_{S_1} f(x_1, x_2, \dots, x_m) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_m \tag{3}$$
$$S_1 : x_l \leq 0 \text{ for all } 1 \leq l \leq m,$$

where $f(x_1, x_2, \dots, x_m)$ is the joint probability density function of random variables $X_l$ for all $1 \leq l \leq m$.

These variables are transformed by $y_1 = x_1$, $y_j = x_j - x_{j-1}$, and $S_1$ is expressed as $S_2$ in terms of $y_j$:

$$S_2 : \sum_{i=1}^{j} y_i \leq 0 \quad \text{for all } 1 \leq j \leq m.$$

The Jacobian for $x_i = \sum_{j=1}^{i} y_j$ for all $1 \leq i \leq m$, which corresponds to $S_1$, is given by

$$\mathscr{J} = \frac{\partial(x_1, x_2, \dots, x_m)}{\partial(y_1, y_2, \dots, y_m)} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = 1.$$

Therefore, the right side of Eq. (3) becomes the following:

$$\iint \cdots \int_{S_2} g(y_1, y_2, \dots, y_m) \, |\mathscr{J}| \, \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_m$$
$$= \int_{-\infty}^{0} \mathrm{d}y_1 \int_{-\infty}^{-y_1} \mathrm{d}y_2 \cdots \int_{-\infty}^{-\sum_{i=1}^{m-1} y_i} g_1(y_1) g_2(y_2) \cdots g_m(y_m) \mathrm{d}y_m, \tag{4}$$

where $g(y_1, y_2, \ldots, y_m)$ is the joint probability density function of random variables

$$Y_1 = T_i^{(1)} - T_{\text{tact}}, \tag{5}$$

$$Y_j = T_i^{(j)} - T_{i+1}^{(j-1)} \quad \text{for } 2 \le j \le m, \tag{6}$$

and each $g_j(y_j)$ is the probability density function (pdf) of $Y_j$. Note that the equality in Eq. (4) holds since all $Y_j$ $(1 \le j \le m)$ are assumed to be independent of each other. As a result, the probability of event $E_i$ occurring, i.e. $\Pr(E_i)$, can be expressed by the right side of Eq. (4).

Moreover, since we assume that all $T_i^{(j)}$ $(i = 1, 2, \ldots, n)$ have the same distribution function, $\Pr(E_1) = \Pr(E_2) = \cdots = \Pr(E_{n-1})$ holds. Although the two events $E_i$ and $E_j$ $(i \ne j)$ are not independent, precisely speaking, we approximate the no-collision probability over all $n$ jobs (i.e. (1)) by the $(n-1)$-th power of the right side of Eq. (4). The approximate probability of collision is then given by

$$1 - \left( \int_{-\infty}^{0} \mathrm{d}y_1 \int_{-\infty}^{-y_1} \mathrm{d}y_2 \cdots \int_{-\infty}^{-\sum_{i=1}^{m-1} y_i} g_1(y_1) g_2(y_2) \cdots g_m(y_m) \mathrm{d}y_m \right)^{n-1}. \tag{7}$$

## 4 Numerical results

In this section, based on the above formula (Eq. (7)), we present the numerical results. For this, we assume that the processing time $T_i^{(j)}$ on machine $M_j$ follows a normal distribution with parameters of expectation $\mu_j$ and standard deviation $\sigma_j$, i.e., $T_i^{(j)} \sim \mathrm{N}(\mu_j, \sigma_j^2)$. Then, by Eq. (5), the pdf $g_1(y_1)$ of $Y_1$ is obtained by translating the pdf of the normal distribution $\mathrm{N}(\mu_1, \sigma_1^2)$ by $-T_{\text{tact}}$, yielding $Y_1 \sim \mathrm{N}(\mu_1 - T_{tact}, \sigma_1^2)$. On the other hand, by Eq. (6), the pdfs $g_j(y_j)$ of $Y_j$ for $2 \le j \le m$ are obtained by the reproductive property of the normal distribution, yielding $Y_j \sim \mathrm{N}(\mu_j - \mu_{j-1}, \sigma_{j-1}^2 + \sigma_j^2)$ for all $2 \le j \le m$. The $g_1(y_1) g_2(y_2) \cdots g_m(y_m)$ in Eq. (7) then becomes as follows:

$$\frac{\exp\left\{ -\frac{1}{2} \left( \frac{(y_1 - \mu_1 + T_{tact})^2}{\sigma_1^2} + \sum_{j=2}^{m} \frac{(y_j - \mu_j + \mu_{j-1})^2}{\sigma_{j-1}^2 + \sigma_j^2} \right) \right\}}{\sigma_1 (\sqrt{2\pi})^m \prod_{j=2}^{m} \sqrt{\sigma_{j-1}^2 + \sigma_j^2}}. \tag{8}$$

