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# A Semantic Investigation of Orthologic and Orthomodular Logic

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## Abstract

The present paper is divided into two parts. In Part I, we show that weak orthologic has the finite model property, and hence that it is decidable. In part II, we construct a new semantics for orthomodularlogic by using Rickart * semigroups, and prove the completeness theorem of orthomodular logic with respect to this semantics.

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# 0 Introduction

When G.Birkhoff and J.von Neumann [1] formulated the logical aspect of the theory of quantum mechanics in 1936, investigations of completely new logical system started under the name of 'quantum logic'. They pointed out that according to the Hilbert-space formalism of quantum mechanics, the physical propositions of a quantum system, which are represented by the closed subspaces of the Hilbert space, form an orthomodular lattice. After their work, algebraic semantics, that is, lattice-theoretic models of quantum logic have been mainly discussed in this area.

In both orthologic and orthomodular logic, in which logical formulas are interpreted by elements of ortholattices and orthomodular lattices respectively, two different notions of logical consequences are considered. Their corresponding logics are called weak logic and strong logic.

The notion of weak consequence was first appeared in 1974 in G.Kalmbach's paper [7]. In the paper, she discussed the definability of implication connectives in weak orthomodular logic, and proved that the deduction theorem does not hold for the logic which has the implication connective defined by her. Later J.Malinowski [9] showed in 1990 that no weak orthomodular logic admits the deduction theorem.

H.Dishkant [2] gave a Kripke-style semantics for orthologic in 1972, that is, the set of all formulas which hold true in all algebraic semantics (ortholattices) equals the set of all formulas which hold true in all his Kripke-style models. In developing this Kripke-style semantics, R.I.Goldblatt [4] treated orthologic as a binary logic, and introduced the notion of strong consequence. By applying filtration technique to his Kripke-style semantics for strong orthologic, he showed in 1974 that it has the finite model property, and hence that it is decidable. In the same paper, he proposed a Kripke-style semantics for strong orthomodular logic, which is given by some restriction of that for strong orthologic, but the decision problem for strong orthomodular logic still remains open. In 1992, J.Malinowski [10] tried to compare several aspects of weak logic and strong logic with each other and proved that the deduction theorem also fails in strong orthologic. In the same paper, he mentioned that the decision problem for weak orthologic was also open.

There is another type of semantics for quantum logic besides lattice-theoretic semantics and Kripke-style semantics, which are stated above. That is a dialog-game semantics which was developed by P.Mittelstaedt [11] and E.W.Stachow [14].

On the other hand, syntactical methods are also useful in quantum logic. In 1980 H.Nishimura [12] translated the formal system of Goldblatt's binary logic for orthologic into a sequential one, like Gentzen's LK or LJ. The cut-elimination theorem does not hold in his system, but several interesting theorems have been proved by using his system from a syntactical point of view. S.Tamura [15] gave a sequential formal system without cut-rule for strong orthologic in 1988, and showed that strong orthologic is decidable by following Gentzen's method. However this technique does not work for strong orthomodular logic.

Goldblatt [5] proved in 1984 that there is no elementary condition on the orthogonality relation that characterises the orthomodular law. Here the orthogonarity relation is an irreflexive and symmetric relation which plays an important role in the Kripke-style semantics for strong orthologic. That seems the reason why orthomodular logic is so intractable.

In study of orthomodular lattice, D.J.Foulis [3] gave in 1960 the representation theorem for orthomodular lattices with a particular kind of semigroups. Moreover, in 1966 M.F.Janowitz [6] extended Foulis's representation theorem to bounded lattices. ( See also [8]). Their representaion theorems may be useful in constructing models for non-classical logics.

In Part I of the present paper, we will try to extend the Kripke-style semantics of Goldblatt to weak orthologic and will show that it is also decidable. In Part II, we will propose a new semantics, which is based on Foulis's represantation theorem and show that the strong orthomodular logic is complete with respect to this semantics.

We will give here some basic notions which will be used in both Part I and Part II.
The language L of our logics consists of :

(i) a countable collection $\{ p_i \mid i < \omega \}$ of propositional variables,

(ii) the connectives $\neg$ and $\wedge$ of negation and conjunction,

(iii) parentheses ( and ).

The set $\Phi$ of formulas of L is defined in the usual way. That is, $\Phi$ is the minimum set which satisfies the following three conditions:

(i) for every $i < \omega$, $p_i \in \Phi$,

(ii) if $\alpha \in \Phi$, then $(\neg\alpha) \in \Phi$,

(iii) if $\alpha, \beta \in \Phi$, then $(\alpha \wedge \beta) \in \Phi$.

The letters $\alpha, \beta$, etc. are used as metavariables ranging over $\Phi$. Parentheses may be omitted by the convention that $\neg$ binds strongly than $\wedge$. The disjunction $\alpha \vee \beta$ of $\alpha$ and $\beta$ can be introduced as the abbreviation of $\neg(\neg\alpha \wedge \neg\beta)$.

In this paper, we will consider logics mainly from a semantical point of view. In order to define orthologics and orthomodular logics, we will use semantical interpretations of our formulas into corresponding lattices, rather than syntactical formal systems. So, at first we introduce the notions of ortholattices and orthomodular lattices.

**Definition 0.1 (Ortholattice and orthomodular lattice)** An *ortholattice* $\mathcal{A}$ is a structure $\langle$ A, $\leq$, $\sqcap$, $\sqcup$, $^\perp$, $\mathbf{1}$, $\mathbf{0}$ $\rangle$, which satisfies the following conditions:

(i) $\langle$ A, $\leq$, $\sqcap$, $\sqcup$, $\mathbf{1}$, $\mathbf{0}$ $\rangle$ is a lattice with $\mathbf{1}$(maximum) and $\mathbf{0}$(minimum). We denote, for any $x, y \in$ A, $\quad x \sqcap y :=$ inf $\{x, y\}$, $\quad x \sqcup y :=$ sup $\{x, y\}$.

(ii) The unary operation $^\perp$ (*orthocomplement*) satisfies the following conditions, (a), (b) and (c): for any $x, y \in$ A,

(a) $x \sqcap x^\perp = \mathbf{0}$

(b) $x^{\perp\perp} = x$

(c) $x \leq y \quad$ implies $\quad y^\perp \leq x^\perp$

It is easy to see that $x \sqcup y = (x^{\perp} \sqcap y^{\perp})^{\perp}$ holds in any ortholattice.

An *orthomodular lattice* $\mathcal{A} = \langle$ A, $\leq$, $\sqcap$, $\sqcup$, $^{\perp}$, $\mathbf{1}$, $\mathbf{0}$ $\rangle$ is an ortholattice which also satisfies the following condition (d).

(d) $\quad x \leq y \quad$ implies $\quad y = x \sqcup (x^{\perp} \sqcap y)$

We call the condition (d) the *orthomodular law.* ∎

It is well-known that 1): every modular ortholattice, that is, an ortholattice satisfying the *modular law* :

$$\text{For any } x, y, z \in \text{A}, \quad x \leq y \quad \text{implies} \quad x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$$

is an orthomodular lattice, but the converse does not always hold. Also, it can be shown that 2): every distributive ortholattice is a modular ortholattice, but the converse does not always hold and that 3): a lattice A is a distributive ortholattice iff it is a Boolean algebra.

Based on these lattice-theoretic notions, we will introduce a semantics, by which we can interpret formulas in L.

**Definition 0.2 (Valuation)** A valuation is a function $v$, which associates with any formula $\alpha \in \Phi$ an element $v(\alpha)$ in an ortholattice $\mathcal{A}$ (or an orthomodular lattice), and satisfies the following conditions:
for any formula $\alpha$, $\beta$,

(i) $\quad v(\neg\alpha) = (v(\alpha))^{\perp}$

(ii) $\quad v(\alpha \wedge \beta) = v(\alpha) \sqcap v(\beta)$

When $\mathcal{A}$ is an ortholattice, we call this $v$ an *orthovaluation*, and when $\mathcal{A}$ is an orthomodular lattice, an *orthomodular valuation.* ∎

It is easy to see that for any valuation $v$ and for any formula $\alpha$, the value $v(\alpha)$ is uniquely determined by the values $v(p_i)$ for propositional variables $p_i$ appearing in $\alpha$.

By using these concepts defined above, we will introduce orthologic and orthomodular logic.

**Definition 0.3 (Orthologic and orthomodular logic)** The *orthologic* OL is the set of pairs of formulas $(\alpha, \beta)$ satisfying the following conditions: for any ortholattice $\mathcal{A}$ and for any orthovaluation $v$ from $\Phi$ to A, $\quad v(\alpha) \leq v(\beta)$.

Similarly, the *orthomodular logic* OML is the set of pairs of formulas $(\alpha, \beta)$ satisfying the following conditions: for any orthomodular lattice $\mathcal{A}$ and for any orthomodular valuation $v$ from $\Phi$ to A, $\quad v(\alpha) \leq v(\beta)$. ∎

To say strictly, the word 'logic' in the above definition means 'strong logic'. But we will often say 'orthologic' or 'orthomodular logic' simply in this sense when no confusion will occur.

# Part I

# Semantics of weak orthologic

In this part I, we concentrate our considerations only on orthologics – weak one and strong one, and we make a distinction between these two notions explicitly.

## 1 Weak orthologic

**Definition 1.1** **(Weak orthologic)** The weak orthologic WOL is the set of pairs of formulas $(\alpha, \beta)$, such that these $\alpha$ and $\beta$ satisfy the following: for any ortholattice $\mathcal{A}$ and for any orthovaluation $v$ from $\Phi$ to A of this $\mathcal{A}$, if $v(\alpha) = \mathbf{1}$, then $v(\beta) = \mathbf{1}$. We use the symbol $\alpha \models_w \beta$ in place of $(\alpha, \beta) \in$ WOL. ∎

We denote here the strong orthologic introduced in Definition 0.3 with SOL explicitly and we use the symbol $\alpha \models_s \beta$ in place of $(\alpha, \beta) \in$ SOL. It is easily seen that if $\alpha \models_s \beta$, then $\alpha \models_w \beta$. So SOL is a subset of WOL. The next result can be shown without difficulty, but this is one of the key observations to show the decidability of weak orthologic, which is the main result of Part I.

**Proposition 1.2** For any formulas $\alpha$ and $\beta$, the following two conditions are equivalent.

(i) $\alpha \models_w \beta$.

(ii) For any ortholattice $\mathcal{A}$, and for any orthovaluation $v : \Phi \to$ A, if $v(\neg(\chi \wedge \neg\chi)) \leq v(\alpha)$, then $v(\neg(\chi \wedge \neg\chi)) \leq v(\beta)$, for any fixed formula $\chi$.

This proposition follows from the fact that for any ortholattice $\mathcal{A}$ and for any orthovaluation $v$, $v(\neg(\chi \wedge \neg\chi)) = \mathbf{1}$ always holds for every formula $\chi$. From this proposition, we will get the suitable semantics of weak orthologic, if for any formulas $\alpha$, $\beta$, $\sigma$, and $\tau$, we have a complete interpretation of the following statement (P):

(P): For any ortholattice $\mathcal{A}$ and any orthovaluation $v : \Phi \to$ A, if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.

We will see that we can have a complete interpretation of (P) by using the same Kripke-style semantics for strong orthologic introduced by Goldblatt in [4]. We will prove the completeness theorem for weak orthologic with respect to that semantics, and show the decidability of weak orthologic by proving the finite model property in almost the same way as in [4].

## 2 Completeness for weak orthologic

We will introduce the semantics which was used to show the completeness and the finite model property of strong orthologic. Here we intend to show the completeness of weak orthologic by using the same semantics.

**Definition 2.1 (Orthoframe)** $\mathcal{F} = \langle \mathrm{X}, \perp \rangle$ is an *orthoframe* if it satisfies the following conditions:

(i) X is a non-empty set called the *carrier* of $\mathcal{F}$.

(ii) $\perp$ is an irreflexive, symmetric binary relation on X. This relation $\perp$ is called an *orthogonality* relation.

∎

**Definition 2.2 (Some notions on orthogonality relation)** Let $\mathcal{F} = \langle \mathrm{X}, \perp \rangle$ be an orthoframe.

(i) For $x, y \in \mathrm{X}$, $x$ is said to be *orthogonal* to $y$ iff $x \perp y$ holds.

(ii) For $x \in \mathrm{X}$ and for $\mathrm{Y} \subseteq \mathrm{X}$, if for any $y \in \mathrm{Y}$ $(x \perp y)$ holds, then $x$ is said to be orthogonal to the subset Y, and this relation is denoted as $x \perp \mathrm{Y}$.

(iii) For $\mathrm{Y} \subseteq \mathrm{X}$, Y is $\perp$-*closed* iff the following condition holds:

$$\forall x \in \mathrm{X}\, [\forall y \in \mathrm{X}\, (y \perp \mathrm{Y} \Rightarrow x \perp y) \implies x \in \mathrm{Y}]$$

∎

We can show the next lemma on the orthogonality relation.

**Lemma 2.3 (Properties of orthogonality relation)** Let $\mathcal{F} = \langle \mathrm{X}, \perp \rangle$ be an orthoframe. Then the following holds.

(i) X and $\emptyset$ are $\perp$-closed.

(ii) If P, Q ($\subseteq$ X) are $\perp$-closed,
then the sets $\mathrm{P} \cap \mathrm{Q}$ and $\mathrm{P}^{\perp} = \{\, x \in \mathrm{X} \mid x \perp \mathrm{P} \}$ are also $\perp$-closed.

**Proof** :

(i) (a) It is obvious that X is $\perp$-closed.

(b) By the definition, for any $x \in \mathrm{X}$, $x \in \emptyset$ does not hold. So, in order to prove that $\emptyset$ is $\perp$-closed, it is enough to show that for every $x \in \mathrm{X}$, there exists some $y \in \mathrm{X}$ such that $y \perp \emptyset$ holds but that $x \perp y$ does not hold. Take this $x$ for $y$. It is obviuos that $x \perp \emptyset$, and by the irreflexivity of relation $\perp$, we have that $x \perp x$ does not hold. Consequently we have shown that $\emptyset$ is $\perp$-closed.

(ii) (a) $\mathrm{P} \cap \mathrm{Q}$: Suppose that $\forall y \in \mathrm{X}(y \perp (\mathrm{P} \cap \mathrm{Q}) \Rightarrow x \perp y)$ holds for an $x$ in X. We suppose that $x \notin \mathrm{P}$. Since P is $\perp$-closed, there exists some $z$ in X such that $z \perp \mathrm{P}$ holds and that $(x \perp z)$ does not hold. Take this $z$ for $y$. Then from the facts that $z \perp \mathrm{P}$ and that $\mathrm{P} \cap \mathrm{Q} \subseteq \mathrm{P}$, we have that $z \perp (\mathrm{P} \cap \mathrm{Q})$. So by the supposition, $x \perp z$ must hold. But this is a contradiction. Therefore we have that $x \in \mathrm{P}$. Similarly we have that $x \in \mathrm{Q}$. Hence $x \in \mathrm{P} \cap \mathrm{Q}$. So we have shown that $\mathrm{P} \cap \mathrm{Q}$ is $\perp$-closed.

(b) $P^{\perp}$: Suppose that $\forall y \in X(y \perp P^{\perp} \Rightarrow x \perp y)$ for an $x$ in X. We suppose that $x$ is not in $P^{\perp}$. Then there exists some $z$ in P such that $x \perp z$ does not hold. Since $z \perp u$ holds for any $u \in P^{\perp}$, we have that $z \perp P^{\perp}$. Take this $z$ for $y$. Then from the supposition $x \perp z$ must hold. But this is a contradiction. Therefore we have that $x \in P^{\perp}$. Consequently we have proved that $P^{\perp}$ is $\perp$-closed.

