

Title	On covering of any point configuration by disjoint unit disks
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Citation	The 23rd Canadian Conference on Computational Geometry (CCCG' 11): 393-397
Issue Date	2011
Type	Conference Paper
Text version	author
URL	<a href="http://hdl.handle.net/10119/10309">http://hdl.handle.net/10119/10309</a>
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Description	

# On covering of any point configuration by disjoint unit disks

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## Abstract

We give a configuration of 53 points that cannot be covered by disjoint unit disks. This improves the previously known configuration of 55 points.

## 1 Introduction

In 2008, a Japanese puzzle designer Naoki Inaba proposed an interesting question [3]: “Given any configuration of 10 points, prove that you can cover all the points by coins. You can use any number of coins, but coins cannot overlap.” That is, he proved the following theorem:

**Theorem 1** *Any configuration of 10 points can be covered by disjoint coins.*

Inaba gave an interesting proof of this theorem based on the probabilistic method. (See appendix; this proof is essentially the same in [4] written in Japanese. The proof can be found in [6] also.) As he mentioned in the answer page [4], this theorem also derives another natural question: How many points arranged appropriately that cannot be covered by disjoint coins? Let  $k$  be the maximum number of points such that any configuration of  $k$  points can be covered by coins. (We note that  $k$  points can be covered by at most  $k$  coins.) Inaba’s theorem shows that  $10 \leq k$ , and trivially there is an upper bound of  $k$ ; if we put sufficiently many points on a fine lattice, disjoint coins cannot cover all of them (Figure 1). This problem spread over the puzzle society in 2010 (at the 9th Gathering 4 Gardner). Peter Winkler took up this problem in his column [5], and he gave a configuration of 60 points that cannot be covered by disjoint coins. Moreover, Peter Winkler improves the lower bound from 10 to 12 [6, 7]. That is,  $12 \leq k \leq 59$ . Recently, Veit Elser improves the upper bound to 54 in 2011 [2]. In this paper, we further improve the upper bound of  $k$  to 52. That is, we give a configuration of 53 points that cannot be covered by disjoint coins. The main theorem is summarized as follows.

**Theorem 2** *Let  $k$  be the maximum number such that any configuration of  $k$  points can be covered by disjoint coins. Then  $12 \leq k \leq 52$ .*

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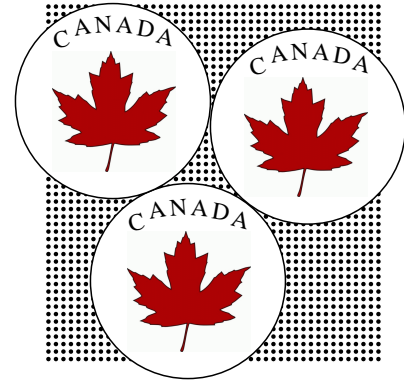


Figure 1: Points cannot be covered by disjoint coins

Hereafter, we assume that each coin is a unit disk of radius 1. To simplify the argument, each unit disk is an open disk. That is, a point on the edge of a unit disk is not covered by the disk. (Using the perturbation technique, our results can be applied to the closed disks.) Let  $L_3$ ,  $L_4$ , and  $L_6$  be a triangular, square, and hexagonal lattice, respectively. The *size* of a lattice is defined by the shortest distance between any pair of two points in  $L_i$  for  $i = 3, 4, 6$  (Figure 2). We sometimes abuse  $L_i$  as a set of lattice points for  $i = 3, 4, 6$ . Our construction of the point configuration consists of two phases.