Eq. (7) with Eq. (8) may not be simplified any further since the integral of the pdf of normal distribution cannot be generally expressed as an elementary function[6]. So, we directly obtain the numeric integration values of Eq. (7) with Eq. (8) by using MATHEMATICA[12].

For our computation in this section, the number of jobs is set to $n = 1{,}000$, $1 \le m \le 4$, and the parameters of the normal distributions are set so that the expectation and the standard deviation of the processing time on each machine become equal to 1 and 0.01, respectively (i.e., $\mu_j = 1$, $\sigma_j = 0.01$). The numerical results are shown in Figs. 2 – 5, for $1 \le m \le 4$, respectively. In those Figs. 2 – 5, the solid lines represent the numeric values based on Eq. (7).

We also carried out the following simulations to evaluate the exact probabilities. The procedure is stated as follows: given the number of jobs $n$, the number of machines $m$, the tact time $T_{tact}$, the parameters of the normal distribution, and a positive integer $c$ (specifying the number of iterations, which is related to the accuracy), derive the collision probability by the following algorithm.

**Simulation Algorithm**

**Step 1:** $loop := 1$.

**Step 2:** Generate the processing time $t_i^{(j)}$ $(1 \leq i \leq n, 1 \leq j \leq m)$ randomly from the normal distribution.

**Step 3:** Based on the condition in Lemma 1, check whether a collision occurs. Let $loop := loop + 1$. If $loop \leq c$, return to Step 2; otherwise go to Step 4.