□

Now we introduce orthomodels and the truth conditons on them.

**Definition 2.4 (Orthomodel)** $\mathcal{M} = \langle X, \perp, V \rangle$ is an *orthomodel* on the frame $\mathcal{F} = \langle X, \perp \rangle$ iff V is a function assigning to each propositional variable $p_i$ a $\perp$-closed subset $V(p_i)$ of X.

The notion of truth in orthomodels is defined inductively as follows: The symbol '$\mathcal{M} \models_x \alpha$' is read as "$\alpha$ is true at $x$ in $\mathcal{M}$".

(i) $\mathcal{M} \models_x p_i$ iff $x \in V(p_i)$,

(ii) $\mathcal{M} \models_x \alpha \wedge \beta$ iff $\mathcal{M} \models_x \alpha$ and $\mathcal{M} \models_x \beta$,

(iii) $\mathcal{M} \models_x \neg\alpha$ iff for any $y \in X$, ($\mathcal{M} \models_y \alpha$ only if $x \perp y$ ).

∎

For each formula $\alpha$, define $\|\alpha\|^{\mathcal{M}} := \{x \in X \mid \mathcal{M} \models_x \alpha\}$. Then we can restate the above conditions in the following way.

(i) $\|p_i\|^{\mathcal{M}} = V(p_i)$,

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$,

(iii) $\|\neg\alpha\|^{\mathcal{M}} = \{x \in X \mid x \perp \|\alpha\|^{\mathcal{M}}\}$.

We need one more notion, that is, the notion of '$\alpha$ implies $\beta$' in an orthomodel.

**Definition 2.5** Let $\alpha$ and $\beta$ be formulas.

(i) $\alpha$ *implies* $\beta$ at $x$ in a model $\mathcal{M}$ ( $\mathcal{M} : \alpha \models_x \beta$ ) iff either $\mathcal{M} \models_x \alpha$ does not hold or else $\mathcal{M} \models_x \beta$ holds.

(ii) $\alpha$ *implies* $\beta$ in a model $\mathcal{M}$ ( $\mathcal{M} : \alpha \models \beta$ ) iff for all $x$ in the model $\mathcal{M}$, $\mathcal{M} : \alpha \models_x \beta$ holds.

∎

It is easily to see that $\mathcal{M} : \alpha \models \beta$ is equivalent to $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$. In order to prove the completeness of weak orthologic with respect to the semantics defined above, it is enough to show that for any formulas $\alpha$, $\beta$, $\sigma$ and $\tau$, the previous statement (P) is equivalent to the follwing statement (Q).

(Q): For any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.

Indeed, for a given ortholattice and a given orthovaluation we can construct the corresponding orthomodel. Conversly, for a given orthomodel we can build up the corresponding ortholattice and orthovaluation. From now on, we will show how to construct, and prove the completeness.

**Lemma 2.6 (From frame to lattice)** For a given orthoframe $\mathcal{F} = \langle \mathrm{X}, \perp \rangle$, let $\mathrm{A}_{\mathcal{M}} = \{ \mathrm{Y} \mid \mathrm{Y} \text{ is a } \perp\text{-closed subset of X} \}$. Then $\mathcal{A}_{\mathcal{M}} = \langle \mathrm{A}_{\mathcal{M}}, \subseteq, \cap, \cup, ^{\perp}, \emptyset, \mathrm{X} \rangle$ is an ortholattice, where $\subseteq$, $\cap$ and $\cup$ mean set inclusion, intersection and union respectively, and $^{\perp}$ is the operation stated in Lemma 2.3, that is, for a $\perp$-closed subset Y of X, $\mathrm{Y}^{\perp} := \{x \in \mathrm{X} | x \perp \mathrm{Y}\}$.

**Proof** : By Lemma 2.3, we have already shown that $\emptyset, \mathrm{X} \in \mathrm{A}_{\mathcal{M}}$ and that they are the minimun and the maximum with respect to the set inclusion in $\mathrm{A}_{\mathcal{M}}$ respectively. By the same lemma we also have that $\mathrm{A}_{\mathcal{M}}$ is closed under the operations $\cap$ and $^{\perp}$. So we have only to check whether the operation $^{\perp}$ satisfies the conditions for orthocomplement (a), (b) and (c) in (ii) of the Definition 0.1, that is, the following holds. For any $\perp$-closed subsets P, Q,
(a) $\mathrm{P} \cap \mathrm{P}^{\perp} = \emptyset$,
(b) $\mathrm{P}^{\perp\perp} = \mathrm{P}$,
(c) If $\mathrm{P} \subseteq \mathrm{Q}$, then $\mathrm{Q}^{\perp} \subseteq \mathrm{P}^{\perp}$.

Proof of (a): Suppose that $\mathrm{P} \cap \mathrm{P}^{\perp}$ is not empty. Then we take $x$ in $\mathrm{P} \cap \mathrm{P}^{\perp}$. Since $x \in \mathrm{P}^{\perp}$, $x \perp y$ holds for any $y$ in P. Then we have that $x \perp x$ because $x \in \mathrm{P}$, which contradicts the the irreflexivity of the relation $\perp$.
Proof of (b): Take any $x$ in $\mathrm{P}^{\perp\perp}$. Then $x \perp y$ holds for any $y$ in $\mathrm{P}^{\perp}$. Now suppose that $x \notin \mathrm{P}$. Then, from the fact that P is $\perp$-closed, there exists some $z$ in X such that $z \perp \mathrm{P}$ holds and that $x \perp z$ does not hold. Of course $z$ is in $\mathrm{P}^{\perp}$, thus we can take this $z$ for $y$. Then $x \perp z$ must hold, but this is a contradiction. Therefore we have that $x$ is in P, and then that $\mathrm{P}^{\perp\perp} \subseteq \mathrm{P}$.
Conversely, take any $x$ in P. Then $x \perp y$ holds for any $y$ in $\mathrm{P}^{\perp}$. Now suppose that $x \notin \mathrm{P}^{\perp\perp}$. Then from the fact that $\mathrm{P}^{\perp\perp}$ is $\perp$-closed, there exists some $z$ in X such that $z \perp \mathrm{P}^{\perp\perp}$ holds and that $x \perp z$ does not hold. Here for any $u$ in X, if $u \perp \mathrm{P}^{\perp}$, then $u \in \mathrm{P}^{\perp\perp}$ and thus $z \perp u$ holds. Because $\mathrm{P}^{\perp}$ is $\perp$-closed, $z \in \mathrm{P}^{\perp}$. Thus we can take this $z$ for $y$, and then $x \perp z$ must hold. But this is a contradiction. Therefore we have that $x$ is in $\mathrm{P}^{\perp\perp}$, and then that $\mathrm{P} \subseteq \mathrm{P}^{\perp\perp}$.
Proof of (c): Suppose that $\mathrm{P} \subseteq \mathrm{Q}$. Take any $x$ in $\mathrm{Q}^{\perp}$. Then $x \perp y$ holds for any $y$ in Q. Now we suppose that $x \notin \mathrm{P}^{\perp}$. Then there exists some $z$ in P such that $x \perp z$ does not hold. This $z$ is in Q since $\mathrm{P} \subseteq \mathrm{Q}$. Let us take $z$ for $y$. Then $x \perp z$ must hold. Contradiction. So we have that $x$ is in $\mathrm{P}^{\perp}$. Therefore we have $\mathrm{Q}^{\perp} \subseteq \mathrm{P}^{\perp}$. □

The above lemma shows how to construct the corresponding ortholattice from a given orthoframe. Further, we can construct the suitable orthovaluation from a given orthomodel, that is, an orthoframe with a function V.

**Corollary 2.7** Let $\mathcal{M} = \langle \mathrm{X}, \perp, \mathrm{V} \rangle$ be an orthomodel, and $\mathcal{A}_{\mathcal{M}}$ the corresponding ortholattice defined in Lemma 2.6 . Then $\|\alpha\|^{\mathcal{M}} = \{x \in \mathrm{X} \mid \mathcal{M} \models_x \alpha\}$ is $\perp$-closed for any formula $\alpha$.
**Proof** : From Lemma 2.3 and Definition 2.4, we can prove this lemma easily by the induction on the construction of the formula $\alpha$. □

**Theorem 2.8** Let $\mathcal{M} = \langle X, \perp, V \rangle$ be an orthomodel.

(i) Let $v_{\mathcal{M}} : \Phi \to A_{\mathcal{M}}$ be an orthovaluation, the values of which is defined only for the case of propositional variables as follows: for any propositional variable $p_i$, $v_{\mathcal{M}}(p_i) := \|p_i\|^{\mathcal{M}}$. Then it turns out that $v_{\mathcal{M}}(\alpha) = \|\alpha\|^{\mathcal{M}}$ holds or every formula $\alpha$.

(ii) For any formulas $\alpha$ and $\beta$, the following two conditions are equivalent.
(1): $\mathcal{M} : \alpha \models \beta$ (2) : $v_{\mathcal{M}}(\alpha) \subseteq v_{\mathcal{M}}(\beta)$.

**Proof** : Proof of (i): By Lemma 2.7 it is assured that the valuation $v_{\mathcal{M}}$ is well-defined. To prove (i) of this theorem, we use the induction on the construction of the formula $\alpha$.

(a) Step $\alpha = p_i$: Trivial from the definition.

(b) Step $\alpha = \sigma \wedge \tau$: By induction hypothesis, we have that $v_{\mathcal{M}}(\sigma) = \|\sigma\|^{\mathcal{M}}$, and that $v_{\mathcal{M}}(\tau) = \|\tau\|^{\mathcal{M}}$. From these facts together with the properties of the orthovaluation $v_{\mathcal{M}}$ and the truth condition of the orthomodel $\mathcal{M}$, we can derive,
$v_{\mathcal{M}}(\sigma \wedge \tau) = v_{\mathcal{M}}(\sigma) \cap v_{\mathcal{M}}(\tau) = \|\sigma\|^{\mathcal{M}} \cap \|\tau\|^{\mathcal{M}} = \|\sigma \wedge \tau\|^{\mathcal{M}}$.

(c) Step $\alpha = \neg\sigma$: By induction hypothesis, $v_{\mathcal{M}}(\sigma) = \|\sigma\|^{\mathcal{M}}$. From this, we can derive,
$v_{\mathcal{M}}(\neg\sigma) = (\|\sigma\|^{\mathcal{M}})^{\perp} = \{x \in X \mid x \perp \|\sigma\|^{\mathcal{M}}\} = \|\neg\sigma\|^{\mathcal{M}}$.

Proof of (ii): First we show the direction ((1) $\Rightarrow$ (2)). Suppose $\mathcal{M} : \alpha \models \beta$ holds. This means that for any $x$ in $\mathcal{M}$, either $\mathcal{M} \not\models_x \alpha$ or $\mathcal{M} \models_x \beta$. So we have that $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$, which shows that $v_{\mathcal{M}}(\alpha) \subseteq v_{\mathcal{M}}(\beta)$.
To show the converse direction ((1) $\Leftarrow$ (2)), suppose that $\mathcal{M} : \alpha \models \beta$ does not hold. This means that there exists some $y$ in $\mathcal{M}$ such that $\mathcal{M} \models_y \alpha$ and $\mathcal{M} \not\models_y \beta$. So we have that $\|\alpha\|^{\mathcal{M}} \not\subseteq \|\beta\|^{\mathcal{M}}$, which shows that $v_{\mathcal{M}}(\alpha) \not\subseteq v_{\mathcal{M}}(\beta)$.
Consequently we have proved that these two conditions are equivalent. □

**Corollary 2.9** ( **(P)⇒(Q) (Soundness)** ) For given formulas $\alpha$, $\beta$, $\sigma$ and $\tau$, let (P) and (Q) be the statements as follows:
(P): For any ortholattice $\mathcal{A}$ and for any orthovaluation $v : \Phi \to A$,
if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.
(Q): For any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.
Then (P) implies (Q).

**Proof** : Suppose that (Q) does not hold. Then there exists an orthomodel $\mathcal{M} = \langle X, \perp, V \rangle$ such that $\mathcal{M} : \sigma \models \alpha$ holds and that $\mathcal{M} : \tau \models \beta$ does not hold. Now we consider the ortholattice $\mathcal{A}_{\mathcal{M}}$ and the orthovaluation $v_{\mathcal{M}}$ which are determined by the orthomodel $\mathcal{M}$ as described in Lemma 2.6 and Theorem 2.8 . Then by Theorem 2.8, we have that $v_{\mathcal{M}}(\sigma) \subseteq v_{\mathcal{M}}(\alpha)$ and that $v_{\mathcal{M}}(\tau) \not\subseteq v_{\mathcal{M}}(\beta)$. So we conclude that (P) does not hold. □

We have already done half of our work. To finish it, we will next show the way to build the suitable orthomodel for a given ortholattice and a given orthovaluation.

**Lemma 2.10 (From lattice to frame)** For a given ortholattice $\mathcal{A} = \langle$ A, $\leq$, $\sqcap$, $\sqcup$, $^{\perp}$, **1**, **0** $\rangle$, let $X_{\mathcal{A}}$ :={ F | F is a proper filter of A } and let $\perp_{\mathcal{A}}$ be such a binary relation that for any F, G $\in X_{\mathcal{A}}$, F$\perp_{\mathcal{A}}$G iff $\exists a \in$ A ( $a^{\perp} \in$ F and $a \in$ G ). Then $\mathcal{F}_{\mathcal{A}} = \langle X_{\mathcal{A}}, \perp_{\mathcal{A}} \rangle$ is the corresponding orthoframe to $\mathcal{A}$.

**Proof** : We have to check the conditions of Definition 2.1 .

(i) Let $F_0 = \{\mathbf{1}\}$. Then $F_0$ is a proper filter. So $X_{\mathcal{A}}$ is not empty.

(ii) We show that the relation $\perp_{\mathcal{A}}$ is irreflexive and symmetric. First suppose that F$\perp_{\mathcal{A}}$F for a filter F $\in X_{\mathcal{A}}$. Then there exists some element $a$, such that $a^{\perp} \in$ F and $a \in$ F. Since F is a filter, we have that F $\ni a^{\perp} \sqcap a = \mathbf{0}$. This contradicts to the fact that F is proper. So F$\perp_{\mathcal{A}}$F does not hold for any F and thus $\perp_{\mathcal{A}}$ is irreflexive.
Second, suppose that F$\perp_{\mathcal{A}}$G for F, G $\in X_{\mathcal{A}}$. Then there exists some element $a$, such that $a^{\perp} \in$ F and $a \in$ G. We put $b$:= $a^{\perp}$. Then $b^{\perp} = a^{\perp\perp} = a$. Now this $b$ satisfies that $b^{\perp} \in$ G and $b \in$ F. Therefore we can conclude that G$\perp_{\mathcal{A}}$F, and hence $\perp_{\mathcal{A}}$ is symmetric.

□

By the above lemma, we can construct the orthoframe which is correspond to a given ortholattice. Moreover, when an ortholattice and an orthovaluation are given to us, we can build up the suitable orthomodel out of them in the following way.

