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Description	

Bipartite Permutation Graphs are Reconstructible

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The graph reconstruction conjecture is a long-standing open problem in graph theory.
The conjecture has been verified for all graphs with at most 11 vertices. Further, the
conjecture has been verified for regular graphs, trees, disconnected graphs, unit interval
graphs, separable graphs with no pendant vertex, outer-planar graphs, and unicyclic
graphs. We extend the list of graph classes for which the conjecture holds. We give a
proof that bipartite permutation graphs are reconstructible.

Keywords: the graph reconstruction conjecture, bipartite permutation graphs

1. Introduction

The graph reconstruction conjecture proposed by Ulam and Kelly[a] has been studied
by many researchers intensively. In order to state the conjecture, we first introduce
some terms. A graph G' is called a *card* of a graph $G = (V, E)$, if G' is isomorphic
to $G-v$ for some $v \in V$, where $G-v$ is a graph obtained from G by removing v and
incident edges. A multi-set of n graphs with $n-1$ vertices for some positive integer
n is called a *deck*. Especially, the multi-set of the $|V|$ cards of G, each of which is
isomorphic to $G - v$ for each $v \in V$, is a deck of G. A graph G is a *preimage* of
a deck D, if D is a deck of G. We also say that a graph G is a preimage of the n
graphs if each card of G is isomorphic to each of them. The graph reconstruction
conjecture is that there is at most one preimage of given n graphs ($n \geq 3$). No one
has found a proof nor a counter example of this conjecture, while it was verified for
small graphs by exhaustive checking [12].

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[a]Determining the first person who proposed the graph reconstruction conjecture is difficult, actu-
ally. See [7] for the detail.

The graph reconstruction conjecture has been verified for some graph classes. Kelly showed that the conjecture is true on regular graphs, trees, and disconnected graphs [9]. Other classes proven to be reconstructible[b] are unit interval graphs [13], separable graphs with no pendant vertex [2], outer-planar graphs [5], unicyclic graphs [11], etc. We extend the list of graph classes for which the conjecture holds; We give a proof that bipartite permutation graphs are reconstructible.

Rimscha showed that unit interval graphs are reconstructible [13]. Unit interval graphs have somewhat path-like structures, and so do bipartite permutation graphs. Further, the representation of a unit interval graph is unique, similar to that of a bipartite permutation graph. Thus, we first thought that we can easily prove that bipartite permutation graphs are reconstructible. There are two differences between the two classes, that make it difficult to prove that bipartite permutation graphs are reconstructible. One is that, bipartite permutation graphs are bipartite. Therefore, we have to determine from which partite a vertex was removed for cards in a deck. The second difference is that, in the case of unit interval graphs, there is no disconnected card obtained by removing a vertex laying at the end of the path structure. In a deck of a bipartite permutation graph, there can be a disconnected card that is obtained by removing a polar vertex which lays at the end of the path structure (We will define a polar vertex later.) Therefore, we had to consider many exceptional cases.

Kelly showed the following lemma.

Lemma 1.1 (Kelley's Lemma [9]). *Let G be any preimage of the given deck, and let H be a graph whose number of vertices is smaller than that of G. Then we can uniquely determine the number of subgraphs in G isomorphic to H from the deck.*

Greenwell and Hemminger extended this lemma to a more general form [6]. We can determine the degree sequence of a preimage of the given deck from these lemmas. Moreover, given a deck of a graph, we can determine the degree of removed vertex for each card in the deck. Note that $\sum_{v:\text{ vertex}} \deg(v) = 2 \times (\# \text{ of edges})$. Thus, we can easily show for example that cycles are reconstructible, since a graph is a cycle if and only if it is connected, and all its vertices have degree exactly two.

Tutte proved that the dichromatic rank and Tutte polynomials are reconstructible (i.e. looking at the deck, they are uniquely determined) [15]. Bollobás showed that almost all graphs are reconstructible from three well-chosen graphs in its deck [1]. About permutation graphs, Rimscha showed that permutation graphs are recognizable in the sense that looking at the deck of G one can determine whether or not G belongs to permutation graphs [13]. To be precise Rimscha showed in the paper that comparability graphs are recognizable. Even's result [4] directly gives a proof in the case of permutation graphs. Rimscha also showed in the same paper that many subclasses of perfect graphs including perfect graphs themselves

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[b]A graph is *reconstructible*, if its deck has only one preimage.

are recognizable, and moreover, some of subclasses, such as unit interval graphs, are reconstructible. There are a lot of papers about the conjecture, and many good surveys about this conjecture. See for example [3,7].