## 2 Configuration of the points in a circle

We first consider point configurations in a large circle. We denote by  $x$  a circle of radius  $r = 2\sqrt{3}/3 - 1 = 0.1547\dots$ . For the circle  $x$ , we have the following lemma:

**Lemma 3** *Let  $C_1$  and  $C_2$  be disjoint two unit disks. We suppose that a circle  $x$  circumscribes both of  $C_1$  and  $C_2$ . Then we cannot arrange any unit disk  $C_3$  with  $C_3 \cap x \neq \emptyset$  that is disjoint from  $C_1$  and  $C_2$ .*

**Proof.** Since  $r = 2\sqrt{3}/3 - 1$ , when  $C_1, C_2, C_3$  touch with each other, the circle  $x$  also touches all of them (Figure 3). Since  $C_1, C_2, C_3$  are disjoint, the lemma follows immediately.  $\square$

Using the circle  $x$  of radius  $r$ , we give the key idea of our point configuration:

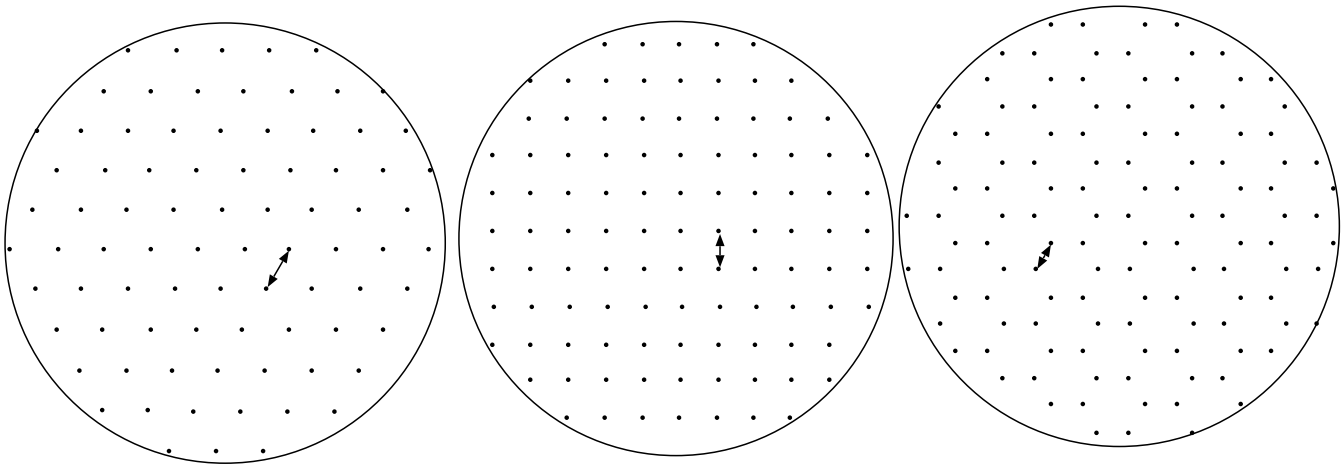


Figure 2: Triangular lattice, square lattice, and hexagonal lattice. Each size is given by the length of the arrow.