**Lemma 2.11** Let $\mathcal{A}$ and $v$ be a given ortholattice and a given orthovaluation and let $V_{\mathcal{A}}(p_i)$:= { F | F is a proper filter of A and $v(p_i) \in$ F } for any propositional variable $p_i$. Then $\mathcal{M}_{\mathcal{A}} = \langle X_{\mathcal{A}}, \perp_{\mathcal{A}}, V_{\mathcal{A}} \rangle$ is an orthomodel, where $\langle X_{\mathcal{A}}, \perp_{\mathcal{A}} \rangle$ is the orthoframe described in the above lemma.

**Proof** : According to Definition 2.2 and 2.4, we have to prove that $V_{\mathcal{A}}(p_i)$ is $\perp_{\mathcal{A}}$-closed.
(a) Case $v(p_i) = \mathbf{0}$: $V_{\mathcal{A}}(p_i) = \emptyset$ is obviously $\perp_{\mathcal{A}}$-closed.
(b) Case $v(p_i) = \mathbf{1}$: $V_{\mathcal{A}}(p_i) = X_{\mathcal{A}}$ is also obviously $\perp_{\mathcal{A}}$-closed.
(c) Case $v(p_i) \neq \mathbf{0}, \mathbf{1}$: We consider any element F in $X_{\mathcal{A}}$ and suppose that for any element G $\in X_{\mathcal{A}}$, [G$\perp_{\mathcal{A}}V_{\mathcal{A}}(p_i) \Longrightarrow$ F$\perp_{\mathcal{A}}$G] holds. Let $G_0 = \{ x \in A \mid x \geq v(p_i)^{\perp} \}$. Then, obviously $G_0$ is a proper filter of A. For any H $\in V_{\mathcal{A}}(p_i)$, $G_0\perp_{\mathcal{A}}$H holds, since $v(p_i)^{\perp} \in G_0$ and $v(p_i) \in$ H. Therefore from our supposition, we have that F$\perp_{\mathcal{A}}G_0$, which says that there exists such an element $a$ in A that $a^{\perp} \in$ F and $a \in G_0$. From the fact that $a \in G_0$, we have that $a \geq v(p_i)^{\perp}$, and thus $a^{\perp} \leq (v(p_i))^{\perp\perp} = v(p_i) \in$ F. So we have shown that F$\in V_{\mathcal{A}}(p_i)$. Hence $V_{\mathcal{A}}(p_i)$ is $\perp_{\mathcal{A}}$-closed. □

We call this $\mathcal{M}_{\mathcal{A}}$ the *canonical orthomodel* for $\mathcal{A}$ and $v$. Because $\mathcal{M}_{\mathcal{A}}$ is proved to be an orthomodel, we can express the truth conditions in the following way by using the notation $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} := \{F \in X_{\mathcal{A}} \mid \mathcal{M}_{\mathcal{A}} \models_F \alpha\}$.

(i) $\|p_i\|^{\mathcal{M}_{\mathcal{A}}} = V_{\mathcal{A}}(p_i)$.

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}_{\mathcal{A}}} = \|\alpha\|^{\mathcal{M}_{\mathcal{A}}} \cap \|\beta\|^{\mathcal{M}_{\mathcal{A}}}$.

(iii) $\|\neg\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \{F \in X_{\mathcal{A}} \mid F\perp\|\alpha\|^{\mathcal{M}_{\mathcal{A}}}\}$.

**Theorem 2.12** Let $\mathcal{A}$ be an ortholattice and $v : \Phi \to \mathrm{A}$ an orthovaluation and $\mathcal{M}_\mathcal{A} = \langle \mathrm{X}_\mathcal{A}, \perp_\mathcal{A}, \mathrm{V}_\mathcal{A} \rangle$ the canonical orthomodel for $\mathcal{A}$ and $v$.

(i) For every formula $\alpha$, $\|\alpha\|^{\mathcal{M}_\mathcal{A}} = \{\ \mathrm{F} \mid \mathrm{F}$ is a proper filter and $v(\alpha) \in \mathrm{F}\ \}$ holds.

(ii) For any formulas $\alpha$ and $\beta$, the following two conditions are equivalent.
(1): $v(\alpha) \leq v(\beta)$. (2) : $\mathcal{M}_\mathcal{A} : \alpha \models \beta$.

**Proof** : Proof of (i): We use the induction on the construction of the formula $\alpha$.

(a) Step $\alpha = p_i$: Trivial by the definition of $\mathrm{V}_\mathcal{A}(p_i)$.

(b) Step $\alpha = \sigma \wedge \tau$: First we notice that the following fact holds, that is: for any filter F in $\mathrm{X}_\mathcal{A}$, $[\, v(\sigma), v(\tau) \in \mathrm{F} \iff v(\sigma \wedge \tau) \in \mathrm{F}\,]$ holds.
So from this fact together with the truth conditions of the model $\mathcal{M}_\mathcal{A}$ and the induction hypothesis,
$\|\sigma \wedge \tau\|^{\mathcal{M}_\mathcal{A}} = \|\sigma\|^{\mathcal{M}_\mathcal{A}} \cap \|\tau\|^{\mathcal{M}_\mathcal{A}}$
$= \{\ \mathrm{F} \mid \mathrm{F}$ is a proper filter and $v(\sigma) \in \mathrm{F}$ and $v(\tau) \in \mathrm{F}\ \}$
$= \{\ \mathrm{F} \mid \mathrm{F}$ is a proper filter and $v(\sigma \wedge \tau) \in \mathrm{F}\ \}$.

(c) Step $\alpha = \neg\sigma$: First we notice that from the induction hypothesis,
$\|\sigma\|^{\mathcal{M}_\mathcal{A}} = \{\ \mathrm{F} \mid \mathrm{F}$ is a proper filter and $v(\sigma) \in \mathrm{F}\ \}$ holds. Then we will show the following fact: For any proper filter F on $\mathrm{X}_\mathcal{A}$, $[\, v(\neg\sigma) \in \mathrm{F} \iff \mathrm{F} \perp_\mathcal{A} \|\sigma\|^{\mathcal{M}_\mathcal{A}}\,]$.
(1) Suppose that $v(\sigma) = \mathbf{0}$. Then $v(\neg\sigma) = \mathbf{1} \in \mathrm{F}$ holds always since F is a filter. Moreover, $\|\sigma\|^{\mathcal{M}_\mathcal{A}} = \emptyset$ holds because no proper filter can have the minimum element $\mathbf{0}$. So $\mathrm{F} \perp_\mathcal{A} \emptyset$ always holds for any proper filter F.
(2) Thus we can assume that $v(\sigma) \neq \mathbf{0}$.
($\Rightarrow$) Suppose $v(\neg\sigma) \in \mathrm{F}$. Take any $\mathrm{G} \in \|\sigma\|^{\mathcal{M}_\mathcal{A}}$. This G is a proper filter and $v(\sigma) \in \mathrm{G}$. So we have that $\mathrm{F} \perp_\mathcal{A} \mathrm{G}$ for any G in $\|\sigma\|^{\mathcal{M}_\mathcal{A}}$. This means $\mathrm{F} \perp_\mathcal{A} \|\sigma\|^{\mathcal{M}_\mathcal{A}}$.
($\Leftarrow$) Suppose that $\mathrm{F} \perp_\mathcal{A} \|\sigma\|^{\mathcal{M}_\mathcal{A}}$. Then for any G in $\|\sigma\|^{\mathcal{M}_\mathcal{A}}$, there exists some $a$ such that $a^\perp \in \mathrm{F}$ and $a \in \mathrm{G}$. Now let $\mathrm{G}_0 = \{x \in \mathrm{A} \mid x \geq v(\sigma)\}$. Since $v(\sigma) \neq \mathbf{0}$, $\mathrm{G}_0$ is a proper filter for which $v(\sigma) \in \mathrm{G}_0$ holds. By our assumption $\mathrm{F} \perp_\mathcal{A} \mathrm{G}_0$. Hence there exists some $b$ such that $b^\perp \in \mathrm{F}$ and $a \in \mathrm{G}_0$. So $b \geq v(\sigma)$, then $b^\perp \leq v(\sigma)^\perp = v(\neg\sigma) \in \mathrm{F}$.
Thus we have:
$\|\neg\sigma\|^{\mathcal{M}_\mathcal{A}} = \{\ \mathrm{F} \in \mathrm{X}_\mathcal{A} \mid \mathrm{F} \perp_\mathcal{A} \|\sigma\|^{\mathcal{M}_\mathcal{A}}\ \}$
$= \{\ \mathrm{F} \mid \mathrm{F}$ is a proper filter and $v(\neg\sigma) \in \mathrm{F}\ \}$.

Proof of (ii): First we show the direction ( $\Rightarrow$ ). Suppose that $v(\alpha) \leq v(\beta)$ and that $\mathcal{M}_\mathcal{A} \models_\mathrm{G} \alpha$ for $\mathrm{G} \in \mathrm{X}_\mathcal{A}$. Then $\mathrm{G} \in \|\alpha\|^{\mathcal{M}_\mathcal{A}}$, that is, G is a proper filter which satisfies $v(\alpha) \in \mathrm{G}$. Since G is a filter and $v(\alpha) \leq v(\beta)$, we have that $v(\beta) \in \mathrm{G}$. Thus $\mathrm{G} \in \|\beta\|^{\mathcal{M}_\mathcal{A}}$. Therefore $\mathcal{M}_\mathcal{A} : \alpha \models \beta$.
To show the converse direction ( $\Leftarrow$ ), suppose that $v(\alpha) \not\leq v(\beta)$. Then $v(\alpha)$ cannot be $\mathbf{0}$. Let $\mathrm{G}_1 = \{x \in \mathrm{A} \mid x \geq v(\alpha)\}$. Then $\mathrm{G}_1$ is a proper filter. Clearly, $\mathrm{G}_1$ satisfies that $\mathcal{M}_\mathcal{A} \models_{\mathrm{G}_1} \alpha$ but that $\mathcal{M}_\mathcal{A} \not\models_{\mathrm{G}_1} \beta$. Consequently we have proved that $\mathcal{M}_\mathcal{A} : \alpha \models \beta$ does not hold. □

**Corollary 2.13** **( (Q)⇒(P) (Completeness) )** For given formulas $\alpha$, $\beta$, $\sigma$ and $\tau$, let (P) and (Q) be the same statements in Corollary 2.9 . That is,

(P): For any ortholattice $\mathcal{A}$ and for any orthovaluation $v : \Phi \rightarrow \mathrm{A}$,
if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.

(Q): For any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.

Then (Q) implies (P).

**Proof** : Suppose that (P) does not hold. So there exists an ortholattice $\mathcal{A}$ and an orthovaluation $v : \Phi \rightarrow \mathrm{A}$ such that $v(\sigma) \leq v(\alpha)$ and $v(\tau) \not\leq v(\beta)$. Let $\mathcal{M}_{\mathcal{A}} = \langle \mathrm{X}_{\mathcal{A}}, \perp_{\mathcal{A}}, \mathcal{V}_{\mathcal{A}} \rangle$ be the canonical orthomodel for $\mathcal{A}$ and $v$. Then by Theorem 2.12, we have that $\mathcal{M}_{\mathcal{A}} : \sigma \models \alpha$ and $\mathcal{M}_{\mathcal{A}} : \tau \not\models \beta$. Thus (Q) does not hold. □

**Theorem 2.14** **(The Completeness Theorem)** For given formulas $\alpha$, $\beta$, $\sigma$, and $\tau$, the statements (P) and (Q) are mutually equivalent, that is

(P): for any ortholattice $\mathcal{A}$ and any orthovaluation $v : \Phi \rightarrow \mathrm{A}$,
if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.

(Q): for any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.

**Proof** : It is obvious by Corollary 2.9, and 2.13 . □

Here recall Proposition 1.2 . Then we can prove the completeness theorem for weak orthologic.

**Corollary 2.15** **( Completeness for weak orthologic)** For given formulas $\alpha$ and $\beta$, the following statements (P') and (Q') are equivalent.

(P'): $\alpha \models_w \beta$.

(Q'): for any orthomodel $\mathcal{M}$, and for any fixed formula $\chi$,
if $\mathcal{M} : \neg(\chi \wedge \neg\chi) \models \alpha$, then $\mathcal{M} : \neg(\chi \wedge \neg\chi) \models \beta$.

# 3 Filtration and decidability

Next we will see that there exists an algorithm which can decide whether the statement (P) holds or not for given formulas $\alpha$, $\beta$, $\sigma$ and $\tau$. In order to complete this work, we will modify Goldblatt's proof in [4] and prove the finite model property as the following form. That is, for given formulas $\alpha$, $\beta$, $\sigma$ and $\tau$, the statement (Q) is equivalent to the following statement (R):

(R): for every orthomodel $\mathcal{N}$ which has at most $2^{k+l}$ points, if $\mathcal{N} : \sigma \models \alpha$, then $\mathcal{N} : \tau \models \beta$, where $k$ is the number of subformulas included in $\alpha$, $\beta$, $\sigma$ and $\tau$ together and $l$ is the number of propositional variables included in $\alpha$, $\beta$, $\sigma$ and $\tau$ torether.

Thus we can show that the weak orthologic has the finite model property and hence that it is decidable. First we will see some notions that we need in this section.

**Definition 3.1 (Admissible set of formulas)** A set $\Psi$ of formulas is *admissible* if it satisfies the following:

(i) If $\alpha \in \Psi$ and $\beta$ is a subformula of $\alpha$, then $\beta \in \Psi$.

(ii) For any propositional variable $p_i$ $(i \in \omega)$, if $p_i \in \Psi$, then $\neg p_i \in \Psi$.

∎

Let $\mathcal{M} = \langle \mathrm{X}, \perp, \mathrm{V} \rangle$ be an orthomodel and $\Psi$ an admissible set of formulas. Define an equivalence relation $\approx$ on the set X as follows:

For $x, y \in \mathrm{X}$, $\quad x \approx y \quad$ iff $\quad$ for any $\alpha \in \Psi$, $( \mathcal{M} \models_x \alpha \Leftrightarrow \mathcal{M} \models_y \alpha )$.

For each $x \in \mathrm{X}$, define $[\ x\ ] := \{\ y \mid x \approx y\ \}$. Let X' be the quotient set $\mathrm{X}/_{\approx}$. Define a binary relation $\perp'$ on the set X' and a valuation function V' as follows:

(i) $[\ x\ ] \perp' [\ y\ ]$ iff there exists a formula $\alpha$ such that $\neg\alpha \in \Psi$, which satisfies the following: either $( \mathcal{M} \models_x \neg\alpha$ and $\mathcal{M} \models_y \alpha )$ or $( \mathcal{M} \models_x \alpha$ and $\mathcal{M} \models_y \neg\alpha )$ holds.

(ii) For every propositional variable $p_i$, $\mathrm{V}'(p_i) := \{\ [\ x\ ] \mid p_i \in \Psi$ and $x \in \mathrm{V}(p_i)\ \}$.

This structure $\mathcal{M}' = \langle \mathrm{X}', \perp', \mathrm{V}' \rangle$ is called the *filtration* of an orthomodel $\mathcal{M}$ through an admissible set$\Psi$.

Here we will acertain that the above definition does make sense for sure.

