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Description	

A predicative completion of a uniform space

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Abstract

We give a predicative construction of a completion of a uniform space in the constructive Zermelo-Fraenkel set theory.

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1 Introduction

In [6, Problems 17 to 21 of Chapter 4], Bishop introduced a constructive concept of a uniform space with a set of pseudometrics, and showed basic theorems, such as, that arbitrary uniform space has a completion (the set of Cauchy filters); see also [7, Problems 22 to 26 of Chapter 4], and [8, 10] for Bishop's constructive mathematics. Although, apparently, Bishop did not actually say explicitly that the completion should have been constructed in this way, since we have to think of the *set* of Cauchy filters, the construction of a completion is problematic from a predicative point of view, such as in the constructive Zermelo-Fraenkel set theory (**CZF**), founded by Aczel [1, 2, 3], without the powerset axiom and the full separation axiom.

Schuster et al. [19] and Bridges and Vîţă [9] employed a set of entourages with an extra condition to define a uniformity. If the discrete uniformity on $\mathbf{R}$ were defined by a set D of pseudometrics, then there would exist $d_1, \ldots, d_n \in D$ and $\epsilon > 0$ such that $\sum_{k=1}^{n} d_k(x, y) < \epsilon$ implies $x = y$ for each

$x, y \in \mathbf{R}$, and hence we would have the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x, y \in \mathbf{R}[x = y \vee \neg(x = y)],$$

which is provably false both in intuitionistic mathematics and in constructive recursive mathematics. Therefore their approach seems more general than the approach with a set of pseudometrics by Bishop; see also a discussion in [6, Appendix A], and [16]. However their approach for uniform spaces has a problem from a predicative point of view, and the extra condition leads to a phenomenon that we find unsatisfactory: namely, that if the real line, taken with the discrete uniform structure, satisfies it, then one can derive the non-constructive principle WLPO; see [13, Remark 3.1].

In this paper, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in a subsystem $\mathbf{CZF}^-$ of the constructive set theory **CZF**; see [12] for a construction of a completion of a uniform space in terms of formal topology [17, 18].

There are other constructive treatments of uniformity: for example, see [11] for uniform spaces in formal topology; see also [4] for general topology and formal topology in **CZF**.

2 The constructive set theory CZF

The constructive set theory **CZF**, founded by Aczel [1, 2, 3], grew out of Myhill's constructive set theory [15] as a formal system for Bishop's constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory [14].

Definition 1. The language of **CZF** contains variables for sets, a constant ω, and the binary predicates $=$ and $\in$. The axioms and rules are the axioms and rules of intuitionistic predicate logic with equality, and the following set theoretic axioms:

1. **Extensionality:** $\forall a \forall b(\forall x(x \in a \Longleftrightarrow x \in b) \Longrightarrow a = b)$.
2. **Pairing:** $\forall a \forall b \exists c \forall x(x \in c \Longleftrightarrow x = a \vee x = b)$.
3. **Union:** $\forall a \exists b \forall x(x \in b \Longleftrightarrow \exists y \in a(x \in y))$.

4. **Restricted Separation:**

$$\forall a \exists b \forall x (x \in b \Longleftrightarrow x \in a \wedge \varphi(x))$$

for every *restricted* formula $\varphi(x)$, where a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

5. **Strong Collection:**

$$\forall a (\forall x \in a \exists y \varphi(x,y) \Longrightarrow \exists b (\forall x \in a \exists y \in b \varphi(x,y) \wedge \forall y \in b \exists x \in a \varphi(x,y)))$$

for every formula $\varphi(x,y)$.

6. **Subset Collection:**

$$\forall a \forall b \exists c \forall u (\forall x \in a \exists y \in b \varphi(x,y,u) \Longrightarrow \\ \exists d \in c (\forall x \in a \exists y \in d \varphi(x,y,u) \wedge \forall y \in d \exists x \in a \varphi(x,y,u)))$$

for every formula $\varphi(x,y,u)$.

7. **Infinity:**

$$(\omega 1) \qquad 0 \in \omega \wedge \forall x (x \in \omega \Longrightarrow x+1 \in \omega),$$
$$(\omega 2) \qquad \forall y (0 \in y \wedge \forall x (x \in y \Longrightarrow x+1 \in y) \Longrightarrow \omega \subseteq y),$$

where $x+1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset = \{x \in \omega \mid \bot\}$.