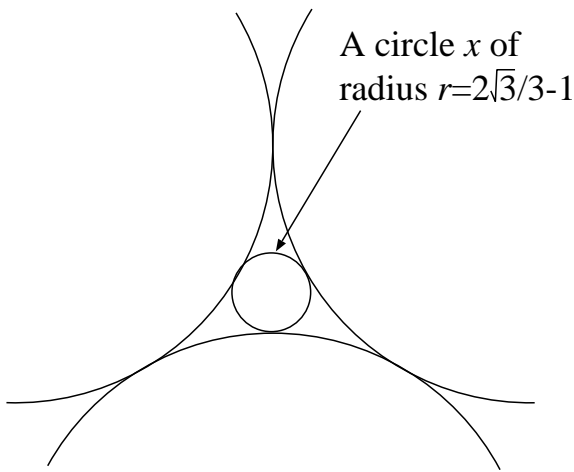


Figure 3: The circle  $x$  in the space surrounded by three unit disks

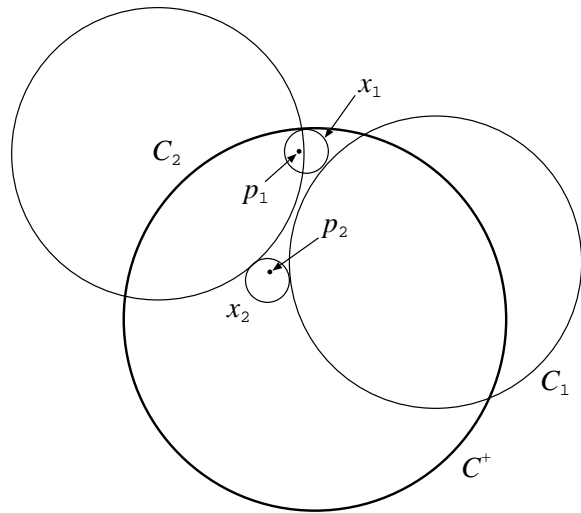


Figure 5: Proof of Lemma 4

**Lemma 4** Let  $C^+$  be a disk of radius  $1 + 2r$ . For  $i = 3, 4, 6$ , let  $L_i$  be the lattice of size  $\sqrt{3}r$ ,  $\sqrt{2}r$ , and  $r$ , respectively (Figure 4). (That is, we make  $x$  the largest empty circle of each  $L_i$ .) Then any point configuration in  $L_i \cap C^+$  cannot be covered by disjoint unit disks.

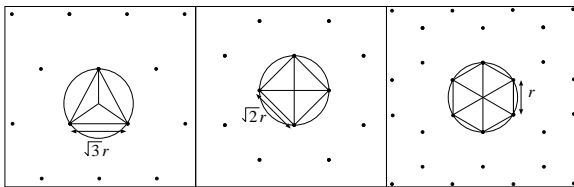


Figure 4: The size of each lattice

**Proof.** We first observe that when we put a closed disk  $x'$  of radius  $r$  in  $L_i \cap C^+$ ,  $x'$  should contain at least one point in  $L_i \cap C^+$  because of the size of  $L_i$ .

Now in order to derive a contradiction, we assume that all the points in  $L_i \cap C^+$  are covered by disjoint unit disks  $C_1, C_2, \dots$ . Without loss of generality,  $C_1 \cap C^+$  contains the largest number of points in  $C^+$  among  $C_i \cap C^+$  (Figure 5). Then we can put a circle  $x_1$  of radius  $r$  in  $C^+ \setminus C_1$  such that  $x_1$  inscribes  $C^+$  and circumscribes  $C_1$ . Then, by the observation,  $x_1$  contains at least one

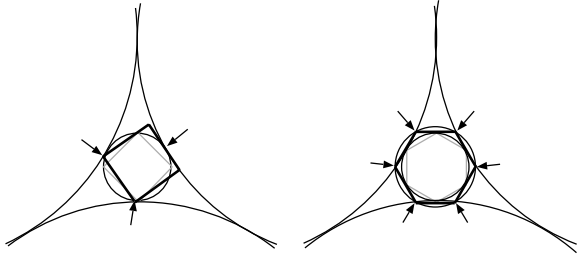


Figure 6: Enlargement of the lattices  $L_4$  and  $L_6$

point  $p_1$  in  $L_i \cap C^+$ . By the assumption, there is a disk  $C_2$  covering the point  $p_1$  in  $x_1$ . Then we can put again a circle  $x_2$  of radius  $r$  in  $C^+ \setminus C_1$  such that  $x_2$  circumscribes  $C_1$  and  $C_2$ , and  $x_2$  contains a point  $p_2$  in  $L_i \cap C^+$ . (Note that  $x_1$  and  $x_2$  may overlap.) Then, by Lemma 3, we cannot cover  $p_2$  by the other unit disks  $C_3, \dots$ . This is a contradiction. Thus the lemma follows.  $\square$