**Proposition 3.2** The binary relation $\perp'$ and the valuation $\mathrm{V}'$ defined above are well-defined.

**Proof** :

(i) For the relation $\perp'$, we have to show that for any $x, y, x', y' \in \mathrm{X}$, if $[\ x\ ] \perp' [\ y\ ]$, $x \approx x'$ and $y \approx y'$, then $[\ x'\ ] \perp' [\ y'\ ]$. Suppose that $[\ x\ ] \perp' [\ y\ ]$, $x \approx x'$ and $y \approx y'$. $[\ x\ ] \perp' [\ y\ ]$ means that there exists some formula $\alpha$, such that $\neg\alpha \in \Psi$ and that
(1): $\mathcal{M} \models_x \neg\alpha$ and $\mathcal{M} \models_y \alpha$ or (2): $\mathcal{M} \models_x \alpha$ and $\mathcal{M} \models_y \neg\alpha$ holds.
From the supposition that $\neg\alpha \in \Psi$, $\alpha$ is also in the admissible set $\Psi$. In case (1), we have that $\mathcal{M} \models_{x'} \neg\alpha$ and that $\mathcal{M} \models_{y'} \alpha$, since $x \approx x'$ and $y \approx y'$. So we can derive that $[\ x'\ ] \perp' [\ y'\ ]$. Clearly the similar argument works for the case (2). So we conclude that $\perp'$ on the set X' is well-defined.

(ii) For the valuation V', we have to prove that for any $x, y \in \mathrm{X}$, if $[\ x\ ] \in \mathrm{V}'(p_i)$ and $x \approx y$, then $[\ y\ ] \in \mathrm{V}'(p_i)$. Suppose that $[\ x\ ] \in \mathrm{V}'(p_i)$ and $x \approx y$. The supposition that $[\ x\ ] \in \mathrm{V}'(p_i)$ means that $p_i \in \Psi$ and $x \in \mathrm{V}(p_i)$. So we have that $\mathcal{M} \models_x p_i$. Since $x \approx y$, we also have that $\mathcal{M} \models_y p_i$, which is equivalent to $y \in \mathrm{V}(p_i)$. Then $[\ y\ ] \in \mathrm{V}'(p_i)$, and therefore the valuation V' is well-defined.

□

**Lemma 3.3** $\mathcal{M}' = \langle \mathrm{X}', \perp', \mathrm{V}' \rangle$ is an orthomodel.

**Proof** : By Definition 2.1, and 2.4, it is enough for us to check the following conditions:

(i) X' is not empty.

(ii) $\perp'$ is an orthogonarity relation on X'.

(iii) $\mathrm{V}'(p_i)$ is $\perp'$-closed.

They can be shown as follows.

(i) $\mathrm{X}'$ is obviously not empty, because X is not empty.

(ii) Symmetricity is trivial from the definition of $\perp'$. We have only to show irreflexivity. Suppose $[\ x\ ]\perp'[\ x\ ]$ for some $x \in \mathrm{X}$. This means that there exists some formula $\alpha$ such that $\neg\alpha \in \Psi$, $\mathcal{M}\models_x\alpha$ and $\mathcal{M}\models_x\neg\alpha$ hold. Since $\mathcal{M}\models_x\neg\alpha$ is equivalent to $\forall y \in \mathrm{X}\ (\mathcal{M}\models_y\alpha \Rightarrow x\perp y)$, by taking $x$ for $y$ we have $x\perp x$. Contradiction. Therefore $\perp'$ is irreflexive.

(iii) Suppose that $\forall[\ y\ ] \in \mathrm{X}'\ (\ [\ y\ ]\perp'\mathrm{V}'(p_i) \Rightarrow [\ x\ ]\perp'[\ y\ ]\ )$ for an arbitrary $[\ x\ ]$ in $\mathrm{X}'$. It is enough to show that the condition $[\ x\ ] \notin \mathrm{V}'(p_i)$ will lead us to a contradiction. This condition means that either $p_i \notin \Psi$, or $x \notin \mathrm{V}(p_i)$. If $p_i \notin \Psi$, then we have that $\mathrm{V}'(p_i) = \emptyset$, so this is $\perp'$-closed. If $p_i \in \Psi$, then $x \notin \mathrm{V}(p_i)$, and we need a further consideration. Since $\mathrm{V}(p_i)$ is $\perp$-closed, there exists some $u \in \mathrm{X}$ such that $u\perp\mathrm{V}(p_i)$ but $x\perp u$ does not hold. Here we assume that $\mathcal{M}\models_z p_i$ for an arbitrary $z$ in X, which is equivalent to $z \in \mathrm{V}(p_i)$. Then from the property of $u$, we have that $u\perp z$. On the other hand, we also have that $\mathcal{M}\models_u\neg p_i$. Since $\neg p_i \in \Psi$, we can derive that $[\ u\ ]\perp'[\ z\ ]$. This argument works well for any $[\ w\ ]$ in $\mathrm{V}'(p_i)$, since $\mathcal{M}\models_w p_i$ for this $w$. Then we have that $[\ u\ ]\perp'\mathrm{V}'(p_i)$. Therefore we can take this $[\ u\ ]$ for $[\ y\ ]$, and conclude that $[\ x\ ]\perp'[\ u\ ]$ from the supposition. This conclusion means that there exists some formula $\alpha$, which satisfies that $\neg\alpha \in \Psi$, and that (1): $\mathcal{M}\models_x \neg\alpha$ and $\mathcal{M}\models_u \alpha$ or (2): $\mathcal{M}\models_x \alpha$ and $\mathcal{M}\models_u \neg\alpha$ holds.
In case (1), $\mathcal{M}\models_x \neg\alpha$ is equivalent to $\forall z \in \mathrm{X}(\ \mathcal{M}\models_z \alpha \Rightarrow x\perp z\ )$. We can take $u$ for this $z$, so it must be true that $x\perp u$. This contradicts to the property of $u$. Clearly similar argument works for the case (2). So we have shown that $\mathrm{V}'(p_i)$ is $\perp'$-closed.

□

**Theorem 3.4 (Filtration Theorerm)** For any $\alpha \in \Psi$ and for any $x \in \mathrm{X}$, the two conditions (1): $\mathcal{M}\models_x\alpha$ and (2): $\mathcal{M}'\models_{[x]}\alpha$ are equivarent.
**Proof** : We use the induction on the construction of the formula $\alpha$.

(i) Step $\alpha = p_i$: We have that $p_i \in \Psi$. Thus,
$\mathcal{M}\models_x p_i \Leftrightarrow x \in \mathrm{V}(p_i) \Leftrightarrow [\ x\ ] \in \mathrm{V}'(p_i) \Leftrightarrow \mathcal{M}'\models_{[x]} p_i$.

(ii) Step $\alpha = \sigma \wedge \tau$:
$\mathcal{M}\models_x \sigma\wedge\tau \Leftrightarrow \mathcal{M}\models_x \sigma$ and $\mathcal{M}\models_x \tau$
( by the induction hypothesis ),
$\Leftrightarrow \mathcal{M}'\models_{[x]} \sigma$ and $\mathcal{M}'\models_{[x]} \tau \Leftrightarrow \mathcal{M}'\models_{[x]} \sigma\wedge\tau$.

(iii) Step $\alpha = \neg\sigma$:
First we show the direction ((1) $\Rightarrow$ (2)). Suppose $\mathcal{M}\models_x \neg\sigma$. For any $y \in \mathrm{X}$ such that $\mathcal{M}'\models_{[y]} \sigma$, we have that $\neg\sigma \in \Psi$, and that $\mathcal{M}\models_y \sigma$ by the induction hypothesis. Since $\neg\sigma \in \Psi$, $\mathcal{M}\models_x \neg\sigma$ and $\mathcal{M}\models_y \sigma$, we have that $[\ x\ ]\perp'[\ y\ ]$. Therefore we have that $\mathcal{M}'\models_{[x]} \neg\sigma$ .
To prove the converse direction ((1) $\Leftarrow$ (2)), suppose that $\mathcal{M}'\models_{[x]} \neg\sigma$ . This means

that $\forall[\,y\,] \in X'\,(\,\mathcal{M}' \models_{[y]} \sigma \Rightarrow [\,x\,] \perp' [\,y\,]\,)$. For any $y \in X$ such that $\mathcal{M} \models_y \sigma$, we have that $\mathcal{M}' \models_{[y]} \sigma$ by induction hypothesis, and so that $[\,x\,] \perp' [\,y\,]$. Therefore there exists some formula $\pi$ such that $\neg\pi \in \Psi$ and that
(1): $\mathcal{M} \models_x \neg\pi$ and $\mathcal{M} \models_y \pi$ or (2): $\mathcal{M} \models_x \pi$ and $\mathcal{M} \models_y \neg\pi$ holds.
In case (1), from $\mathcal{M} \models_x \neg\pi$, we have that $\forall z \in X(\mathcal{M} \models_z \pi \Rightarrow x \perp z)$. We can take $y$ for this $z$, then we have that $x \perp y$. In case (2), we can also conclude that $x \perp y$ by similar argment. Therefore we conclude that $\mathcal{M} \models_x \neg\sigma$.

□

**Theorem 3.5** **((Q) ⇔ (R))** Let $\alpha$, $\beta$, $\sigma$ and $\tau$ be formulas which include $k$ subformulas and $l$ propositional variables in total. Then following statements (Q) and (R) are equivalent.

(Q): for any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.

(R): for every orthomodel $\mathcal{N}$ which has at most $2^{k+l}$ points, if $\mathcal{N} : \sigma \models \alpha$, then $\mathcal{N} : \tau \models \beta$.

**Proof** : To show the direction (Q) ⇒ (R) is trivial. We have only to show the other direction (Q) ⇐ (R).
Suppose (Q) does not hold. Then there exists some orthomodel $\mathcal{M} = \langle X, \perp, V \rangle$ such that $\mathcal{M} : \sigma \models \alpha$ and $\mathcal{M} : \tau \not\models \beta$. In other words, $\forall x \in X(\ \mathcal{M} \not\models_x \sigma$ or $\mathcal{M} \models_x \alpha\ )$ and $\exists y \in X(\ \mathcal{M} \models_y \tau$ and $\mathcal{M} \not\models_y \beta\ )$ hold.

Now let $\Psi$ be the smallest admissible set of formulas which has $\alpha$, $\beta$, $\sigma$ and $\tau$, and consider the filtration $\mathcal{M}' = \langle X', \perp', V' \rangle$ of the orthomodel $\mathcal{M}$ through this $\Psi$. Of course $\alpha, \beta, \sigma, \tau \in \Psi$. Then by Theorem 3.4, we have that $\forall[\,x\,] \in X'(\ \mathcal{M}' \not\models_{[x]} \sigma$ or $\mathcal{M}' \models_{[x]} \alpha\ )$ and $\exists[\,y\,] \in X'(\ \mathcal{M}' \models_{[y]} \tau$ and $\mathcal{M}' \not\models_{[y]} \beta\ )$ hold. This means that $\mathcal{M}' : \sigma \models \alpha$ and $\mathcal{M}' : \tau \not\models \beta$.

Here we count the number of points in $\mathcal{M}'$. By the definition of admissible set, we have that $\Psi$ has at most $k + l$ formulas, and hence that $\Psi$ has at most $2^{k+l}$ subsets. Now we denote a subset $\{\pi \mid \mathcal{M} \models_x \pi\}$ of $\Psi$ by $\Psi_x$, then it is easy to check that the two conditions $[\,x\,] \neq [\,y\,]$ and $\Psi_x \neq \Psi_y$ are equivalent. Therefore the number of different points in the model $\mathcal{M}'$ is smaller than or equal to the number of subset of $\Psi$. Then the model $\mathcal{M}'$ contains at most $2^{k+l}$ points. Consequently we have shown that (R) does not hold. □

According to all the arguments mentioned above in Part I, we can conclude the decidability of weak orthologics.

**Corollary 3.6** **( Decidability of weak orthologic )**

(i) There is an algorithum by which we can decide whether the following statement holds or not: for given formulas $\alpha$, $\beta$, $\sigma$, and $\tau$, and for any ortholattice $\mathcal{A}$ and any orthovaluation $v : \Phi \to A$, if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.

(ii) Weak orthologic is decidable.

# Part II
# Semigroup semantics for orthomodular logic

In Part II, we will discuss about strong orthomodular logic only, so we call it orthomodular logic simply. There is a representation theorem for orthomodular lattices established by D.J.Foulis [3] in 1960. His theorem was based on a particular kind of semigroups.

C.E.Rickart [13] first introduced 'Rickart * rings' in developing the theory of operator algebras in 1946. We adopt the terminology of S.Maeda's book [8] and will call our semigroups 'Rickart * semigroups', though Foulis did not use this term in his paper. Here we propose a semantics for orthomodular logic based on Foulis's Representation theorem and will show the completeness theorem for orthomodular logic with respect to this semantics.

## 4 Rickart * semigroups

First we introduce a special type of semigroups called Rickart * semigroups and lead some properties of them. A Rickart * semigroup is a structure $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ which satisfies the following conditions (i), (ii), (iii) and (iv).

(i) $\langle \mathrm{G}, \cdot \rangle$ is a semigroup, that is,
  (a) $\cdot$ is a binary operation on G.
  (b) For any $x, y, z \in \mathrm{G}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(ii) There exists the unique element 0 ( zero element ) in G such that $0 \cdot x = x \cdot 0 = 0$ holds for any $x \in \mathrm{G}$.

(iii) $*$ is a unary operation on G, which satisfies the following:
For any $x, y \in \mathrm{G}$, (a): $(x^*)^* = x$. (b): $(x \cdot y)^* = y^* \cdot x^*$.

Before introducing the conditon (iv), it is necessary to introduce some other notions.

- An element $e \in \mathrm{G}$ is called a *projection* iff it satisfies $e^* = e \cdot e = e$.
  We denote the set of all projections in G by P(G).
- For an element $x \in \mathrm{G}$, the set $\{ x \}^{(\mathrm{r})} := \{ y \in \mathrm{G} \mid x \cdot y = 0 \}$ is called the *right annihilator* for $x$.

By using these two notions, we formulate the conditon (iv) as follows:

(iv) For any $x \in \mathrm{G}$, there exists a projection $e$ such that the right annihilator for $x$ can be expressed as : $\{ x \}^{(\mathrm{r})} = e \cdot \mathrm{G} = \{ e \cdot y \mid y \in \mathrm{G} \}$. We call this $e$ a *right annihilating projection* for $x$.

From now on, we will see some properties of this Rickart * semigroups. ( See e.g. [8].)

**Lemma 4.1 (Properties of P(G))** Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup.

(i) For any $x \in \mathrm{G}$, the right annihilating projection for $x$ is uniquely determined. Hereafter, this will be written as $x^{\mathrm{r}}$.

(ii) There is the unit element in G, that is, an element 1 satisfying that for any $x \in \mathrm{G}$, $x \cdot 1 = 1 \cdot x = x$.

(iii) Both 0 and 1 are projections.

(iv) For any $e, f \in \mathrm{P(G)}$, the following three conditions are equivalent.
(a) $e \cdot f = e$.
(b) $f \cdot e = e$.
(c) $e \cdot \mathrm{G} \subseteq f \cdot \mathrm{G}$.

**Proof** :

(i) Suppose that there exist two projections $e_1$ and $e_2$ such that $\{x\} = e_1 \cdot \mathrm{G} = e_2 \cdot \mathrm{G}$. Then since $e_1 = e_1 \cdot e_1 \in e_1 \cdot \mathrm{G} = e_2 \cdot \mathrm{G}$, there exists some $s \in \mathrm{G}$ which satisfies $e_1 = e_2 \cdot s$. So we have that $e_2 \cdot e_1 = e_2 \cdot e_2 \cdot s = e_2 \cdot s = e_1$. Similar argument will give us that $e_1 \cdot e_2 = e_2$. By operating * to the former equation, we have that $e_1 = {e_1}^* = (e_2 \cdot e_1)^* = {e_1}^* \cdot {e_2}^* = e_1 \cdot e_2 = e_2$.