8. **$\in$-Induction:**

$$(\mathrm{IND}_{\in}) \qquad \forall a (\forall x \in a \varphi(x) \Longrightarrow \varphi(a)) \Longrightarrow \forall a \varphi(a)$$

for every formula $\varphi(a)$.

A subsystem $\mathbf{CZF}^-$ is obtained by removing $\in$-Induction from $\mathbf{CZF}$. Let a and b be sets. Using Strong Collection, the *cartesian product* $a \times b$ of a and b consisting of the ordered pairs $(x,y) = \{\{x\},\{x,y\}\}$ with $x \in a$ and $y \in b$ can be introduced in $\mathbf{CZF}^-$. A *relation* r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is *total* (or is a *multivalued function*) if for every $x \in a$ there exists $y \in b$ such that $(x,y) \in r$. The class of total relations between a and b is denoted by $\mathrm{mv}(a,b)$, or more formally

$$r \in \mathrm{mv}(a,b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b ((x,y) \in r).$$

A *function* from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x,y) \in f$. The class of functions from a to b is denoted by b^a, or more formally

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a,b) \wedge \forall x \in a \forall y,z \in b((x,y) \in f \wedge (x,z) \in f \Longrightarrow y = z).$$

In **CZF**$^-$, we can prove

Fullness: $\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a,b) \wedge \forall r \in \mathrm{mv}(a,b) \exists s \in c(s \subseteq r))$,

and, as a corollary, we see that b^a is a set, that is

Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \Longleftrightarrow f \in b^a)$.

For more details of **CZF**, see [5].

3 A completion of a uniform space

In this section, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in **CZF**$^-$.

A *uniform space* $(X, \mathcal{U})$ is a pair of a set X and a set $\mathcal{U}$ of subsets of $X \times X$ such that

Ub1. $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U}(W \subseteq U \cap V)$,

Ub2. $\forall U \in \mathcal{U}(\Delta \subseteq U)$,

Ub3. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \subseteq U^{-1})$,

Ub4. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \circ V \subseteq U)$.

Here $\Delta = \{(x,x) \mid x \in X\}$, and $U^{-1} = \{(x,y) \mid (y,x) \in U\}$ and $U \circ V = \{(x,z) \mid \exists y((x,y) \in V \wedge (y,z) \in U)\}$ for each $U, V \subseteq X \times X$. Note that $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. We set $U^0 = \Delta$ and $U^{n+1} = U^n \circ U$.

A uniform space $(X, \mathcal{U})$ is T_1 if

$$\forall x, y \in X[\forall U \in \mathcal{U}((x,y) \in U) \Longrightarrow x = y].$$

Remark 2. Let D be a set of pseudometrics on a set X, and let $\mathcal{U}_D$ be the set of subsets of $X \times X$ of the form

$$U_{d_1,\ldots,d_n}(\epsilon) = \{(x, y) \in X \times X \mid \sum_{k=1}^{n} d_k(x, y) < \epsilon\},$$

where $d_1, \ldots, d_n \in D$ ($n \geq 0$) and $\epsilon > 0$. Then it is straightforward to see that the pair $(X,\mathcal{U}_D)$ forms a uniform space, and it is T_1 if

$$\forall x, y \in X[\forall d \in D(d(x, y) = 0) \Longrightarrow x = y].$$

Especially, for a metric space (X, d), the pair $(X,\mathcal{U}_d)$ forms a T_1 uniform space, where $\mathcal{U}_d = \{U_n \mid n \in \mathbf{N}\}$ and $U_n = \{(x, y) \in X \times X \mid d(x, y) < 2^{-n}\}$.

