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The uniform boundedness theorem and a boundedness principle

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Abstract

We deal with a form of the uniform boundedness theorem (or the Banach-Steinhaus theorem) for topological vector spaces in Bishop's constructive mathematics, and show that the form is equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics.

Keywords: constructive mathematics, topological vector space, the uniform boundedness theorem, boundedness principle.
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1 Introduction

The notion of a topological vector space, as a generalization of the notion of a normed space, is a very important notion to investigate function spaces, such as the space of test functions, which do not form Banach spaces (see, for example, [23]). Nonetheless little investigation on topological vector spaces has been done in Bishop's constructive mathematics [6, 7, 8, 10]; see also a discussion in [6, Appendix A]. We can only find, noting that a topological vector space is a uniform space, a constructive concept of a uniform space with a set of pseudometrics, and basic theorems, such as, that arbitrary uniform space has a completion, in [6, Problems 17 to 21 of Chapter 4]; see also [7, Problems 22 to 26 of Chapter 4], and [9, 24, 12, 11, 17, 5] for other constructive treatments of a uniform space.

However, using the notion of a neighbourhood space [6, 3.3] (see also [7, 3.3]) introduced by Bishop, we can naturally define a notion of a topological vector space in Bishop's constructive mathematics as follow.

A *neighbourhood space* is a pair (X, τ) consisting of a set X and a set τ of subsets of X such that

NS1. $\forall x \in X \exists U \in \tau (x \in U)$,

NS2. $\forall x \in X \forall U, V \in \tau [x \in U \cap V \implies \exists W \in \tau (x \in W \subseteq U \cap V)]$.

The set τ is an *open base* on X, and an element of τ is a *basic open set*. A subset of X is *open* if it is a union of basic open sets. A *neighbourhood* of a point $x \in X$ is a subset $A \subseteq X$ such that $x \in U \subseteq A$ for some $U \in \tau$. An open base σ on X is *compatible* with τ if each neighbourhood in σ is a neighbourhood in τ, and vice versa. An open base is *compatible with a metric* d if it is compatible with the open base induced by open balls. A function f between neighbourhood spaces (X, τ) and (Y, σ) is *continuous* if $f^{-1}(V)$ is open for each $V \in \sigma$.

A *topological vector space* is a vector space E equipped with an open base τ such that the vector space operations, addition $(x, y) \mapsto x + y$ and scalar multiplication $(a, x) \mapsto ax$, are continuous, that is, if U is a neighbourhood of $x + y$, then there exist neighbourhoods V and V' of x and y, respectively, such that $V + V' = \{v + v' \mid v \in V, v' \in V'\} \subseteq U$, and if U is a neighbourhood of ax, then for some $\delta > 0$ and some neighbourhood V of x we have $bV = \{bv \mid v \in V\} \subseteq U$ whenever $|a - b| < \delta$. It is *metrizable* if τ is compatible with some metric d, and is an *F-space* if its open base τ is compatible with a complete invariant metric d. Here a metric d on a vector space E is *invariant* if $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in E$.

In this paper, we deal with the following form of the uniform boundedness theorem (or the Banach-Steinhaus theorem) for topological vector spaces [23, 2.6] in Bishop's constructive mathematics.

The Uniform Boundedness Theorem. *If $(T_m)_m$ is a sequence of continuous linear mappings from an F-space E into a topological vector space F such that the set*

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded in F for each $x \in E$, then $(T_m)_m$ is equicontinuous.

Here a subset A of a topological vector space E is *bounded* if for each neighbourhood V of 0 in E there exists a positive integer K such that $A \subseteq tV$ for each $t \geq K$, and a set Γ of continuous linear mappings between topological vector spaces E and F is *equicontinuous* if for each neighbourhood V of 0 in F there exists a neighbourhood U of 0 in E such that $T(U) \subseteq V$ for each $T \in \Gamma$.

We know that a (contrapositive) form of the uniform boundedness theorem for normed spaces has a constructive proof [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]), and a corollary [23, Theorem 2.8] of the uniform boundedness theorem for a sequence of *sequentially continuous* linear mappings from a separable Banach space into a normed space holds constructively [14, Theorem 7]. However, the corollary for a sequence of *continuous* linear mappings not only implies, but also is equivalent to the following boundedness principle (BD-N) [15, Theorem 21].

BD-N. Every pseudobounded countable subset of $\mathbf{N}$ is bounded.