(ii) We will show that $0^{\mathrm{r}}$ is the unit element 1. We note first that $\{0\}^{(\mathrm{r})} = 0^{\mathrm{r}} \cdot \mathrm{G} = \{x \in \mathrm{G} \mid 0 \cdot x = 0\} = \mathrm{G}$. So for any $y \in \mathrm{G} = 0^{\mathrm{r}} \cdot \mathrm{G}$, there exists some $s \in \mathrm{G}$ such that $y = 0^{\mathrm{r}} \cdot s$. By multiplying $0^{\mathrm{r}}$ to both sides of this equation from the left, we have $0^{\mathrm{r}} \cdot y = 0^{\mathrm{r}} \cdot 0^{\mathrm{r}} \cdot s = 0^{\mathrm{r}} \cdot s = y$. This means that $0^{\mathrm{r}} \cdot y = y$ holds for any $y \in \mathrm{G}$. Moreover we can see that for any $z \in \mathrm{G}$, $z \cdot 0^{\mathrm{r}} = (z \cdot 0^{\mathrm{r}})^{**} = (0^{\mathrm{r}*} \cdot z^*)^* = (0^{\mathrm{r}} \cdot z^*)^* = (z^*)^* = z$. Thus we have that for any $x \in \mathrm{G}$, $x \cdot 0^{\mathrm{r}} = 0^{\mathrm{r}} \cdot x = x$.

(iii) As for the zero element 0, clearly $0 \cdot 0 = 0$ holds. So it is enough to prove that $0^* = 0$. For any $x \in \mathrm{G}$, we have that $0 = 0 \cdot x$. Here by taking $0^*$ for $x$, we can derive that $0 = 0 \cdot 0^*$. By operating $*$ to both sides of this equation, we conclude that $0^* = (0 \cdot 0^*)^* = 0^{**} \cdot 0^* = 0 \cdot 0^* = 0$.
Similarly, we can show that 1 is a projection.

(iv) (a) $\Leftrightarrow$ (b): We will prove the direction (a) $\Rightarrow$ (b) first. By operating $*$ to the both sides of $e \cdot f = e$, we have $e = e^* = (e \cdot f)^* = f^* \cdot e^* = f \cdot e$. The other direction can be shown quite similarly.
(b) $\Leftrightarrow$ (c): To prove the direction (b) $\Rightarrow$ (c), we will take an arbitrary element $p \in e \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$ satisfying that $p = e \cdot s$. So by the assumption (b), $p = e \cdot s = f \cdot e \cdot s \in f \cdot \mathrm{G}$. So we have that $e \cdot \mathrm{G} \subseteq f \cdot \mathrm{G}$.
Conversely, suppose that $e \cdot \mathrm{G} \subseteq f \cdot \mathrm{G}$. Since $e$ is in $e \cdot \mathrm{G}$ and hence in $f \cdot \mathrm{G}$, there exists some $s \in \mathrm{G}$ such that $e = f \cdot s$. By multiplying $f$ to this equation, we have that $f \cdot e = f \cdot f \cdot s = f \cdot s = e$.

□

The above Lemma 4.1 (iv) assures us the possibility of introducing a partially order on P(G).

**Definition 4.2 (Order on P(G))** Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup. Define a partial order $\leq$ on P(G) as follows: for $e, f \in \mathrm{P(G)}$, $\quad e \leq f \quad$ iff $\quad e \cdot f = e$. ∎

It is obvious that 1 is the maximum and that 0 is the minimum with respect to this order. Hence P(G) can be regarded as a bounded partial ordered set.

In the proof of Lemma 4.1, we have defined the unary operation $^{\mathrm{r}}$ from G to P(G). Here we will see some of the basic properties of the operation $^{\mathrm{r}}$in detail, which will be used in the later discussion.

**Lemma 4.3 (Properties of the operation $^{\mathrm{r}}$)** Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup. For any $x, y \in \mathrm{G}$ and for any $e, f \in \mathrm{P(G)}$, the following statements can be verified.

(i) $0^{\mathrm{r}} = 1$, and $1^{\mathrm{r}} = 0$.

(ii) $x \cdot x^{\mathrm{r}} = 0$, and $x^{\mathrm{r}} \cdot x^* = 0$.

(iii) If $x \cdot e = 0$, then $e \leq x^{\mathrm{r}}$.

(iv) $x^{\mathrm{r}} \leq (y \cdot x)^{\mathrm{r}}$.

(v) If $e \leq f$, then $f^{\mathrm{r}} \leq e^{\mathrm{r}}$.

(vi) $x = x \cdot x^{\mathrm{rr}}$, and $e \leq e^{\mathrm{rr}}$.

(vii) $x^{\mathrm{r}} = x^{\mathrm{rrr}}$.

(viii) If $e \cdot x = x \cdot e$, then $e^{\mathrm{r}} \cdot x = x \cdot e^{\mathrm{r}}$.

**Proof** :

(i) The first equation was already proved in Lemma 4.1 .Similarly, from the fact that $\{1\}^{(\mathrm{r})} = 1^{\mathrm{r}} \cdot \mathrm{G} = \{0\}$, it follows that $1^{\mathrm{r}} = 0$.

(ii) We have that $x^{\mathrm{r}} \in \{x\}^{(\mathrm{r})} = x^{\mathrm{r}} \cdot \mathrm{G}$. So $x \cdot x^{\mathrm{r}} = 0$. Next, by operating * to the both sides of this equation, we also have that $x^{\mathrm{r}} \cdot x^* = 0$.

(iii) Suppose that $x \cdot e = 0$. We have only to show that $e \cdot \mathrm{G} \subseteq x^{\mathrm{r}} \cdot \mathrm{G}$. Take any $p \in e \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$ such that $p = e \cdot s$. So $x \cdot p = x \cdot e \cdot s = 0$. Therefore $p \in \{x\}^{(\mathrm{r})} = x^{\mathrm{r}} \cdot \mathrm{G}$.

(iv) It is enough to show that $x^{\mathrm{r}} \cdot \mathrm{G} \subseteq (y \cdot x)^{\mathrm{r}} \cdot \mathrm{G}$. Take any $p \in x^{\mathrm{r}} \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$ such that $p = x^{\mathrm{r}} \cdot s$. So we have $x \cdot p = x \cdot x^{\mathrm{r}} \cdot s = 0$ by (ii). Thus $(y \cdot x) \cdot p = 0$ for any $y \in \mathrm{G}$, which means that $p \in (y \cdot x)^{\mathrm{r}} \cdot \mathrm{G}$. So $x^{\mathrm{r}} \cdot \mathrm{G} \subseteq (y \cdot x)^{\mathrm{r}} \cdot \mathrm{G}$.

(v) Suppose $e \leq f$. Then we have to show that $f^{\mathrm{r}} \cdot \mathrm{G} \subseteq e^{\mathrm{r}} \cdot \mathrm{G}$. Take any $p \in f^{\mathrm{r}} \cdot \mathrm{G}$, then there exists some $s \in \mathrm{G}$ such that $p = f^{\mathrm{r}} \cdot s$. So we have that $f \cdot p = f \cdot f^{\mathrm{r}} \cdot s = 0$. Therefore $e \cdot p = (e \cdot f) \cdot p = 0$. This means that $p \in \{e\}^{(\mathrm{r})} = e^{\mathrm{r}} \cdot \mathrm{G}$.

(vi) By (ii), $x^* \in \{x^{\mathrm{r}}\}^{(\mathrm{r})} = x^{\mathrm{rr}} \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$, such that $x^* = x^{\mathrm{rr}} \cdot s$. By operating * to this equation, we have that $x = x^{**} = s^* \cdot x^{\mathrm{rr}*} = s^* \cdot x^{\mathrm{rr}}$. Further operating $x^{\mathrm{rr}}$ from the right to the equation $x = s^* \cdot x^{\mathrm{rr}}$, we can derive that $x \cdot x^{\mathrm{rr}} = (s^* \cdot x^{\mathrm{rr}}) \cdot x^{\mathrm{rr}} = s^* \cdot x^{\mathrm{rr}} = x$. In particular, when $x$ is equal to a projection $e$, we have that $e \cdot e^{\mathrm{rr}} = e$, that is, $e \leq e^{\mathrm{rr}}$.

(vii) Since $x^{\mathrm{r}}$ is a projection, we have $x^{\mathrm{r}} \leq x^{\mathrm{rrr}}$ by (vi). Conversely, take any $p \in x^{\mathrm{rrr}} \cdot \mathrm{G} = \{x^{\mathrm{rr}}\}^{(\mathrm{r})}$. Then $p$ satisfies that $x^{\mathrm{rr}} \cdot p = 0$. Again by (vi), $x \cdot p = (x \cdot x^{\mathrm{rr}}) \cdot p = 0$. Thus $p \in \{x\}^{(\mathrm{r})} = x^{\mathrm{r}} \cdot \mathrm{G}$. Therefore we conclude that $x^{\mathrm{r}} = x^{\mathrm{rrr}}$.

(viii) Suppose that $e \cdot x = x \cdot e$. Then we have $e \cdot x \cdot e^{\mathrm{r}} = x \cdot e \cdot e^{\mathrm{r}} = 0$, since $e \cdot e^{\mathrm{r}} = 0$. So $x \cdot e^{\mathrm{r}} \in \{e\}^{(\mathrm{r})} = e^{\mathrm{r}} \cdot \mathrm{G}$, and there exists some $s \in \mathrm{G}$ satisfying that $x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot s$. By multiplying $e^{\mathrm{r}}$ from the left to both sides of this equation, we have that

$e^{\mathrm{r}} \cdot x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot e^{\mathrm{r}} \cdot s = e^{\mathrm{r}} \cdot s = x \cdot e^{\mathrm{r}}$ $\cdots (1)$

On the other hand, by operating $*$ to the supposition $e \cdot x = x \cdot e$, so we have that $x^* \cdot e = e \cdot x^*$. Then $e \cdot x^* \cdot e^{\mathrm{r}} = x^* \cdot e \cdot e^{\mathrm{r}} = 0$, which means that $x^* \cdot e^{\mathrm{r}} \in \{e\}^{(\mathrm{r})} = e^{\mathrm{r}} \cdot \mathrm{G}$. So there exists some $t \in \mathrm{G}$ such that $x^* \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot t$. By multiplying $e^{\mathrm{r}}$ from the left to both sides of this equation, we have that $e^{\mathrm{r}} \cdot x^* \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot e^{\mathrm{r}} \cdot t = e^{\mathrm{r}} \cdot t = x^* \cdot e^{\mathrm{r}}$. Further operating $*$ again, we get that

$e^{\mathrm{r}} \cdot x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot x$ $\cdots (2)$.

From (1) and (2), we can conclude that $x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot x$.

□

These results will be used many times in the rest of this section. Now we will consider a particular class of projections, called *closed projections.*

**Definition 4.4 (Closed projection)** A projection $f \in \mathrm{P(G)}$ is called *closed* iff there exists an element $x \in \mathrm{G}$ such that $f$ is the right annihilating projection for $x$. This means that a closed projection $f$ can be written as $f = x^{\mathrm{r}}$ for some element $x \in \mathrm{G}$. We denote the set of all closed projections in G by $\mathrm{P}_c(\mathrm{G})$. ∎

In other words, the set $\mathrm{P}_c(\mathrm{G})$ is the range of the function $^{\mathrm{r}}$ from G to P(G). We show here a necessary and sufficient condition on a projection to be closed.

**Proposition 4.5** For any $e \in \mathrm{P(G)}$, $e \in \mathrm{P}_c(\mathrm{G})$ if and only if $e^{\mathrm{rr}} = e$.

**Proof** : Suppose $e \in \mathrm{P}_c(\mathrm{G})$. Then there exists some $x \in \mathrm{G}$, such that $e = x^{\mathrm{r}}$. So by Lemma 4.3 (vii), $e^{\mathrm{rr}} = x^{\mathrm{rrr}} = x^{\mathrm{r}} = e$.

Conversely suppose $e^{\mathrm{rr}} = e$. Then $\{e^{\mathrm{r}}\}^{(\mathrm{r})} = e^{\mathrm{rr}} \cdot \mathrm{G} = e \cdot \mathrm{G}$. Therefore $e$ is the right annihilating element for $e^{\mathrm{r}} \in \mathrm{G}$. □

We will show that in $\mathrm{P}_c(\mathrm{G})$ we can always find the supremum and the infimum of any two elements of it and hence this partially ordered set forms a lattice. Moreover we can show that $\mathrm{P}_c(\mathrm{G})$ is an orthomodular lattice, whose fact is one of the keys of Part II. First we will show the existence of the infimum ( meet ) of two closed projections in $\mathrm{P}_c(\mathrm{G})$.

**Lemma 4.6 (Existence of meet in $\mathrm{P}_c(\mathrm{G})$)**

(i) For any closed projections $e$ and $f$ such that $e \cdot f = f \cdot e$, $e \cdot f \in \mathrm{P}_c(\mathrm{G})$ holds, and there exists the infimum $(e \sqcap f)$ of $e, f$, which satisfies the equation $e \sqcap f = e \cdot f$.

(ii) In general, for any closed projections $e$ and $f$, there exists the infimum $(e \sqcap f)$ of $e, f$ and the equation $e \sqcap f = e \cdot (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} \cdot e = e \sqcap (f^{\mathrm{r}} \cdot e)^{\mathrm{r}}$ holds.

**Proof** :

(i) Suppose that $e \cdot f = f \cdot e$. First we show that $e \cdot f \in \mathrm{P}_c(\mathrm{G})$. Since $e, f \in \mathrm{P(G)}$ and $e \cdot f = f \cdot e$, we can derive:

$$(e \cdot f)^* = f^* \cdot e^* = f \cdot e = e \cdot f, \quad \text{and} \quad (e \cdot f) \cdot (e \cdot f) = e \cdot e \cdot f \cdot f = e \cdot f.$$

Thus, $e, f \in \mathrm{P(G)}$. To prove that $e \cdot f \in \mathrm{P}_c(\mathrm{G})$, by Proposition 4.4, it is enough to show that $(e \cdot f)^{\mathrm{rr}} = e \cdot f$. Then we have only to show that $(e \cdot f)^{\mathrm{rr}} \le e \cdot f$ as the converse inequality holds always by Lemma 4.3 (vi). Considering the Lemma 4.3 (iv), we have that $e^{\mathrm{r}} \le (e \cdot f)^{\mathrm{r}}$. Then by the Lemma 4.3 (v), we can derive that $(e \cdot f)^{\mathrm{rr}} \le e^{\mathrm{rr}} = e$, which means $e \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Similarly we can derive that $f \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Therefore $e \cdot f \cdot (e \cdot f)^{\mathrm{rr}} = e \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Thus $(e \cdot f)^{\mathrm{rr}} \le e \cdot f$.

Second we will show that $e \cdot f$ is the infimum of $e$ and $f$. The infimum $e \sqcap f$ of $e$ and $f$ must satisfy the following conditions:

(a) $e \sqcap f \le e$ and $e \sqcap f \le f$.

(b) For every $g \in \mathrm{P}_c(\mathrm{G})$ such that $g \le e$ and $g \le f$, $\quad g \le e \sqcap f$.