Let $\ll_n$ be a relation on $\mathcal{U}$ defined by

$$V \ll_n U \Leftrightarrow \exists W \in \mathcal{U}(V \subseteq W \cap W^{-1} \wedge W^n \subseteq U).$$

Lemma 3. *For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \ll_n U$, and if $V \ll_n U$, then $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq U$ for each $k_1, \ldots, k_n \in \{-1, 1\}$.*

Proof. Let $U \in \mathcal{U}$, and let m be a natural number with $n \leq 2^m$. Then, using (Ub4) m times, there exists $W \in \mathcal{U}$ such that $W^{2^m} \subseteq U$, and hence we have $W^n \subseteq W^{2^m} \subseteq U$, by using (Ub2) if necessary. There exists $W' \in \mathcal{U}$ such that $W' \subseteq W^{-1}$, by (Ub3), and hence there exists $V \in \mathcal{U}$ such that $V \subseteq W \cap W' \subseteq W \cap W^{-1}$, by (Ub1). If $V \ll_n U$, then there exists $W \in \mathcal{U}$ such that $V \subseteq W \cap W^{-1}$ and $W^n \subseteq U$, and therefore, since $V^k \subseteq W$ for each $k \in \{-1, 1\}$, we have $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq W^n \subseteq U$ for each $k_1, \ldots, k_n \in \{-1, 1\}$. □

A set $\mathcal{F}$ of subsets of X is a *filter* if

Fb1. $\forall A \in \mathcal{F} \exists x \in X(x \in A)$,

Fb2. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F}(C \subseteq A \cap B)$.

A filter $\mathcal{F}$ on X *converges* to x in X if for each $U \in \mathcal{U}$ there exists $A \in \mathcal{F}$ such that $A \subseteq U(x) = \{y \in X \mid (x, y) \in U\}$. A filter $\mathcal{F}$ on X is a *Cauchy filter* if

FbC. $\forall U \in \mathcal{U} \exists A \in \mathcal{F}(A \times A \subseteq U)$.

A uniform space $(X,\mathcal{U})$ is *complete* if every Cauchy filter on X converges.

Let $(X,\mathcal{U})$ be a T_1 uniform space. Then, since X and $\mathcal{U}$ are sets, by Fullness, there exists a set R such that $R \subseteq \mathrm{mv}(\mathcal{U}, X)$ and

$$\forall r \in \mathrm{mv}(\mathcal{U}, X) \exists s \in R(s \subseteq r). \tag{1}$$

Let φ be a restricted formula defined by

$$\varphi(r) \Leftrightarrow \forall U, V \in \mathcal{U} \forall x, y \in X[(U,x) \in r \wedge (V,y) \in r \Longrightarrow (x,y) \in V^{-1} \circ U].$$

Note that

$$\varphi(r) \wedge s \subseteq r \Longrightarrow \varphi(s). \tag{2}$$

Using Restricted separation, define a set $\widetilde{X}$ by

$$\widetilde{X} = \{r \in R \mid \varphi(r)\}.$$

For each $U \in \mathcal{U}$, define a subset $\widetilde{U}$ of $\widetilde{X} \times \widetilde{X}$, using Restricted Separation, as follows:

$$\begin{aligned}\widetilde{U} = \{(r,s) \mid \exists U_1, U_2 \in \mathcal{U} \exists x_1, x_2 \in X (&U_1 \subseteq U \wedge U_2 \subseteq U \wedge \\ &(U_1, x_1) \in r \wedge (U_2, x_2) \in s \wedge (x_1, x_2) \in U)\}.\end{aligned}$$

By Strong Collection, let

$$\widetilde{\mathcal{U}} = \{\widetilde{U} \mid U \in \mathcal{U}\}.$$

The equality $=_{\widetilde{X}}$ on $\widetilde{X}$ is defined by

$$r =_{\widetilde{X}} s \Leftrightarrow \forall \widetilde{U} \in \widetilde{\mathcal{U}}((r,s) \in \widetilde{U}).$$

Remark 4. We may think of a multivalued function $r \in \mathrm{mv}(\mathcal{U}, X)$ as a multivalued *net* in X indexed by the directed set $\mathcal{U}$, and the formula $\varphi(r)$ as expressing a *regularity* of r. Then the set $\widetilde{X}$ is a set of regular multivalued nets in X indexed by the specific directed set $\mathcal{U}$; a similar trick can be found in the proof that the class of points of a complete uniform formal topology is a set in [11]. If $\mathcal{U}$ is countable, then, in the presence of the axiom of countable choice, we may define $\widetilde{X}$ as the set of regular *sequences* (singlevalued functions on $\mathbf{N}$) in X. In the uniform space $(X, \mathcal{U}_d)$ induced by a metric space (X,d), each regular sequence $(x_n)_n$ in $(X, \mathcal{U}_d)$ is a regular sequence in the metric space (X, d) in the sense that

$$d(x_m, x_n) < 2^{-m} + 2^{-n}$$

for each $m, n \in \mathbf{N}$. On the other hand, for each regular sequence $(x_n)_n$ in (X, d), the sequence $(x_{n+1})_n$ is a regular sequence in $(X, \mathcal{U}_d)$.