**Step 4:** Output the collision probability (the number of collisions observed in Step 3)$/c$.

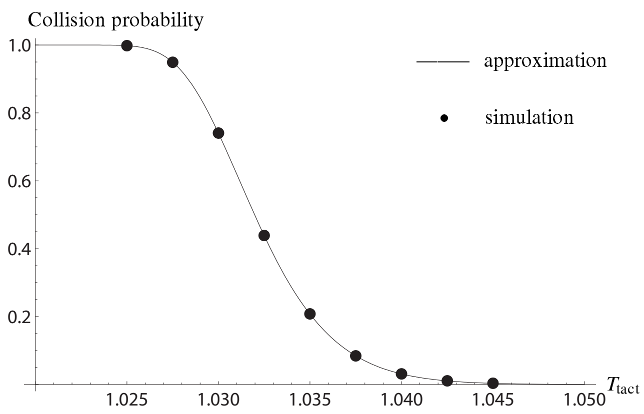


**Fig. 2** Collision probability evaluated by approximate formula and simulation when $m = 1$.

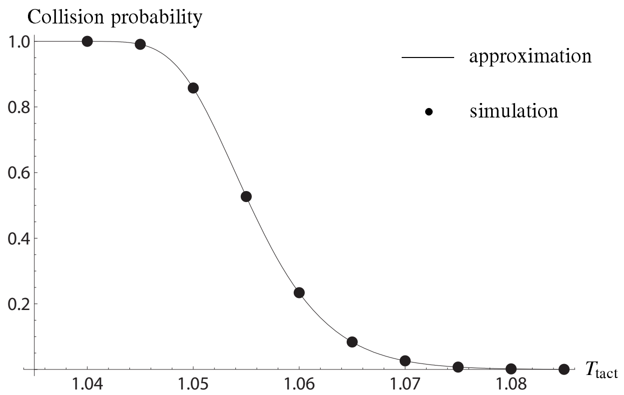


**Fig. 3** Collision probability evaluated by approximate formula and simulation when $m = 2$.

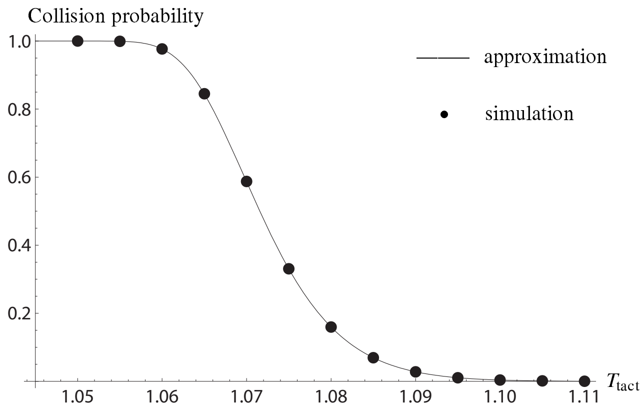


**Fig. 4** Collision probability evaluated by approximate formula and simulation when $m = 3$.

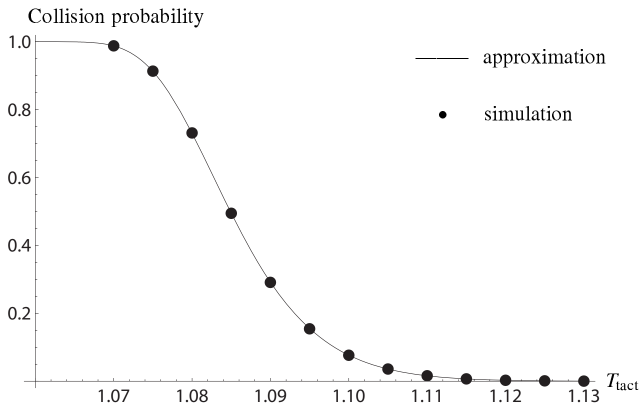


**Fig. 5** Collision probability evaluated by approximate formula and simulation when $m = 4$.

The computation time is $\Theta(cmn)$. Throughout all the simulations, we use Mersenne Twister[9] as the pseudorandom generator, and the number of iterations is set to $c = 1{,}000{,}000$. The simulation results are shown in Figs. 2 – 5, for $1 \le m \le 4$, respectively. In those Figs. 2 – 5, the black dots represent the results of the simulations.

Figs. 2 – 5 show that, as the tact time increases, the collision probability decreases, clearly exhibiting the trade-off between the tact time and the collision probability. We also confirmed that the collision probability increases with $m$. We may conclude that the numerical and simulation results are reasonably close in most cases.

## 5 Simplification of approximate formula

In this section, we show that the approximate formula Eq. (7) shown in Sect. 3 can be simplified when the processing time $T_i^{(j)}$ on machine $M_j$ follows an exponential distribution with a parameter $\lambda_j$ which is a positive real number. The pdf of the exponential distribution is defined as follows:

$$f(x;\lambda_j) = \lambda_j \mathrm{e}^{-\lambda_j x} \quad \text{for } x \geq 0.$$

Using exponential distribution, we show a closed form formula.

## 5.1 Closed form for $m = 1$

By Lemma 1 for $m = 1$, the no-collision probability becomes as follows:

$$\begin{aligned}
&\Pr\left(T_i^{(1)} \leq T_{\text{tact}} : 1 \leq i \leq n-1\right) \\
&= \left(\Pr\left(T_1^{(1)} \leq T_{\text{tact}}\right)\right)^{n-1} \qquad \text{(since the } T_i^{(1)} \text{ are i.i.d.)} \\
&= \left(\int_0^{T_{\text{tact}}} \lambda_1 \mathrm{e}^{-\lambda_1 x} \mathrm{d}x\right)^{n-1} \\
&= \left(1 - \mathrm{e}^{-\lambda_1 T_{\text{tact}}}\right)^{n-1}.
\end{aligned}$$

Therefore, the collision probability for $m = 1$ is given by the following closed form:

$$1 - \left(1 - \mathrm{e}^{-\lambda_1 T_{\text{tact}}}\right)^{n-1}.$$

Note that this is not an approximation but an exact formula.