By Lemma 4, we can use  $L_3 \cap C^+$  of size  $\sqrt{3}r$ ,  $L_4 \cap C^+$  of size  $\sqrt{2}r$ , and  $L_6 \cap C^+$  of size  $r$ , where  $r = 2\sqrt{3}/3 - 1$ , as point configurations that give upper bounds of  $k$ , respectively. Among them, the upper bound  $k < 82$  given by  $L_3 \cap C^+$  is much better than the others (the leftmost one in Figure 2). For  $L_4$  and  $L_6$ , we can slightly enlarge the size of the lattices than that of Lemma 4 with careful analyses. For  $L_4$ , when four points around on  $x$  in Figure 4, at most one point on  $x$  touches surrounding unit disks. Hence we can enlarge  $L_4$  until at most three points of the square touch surrounding unit disks (Figure 6). (More precisely, we can enlarge to the minimum square of all the squares of which three points of it touch the surrounding disks.) For  $L_6$ , we can enlarge  $L_6$  in Figure 4 until all of 6 points are on surrounding unit disks as in Figure 6. However, these enlargements cannot catch up with the case of  $L_3$  at all. Even using the enlargement technique, our best achievements of the cases of  $L_4$  and  $L_6$  are  $k < 102$  and  $k < 119$ , respectively. (The point configurations after enlargements are given in Figure 2.) Hence we omit the details of these enlargements.

### 3 Improvement of the point configuration

Hereafter, we fix the lattice  $L_3$  of size  $\sqrt{3}r$ . Carefully checking the proof of Lemma 4, we can see that  $C^+$  is redundant. We first cut off the top and the bottom of  $C^+$  as in Figure 7. More precisely, the lines  $AB$  and  $EF$  are straight line segments in parallel, and the distance between  $AB$  and the center of  $C^+$  is equal to the distance between  $EF$  and the center of  $C^+$ . The distance between  $AB$  and  $EF$  is  $1 + 3r$ . The curves  $HA, BC, DE,$  and  $FG$  are arcs of the circles of radius  $r$ . The curves  $CD$  and  $GH$  are arcs of the circle  $C^+$

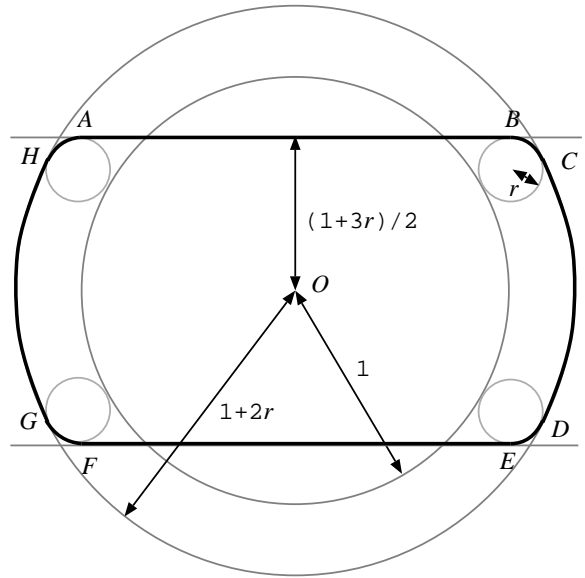


Figure 7: The oval-like form  $\Theta$

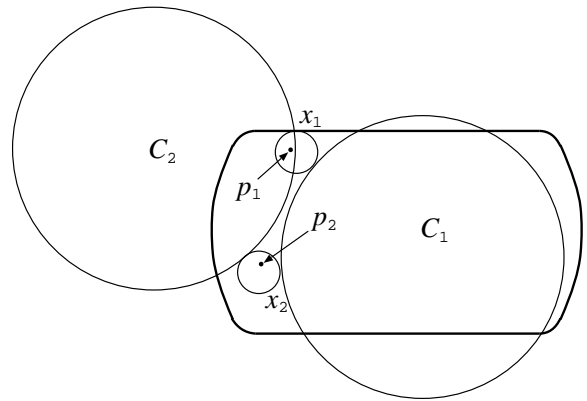


Figure 8: Proof of Lemma 5

of radius  $1 + 2r$ . Let  $\Theta$  be the closed area surrounded by the resulting oval-like form  $ABCDEFGH$ . We now refine Lemma 4:

**Lemma 5** *Let  $\Theta$  be the closed area given by the oval in Figure 7. Let  $L_3$  be the lattice of size  $\sqrt{3}r$ . Then any point configuration in  $L_3 \cap \Theta$  cannot be covered by disjoint unit disks.*