Proposition 5. *$(\widetilde{X}, \widetilde{\mathcal{U}})$ is a* T_1 *uniform space.*

Proof. (Ub1): Let $U, V \in \mathcal{U}$. Then there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$, and it is straightforward to see that $\widetilde{W} \subseteq \widetilde{U} \cap \widetilde{V}$.

(Ub2): Let $U \in \mathcal{U}$ and $r \in \widetilde{X}$. Then, since $r \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(U, x) \in r$, and therefore, since $(x, x) \in U$, we have $(r, r) \in \widetilde{U}$.

(Ub3): Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \subseteq U^{-1}$, and it is straightforward to see that $\widetilde{V} \subseteq \widetilde{U}^{-1}$.

(Ub4): Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_4 U$, by Lemma 3. Let $(r, s) \in \widetilde{V}$ and $(s, t) \in \widetilde{V}$. Then there exist $V_1, V_2, W_1, W_2 \in \mathcal{U}$ and $x_1, x_2, y_1, y_2 \in X$ such that $V_1, V_2, W_1, W_2 \subseteq V$, $(V_1, x_1) \in r$, $(V_2, x_2) \in s$, $(W_1, y_1) \in s$, $(W_2, y_2) \in t$, $(x_1, x_2) \in V$ and $(y_1, y_2) \in V$. Since $(V_2, x_2) \in s$, $(W_1, y_1) \in s$ and $\varphi(s)$, we have $(x_2, y_1) \in W_1^{-1} \circ V_2$, and hence

$$(x_1, y_2) \in V \circ W_1^{-1} \circ V_2 \circ V \subseteq V \circ V^{-1} \circ V \circ V \subseteq U,$$

by Lemma 3. Therefore, since $V_1, W_2 \subseteq V \subseteq U$, we have $(r, t) \in \widetilde{U}$.

The uniform space $(\widetilde{X}, \widetilde{\mathcal{U}})$ is T_1 by the definition of equality. □

Let $\mathcal{F}$ be a Cauchy filter on $\widetilde{X}$. Define a subset r of $\mathcal{U} \times X$, by Restricted Separation, as follows:

$$r = \{(U, x) \mid \exists V \in \mathcal{U} \exists A \in \mathcal{F} \exists s \in A (V \ll_4 U \wedge A \times A \subseteq \widetilde{V} \wedge (V, x) \in s)\}.$$

Lemma 6. *$r \in \mathrm{mv}(\mathcal{U}, X)$ and $\varphi(r)$.*

Proof. Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_4 U$, by Lemma 3. Since $\mathcal{F}$ is a Cauchy filter, there exists $A \in \mathcal{F}$ such that $A \times A \subseteq \widetilde{V}$, by (FbC), and hence there exists $s \in A$, by (Fb1). Since $s \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(V, x) \in s$, and hence $(U, x) \in r$. Therefore $r \in \mathrm{mv}(\mathcal{U}, X)$.

Let $(U, x) \in r$ and $(V, y) \in r$. Then there exist $U_0, V_0 \in \mathcal{U}$, $A, B \in \mathcal{F}$, $s \in A$ and $s' \in B$ such that $U_0 \ll_4 U$, $V_0 \ll_4 V$, $A \times A \subseteq \widetilde{U_0}$, $B \times B \subseteq \widetilde{V_0}$, $(U_0, x) \in s$ and $(V_0, y) \in s'$. Since $\mathcal{F}$ is a filter, there exist $C \in \mathcal{F}$ and $t \in C$ such that $t \in C \subseteq A \cap B$, by (Fb2) and (Fb1). Since $(s, t) \in \widetilde{U_0}$ and $(s', t) \in \widetilde{V_0}$, there exist $U_1, U_2, V_1, V_2 \in \mathcal{U}$ and $x_1, x_2, y_1, y_2 \in X$ such that $U_1, U_2 \subseteq U_0$, $V_1, V_2 \subseteq V_0$, $(U_1, x_1) \in s$, $(U_2, x_2) \in t$, $(V_1, y_1) \in s'$, $(V_2, y_2) \in t$, $(x_1, x_2) \in U_0$ and $(y_1, y_2) \in V_0$. Since $(U_0, x), (U_1, x_1) \in s$, $(V_1, y_1), (V_0, y) \in s'$ and $(U_2, x_2), (V_2, y_2) \in t$, we have $(x, x_1) \in U_1^{-1} \circ U_0$, $(y_1, y) \in V_0^{-1} \circ V_1$, and $(x_2, y_2) \in V_2^{-1} \circ U_2$, and hence