Here a subset S of $\mathbf{N}$ is *countable* if it is a range of $\mathbf{N}$, *pseudobounded* if $\lim_{n\to\infty} s_n/n = 0$ for each sequence $(s_n)_n$ in S, and *bounded* if there exists a positive integer K such that $s < K$ for each $s \in S$; see [13, 18, 22] for pseudobounded sets.

The boundedness principle BD-N is equivalent to the statement "every sequentially continuous mapping from a separable metric space into a metric space is continuous" [13, Theorem 4], is derivable in intuitionistic mathematics with a continuity principle [13, Proposition 3] and in constructive recursive mathematics with Church's thesis and Markov's principle [13, Proposition 4], and is not provable in $\mathbf{HA}^\omega$ with axiom of choice for all finite types [19].

In the following, we show that the uniform boundedness theorem for topological vector spaces with a sequence of continuous linear mappings is also equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics [16, 20, 25].

Although the result is presented in informal Bishop-style constructive mathematics, it is possible to formalize it in constructive Zermero-Fraenkel set theory (**CZF**), founded by Aczel [1, 2, 3], with the dependent choice axiom (DC), which permits a quite natural interpretation in Martin-Löf type theory [21]. Note that the axiom of countable choice (AC_ω) follows from the dependent choice axiom in **CZF**; see [4, Section 8].

2 The main results

A topological vector space E is *separated* if for each neighbourhood U of 0 there exist a neighbourhood V of 0 and an open set W such that $E = U \cup W$ and $V \cap W = \emptyset$.

Each topological vector space E whose open base is *compatible with a set* $\{d_i \mid i \in I\}$ *of pseudometrics*, that is, compatible with the open base consisting of the sets $B_{i_1,\ldots,i_n}(x, \epsilon) = \{y \in E \mid \sum_{k=1}^{n} d_{i_k}(x, y) < \epsilon\}$, is separated. In fact, for each neighbourhood U of 0, there exist $i_1, \ldots, i_n \in I$ and $\epsilon > 0$ such that $B_{i_1,\ldots,i_n}(0, \epsilon) \subseteq U$, and hence, taking $V = B_{i_1,\ldots,i_n}(0, \epsilon/2)$ and $W = \{y \in E \mid \epsilon/2 < \sum_{k=1}^{n} d_{i_k}(0, y)\}$, we have $E = U \cup W$ and $V \cap W = \emptyset$.

On the other hand, suppose that a vector space E with the discrete topology is separated. Then, since $\{0\}$ is a neighbourhood of 0, there exist a neighbourhood V of 0 and an open set W such that $E = \{0\} \cup W$ and $V \cap W = \emptyset$. Hence for each $x \in E$, either $x \in \{0\}$ or $x \in W$: in the former case, we have $x = 0$; in the latter case, we have $\neg(x = 0)$. If $E = \mathbf{R}$, then this is equivalent to the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x \in \mathbf{R}[x = 0 \vee \neg(x = 0)].$$

Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot find out whether the topological vector space $\mathbf{R}$ with the discrete topology is separated.

A subset A of a topological vector space E is *unbounded* if there exits a neighbourhood V of 0 in E such that for each positive integer k there exist $t \geq k$ and $x \in A$ such that $x \notin tV$.

The following theorem generalizes the constructive version of the uniform boundedness theorem [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]) to topological vector spaces.

Theorem 1. *Let $(T_n)_n$ be a sequence of continuous linear mappings from an F-space E into a separated topological vector space F. If there exists a bounded sequence $(x_n)_n$ in E such that $\{T_n x_n \mid n \in \mathbf{N}\}$ is unbounded, then $\{T_n x \mid n \in \mathbf{N}\}$ is unbounded for some $x \in E$.*

Proof. Suppose that $(x_n)_n$ is a bounded sequence in E such that $\{T_n x_n \mid n \in \mathbf{N}\}$ is unbounded. Then there exists a neighbourhood V_0 of 0 in F such that for each k, $T_n x_n \notin tV_0$ for some $t \geq k$ and n. Since $(a, x) \mapsto ax$ is continuous at $(0, 0)$, there exist N and a neighbourhood V_1 of 0 in F such that $a'V_1 \subseteq V_0$

for each a' with $|a'| \leq 1/N$. Furthermore, since $(x, y) \mapsto x - y$ is continuous at $(0, 0)$, there exists a neighbourhood V_2 of 0 in F such that $V_2 - V_2 \subseteq V_1$. Since F is separated, there exist a neighbourhood V_3 of 0 and an open set W in F such that $F = V_2 \cup W$ and $V_3 \cap W = \emptyset$.