It is obvious that $e \cdot f \le e$ and that $e \cdot f \le f$, because $e, f$ are projections. Now take any $g \in \mathrm{P}_c(\mathrm{G})$ such that $g \cdot e = g$ and $g \cdot f = g$. Then $g \cdot (e \cdot f) = (g \cdot e) \cdot f = g \cdot f = g$. Therefore $g \le e \cdot f$. Thus $e \sqcap f = e \cdot f$.

(ii) We put $u := f^{\mathrm{r}} \cdot e$. By Lemma 4.3 (iv), we have that $e^{\mathrm{r}} \le (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = u^{\mathrm{r}}$. This means that $e^{\mathrm{r}} \cdot u^{\mathrm{r}} = e^{\mathrm{r}} = u^{\mathrm{r}} \cdot e^{\mathrm{r}}$. By applying Lemma 4.3 (viii), we have that $e \cdot u^{\mathrm{r}} = u^{\mathrm{r}} \cdot e$. Then by (i) of the present lemma, we can conclude that $e \cdot u^{\mathrm{r}} \in \mathrm{P}_c(\mathrm{G})$, and that $e \sqcap u^{\mathrm{r}} = e \cdot u^{\mathrm{r}}$. So it remains to show that $e \sqcap f = e \cdot u^{\mathrm{r}}$.

(a) Clearly, $e \cdot (e \cdot u^{\mathrm{r}}) = e \cdot u^{\mathrm{r}}$. So we have $e \cdot u^{\mathrm{r}} \le e$. On the other hand, $f^{\mathrm{r}} \cdot e \cdot u^{\mathrm{r}} = f^{\mathrm{r}} \cdot e \cdot (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = 0$. So from Lemma 4.3 (iii), we derive that $e \cdot u^{\mathrm{r}} \le f^{\mathrm{rr}} = f$. Thus $e \cdot u^{\mathrm{r}}$ is a lower bound of $\{e, f\}$.

(b) Take any $g \in \mathrm{P}_c(\mathrm{G})$ such that $g \cdot e = g$ and $g \cdot f = g$. Then because $f \cdot f^{\mathrm{r}} = 0$, we have that $g \cdot f \cdot f^{\mathrm{r}} \cdot e = 0$. By our assumption on $g$, $\quad g \cdot f^{\mathrm{r}} \cdot e = 0$, which means that $g \cdot u = 0$. By Lemma 4.3 (iii), we can derive that $u \le g^{\mathrm{r}}$. So by Lemma 4.3 (v), $g = g^{\mathrm{rr}} \le u^{\mathrm{r}}$. This is equivalent to $g \cdot u^{\mathrm{r}} = g$. Again using the assumption on $g$, $\quad g \cdot e \cdot u^{\mathrm{r}} = g$. So we have derived that $g \le e \cdot u^{\mathrm{r}}$.

Thus we have shown that $e \sqcap f = e \cdot u^{\mathrm{r}}$.

□

The above lemma shows that for any pair of closed projections, their infimum exists always in the set $\mathrm{P}_c(\mathrm{G})$. It is easy to see that this fact is equivalent to the following proposition.

**Proposition 4.7** For any $e, f \in \mathrm{P}_c(\mathrm{G})$, the following equation holds:

$$e \cdot \mathrm{G} \cap f \cdot \mathrm{G} = (e \sqcap f) \cdot \mathrm{G}.$$

Next we will see that $\mathrm{P}_c(\mathrm{G})$ is an orthomodular lattice. In its proof, the existence of the supremum can be assured because of the existence of the orthocomplements.

**Theorem 4.8** $\mathrm{P}_c(\mathrm{G})$ forms an orthomodular lattice, where the orthocomplement is the operation $^{\mathrm{r}}$.

**Proof** : We will check the conditions in Definition 0.1 . But many of them follows immediately from what we have shown already.

(i) $0^{\mathrm{rr}} = 0$ and $1^{\mathrm{rr}} = 1$ by Lemma 4.3 (i). So $1, 0 \in \mathrm{P}_c(\mathrm{G})$. It is obvious that 1 is the maximum closed projection, and that 0 is the minimum closed projection.

(ii) By Lemma 4.6, for any $e, f \in \mathrm{P}_c(\mathrm{G})$ there exists the infimum $e \sqcap f$ in $\mathrm{P}_c(\mathrm{G})$.

(iii) By Lemma 4.6 (ii), we can derive that for any $e \in \mathrm{P}_c(\mathrm{G})$, $e \sqcap e^{\mathrm{r}} = e \cdot (e^{\mathrm{rr}} \cdot e)^{\mathrm{r}} = 0$, since $e^{\mathrm{rr}} = e$ and $e \cdot e^{\mathrm{r}} = 0$.

(iv) By Proposition 4.5, we have that for any $e \in \mathrm{P}_c(\mathrm{G})$, $e^{\mathrm{rr}} = e$.

(v) By Lemma 4.3 (v), for any $e, f \in \mathrm{P}_c(\mathrm{G})$, if $e \leq f$ then $f^{\mathrm{r}} \leq e^{\mathrm{r}}$.

(vi) Since we have now (ii), (iv) and (v) in the above, we can derive $e \sqcup f = (e^{\mathrm{r}} \sqcap f^{\mathrm{r}})^{\mathrm{r}}$ for any $e, f \in \mathrm{P}_c(\mathrm{G})$. So we conclude that for any $e, f \in \mathrm{P}_c(\mathrm{G})$, there exists the supremum $e \sqcup f$ in $\mathrm{P}_c(\mathrm{G})$.

From these facts, we can conclude that $\mathrm{P}_c(\mathrm{G})$ is an ortholattice. So it remains only to show that $\mathrm{P}_c(\mathrm{G})$ satisfies the orthomodular law.

(vii) Suppose that $e \leq f$. This means that $e \cdot f = e = f \cdot e$. By applying Lemma 4.3 (viii), we have that $e^{\mathrm{r}} \cdot f = f \cdot e^{\mathrm{r}}$. Therefore we have that $f \sqcap e^{\mathrm{r}} = f \cdot e^{\mathrm{r}}$. Then by Lemma 4.7 (ii), $e^{\mathrm{r}} \sqcap f^{\mathrm{r}} = e^{\mathrm{r}} \sqcap (f^{\mathrm{rr}} \cdot e^{\mathrm{r}})^{\mathrm{r}} = e^{\mathrm{r}} \sqcap (f \cdot e^{\mathrm{r}})^{\mathrm{r}} = e^{\mathrm{r}} \sqcap (f \sqcap e^{\mathrm{r}})^{\mathrm{r}}$.
So we have $f = e \sqcup f = (e^{\mathrm{r}} \sqcap f^{\mathrm{r}})^{\mathrm{r}} = (e^{\mathrm{r}} \sqcap (f \sqcap e^{\mathrm{r}})^{\mathrm{r}})^{\mathrm{r}} = e \sqcup (f \sqcap e^{\mathrm{r}})$.

Thus $\mathrm{P}_c(\mathrm{G})$ forms an orthomodular lattice. □

Next, in Section 5, we will introduce a semantics for orthomodular logic by using Rickart * semigroups, and prove the soundness.

# 5 Semigroup semantics and soundness theorem

**Definition 5.1 (Orthomodular model)** $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ is a *orthomodular model* ( OM model for short ) iff $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ is a Rickart * semigroup and u is a function assigning to each propositional variable $p_i$ an element $\mathrm{u}(p_i)$ of $\mathrm{P}_c(\mathrm{G})$.

The notion of truth in OM models is defined inductively as follows: the symbol '$(\mathcal{M}, x) \models \alpha$' is read as " a formula $\alpha$ is true at $x$ in $\mathcal{M}$".

(i) $(\mathcal{M}, x) \models p_i$ iff $x \in \mathrm{u}(p_i) \cdot \mathrm{G}$.

(ii) $(\mathcal{M}, x) \models \alpha \wedge \beta$ iff $(\mathcal{M}, x) \models \alpha$ and $(\mathcal{M}, x) \models \beta$.

(iii) $(\mathcal{M}, x) \models \neg\alpha$ iff $\forall y \in \mathrm{G}$, [ $(\mathcal{M}, y) \models \alpha$ only if $y^* \cdot x = 0$ ].

∎

Comparing orthomodular models with the Kripke-type semantics for orthomodular logic discussed in the Goldblatt's paper [4], we notice that $\mathcal{G}$ plays the role of, so to say, a frame. For each formula $\alpha$, define $\|\alpha\|^{\mathcal{M}} := \{x \in \mathrm{G} \mid (\mathcal{M}, x) \models \alpha\}$. Then we can restate the above conditions in the following way:

(i) $\|p_i\|^{\mathcal{M}} = \mathrm{u}(p_i) \cdot \mathrm{G}$.

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$.

(iii) $\|\neg\alpha\|^{\mathcal{M}} = \{x \in \mathrm{G} \mid \forall y \in \|\alpha\|^{\mathcal{M}} \ (y^* \cdot x = 0)\ \}$.

**Definition 5.2** Let $\alpha$ and $\beta$ be formulas.

(i) $\alpha$ *implies* $\beta$ at $x$ in an OM model $\mathcal{M}$ ( $(\mathcal{M}, x) : \alpha \models \beta$ ) iff either $(\mathcal{M}, x) \models \alpha$ does not hold or $(\mathcal{M}, x) \models \beta$ holds.

(ii) $\alpha$ *implies* $\beta$ in an OM model $\mathcal{M}$ ( $\mathcal{M} : \alpha \models \beta$ ) iff for all $x$ in the model $\mathcal{M}$, $(\mathcal{M}, x) : \alpha \models \beta$ holds.

∎

It is easy to see that $\mathcal{M} : \alpha \models \beta$ is equivalent to $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$. We will show that the following two statements (S) and (T) are mutually equivalent.

(S): For given formulas $\alpha$ and $\beta$, for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$.

(T): For given formulas $\alpha$ and $\beta$, and for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

First we will show that the direction ( (S) $\Rightarrow$ (T) ). In order to do this, we need the following lemma.

**Lemma 5.3** Let $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ be an orthomodular model and $e$ such an orthomodular valuation from $\Phi$ to $\mathrm{P_c(G)}$ that $e(p_i) = \mathrm{u}(p_i)$ holds for all propositional variables. Then for any formula $\alpha$, $\|\alpha\|^{\mathcal{M}} = e(\alpha) \cdot \mathrm{G}$ holds.
**Proof** : We use the induction on the construction of the formula $\alpha$.

(i) Step $\alpha = p_i$: It is obvious from $e(p_i) = \mathrm{u}(p_i)$.

(ii) Step $\alpha = \sigma \wedge \tau$: By the induction hypothesis, we have that $\|\sigma\|^{\mathcal{M}} = e(\sigma) \cdot \mathrm{G}$ and $\|\tau\|^{\mathcal{M}} = e(\tau) \cdot \mathrm{G}$. Then by Lemma 4.7 and Proposition 4.8, we can derive that $(e(\sigma) \sqcap e(\tau)) \in \mathrm{P_c(G)}$ and that $e(\sigma) \cdot \mathrm{G} \cap e(\tau) \cdot \mathrm{G} = (e(\sigma) \sqcap e(\tau)) \cdot \mathrm{G}$. Thus we have $\|\sigma \wedge \tau\|^{\mathcal{M}} = e(\sigma \wedge \tau) \cdot \mathrm{G}$ since $e$ is an orthomodular valuation.

(iii) Step $\alpha = \neg\sigma$: Since $e$ is an orthomodular valuation, we have that $e(\neg\sigma) = e(\sigma)^{\mathrm{r}}$. Hence it is enough to show that $\|\neg\sigma\|^{\mathcal{M}} = e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$. Take an arbitrary $p \in \|\neg\sigma\|^{\mathcal{M}} = \{x \in \mathrm{G} \mid \forall y \in \|\sigma\|^{\mathcal{M}} \ (y^* \cdot x = 0)\ \}$. By the induction hypothesis, we have that $\|\sigma\|^{\mathcal{M}} = e(\sigma) \cdot \mathrm{G}$. This $p$ satisfies that $y^* \cdot p = 0$ for all $y \in \|\sigma\|^{\mathcal{M}} = e(\sigma) \cdot \mathrm{G}$. In particular, if we take $e(\sigma)$ for $y$, then we have that $e(\sigma) \cdot p = e(\sigma)^* \cdot p = 0$. Therefore $p \in \{e(\sigma)\}^{(\mathrm{r})} = e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$. Thus $\|\neg\sigma\|^{\mathcal{M}} \subseteq e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$.

Conversely take an arbitrary $q \in e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$. Then we have that $e(\sigma) \cdot q = 0$. Now we consider any $y \in \|\sigma\|^{\mathcal{M}} = e(\sigma) \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$ such that $y = e(\sigma) \cdot s$. By multiplying $e(\sigma)\cdot$ from the left, we derive that $e(\sigma) \cdot y = e(\sigma) \cdot e(\sigma) \cdot s = e(\sigma) \cdot s = y$. So we have that $y^* \cdot q = y^* \cdot e(\sigma)^* \cdot q = y^* \cdot e(\sigma) \cdot q = 0$. Therefore $q \in \|\neg\sigma\|^{\mathcal{M}}$. Thus $\|\neg\sigma\|^{\mathcal{M}} \supseteq e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$ and consequently we have shown that $\|\neg\sigma\|^{\mathcal{M}} = e(\sigma)^{\mathrm{r}} \cdot \mathrm{G}$.

□

Now we can prove the soundness theorem.

**Theorem 5.4 ((S) ⇒ (T)(Soundness))** For given formulas $\alpha$ and $\beta$, let (S) and (T) be the statements as follows:
(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$.
(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.
Then (S) implies (T).
**Proof** : Suppose (T) does not hold. Then there exists some OM model $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ and some point $x$ in $\mathcal{M}$ such that $(\mathcal{M}, x) \models \alpha$ and $(\mathcal{M}, x) \not\models \beta$ hold. This means $\|\alpha\|^{\mathcal{M}} \not\subseteq \|\beta\|^{\mathcal{M}}$, that is, $e(\alpha) \cdot \mathrm{G} \not\subseteq e(\beta) \cdot \mathrm{G}$. So by the definition of the order on $\mathrm{P}_c(\mathrm{G})$, this is equivalent to $e(\alpha) \not\leq e(\beta)$. Since $\mathrm{P}_c(\mathrm{G})$ is an orthomodular lattice and $e$ is an orthomodular valuation, we can conclude that (S) does not hold. □

# 6 Monotone, residuated maps on an ordered set

Next, we will prove the Completeness Theorem. To show the direction ((S) ⇐ (T)), we need to know how to build up an orthomodular model from a given orthomodular lattice. To do this, we need some preparations.

**Definition 6.1 (Residuated, monotone maps on an ordered set)** Let $\langle \mathrm{A}, \leq \rangle$ be an ordered set.

(i) A map $\varphi$ from A to A is called *monotone* iff it satisfies the following condition: for any $x, y \in \mathrm{A}$, if $x \leq y$, then $\varphi(x) \leq \varphi(y)$.

We denote the set of all monotone maps from A to A by $\overline{\mathrm{G}}(\mathrm{A})$.

(ii) A map $\varphi \in \overline{\mathrm{G}}(\mathrm{A})$ is called *residuated* iff there exists a map $\varphi^\sharp \in \overline{\mathrm{G}}(\mathrm{A})$ such that for any $x \in \mathrm{A}$, $\varphi^\sharp(\varphi(x)) \geq x$ and $\varphi(\varphi^\sharp(x)) \leq x$.

