Common Developments Folding into Two or More Convex Polyhedra

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Developments have been investigated as a part of geometry since the 1500s. It is the area that there are a lot of open problems even now and they are investigated actively. Since Lubiw and O'Rourke posed the problem in 1996, polygons that can fold into a polyhedron have been investigated. Demaine and O'Rourke published a book about geometric folding algorithms that includes many results about such polygons in 2007. Such polygons have applications in the form of toys and puzzles. For example, the puzzle "cubigami" is developed by Miller and Knuth. It is a common development that fold into seven different tetracubes.

In this paper, we study common developments that fold into two or more incongruent orthogonal boxes. The developments are simple polygons that consist of unit squares of four connected. It is allowed to fold the developments only in unit square. Therefore, the developments that consist of n unit squares fold the orthogonal boxes of the surface area n. In the problems of development and folding, the following question has been investigated: "Is there a polygon that fold into two incongruent orthogonal boxes?" For such a question, Biedl et al. showed such two developments concretely.

If an orthogonal box is developed simply, it is known that the development may have an overlap or a hole. Such developments are not so desirable practically. Thus, if there is a hole in the inside of developments, we do not regard such polygons as developments in this paper. Also if there is a cut in the inside of developments, such polygons cannot fold another box using the cut. Therefore, we consider developments without such a cut in this paper.

In order to search for common developments that fold into two or more incongruent orthogonal boxes, first, it is necessary to consider a set of boxes whose surface areas are equal and sides are different. Specifically, it is enough to investigate the combinations of a, b, c, a', b', c' that satisfy $2(ab+bc+ca) = 2(a'b'+b'c'+c'a')$. Only such the combination allows us to make a common dovelopment of two different boxes of sizes $a\times b\times c$ and $a'\times b'\times c'$. Using a computer, we can find them easily by a brute force if $1 \leq a, b, c, a', b', c' \leq 50$. For the surface area 22, there are orthogonal boxes of size $1\times 1\times 5$ and $1\times 2\times 3$. It is the smallest surface area that different orthogonal boxes can appear. For looking for common developments that fold into three or more different orthogonal boxes, it is necessary to investigate at least 46 or more surface areas.

Tens of thousands of common developments that fold into two incongruent orthogonal boxes were found by the computer experiment of Mitani and Uehara in 2008. By using the result, it is proved constitutively that such developments exist infinitely. Furthermore, they searched not only for two different orthogonal boxes but also for three such boxes. However, they were not able to find common developments that fold into three different orthogonal boxes.

In their paper, they searched for common developments that fold into two or more different orthogonal boxes by two algorithms. The first algorithm cuts the edges on an orthogonal box at random and develops it. It is the method of finding common developments of the same form from different orthogonal boxes by generating many random developments. The second algorithm connects a unit square at a time at random from scratch. It is the method of generating common developments until it becomes two orthogonal boxes. Since they are randomized algorithms both, the developments are generated at random. Therefore, the existing algorithms do not necessarily enumerate all such developments.

In this paper, first we consider for enumeration algorithms that enumerate all common developments that fold into two given different orthogonal boxes. The algorithm that we considered is first started from one unit square. By connecting unit square at each edge, many partial common developments are generated. Finally unit square is increased to the size of the surface area of given orthogonal boxes.

For two boxes of surface areas 22 of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$, we implemented this enumeration algorithm. As a result, we enumerated all developments that fold into such boxes. The number of common developments of these boxes is 2263. In the result, there is only one interesting common development. The development can also fold into box of $0 \times 1 \times 11$ (of volume 0) further. Such a box is sometimes called a "doubly covered rectangle". That is, we can fold into three different boxes by admiting a box to have volume 0. Also the development has a tiling property.

Unfortunately, we did not reach to find common developments that fold into three incongruent orthogonal boxes. After this research, such developments were found by

Shirakawa and Uehara in 2012. Now the minimum surface area of such developments is 532. It is an open problem how many minimum surface area of developments that fold into three orthogonal boxes is. In particular, it is a future work whether there is a common development that can fold into three boxes of surface area 46 of size $1 \times 1 \times 11$, $1 \times 2 \times 7$ and $1 \times 3 \times 5$.