**Proof.** In order to derive a contradiction, we assume that all points in  $L_3 \cap \Theta$  are covered by disjoint unit disks  $C_1, C_2, \dots$ . Without loss of generality,  $C_1 \cap \Theta$  contains the largest number of points in  $L_3 \cap \Theta$  among  $C_i \cap \Theta$ . Then we can put a circle  $x_1$  of radius  $r$  in  $\Theta \setminus C_1$  such that  $x_1$  inscribes  $\Theta$  and circumscribes  $C_1$  (Figure 8). Then  $x_1$  contains at least one point  $p_1$  in  $L_3 \cap \Theta$ . By the assumption, there is a disk  $C_2$  covering

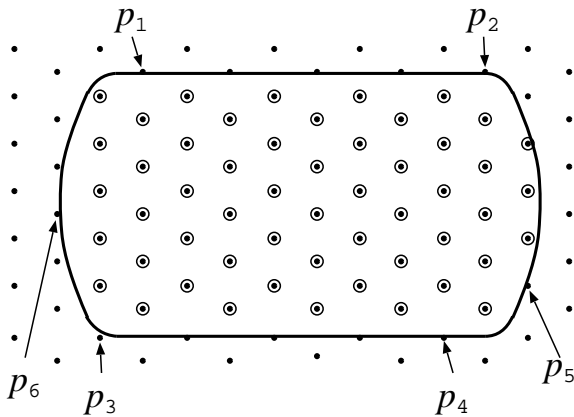


Figure 9: A point configuration in  $\Theta$ ; the circled points are in  $\Theta$ .

the point  $p_1$ . Then we can put again a circle  $x_2$  of radius  $r$  in  $\Theta \setminus C_1$  such that  $x_2$  circumscribes  $C_1$  and  $C_2$ , and  $x_2$  contains a point  $p_2$  in  $L_3 \cap \Theta$ . Then, by Lemma 3, we cannot put any unit disk that covers  $p_2$ . This is a contradiction. Hence the lemma follows.  $\square$

Now we minimize the number of points in  $L_3 \cap \Theta$ , where  $L_3$  has size  $\sqrt{3}r$ . Our best achievement is given in Figure 9. In this point configuration, we have two criteria for the points  $p_1, p_2, \dots, p_6$  in Figure 9.

1. The line  $\ell_1$  joining  $p_1$  and  $p_2$  and the line  $\ell_2$  joining  $p_3$  and  $p_4$  have enough distance to put  $\Theta$  between them; the distance between  $\ell_1$  and  $\ell_2$  is equal to  $5.5\sqrt{3}r = 5.5\sqrt{3}(2\sqrt{3}/3 - 1) = 5.5(2 - \sqrt{3}) = 1.4737\dots$ . On the other hand, the corresponding width of  $\Theta$  is equal to  $1 + 3r = 1 + 3(2\sqrt{3}/3 - 1) = 2\sqrt{3} - 2 = 1.4641\dots$ . Hence we can put  $\Theta$  between  $\ell_1$  and  $\ell_2$  such that all the points on  $\ell_1$  or  $\ell_2$  are outside of  $\Theta$ .
2. In Figure 9, the closest points on the right and left sides of  $\Theta$  are  $p_5$  and  $p_6$ , respectively. We show that we can put  $\Theta$  between them. To simplify the argument, we assume that we put  $\Theta$  on the line  $\ell_2$  (joining  $p_3$  and  $p_4$ ) as in Figure 9, and we take the coordinate with the center  $O = (0, 0)$  of the  $\Theta$ . Let  $p_5 = (x_5, y_5)$  and  $p_6 = (x_6, y_6)$ . Then we have  $p_5 = (x_5, -7\sqrt{3}r/4)$ ,  $p_6 = (x_6, -\sqrt{3}r/4)$ , and  $|x_5 - x_6| = 33r/2 = 11\sqrt{3} - 33/2 = 2.5525\dots$ . Let  $p'_5$  be the point on the edge of  $\Theta$  such that  $p'_5$  has the same height of  $p_5$  (and closest one of two such points). Let  $p'_6$  be the point on the edge of  $\Theta$  defined similarly for  $p_6$ . That is, we can let  $p'_5 = (x'_5, -7\sqrt{3}r/4)$ , and  $p'_6 = (x'_6, -\sqrt{3}r/4)$ . Since  $x_i'^2 + y_i'^2 = (1 + 2r)^2$  for  $i = 5, 6$ , we can obtain  $|x'_5 - x'_6| = \sqrt{115/(4\sqrt{3}) - 725/48} +$

$\sqrt{283/48 - 29/(4\sqrt{3})} = 2.5302\dots$ . Therefore, we can put  $\Theta$  between  $p_5$  and  $p_6$  such that they are outside of  $\Theta$ .