We call this map $\varphi^\sharp$ a residual map for $\varphi$, and denote the set of all residuated, monotone maps on A by G(A). ▮

It will be shown later that the set G(A) can be the base set of a Rickart * semigroup, if we define suitable operations for $\cdot$ and *. First we will see the relation between $\varphi$ and its residual map $\varphi^\sharp$.

**Lemma 6.2** Let $\langle A, \leq \rangle$ be an ordered set. Then the following holds.

(i) For any $\varphi \in G(A)$, the residual map for $\varphi$ is uniquely determined.

(ii) For any $\varphi, \psi \in G(A)$, $(\varphi \cdot \psi)^\sharp = \psi^\sharp \cdot \varphi^\sharp$ holds, where $\cdot$ means the composition operator for maps. Therefore G(A) is closed under this operation $\cdot$ .

**Proof** :

(i) Let $\psi_1$ and $\psi_2$ be residual maps for a monotone map $\varphi$. Then the following equations hold for any $x \in A$.

$$[\psi_i(\varphi(x)) \geq x \text{ and } \varphi(\psi_i(x)) \leq x] \quad ( i = 1, 2 ) \quad \cdots (\star)$$

By $(\star)$, we have $\psi_1(\varphi(x)) \geq x$. Thus in particular, $\psi_1(\varphi(\psi_2(x))) \geq \psi_2(x)$ holds. Again by $(\star)$, we have $\varphi(\psi_2(x)) \leq x$. Since $\psi_1$ is monotone, $\psi_1(\varphi(\psi_2(x))) \leq \psi_1(x)$ holds. Therefore we have $\psi_1(x) \geq \psi_2(x)$. Similarly we can show that $\psi_1(x) \leq \psi_2(x)$. Thus $\psi_1(x) = \psi_2(x)$ holds for any $x$.

(ii) Let $\varphi^\sharp$ and $\psi^\sharp$ be the resudual maps for $\varphi$ and $\psi$ respectively. Then for any $x \in A$, we have that (1): $\varphi^\sharp(\varphi(x)) \geq x$ and that (2): $\psi(\psi^\sharp(x)) \leq x$.

From (1), we have $\varphi^\sharp(\varphi(\psi(x))) \geq \psi(x)$ in particular. Hence by (2) and the monotonicity of $\psi^\sharp$, we have $\psi^\sharp(\varphi^\sharp(\varphi(\psi(x)))) \geq \psi^\sharp(\psi(x)) \geq x$, which means that $(\psi^\sharp \cdot \varphi^\sharp) \cdot (\varphi \cdot \psi)(x) \geq x$.

Similarly by (2), we have $\psi(\psi^\sharp(\varphi^\sharp(x))) \leq \varphi^\sharp(x)$ in particular. Hence by (1) and the monotonicity of $\varphi$, we have $\varphi(\psi(\psi^\sharp(\varphi^\sharp(x)))) \leq \varphi(\varphi^\sharp(x)) \leq x$, which means that $(\varphi \cdot \psi) \cdot (\psi^\sharp \cdot \varphi^\sharp)(x) \leq x$.

So $\psi^\sharp \cdot \varphi^\sharp$ is the residual map for $\varphi \cdot \psi$, that is, $(\varphi \cdot \psi)^\sharp = \psi^\sharp \cdot \varphi^\sharp$. Therefore the set G(A) is closed under the composition operator $\cdot$.

□

It is guaranteed by (i) of Lemma 6.2 that we can write the resudual map for $\varphi$ as $\varphi^\sharp$. And (ii) of Lemma 6.2 means that G(A) is a semigroup with respect to the operation $\cdot$. We will show next that there exists the zero element in G(A).

**Lemma 6.3** Let $\langle A, \leq, \mathbf{0}, \mathbf{1} \rangle$ be an ordered set with the minimun element $\mathbf{0}$ and the maximum element $\mathbf{1}$ and let $\theta$ be a map defined by the condition: for all $x \in A$, $\theta(x) = \mathbf{0}$. Then $\theta$ is the zero element in the semigroup G(A).
**Proof** : Clearly $\theta$ is monotone, and the residual map $\iota$ for $\theta$ is given by the definition: for any $x \in A$, $\iota(x) = \mathbf{1}$ Clearly $\iota$ is monotone, and both $\iota(\theta(x)) = \mathbf{1} \geq x$ and $\theta(\iota(x)) = \mathbf{0} \leq x$ hold for any $x \in A$. Thus $\theta \in G(A)$. It is obvious that for all $\varphi \in G(A)$, $\varphi \cdot \theta = \theta \cdot \varphi = \theta$ holds. □

Next we must think about the unary operator * in G(A) when A is an ortholattice. What kind of unary operators on G(A) satisfies the conditions of the operator * in Rickart * semigroups?

**Lemma 6.4** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, ^{\perp}, \mathbf{1}, \mathbf{0} \rangle$ be an ortholattice. Let * be defined by the following: for any $\varphi \in \mathrm{G(A)}$, $\varphi^*(x) := (\varphi^\sharp(x^\perp))^\perp$ for any $x \in \mathrm{A}$. Then $\varphi^* \in \mathrm{G(A)}$. Moreover the following condtions hold for every $\varphi, \psi \in \mathrm{G(A)}$.

(a) $\varphi^{**} = \varphi$.

(b) $(\varphi \cdot \psi)^* = \psi^* \cdot \varphi^*$.

**Proof** : We put $\psi(x) := (\varphi(x^\perp))^\perp$ for any $x \in \mathrm{A}$ and show that $\psi = \varphi^{*\sharp}$.

(i) First we will show that $\psi$ is monotone. Suppose that $x \leq y$ for $x, y \in \mathrm{A}$. Then by the properties of the operation $^\perp$, we have $x^\perp \geq y^\perp$. Since $\varphi$ is monotone, we have $\varphi(x^\perp) \geq \varphi(y^\perp)$. Again by the properties of $^\perp$, we have $(\varphi(x^\perp))^\perp \leq (\varphi(y^\perp))^\perp$, which means $\psi(x) \leq \psi(y)$. Therefore $\psi$ is monotone.

(ii) Next we will show that $\psi$ is the residual map for $\varphi$. By the properties of the operation $^\perp$ and the properties of $\varphi^\sharp$, we can derive: $\psi \cdot \varphi^*(x) = \psi \cdot (\varphi^\sharp(x^\perp))^\perp = [\varphi(\varphi^\sharp(x^\perp))^{\perp\perp}]^\perp = [\varphi(\varphi^\sharp(x^\perp))]^\perp \geq x^{\perp\perp} = x$. So we have $\psi \cdot \varphi^*(x) \geq x$. Similarly we can derive: $\varphi^* \cdot \psi(x) = \varphi^* \cdot (\varphi(x^\perp))^\perp = [\varphi^\sharp(\varphi(x^\perp))^{\perp\perp}]^\perp = [\varphi^\sharp(\varphi(x^\perp))]^\perp \leq x^{\perp\perp} = x$. So we have $\varphi^* \cdot \psi(x) \leq x$.

Hence we can conclude that $\psi = \varphi^{*\sharp}$ since the residual map of $\varphi^*$ is unique. By (i) and (ii) in the above, we have that $\varphi^* \in \mathrm{G(A)}$. Thus * is a unary operator on G(A). Now we will check the conditions (a) and (b). By the properties of the operation $\perp$, and the definition of $\varphi^*$, we calculate as follows: for any $\varphi, \psi$, and for any $x \in \mathrm{A}$,

(a): $\varphi^{**}(x) = [\varphi^{*\sharp}(x^\perp)]^\perp = [(\varphi(x^{\perp\perp})^\perp]^\perp = \varphi(x)$.

(b): $\psi^* \cdot \varphi^*(x) = \psi^*(\varphi^\sharp(x^\perp))^\perp = [\psi^\sharp(\varphi^\sharp(x^\perp))^{\perp\perp}]^\perp = [\psi^\sharp \cdot \varphi^\sharp(x^\perp)]^\perp = [(\varphi \cdot \psi)^\sharp(x^\perp)]^\perp = (\varphi \cdot \psi)^*(x)$.

Consequently this * satisfies conditions for the operator * in Rickart * semigroups. □

From the above consideration, we can define the notions of projection, closed projection and right annihilator for an element in G(A). In order to get a Rickart * semigroup from G(A), we must show that for any element $\varphi \in \mathrm{G(A)}$, there exists some closed projection $\mu$ such that $\{\varphi\}^{(\mathrm{r})} := \{\psi \in \mathrm{G(A)} \mid \varphi \cdot \psi = \theta\} = \mu \cdot \mathrm{G(A)}$.

**Lemma 6.5** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, ^{\perp}, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice. For each $a \in \mathrm{A}$, define a map $\gamma_a$ by $\gamma_a(x) := (x \sqcup a^\perp) \sqcap a$ for every $x \in \mathrm{A}$.

(i) $\gamma_a$ is a projection in G(A) for any $a \in \mathrm{A}$.

(ii) For any $\varphi \in \mathrm{G(A)}$, if we put $a := \varphi^\sharp(\mathbf{0})$, then $\{\varphi\}^{(\mathrm{r})} = \gamma_a \cdot \mathrm{G(A)}$ holds.

**Proof** : By our assumption, the following orthomodular law holds. For $a, b, c \in \mathrm{A}$,
(1) $a \leq b$ implies $b = (b \sqcap a^\perp) \sqcup a$. (2) $c \leq a$ implies $c = (c \sqcup a^\perp) \sqcap a$.
It is easy to see that (2) follows from (1) and vice versa.

(i) First we will show that $\gamma_a \in \mathrm{G(A)}$. It is obvious that$\gamma_a$ is monotone. We put $\psi(x) := (x \sqcap a) \sqcup a^\perp$ for any $x$ in A. Clearly $\psi$ is also monotone. Moreover, as shown below, it is the residual map for $\gamma_a$.

$$\begin{aligned} \gamma_a \cdot \psi(x) &= [((x \sqcap a) \sqcup a^\perp) \sqcup a^\perp] \sqcap a \\ &= [(x \sqcap a) \sqcup a^\perp] \sqcap a \\ &= x \sqcap a \le x. \end{aligned}$$

In the last equation in the above, we used (2) since $x \sqcap a \le a$.

$$\begin{aligned} \psi \cdot \gamma_a(x) &= [((x \sqcup a^\perp) \sqcap a) \sqcap a] \sqcup a^\perp \\ &= [(x \sqcup a^\perp) \sqcap a] \sqcup a^\perp \\ &= x \sqcup a^\perp \ge x \end{aligned}$$

Also, we used (1) since $x \sqcup a^\perp \ge a^\perp$.
Therefore $\gamma_a{}^\sharp(x) = \psi(x) = (x \sqcap a) \sqcup a^\perp$. So $\gamma_a \in \mathrm{G(A)}$.

Next we will show that $\gamma_a$ satisfies the conditions for projections.

$$\begin{aligned} \gamma_a{}^* = (\gamma_a{}^\sharp(x^\perp))^\perp &= [(x^\perp \sqcap a) \sqcup a^\perp]^\perp \\ &= (x^\perp \sqcap a)^\perp \sqcap a^{\perp\perp} \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x) \end{aligned}$$

$$\begin{aligned} \gamma_a \cdot \gamma_a(x) &= [\{(x \sqcup a^\perp) \sqcap a\} \sqcup a^\perp] \sqcap a \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x) \end{aligned}$$

Since $(x \sqcup a^\perp) \sqcap a \le a$, we used (2) in the above calculation. Thus $\gamma_a$ is a projection.

(ii) First we will prove that $\gamma_a \cdot \mathrm{G(A)} \subseteq \{\varphi\}^{(\mathrm{r})}$. Take any $\psi \in \gamma_a \cdot \mathrm{G(A)}$. Then there exists some element $\lambda \in \mathrm{G(A)}$ such that $\psi = \gamma_a \cdot \lambda$. For any $x \in \mathrm{A}$, $\gamma_a(x) = (x \sqcup a^\perp) \sqcap a \le a = \varphi^\sharp(\mathbf{0})$. So by the monotonicity of $\varphi$, we have that $\varphi \cdot \gamma_a(x) \le \varphi \cdot \varphi^\sharp(\mathbf{0}) \le \mathbf{0}$. This means that $\varphi \cdot \gamma_a = \theta$. Then $\varphi \cdot \psi = \varphi \cdot \gamma_a \cdot \lambda = \theta$, that is $\psi \in \{\varphi\}^{(\mathrm{r})}$.
Thus we conclude that $\gamma_a \cdot \mathrm{G(A)} \subseteq \{\varphi\}^{(\mathrm{r})}$. Next we will show that $\{\varphi\}^{(\mathrm{r})} \subseteq \gamma_a \cdot \mathrm{G(A)}$. Take any $\psi \in \{\varphi\}^{(\mathrm{r})}$.
Then $\psi$ satisfies that $\varphi \cdot \psi = \theta$, which means that for any $x \in \mathrm{A}$, we have that $\varphi \cdot \psi(x) = \mathbf{0}$. Taking $\mathbf{1}$ for $x$, we have $\varphi \cdot \psi(\mathbf{1}) = \mathbf{0}$, and hence $a = \varphi^\sharp(\mathbf{0}) = \varphi^\sharp \cdot \varphi \cdot \psi(\mathbf{1}) \ge \psi(\mathbf{1})$. Therefore we have that for any $x \in \mathrm{A}$, $\psi(x) \le \psi(\mathbf{1}) \le a$. By combining this result with the orthomodular law (2), we have that $\gamma_a \cdot \psi(x) = (\psi(x) \sqcup a^\perp) \sqcap a = \psi(x)$. Consequently $\psi = \gamma_a \cdot \psi \in \gamma_a \cdot \mathrm{G(A)}$.
Thus we have proved $\{\varphi\}^{(\mathrm{r})} = \gamma_a \cdot \mathrm{G(A)}$.

□

Moreover, we can show the following lemma on the set of maps $\gamma_a$.

**Lemma 6.6** For any orthomodular lattice $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$, the relation $\mathrm{P}_c(\mathrm{G}(\mathrm{A})) = \{\gamma_a \mid a \in \mathrm{A}\}$ holds.

**Proof** : Take any $\lambda \in \mathrm{P}_c(\mathrm{G}(\mathrm{A}))$. Then there exists some $\mu \in \mathrm{G}(\mathrm{A})$ such that $\{\mu\}^{(\mathrm{r})} = \lambda \cdot \mathrm{G}(\mathrm{A})$. Now putting $b := \mu^{\sharp}(\mathbf{0})$, we have $\{\mu\}^{(\mathrm{r})} = \gamma_b \cdot \mathrm{G}(\mathrm{A})$ by Lemma 6.5 (ii). So the uniqueness of the right annihilating projection gives us that $\lambda = \gamma_b \in \{\gamma_a \mid a \in \mathrm{A}\}$.
Conversely, consider $\gamma_a$ for $a \in \mathrm{A}$. Since $\gamma_a$ is a projection, $\gamma_a = \gamma_a \cdot \gamma_a = {\gamma_a}^*$ holds. We have that $\gamma_a \cdot {\gamma_a}^{\mathrm{r}} = \theta$. So by operating $*$ to this equation, we get ${\gamma_a}^{\mathrm{r}} \cdot \gamma_a = \theta$. Then of course, ${\gamma_a}^{\mathrm{r}} \cdot \gamma_a \cdot \lambda = \theta$ for any $\lambda \in \mathrm{G}(\mathrm{A})$ holds. Therefore we get $\{{\gamma_a}^{\mathrm{r}}\}^{(\mathrm{r})} = \gamma_a \cdot \mathrm{G}(\mathrm{A})$. Thus $\gamma_a \in \mathrm{P}_c(\mathrm{G}(\mathrm{A}))$.
Consequently we have proved that $\mathrm{P}_c(\mathrm{G}(\mathrm{A})) = \{\gamma_a \mid a \in \mathrm{A}\}$. □

By all the lemmas 6.2, 6.3, 6.4 and 6.5, we can prove the following theorem.