$$
\begin{aligned}
(x, y) &\in V_0^{-1} \circ V_1 \circ V_0^{-1} \circ V_2^{-1} \circ U_2 \circ U_0 \circ U_1^{-1} \circ U_0 \\
&\subseteq (V_0 \circ V_0 \circ V_0^{-1} \circ V_0)^{-1} \circ (U_0 \circ U_0 \circ U_0^{-1} \circ U_0) \subseteq V^{-1} \circ U,
\end{aligned}
$$

by Lemma 3. Therefore $\varphi(r)$. □

By (1), there exists $r_{\mathcal{F}} \in R$ such that $r_{\mathcal{F}} \subseteq r$. Since $\varphi(r_{\mathcal{F}})$, by Lemma 6 and (2), we have $r_{\mathcal{F}} \in \widetilde{X}$.

Lemma 7. *$\mathcal{F}$ converges to $r_{\mathcal{F}}$.*

Proof. Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_2 U$, and there exists $W \in \mathcal{U}$ such that $W \ll_4 V$, by Lemma 3. Since $\mathcal{F}$ is a Cauchy filter, there exists $A \in \mathcal{F}$ such that $A \times A \subseteq \widetilde{W}$. Let $s \in A$. Since $s \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(W, x) \in s$, and hence $(V, x) \in r$. Since $r_{\mathcal{F}} \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x' \in X$ such that $(V, x') \in r_{\mathcal{F}} \subseteq r$. Since $\varphi(r)$ by Lemma 6, we have $(x', x) \in V^{-1} \circ V \subseteq U$, and therefore, since $V, W \subseteq U$, we have $(r_{\mathcal{F}}, s) \in \widetilde{U}$. Thus $A \subseteq \widetilde{U}(r_{\mathcal{F}})$. □

Thus we have the following proposition.

Proposition 8. *$(\widetilde{X}, \widetilde{\mathcal{U}})$ is complete.*

For each $x \in X$, define a subset $\tilde{x}$ of $\mathcal{U} \times X$ by

$$\tilde{x} = \{(U, x) \mid U \in \mathcal{U}\}.$$

Then $\tilde{x}$ is a constant function on $\mathcal{U}$, and, since for each $(U, x), (V, x) \in \tilde{x}$, we have $(x, x) \in V^{-1} \circ U$, we have $\tilde{x} \in \widetilde{X}$.

Lemma 9. *For each $U \in \mathcal{U}$ and $x, y \in X$, $(x, y) \in U$ if and only if $(\tilde{x}, \tilde{y}) \in \widetilde{U}$.*

Proof. Since $(U, x) \in \tilde{x}$ and $(U, y) \in \tilde{y}$, if $(x, y) \in U$, then $(\tilde{x}, \tilde{y}) \in \widetilde{U}$. If $(\tilde{x}, \tilde{y}) \in \widetilde{U}$, then there exist $V, W \in \mathcal{U}$ such that $V, W \subseteq U$, $(V, x) \in \tilde{x}$, $(W, y) \in \tilde{y}$ and $(x, y) \in U$, and so $(x, y) \in U$. □

A mapping f between uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ is *uniformly continuous* if for each $V \in \mathcal{U}'$ there exists $U \in \mathcal{U}$ such that

$$(x, y) \in U \Longrightarrow (f(x), f(y)) \in V$$

for each $x, y \in X$.

Let i be the mapping from $(X, \mathcal{U})$ into $(\widetilde{X}, \widetilde{\mathcal{U}})$ such that

$$i : x \mapsto \tilde{x}.$$

Thus, by Lemma 9, we immediately have the following proposition.