Based on these criteria, we can put  $\Theta$  as in Figure 9, and the only circled points are in  $\Theta$ . The number of the circled points is 53, that concludes the proof of Theorem 2.

#### 4 Concluding remarks

We give an upper bound 52 of the maximum number  $k$  such that any configuration of  $k$  points can be covered by disjoint coins. In the oval  $\Theta$ , it is essentially required that the radius of the largest empty circle is bounded by  $r = 2\sqrt{3}/3 - 1$ . Hence some computational power may improve the upper bound. But smart proof seems to be better; recently, Aloupis develops another technique, and gives a better upper bound [1]. Applying his technique to the point configuration in Figure 9, it seems that we can remove a few more points. Our idea is based on the uniform point configurations. The upper bound based on some nonuniform point configurations would be interesting.

We still have a big gap between 12 and 52. Improvement of the lower bound is also interesting. In appendix, we give the proof of the lower bound 10 by the probabilistic method. Indeed, the proof states stronger result: any configuration of 10 points can be covered by the sheet in Figure 10. That is, the arrangement of the coins are fixed. Moreover, the bound given by the probabilistic method seems to be not tight. Hence the gap between the lower bound and the real value seems to be larger than the gap between the upper bound and the real value.

#### Acknowledgements

The authors are grateful to Hirokazu Iwasawa and Naoki Inaba for their fruitful discussion on this topic. The authors also thank Peter Winkler, Veit Elser, János Pach, and Joseph Mitchell for their helpful comments.

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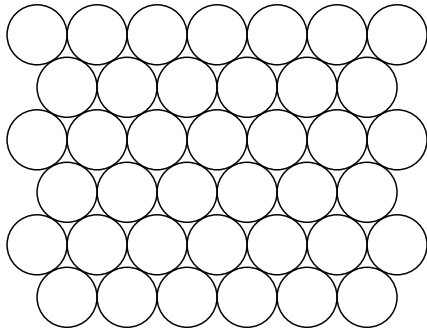


Figure 10: A sheet of infinitely many coins

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#### A Proof of Inaba’s theorem by the probabilistic method

Let  $P$  be any configuration of 10 points  $p_1, p_2, \dots, p_{10}$ . We put randomly a sheet of infinitely many coins arranged like Figure 10 on  $P$ . For  $i = 1, 2, \dots, 10$ , let  $A_i$  be the event that the point  $p_i$  is covered by a coin. Then,  $\Pr\{A_i\} = (\sqrt{3}-\pi/2)/\sqrt{3} > 0.093$  by a simple calculation of ratios of areas of coins and the background. Hence the probability that all points are covered is given as follows:

$$\begin{aligned}
 & \Pr\{A_1 \wedge A_2 \wedge A_3 \wedge \dots \wedge A_{10}\} \\
 &= 1 - (\Pr\{\overline{A_1} \wedge \overline{A_2} \wedge \overline{A_3} \wedge \dots \wedge \overline{A_{10}}\}) \\
 &= 1 - (\Pr\{\overline{A_1} \vee \overline{A_2} \vee \overline{A_3} \vee \dots \vee \overline{A_{10}}\}) \\
 &\geq 1 - (\Pr\{\overline{A_1}\} + \Pr\{\overline{A_2}\} + \Pr\{\overline{A_3}\} + \dots + \Pr\{\overline{A_{10}}\}) \\
 &> 1 - 10 \cdot 0.093 = 0.07 > 0.
 \end{aligned}$$

Since the all points are covered with positive probability, there exists a way to put the sheet to cover all the points.