For each $m \geq 1$, define a subset G_m of E by

$$G_m = \{x \in E \mid T_n x \in mW \text{ for some } n\}.$$

Then G_m is open. Let $y \in E$ and let U be a neighbourhood of y in E. Then, since the addition is continuous, there exists a neighbourhood U' of 0 in E such that $y + U' = \{y + u \mid u \in U'\} \subseteq U$. Since $(x_n)_n$ is bounded, there exists k_0 such that $x_n \in k_0 U'$ for all n. Set $k = mk_0 N$. Then there exist $t \geq k$ and n such that $T_n x_n \notin tV_0$. Either $T_n y/m \in W$ or $T_n y/m \in V_2$. In the former case, setting $z = y$, we have $z \in G_m \cap U$. In the latter case, if $T_n z/m \in V_2$ where $z = y + x_n/k_0$, then, since $mk_0/t \leq mk_0/k = 1/N$, we have

$$\frac{T_n x_n}{t} = \frac{mk_0}{t}\left(\frac{T_n z}{m} - \frac{T_n y}{m}\right) \in \frac{mk_0}{t}(V_2 - V_2) \subseteq \frac{mk_0}{t} V_1 \subseteq V_0,$$

a contradiction; whence $T_n z/m \in W$, and therefore, since $z \in y + U' \subseteq U$, we have $z \in G_m \cap U$. Thus G_m is dense in E.

By applying the constructive version of Baire's theorem [6, Thorem 4 of Chapter 4] (see also [7, Theorem 3.9 of Chapter 4] and [8, Theorem 1.3 of Chapter 2]), we can find a point $x \in E$ such that $x \in G_m$ for all m, that is, for each m there exists n such that $T_n x \notin mV_3$. Therefore $\{T_n x \mid n \in \mathbf{N}\}$ is unbounded. □

We will need the following general lemma later.

Lemma 2. *Each convergent sequence in a topological vector space is bounded.*

Proof. Let $(x_n)_n$ be a sequence in a topological vector space E converging to a limit x in E, and let V be a neighbourhood of 0 in E. Since $(a, x) \mapsto ax$ is continuous at $(0, 0)$, there exist a positive integer M and a neighbourhood V_0 such that $a'V_0 \subseteq V$ for each a' with $|a'| \leq 1/M$. Since $(x_n)_n$ converges to x, there exists a positive integer N such that $x_n - x \in V_0$ for each $n \geq N$. Note that for each $y \in E$, since $a \mapsto ay$ is continuous at 0, there exists a positive integer M' such that $y/M' \in V_0$. Then there exist $M', M'_1, \ldots, M'_{N-1}$ such that $x/M' \in V_0$ and $x_n/M'_n \in V_0$ for $n = 1, \ldots, N-1$. For each $n \geq N$, since $x_n = (x_n - x) + x \in V_0 + M'V_0 \subseteq (M' + 1)V_0$, we have

$x_n/(M'+1) \in V_0$. Let $K = M \max\{M'+1, M'_1, \ldots, M'_{N-1}\}$, and let $t \geq K$. Then for each n, if $n \geq N$, then, since $t^{-1}(M'+1) \leq 1/M$, we have $t^{-1}x_n = (t^{-1}(M'+1))(x_n/(M'+1)) \in V$; or else $n < N$ and, since $t^{-1}M'_n \leq 1/M$, we have $t^{-1}x_n = (t^{-1}M'_n)(x_n/M'_n) \in V$. Therefore $x_n \in tV$ for each n. □

Let E be a separable F-space with a dense sequence $(z_i)_i$, and F be a separated topological vector space. Let $(T_m)_m$ be a sequence of continuous linear mappings from E into F such that for each $x \in E$ its orbit $\{T_m x \mid m \in \mathbf{N}\}$ is bounded. Then for each neighbourhood V of 0 in F, since F is separated, there exist a neighbourhood V_0 of 0 and an open set W_0 such that $F = V \cup W_0$ and $V_0 \cap W_0 = \emptyset$, and there exist a neighbourhood V_1 of 0 and an open set W_1 such that $F = V_0 \cup W_1$ and $V_1 \cap W_1 = \emptyset$. Construct a binary triple sequence α such that

$$\begin{aligned} \alpha(i,m,k) = 0 \quad &\Longrightarrow \quad 1/(k+1)^2 < d(0,z_i) \vee T_m z_i \in V_0 \\ \alpha(i,m,k) = 1 \quad &\Longrightarrow \quad d(0,z_i) < 1/k^2 \wedge T_m z_i \in W_1, \end{aligned}$$

and define a countable subset S of $\mathbf{N}$ by

$$S = \{k \mid \alpha(i,m,k) = 1 \text{ for some } i \text{ and } m\} \cup \{0\}.$$