## 5.2 Closed form for $m = 2$

We sketch a derivation of approximate collision probability for $m = 2$. By Eq. (5), the pdf $g_1(y_1)$ of $Y_1$ is obtained by translating the pdf of the exponential distribution with parameter $\lambda_1$ by $-T_{tact}$. Therefore, we have

$$g_1(y_1) = \begin{cases} \lambda_1 \mathrm{e}^{-\lambda_1 (y_1 + T_{\text{tact}})} & (y_1 \geq -T_{\text{tact}}), \\ 0 & (y_1 < -T_{\text{tact}}). \end{cases} \tag{9}$$

By Eq. (6) for $m = 2$, the $Y_2$ is the sum of two independent random variables $T_i^{(2)}$ and $-T_{i+1}^{(1)}$, with pdfs $h_1$ and $h_2$ respectively. Therefore, the pdf $g_2(y_2)$ of $Y_2$ is given by the convolution of $h_1$ and $h_2$. The pdf of $T_i^{(2)}$ follows the exponential distribution with parameter $\lambda_2$, i.e.,

$$h_1(x) = \begin{cases} \lambda_2 \mathrm{e}^{-\lambda_2 x} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

The graph of the pdf of $-T_{i+1}^{(1)}$ is a reflection of the pdf of $T_{i+1}^{(1)}$ with parameter $\lambda_1$ with respect to the line $y=0$,

$$h_2(y) = \begin{cases} 0 & (y > 0), \\ \lambda_1 \mathrm{e}^{\lambda_1 y} & (y \le 0). \end{cases}$$

Then, by calculating the convolution of the above $h_1$ and $h_2$, the pdf $g_2(y_2)$ of $Y_2$ becomes as follows:

Case 1: $y_2 \ge 0$

$$\begin{aligned} g_2(y_2) &= \int_{-\infty}^{\infty} h_1(x) h_2(y_2 - x) \mathrm{d}x \\ &= \int_{y_2}^{\infty} \lambda_2 \mathrm{e}^{-\lambda_2 x} \cdot \lambda_1 \mathrm{e}^{\lambda_1 (y_2 - x)} \mathrm{d}x \\ &= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \mathrm{e}^{-\lambda_2 y_2}. \end{aligned} \tag{10}$$

Case 2: $y_2 < 0$

$$\begin{aligned} g_2(y_2) &= \int_{-\infty}^{\infty} h_1(x) h_2(y_2 - x) \mathrm{d}x \\ &= \int_{0}^{\infty} \lambda_1 \lambda_2 \mathrm{e}^{\lambda_1 y_2} \mathrm{e}^{-(\lambda_1 + \lambda_2) x} \mathrm{d}x \\ &= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \mathrm{e}^{\lambda_1 y_2}. \end{aligned} \tag{11}$$

Using Eqs. (9) – (11) we can simplify Eq. (4) (i.e. $\Pr(E_i)$) for $m = 2$ in the following equation. Since $y_1 \le 0$ holds in Eq. (4), we have

$$\begin{aligned} &\int_{-\infty}^{0} g_2(y_2) \mathrm{d}y_2 + \int_{0}^{-y_1} g_2(y_2) \mathrm{d}y_2 \\ =& 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} \mathrm{e}^{\lambda_2 y_1}. \end{aligned}$$

Therefore, by denoting the right side of the above equation as $A$, we obtain

$$\begin{aligned} \Pr(E_i) &= \int_{-\infty}^{0} g_1(y_1) \cdot A \mathrm{d}y_1 \\ &= \int_{-\infty}^{-T_{\mathrm{tact}}} g_1(y_1) \cdot A \mathrm{d}y_1 + \int_{-T_{\mathrm{tact}}}^{0} g_1(y_1) \cdot A \mathrm{d}y_1 \\ &= \int_{-T_{\mathrm{tact}}}^{0} \lambda_1 \mathrm{e}^{-\lambda_1 (y_1 + T_{\mathrm{tact}})} \cdot A \mathrm{d}y_1 \\ &= \begin{cases} 1 + \frac{\mathrm{e}^{-\lambda_1 T_{\mathrm{tact}}} \lambda_2^2}{\lambda_1^2 - \lambda_2^2} + \frac{\mathrm{e}^{-\lambda_2 T_{\mathrm{tact}}} \lambda_1^2}{\lambda_2^2 - \lambda_1^2} & (\lambda_1 \ne \lambda_2), \\ 1 - \mathrm{e}^{-\lambda T_{\mathrm{tact}}} - \frac{\lambda}{2} \mathrm{e}^{-\lambda T_{\mathrm{tact}}} \cdot T_{\mathrm{tact}} & (\lambda = \lambda_1 = \lambda_2). \end{cases} \end{aligned}$$

This is a closed form formula of $\Pr(E_i)$, from which the approximate collision probability $1 - \Pr(E_i)^{n-1}$ is also obtained in a closed form formula.