Proposition 10. *$i : (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$ is a uniformly continuous injection.*

Let $(Y, \mathcal{V})$ be a complete T_1 uniform space, and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly continuous. Let $r \in \widetilde{X}$. For each $U \in \mathcal{U}$, define a subset A^r_U of Y by

$$A^r_U = \{f(x) \mid \exists V \in \mathcal{U}(V \subseteq U \wedge (V, x) \in r)\}.$$

By Strong Collection, let

$$\mathcal{F}_r = \{A^r_U \mid U \in \mathcal{U}\}.$$

Lemma 11. *$\mathcal{F}_r$ is a Cauchy filter on Y.*

Proof. For each $U \in \mathcal{U}$, since $(U, x) \in r$ for some $x \in X$, we have $f(x) \in A^r_U$. Since for each $U, V \in \mathcal{U}$ if $V \subseteq U$, then $A^r_V \subseteq A^r_U$, we have for each $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $A^r_W \subseteq A^r_U \cap A^r_V$ by (Ub1). Let $U \in \mathcal{V}$. Then, since f is uniformly continuous, there exists $V \in \mathcal{U}$ such that $(x, y) \in V \Longrightarrow (f(x), f(y)) \in U$ for each $x, y \in X$, and there exists $W \in \mathcal{U}$ such that $W \ll_2 V$. Suppose that $(f(x), f(y)) \in A^r_W \times A^r_W$. Then there exists $W_1, W_2 \in \mathcal{U}$ such that $W_1, W_2 \subseteq W$, $(W_1, x) \in r$ and $(W_2, y) \in r$, and hence $(x, y) \in W_2^{-1} \circ W_1 \subseteq W^{-1} \circ W \subseteq V$. Thus $(f(x), f(y)) \in U$. Therefore $A^r_W \times A^r_W \subseteq U$. □

Since $(Y, \mathcal{V})$ is complete, $\mathcal{F}_r$ converges to a point $\tilde{f}(r)$ in Y.

Lemma 12. *For each $U \in \mathcal{V}$ there exists $V \in \mathcal{U}$ such that*

$$(r, s) \in \widetilde{V} \Longrightarrow (\tilde{f}(r), \tilde{f}(s)) \in U$$

for each $r, s \in \widetilde{X}$.

Proof. Let $U \in \mathcal{V}$. Then there exists $U_0 \in \mathcal{V}$ such that $U_0 \ll_3 U$, and, since f is uniformly continuous, there exists $V_0 \in \mathcal{U}$ such that $(x, y) \in V_0 \Longrightarrow (f(x), f(y)) \in U_0$ for each $x, y \in X$. By Lemma 3, there exists $V \in \mathcal{U}$ such that $V \ll_5 V_0$. Suppose that $(r, s) \in \widetilde{V}$. Then there exist $V_1, V_2 \in \mathcal{U}$ and $x_1, x_2 \in X$ such that $V_1, V_2 \subseteq V$, $(V_1, x_1) \in r$, $(V_2, x_2) \in s$ and $(x_1, x_2) \in V$. Since $\mathcal{F}_r$ and $\mathcal{F}_s$ converge to $\tilde{f}(r)$ and $\tilde{f}(s)$, respectively, we can find $W \in \mathcal{U}$ such that $W \subseteq V$, $A^r_W \subseteq U_0(\tilde{f}(r))$ and $A^s_W \subseteq U_0(\tilde{f}(s))$, and, since A^r_W and A^s_W are inhabited, there exist $x, y \in X$ such that $f(x) \in A^r_W$ and $f(y) \in A^s_W$. Hence there exist $W_1, W_2 \in \mathcal{U}$ such that $W_1, W_2 \subseteq W$, $(W_1, x) \in r$ and $(W_2, y) \in s$. Since $(W_1, x), (V_1, x_1) \in r$ and $(V_2, x_2), (W_2, y) \in s$, we have $(x, x_1) \in V_1^{-1} \circ W_1$ and $(x_2, y) \in W_2^{-1} \circ V_2$, and therefore

$$\begin{aligned}(x, y) &\in W_2^{-1} \circ V_2 \circ V \circ V_1^{-1} \circ W_1 \subseteq W^{-1} \circ V \circ V \circ V^{-1} \circ W \\ &\subseteq V^{-1} \circ V \circ V \circ V^{-1} \circ V \subseteq V_0.\end{aligned}$$