Lemma 3. *For each sequence $(s_n)_n$ in S, either $s_n < n$ for all n or $s_n \geq n$ for some n.*

Proof. Let $(s_n)_n$ be a sequence in S, and construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 \quad &\Longrightarrow \quad s_{n'} < n' \text{ for all } n' \leq n \\ \lambda_n = 1 \quad &\Longrightarrow \quad s_{n'} \geq n' \text{ for some } n' \leq n. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define a sequence $(x_n)_n$ in E as follow: if $\lambda_n = 0$, then set $x_n = 0$; if $\lambda_n = 1 - \lambda_{n-1}$, then pick i and m such that $d(0,z_i) < 1/s_n^2 \leq 1/n^2$ and $T_m z_i \in W_1$, and set $x_j = nz_i$ for all $j \geq n$ (and note that $d(0,x_j) = d(0,nz_i) \leq \sum_{k=1}^{n} d((k-1)z_i, kz_i) = nd(0,z_i) < 1/n$). Then (x_n) is a Cauchy sequence, and hence converges to a limit x in E. Since $\{T_m x \mid m \in \mathbf{N}\}$ is bounded, there exists a positive integer K such that $\{T_m x \mid m \in \mathbf{N}\} \subseteq tV_1$ for each $t \geq K$. If $\lambda_n = 1 - \lambda_{n-1}$ for some $n \geq K$, then there exist i and m such that $x = nz_i$ and $T_m z_i \in W_1$, and hence $T_m x \notin nV_1$, a contradiction. Therefore $\lambda_n = \lambda_{n-1}$ for all $n \geq K$, and so either $\lambda_n = 1$ for some n or $\lambda_n = 0$ for all n. □

Lemma 4. *For each sequence $(s_n)_n$ in S, either $s_n < n$ for all sufficiently large n or $s_n \geq n$ for infinitely many n.*

Proof. Let $(s_n)_n$ be a sequence in S, and, by applying Lemma 3 to subsequences $(s_{n'})_{n' \geq n}$, construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 &\quad \Longrightarrow \quad s_{n'} \geq n' \text{ for some } n' \geq n \\ \lambda_n = 1 &\quad \Longrightarrow \quad s_{n'} < n' \text{ for all } n' \geq n. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define a sequence $(x_n)_n$ in E as follow: if $\lambda_n = 0$, then pick $n' \geq n$, i and m such that $d(0, z_i) < 1/s_{n'}^2 \leq 1/n^2$ and $T_m z_i \in W_1$, and set $x_n = n z_i$ (and note that $d(0, x_n) < 1/n$); if $\lambda_n = 1$, then set $x_n = x_{n-1}$. Then (x_n) is a Cauchy sequence, and hence converges to a limit x in E. Since $\{T_m x \mid m \in \mathbf{N}\}$ is bounded, there exists a positive integer K such that $\{T_m x \mid m \in \mathbf{N}\} \subseteq t V_1$ for each $t \geq K$. If $\lambda_n = 1 - \lambda_{n-1}$ for some $n > K$, then there exist i and m such that $x = (n-1) z_i$ and $T_m z_i \in W_1$, and hence $T_m x \notin (n-1) V_1$, a contradiction. Therefore $\lambda_n = \lambda_{n-1}$ for all $n > K$, and so either $\lambda_n = 1$ for some n or $\lambda_n = 0$ for all n. □

Proposition 5. *The set S is pseudobounded.*

Proof. By [18, Lemma 3], it suffices to show that for each sequence $(s_n)_n$ in S, $s_n < n$ for all sufficiently large n. Let $(s_n)_n$ be a sequence in S. Then, by Lemma 4, either $s_n < n$ for all sufficiently large n or $s_n \geq n$ for infinitely many n. Suppose that $s_n \geq n$ for infinitely many n. Then for each n there exists $n' \geq n$ such that $s_{n'} \geq n'$, and hence there exist i_n and m_n such that $d(0, z_{i_n}) < 1/s_{n'}^2 \leq 1/n^2$ and $T_{m_n} z_{i_n} \in W_1$. Therefore $n T_{m_n} z_{i_n} \notin n V_1$ for each n, and so $\{T_{m_n}(n z_{i_n}) \mid n \in \mathbf{N}\}$ is unbounded. Since $(n z_{i_n})_n$ converges to 0, it is bounded, by Lemma 2, and hence $\{T_{m_n} x \mid n \in \mathbf{N}\}$ is unbounded for some $x \in E$, by Theorem 1. This contradicts with the fact that $\{T_m x \mid m \in \mathbf{N}\}$ is bounded for all $x \in E$. Therefore $s_n < n$ for all sufficiently large n. □