### 5.3 Closed form for the general case of $m$ machines

It is possible to extend the above derivation for $m = 2$ to the general $m$ machines model in a straightforward manner. Therefore, we omit the details. However, there are cases in which Eq. (4) ($\Pr(E_i)$) can be written as a simple expression depending on the condition of parameters $\lambda_j$ $(1 \le j \le m)$. Such cases are given as follows:

*Remark 1* When $\lambda_j$ $(1 \le j \le m)$ are distinct, $\Pr(E_i)$ can be written as

$$1+(-1)^m \sum_{k=1}^{m} \left( \mathrm{e}^{-\lambda_k T_{\mathrm{tact}}} \prod_{j \in I_m \setminus \{k\}} \frac{\lambda_j^2}{\lambda_k^2 - \lambda_j^2} \right),$$

where $I_m$ denotes the set $\{1,2,\cdots,m\}$.

*Remark 2* When $\lambda_j$ $(1 \le j \le m) = \lambda$, $\Pr(E_i)$ can be written as

$$1-\mathrm{e}^{-\lambda T_{\mathrm{tact}}} - \frac{\mathrm{e}^{-\lambda T_{\mathrm{tact}}}}{a_m^{(m)}} \sum_{j=1}^{m-1} a_m^{(m-j)} \lambda^j T_{\mathrm{tact}}^j,$$

where coefficients $a_m^{(j)}$ $(j = 1,2,\ldots,m)$ are represented by the following recursive expressions:

$$\begin{aligned} a_m^{(1)} &= 1, \\ a_m^{(j)} &= a_{m-1}^{(j)} + (m+j-2)a_{m-1}^{(j-1)} \quad \text{for all} \quad 2 \le j \le m-1, \\ a_m^{(m)} &= 2(m-1)a_{m-1}^{(m-1)}. \end{aligned}$$

## 6 Tact time minimization with collision probability

In this section, we consider an optimization problem which reflects an actual problem in the field of manufacture. In manufacturing practice, the setting value for the tact time is important since the production rate can expect to increase drastically as the tact time is shortened. However, the collision probability also increases as the tact time is shortened. Therefore, we parameterize the collision probability and present a problem to find an optimal tact time. Concretely speaking, the problem is as follows:

**Input:** The number of jobs $n$, the number of machines $m$, the probability distribution on the processing time, a positive real value $\alpha$ $(0 \le \alpha \le 1)$.
**Output:** Tact time, such that the collision probability is less than or equal to $\alpha$.
**Objective function:** $T_{\mathrm{tact}} \longrightarrow \min$.

The collision probability decreases monotonically with the tact time. By using this property, we can calculate an optimal tact time efficiently using a binary search.

## 7 Conclusions

Our main contribution in this paper was to present a new evaluation item (i.e., collision probability) for the simple model discussed in queuing theory, which has applications in the manufacture of FPDs and semiconductors, and to analyze it theoretically. We have derived an approximation formula of collision probability and shown numerical results, as well as computer simulation results, when the processing time follows a normal distribution. Moreover, we have shown cases in which our formula can be expressed by a closed form. Finally, we considered how to minimize the tact time by including the collision probability as part of the input.

We assumed that exactly two consecutive jobs flow in the in-line model, in the process by which our approximate formula of the collision probability was derived. Working without this assumption is a future area of investigation. In the numerical results section, we showed some results for instances where the number of machines is small. Seeing how the simulated and approximated results behave when the number of machines is larger is another valuable area of research. In order to do this, ideas for working out multiple integrals might be needed.

Although the in-line $m$ machines model in this paper doesn't have any buffer space between machines, such space may be effective in avoiding collisions between jobs. However, it appears hard to analyze the collision probability with buffer space included. In [1], the collision probability with buffer space included was calculated by computer simulation, and a method was presented to minimize the total number of buffers.

**Acknowledgements** The authors would like to thank the anonymous referees for their valuable comments on some key areas where this work could be improved.

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