Thus $(f(x), f(y)) \in U_0$. Since $(\tilde{f}(r), f(x)) \in U_0$ and $(\tilde{f}(s), f(y)) \in U_0$, we have $(\tilde{f}(r), \tilde{f}(s)) \in U_0^{-1} \circ U_0 \circ U_0 \subseteq U$. ☐

Since $(Y, \mathcal{V})$ is T_1, we have $\tilde{f}(r) = \tilde{f}(s)$ whenever $r =_{\widetilde{X}} s$, by Lemma 12. Hence $\tilde{f}$ is a function on $\widetilde{X}$, and it is uniformly continuous, by Lemma 12. Since $A^{\tilde{x}}_U = \{f(x)\}$ for each $U \in \mathcal{U}$, $\mathcal{F}_{\tilde{x}}$ converges to $f(x)$, and therefore we have the following lemma.

Lemma 13. $f = \tilde{f} \circ i$.

The function $\tilde{f}$ is unique in the following sense.

Lemma 14. *If $h : (\widetilde{X}, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ is uniformly continuous with $f = h \circ i$, then $h = \tilde{f}$.*

Proof. Let $r \in \widetilde{X}$, and let $U \in \mathcal{V}$. Then there exists $U_0 \in \mathcal{V}$ such that $U_0 \ll_2 U$, and since h is uniformly continuous, there exists $V \in \mathcal{U}$ such that $(s, t) \in \widetilde{V} \Longrightarrow (h(s), h(t)) \in U_0$ for each $s, t \in \widetilde{X}$. Since $\mathcal{F}_r$ converges to $\tilde{f}(r)$, we can find $W \in \mathcal{U}$ such that $W \subseteq V$ and $A^r_W \subseteq U_0(\tilde{f}(r))$, and, since A^r_W is inhabited, there exists $x \in X$ such that $f(x) \in A^r_W$. Hence there exists $W' \in \mathcal{U}$ such that $W' \subseteq W \subseteq V$ and $(W', x) \in r$, and therefore, since $(V, x) \in \tilde{x}$ and $(x, x) \in V$, we have $(r, \tilde{x}) \in \widetilde{V}$. Thus $(h(r), h(\tilde{x})) = (h(r), f(x)) \in U_0$. Since $(\tilde{f}(r), f(x)) \in U_0$, we have $(h(r), \tilde{f}(r)) \in U_0^{-1} \circ U_0 \subseteq U$. Therefore, since $(Y, \mathcal{V})$ is T_1, we have $h(r) = \tilde{f}(r)$. □

Now we have shown the following theorem.

Theorem 15. *Let $(Y, \mathcal{V})$ be a complete T_1 uniform space, and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly continuous. Then there exists a unique uniformly continuous $\tilde{f} : (\widetilde{X}, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ such that $f = \tilde{f} \circ i$.*

Remark 16. Let $\mathcal{F}$ be a Cauchy filter on a uniform space $(X, \mathcal{U})$, and let

$$r = \{(U, x) \mid \exists A \in \mathcal{F}(A \times A \subseteq U \wedge x \in A)\}.$$

Then $r \in \mathrm{mv}(\mathcal{U}, X)$ and $\varphi(r)$, and hence there exists $r_{\mathcal{F}} \in \widetilde{X}$ such that $r_{\mathcal{F}} \subseteq r$. On the other hand, for each $r \in \widetilde{X}$, let $\mathcal{F}_r = \{B^r_U \mid U \in \mathcal{U}\}$, where

$$B^r_U = \{x \mid \exists V \in \mathcal{U}(V \subseteq U \wedge (V, x) \in r)\}.$$

Then $\mathcal{F}_r$ is a Cauchy filter on $(X, \mathcal{U})$. In the presence of the powerset axiom, it is straightforward to show that these correspondences $r \mapsto \mathcal{F}_r$ and $\mathcal{F} \mapsto r_{\mathcal{F}}$ between the completion $(\widetilde{X}, \widetilde{\mathcal{U}})$ and the uniform space of the set of all Cauchy filters on $(X, \mathcal{U})$ (the completion of $(X, \mathcal{U})$ in the sense of Bishop) form a uniform isomorphism.

For a metric space (X, d), as mentioned in Remark 4, in the presence of the axiom of countable choice, there is a uniform isomorphism between the completion $(\widetilde{X}, \widetilde{\mathcal{U}_d})$ and Bishop's metric completion.

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