Suppose that S is bounded, that is, there exists a positive integer K such that $k < K$ for each $k \in S$. Let $x \in U = \{x \in E \mid d(0, x) < 1/(K+1)^2\}$ and assume that $T_m x \in W_0$ for some m. Then, since T_m is continuous and W_0 is open, there exists i such that $d(0, z_i) < 1/(K+1)^2$ and $T_m z_i \in W_0$, and hence $T_m z_i \notin V_0$. Therefore $\alpha(i, m, K)$ must be 1, and so $K \in S$, a contradiction. Thus $T_m x \in V$ for each $x \in U$ and $m \in \mathbf{N}$.

We have shown the following theorem.

Theorem 6. *Assume* BD-N*. If $(T_m)_m$ is a sequence of continuous linear mappings from a separable F-space E into a separated topological vector space F such that the set*

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded for each $x \in E$, then $(T_m)_m$ is equicontinuous.

Let $(T_m)_m$ be a sequence of continuous linear mappings from a separable F-space E into a separated topological vector space F such that the limit

$$Tx = \lim_{m \to \infty} T_m x$$

exists for each $x \in E$. Then for each $x \in E$ its orbit $\{T_m x \mid m \in \mathbf{N}\}$ is bounded, by Lemma 2. Therefore, by applying the uniform boundedness theorem, $(T_m)_m$ is equicontinuous.

Let V be a neighbourhood of 0 in F. Then, since $(x, y) \mapsto x - y$ is continuous at $(0, 0)$, there exists a neighbourhood W of 0 in F such that $W - W \subseteq V$. Note that $\overline{W} \subseteq V$: in fact, if $x \in \overline{W}$, then there exist $y, z \in W$ such that $x + y = z$, and hence $x = z - y \in W - W \subseteq V$. Since $(T_m)_m$ is equicontinuous, there exists a neighbourhood U of 0 in E such that $T_m(U) \subseteq W$ for each m, and hence $T(U) \subseteq \overline{W} \subseteq V$. Therefore (being obviously linear) T is continuous.

Let $S = \{s_n \mid n \in \mathbf{N}\}$ be a pseudobounded countable subset of $\mathbf{N}$, and define a sequence $(T_m)_m$ of continuous linear mappings from the Hilbert space l_2 of square summable sequences into itself by

$$T_m x = \sum_{n=1}^{m} s_n \langle x, e_n \rangle e_n,$$

where $(e_n)_n$ is an orthonormal basis of l_2. Then, since S is pseudobounded, as in the proof of [15, Lemma 20], we can show that the limit

$$Tx = \lim_{m \to \infty} T_m x = \sum_{n=1}^{\infty} s_n \langle x, e_n \rangle e_n$$

exists for each $x \in l_2$. If T is continuous, then, since $s_n = \|Te_n\|$ for each n, we see that S is bounded.

Thus we have the following theorem.

Theorem 7. *The following are equivalent*

1. BD-N.

2. *If* $(T_m)_m$ *is a sequence of continuous linear mappings from a separable* F*-space* E *into a separated topological vector space* F *such that the set*

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded for each $x \in E$*, then* $(T_m)_m$ *is equicontinuous.*

3. *If* $(T_m)_m$ *is a sequence of continuous linear mappings from a separable* F*-space* E *into a separated topological vector space* F *such that*

$$Tx = \lim_{m\to\infty} T_m x$$

exists for each $x \in E$*, then* T *is continuous.*

Remark 8. The sequence $(T_m)_m$ of continuous linear mappings from l_2 into itself constructed from a countable subset S of $\mathbf{N}$ before Theorem 7 is a sequence of *compact self-adjoint* operators on l_2. Therefore BD-N actually follows from the weaker statement: If $(T_m)_m$ is a sequence of compact self-adjoint operators on l_2 such that $Tx = \lim_{m\to\infty} T_m x$ exists for each $x \in l_2$, then T is continuous.

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