**Theorem 6.7** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle \mathrm{G}(\mathrm{A}), \cdot, * \rangle$ is a Rickart * semigroup, where $\cdot$ is a composition operator of maps and * is a unary operator defined in Lemma 6.3 .

# 7 Canonical model and Completeness Theorem

Now we have prepared all the notions for constructing the canonical model for orthomodular logic.

**Definition 7.1 (Canonical model)** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice, and $v : \Phi \to \mathrm{A}$ an orthomodular valuation. The canonical model for $\mathcal{A}$ and $v$ is the structure $\mathcal{M}_{\mathcal{A}} = \langle \mathrm{G}(\mathrm{A}), \cdot, *, \mathrm{u}_{\mathcal{A}} \rangle$, where

(i) G(A) is the set of all residuated monotone maps on A,

(ii) $\cdot$ is the composition operator of maps on A,

(iii) $*$ is the unary operator on G(A) defined in Lemma 6.4, that is,
for any $\varphi \in \mathrm{G}(\mathrm{A})$, $\varphi^*(x) := (\varphi^{\sharp}(x^{\perp}))^{\perp}$ for all $x \in \mathrm{A}$,

(iv) $\mathrm{u}_{\mathcal{A}}$ is a function assigning to each propositional variable $p_i$ an element of the set $\{\gamma_a \mid a \in \mathrm{A}\}$, such that, $\mathrm{u}_{\mathcal{A}}(p_i) := \gamma_{v(p_i)}$.

∎

**Lemma 7.2** Let $\mathcal{A}$ be an orthomodular lattice and $v$ an orthomodular valuation. Then the canonical model $\mathcal{M}_{\mathcal{A}} = \langle \mathrm{G}(\mathrm{A}), \cdot, *, \mathrm{u}_{\mathcal{A}} \rangle$ is an orthomodular model.

**Proof** : By Theorem 6.7, we have that $\mathcal{G}(\mathcal{A})$ is a Rickart * semigroup. We can also see that $\mathrm{u}_{\mathcal{A}}$ is a function from $\Phi$ to the set of closed projections in G(A) by Lemma 6.6 . So $\mathcal{M}_{\mathcal{A}}$ satisfies all the conditions for an orthomodular model in Definition 5.1 . □

Since $\mathcal{M}_\mathcal{A}$ is an orthomodular model, the notion of truth in $\mathcal{M}_\mathcal{A}$ can be defined similarly in Definition 5.1 as follows Let $\alpha$, $\beta$ be formulas, $\varphi$, $\psi$ elements in G(A). Then:

(i) $(\mathcal{M}_\mathcal{A}, \varphi) \models p_i$ iff $p_i \in \mathrm{u}(p_i) \cdot \mathrm{G(A)}$.

(ii) $(\mathcal{M}_\mathcal{A}, \varphi) \models \alpha \wedge \beta$ iff $(\mathcal{M}_\mathcal{A}, \varphi) \models \alpha$ and $(\mathcal{M}_\mathcal{A}, \varphi) \models \beta$.

(iii) $(\mathcal{M}_\mathcal{A}, \varphi) \models \neg\alpha$ iff $\forall \psi \in \mathrm{G(A)}$, [ $(\mathcal{M}_\mathcal{A}, \varphi) \models \alpha$ only if $\psi^* \cdot \varphi = 0$ ].

By denoting $\|\alpha\|^{\mathcal{M}_\mathcal{A}} := \{\varphi \in \mathrm{G(A)} \mid (\mathcal{M}_\mathcal{A}, \varphi) \models \alpha\}$, we can restate the above conditions in the following way.

(i) $\|p_i\|^{\mathcal{M}_\mathcal{A}} = \mathrm{u}(p_i) \cdot \mathrm{G(A)}$.

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}_\mathcal{A}} = \|\alpha\|^{\mathcal{M}_\mathcal{A}} \cap \|\beta\|^{\mathcal{M}_\mathcal{A}}$.

(iii) $\|\neg\alpha\|^{\mathcal{M}_\mathcal{A}} = \{\varphi \in \mathrm{G(A)} \mid \forall \psi \in \|\alpha\|^{\mathcal{M}_\mathcal{A}} \ (\psi^* \cdot \varphi = 0)\ \}$.

Here we will make a comment about the order on $\mathrm{P}_c(\mathrm{G(A)})$, where A is an orthomodular lattice. Because $\gamma_a \in \mathrm{P}_c(\mathrm{G})$ is a projection, the order on the set $\{\gamma_a \mid a \in \mathrm{A}\}$ is defined as in Definition 4.3, that is,

$$\text{For } a, b \in \mathrm{A}, \qquad \gamma_a \leq \gamma_b \quad \text{iff} \quad \gamma_a \cdot \gamma_b = \gamma_a$$

By Lemma 4.2, we have that $\gamma_a \leq \gamma_b$ is equivalent to $\gamma_a \cdot \mathrm{G(A)} \subseteq \gamma_b \cdot \mathrm{G(A)}$.
We can show the following lemma on this order relation.

**Lemma 7.3** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, {}^\perp, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice. Then the following two conditions are equivalent.

(i) $a \leq b$ on A.

(ii) $\gamma_a \leq \gamma_b$ on $\mathrm{P}_c(\mathrm{G(A)})$.

**Proof** : ( (i)$\Rightarrow$(ii) ): Suppose that $a \leq b$. Then, for all $x \in \mathrm{A}$ the following holds:

$$\begin{aligned} \gamma_b \cdot \gamma_a(x) &= [\{(x \sqcup a^\perp) \sqcap a\} \sqcup b^\perp] \sqcap b \\ &= (x \sqcup a^\perp) \sqcap a = \gamma_a(x) \end{aligned}$$

Since we have $(x \sqcup a^\perp) \sqcap a \leq a \leq b$, we used the orthomodular law (2) in the proof of Lemma 6.5 . Thus we conclude that $\gamma_a \leq \gamma_b$.
( (i)$\Leftarrow$(ii) ): Suppose that $\gamma_a \leq \gamma_b$. This means that $\gamma_a \cdot \gamma_b = \gamma_b \cdot \gamma_a = \gamma_a$. Since $\gamma_a(\mathbf{1}) \leq \mathbf{1}$, $\gamma_a(\mathbf{1}) = \gamma_b \cdot \gamma_a(\mathbf{1}) = \gamma_b(\gamma_a(\mathbf{1})) \leq \gamma_b(\mathbf{1})$. Recall here that $\gamma_a(x) := (x \sqcup a^\perp) \sqcap a$ for any $x \in \mathrm{A}$, then we have that $a = \gamma_a(\mathbf{1}) \leq \gamma_b(\mathbf{1}) = b$. $\square$

As in Lemma 5.3, we can also extend the domain of valuation function $\mathrm{u}_\mathcal{A}$ from the set of propositional variables to the set of all formulas $\Phi$.

**Lemma 7.4** Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, {}^{\perp}, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice and $v$ an orthomodular valuation. Let $\mathcal{M}_{\mathcal{A}}$ be the canonical orthomodular model corresponding to $\mathcal{A}$. Then for any formula $\alpha$, $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\alpha)} \cdot \mathrm{G(A)}$.

**Proof** : We use the induction on the construction of the formula $\alpha$.

(i) Step $\alpha = p_i$: Trivial from the definition of $\mathrm{u}_{\mathcal{A}}$.

(ii) Step $\alpha = \sigma \wedge \tau$: By induction hypothesis, we have that $\|\sigma\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\sigma)} \cdot \mathrm{G(A)}$, and that $\|\tau\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\tau)} \cdot \mathrm{G(A)}$. So by Proposition 4.8, we can derive that $\|\sigma \wedge \tau\|^{\mathcal{M}_{\mathcal{A}}} = \|\sigma\|^{\mathcal{M}_{\mathcal{A}}} \cap \|\tau\|^{\mathcal{M}_{\mathcal{A}}} = (\gamma_{v(\sigma)} \sqcap \gamma_{v(\tau)}) \cdot \mathrm{G(A)}$. Therefore we have only to show that $\gamma_{v(\sigma)} \sqcap \gamma_{v(\tau)} = \gamma_{v(\sigma\wedge\tau)}$. But it is obvious by checking that $\gamma_{v(\sigma\wedge\tau)}$ satisfies the conditions for the infimum of $\gamma_{v(\sigma)}$ and $\gamma_{v(\tau)}$, since we have Lemma 7.3 .

(iii) Step $\alpha = \neg\sigma$: First we show that $\gamma_{v(\sigma)^{\perp}} = \gamma_{v(\sigma)}{}^{\mathrm{r}}$. Now we consider a map $\gamma_{v(\sigma)}{}^{\sharp}(x) = (x \sqcap v(\sigma)) \sqcup v(\sigma)^{\perp}$ for $x$ in A. So $\gamma_{v(\sigma)}{}^{\sharp}(\mathbf{0}) = v(\sigma)^{\perp}$. Then by Lemma 6.5 (ii), we can conclude that $\{\gamma_{v(\sigma)}\}^{(\mathrm{r})} = \gamma_{v(\sigma)^{\perp}} \cdot \mathrm{G(A)}$. The uniqueness of the right annihilating projection gives us that $\gamma_{v(\sigma)^{\perp}} = \gamma_{v(\sigma)}{}^{\mathrm{r}}$.

By the induction hypothesis, we have that $\|\sigma\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\sigma)} \cdot \mathrm{G(A)}$. By the above result and the properties of orthomodular valuation $v$, we have that $\gamma_{v(\neg\sigma)} = \gamma_{v(\sigma)^{\perp}} = \gamma_{v(\sigma)}{}^{\mathrm{r}}$. So it is enough to show that $\|\neg\sigma\|^{\mathcal{M}_{\mathcal{A}}} \subseteq \gamma_{v(\sigma)^{\mathrm{r}}} \cdot \mathrm{G(A)}$ and that $\gamma_{v(\sigma)^{\mathrm{r}}} \cdot \mathrm{G(A)} \subseteq \|\neg\sigma\|^{\mathcal{M}_{\mathcal{A}}}$.

Take an arbitrary $\varphi \in \|\neg\sigma\|^{\mathcal{M}_{\mathcal{A}}} = \{\varphi \in \mathrm{G(A)} \mid \forall\psi \in \|\sigma\|^{\mathcal{M}_{\mathcal{A}}} \, (\psi^* \cdot \varphi = \theta)\}$. Then $\varphi$ satisfies that $\psi^* \cdot \varphi = \theta$ for any $\psi \in \|\sigma\|^{\mathcal{M}_{\mathcal{A}}}$. By taking $\gamma_{v(\sigma)}$ for $\psi$, we have $\gamma_{v(\sigma)}{}^* \cdot \varphi = \gamma_{v(\sigma)} \cdot \varphi = \theta$. Therefore we conclude that $\varphi \in \{\gamma_{v(\sigma)}\}^{(\mathrm{r})} = \gamma_{v(\sigma)^{\mathrm{r}}} \cdot \mathrm{G(A)}$

Conversely take an arbitrary $\varphi \in \gamma_{v(\sigma)^{\mathrm{r}}} \cdot \mathrm{G(A)}$. Then we have that $\gamma_{v(\sigma)} \cdot \varphi = \theta$. Now for any $\psi \in \|\sigma\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\sigma)} \cdot \mathrm{G(A)}$, there exists some $\lambda$ such that, $\psi = \gamma_{v(\sigma)} \cdot \lambda$ . Thus $\psi^* \cdot \varphi = (\gamma_{v(\sigma)} \cdot \lambda)^* \cdot \varphi = \lambda^* \cdot \gamma_{v(\sigma)} \cdot \varphi = \theta$. Therefore we conclude that $\varphi \in \|\neg\sigma\|^{\mathcal{M}_{\mathcal{A}}}$. Consequently we have proved that $\|\neg\sigma\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\neg\sigma)^{\mathrm{r}}} \cdot \mathrm{G(A)}$.

□

We have now reached the following Completeness Theorem.

**Theorem 7.5** **((S)⇐(T)(Completeness))** For given formulas $\alpha$ and $\beta$, let (S) and (T) be the same statements in Theorem 5.4 . That is,

(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$.

(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

Then (T) implies (S).

**Proof** : Suppose (S) does not hold. Then there exists an orthomodular lattice $\mathcal{A}$, and an orthomodular valuation$v$ such that $v(\alpha) \not\leq v(\beta)$. Take the canonical orthomodular model $\mathcal{M}_{\mathcal{A}}$ for $\mathcal{A}$ and $v$. Then by the above relation, we have that $\gamma_{v(\alpha)} \not\leq \gamma_{v(\beta)}$. This is equivalent to $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\alpha)} \cdot \mathrm{G(A)} \not\subseteq \gamma_{v(\beta)} \cdot \mathrm{G(A)} = \|\beta\|^{\mathcal{M}_{\mathcal{A}}}$. So there exists some point $\varphi$ in the OM model $\mathcal{M}_{\mathcal{A}}$ such that $\varphi \in \|\alpha\|^{\mathcal{M}_{\mathcal{A}}}$, but $\varphi \notin \|\beta\|^{\mathcal{M}_{\mathcal{A}}}$. This $\varphi$ satisfies that $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha$, and $(\mathcal{M}_{\mathcal{A}}, \varphi) \not\models \beta$. Therefore for this orthomodular model $\mathcal{M}_{\mathcal{A}}$,

$\mathcal{M}_{\mathcal{A}} : \alpha \not\models \beta$. Consequently (T) does not hold. □

# 8 Concluding Remarks

The present work is based mainly on Goldblatt's paper [4], and the following representation theorem by Foulis.

**Theorem 8.1 (Foulis's representation theorem)** Let $\mathcal{A}$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle \mathrm{G(A)}, \cdot, * \rangle$ is a Rickart * semigroup and A is isomorphic to $\mathrm{P}_c(\mathrm{G(A)})$.

□

In the above theorem, $\mathcal{G}(\mathcal{A})$ is defined in Theorem 6.7, and $\mathrm{P}_c(\mathrm{G(A)})$ is the set of all closed projections in G(A).

As we mentioned, Goldblatt proposed a Kripke-style semantics for orthomodularf logic in [4]. He restricted his orthomodel in some way and proved the completeness theorem of orthomodular logic with respect to this restricted model. He called it *quantum model.* Indeed, he used the same frame as is described in Definition 2.1, and made some restriction on the range of the function V. Therefore the carrier X, the relation $\perp$ and the notion of truth of quantum models are quite the same as those of orthomodels.

Comparing our orthomodular models with his quantum models, we can easily see that restriction on the range of V in orthomodel is correspond to restriction on the range of u into $\mathrm{P}_c(\mathrm{G})$ in orthomodular model. So situation is similar. Moreover we notice that the corresponding binary relation R on Rickart * semigroups to the relation $\perp$ is as follows: for $x, y \in \mathrm{G}$, $x\mathrm{R}y$ iff $y^* \cdot x = 0$. This relation R is symmetric and irreflexive without the zero element of G. Indeed it is easy to show that for an $x \in \mathrm{G}$, if $x\mathrm{R}x$, then $x = 0$.

Therefore our orthomodular models are special members of Goldblatt's quantum models.

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