We explain about bipartite permutation graphs in the next section. Then, we prove the statement in Section 3. The proof has two subsections. In the first subsection, we give the main idea of the proof. In the second subsection, we consider some exceptional cases. The proof uses some lemmas on bipartite permutation graphs. Since we think that checking these lemmas one by one may make readers lose the way, we write the proofs of some of them in Section 4.

2. Bipartite Permutation Graphs

All the graphs in this paper are simple and undirected unless stated otherwise. Given a graph $G = (V, E)$ and a vertex $v \in V$, we denote by $\deg_G(v)$ the degree of v in G.

2.1. *Permutation diagram*

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $1, \ldots, n$. We denote $(\pi_n, \pi_{n-1}, \ldots, \pi_1)$ by $\overline{\pi}$.

We call a set L of line segments connecting two horizontal parallel lines on Euclidean plane a *permutation diagram*. A permutation diagram represents a permutation. Let $l_1, l_2, \ldots, l_{|L|}$ be the line segments in L. We assume that the end-points of them appear in this order from left to right on the upper horizontal line. Then, the permutation represented by L is $(\pi_1, \pi_2, \ldots, \pi_{|L|})$, where the end-points of $l_1, l_2, \ldots, l_{|L|}$ appear in the order of $\pi_1, \ldots, \pi_n$ on the lower horizontal line. Equivalently, the ith left-most end-point among those of the segments in L is that of $l_{\pi_i^{-1}}$, on the lower horizontal line, for each $i \in \{1, 2, \ldots, |L|\}$. See Fig. 1(a) for example. The permutation diagram represents $(2, 7, 3, 5, 1, 6, 4) = (5, 1, 3, 7, 4, 6, 2)^{-1}$.

Let P be a permutation diagram. We denote by P^{H} a permutation diagram obtained by reversing P horizontally. See Fig. 1(b) for example. The permutation diagram is obtained by reversing (a) horizontally. Similarly, we denote by P^{V} and P^{R} permutation diagrams obtained by reversing P vertically, and by rotating P $180°$, respectively.[c]

2.2. *Bipartite permutation graphs*

Let π be a permutation of the numbers $1, 2, \ldots, n$. $G_\pi = (V_\pi, E_\pi)$ is a graph satisfying that

- $V_\pi = \{1, \ldots, n\}$, and

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[c]Let P be a permutation diagram representing a permutation π. For those who want concrete expressions, it is not difficult to check that P^{V} represents $\pi^{\mathrm{V}} = \pi^{-1}$, P^{H} represents $\pi^{\mathrm{H}} = \overline{\pi^{-1}}^{-1}$, and P^{R} represents $\pi^{\mathrm{R}} = \overline{\pi^{-1}}$.
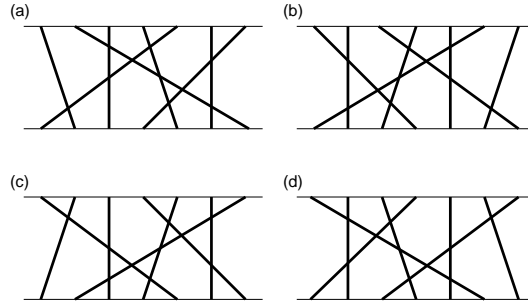
Fig. 1. (a) is an example of a permutation diagram. (b), (c), and (d) are permutation diagrams obtained from (a) by reversing horizontally, reversing vertically, and rotating $180°$, respectively. They represent permutations (2,7,3,5,1,6,4), (4,2,7,3,5,1,6), (5,1,3,7,4,6,2), and (6,2,4,1,5,7,3), respectively.

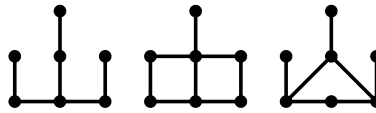Fig. 2. Forbidden graphs of bipartite permutation graphs are these graphs, K_3, and cycles of length more than four.

- $\{i, j\} \in E_\pi \Leftrightarrow (i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0.$

A graph G is called a *permutation graph* if there exists a permutation π such that G is isomorphic to G_π. Equivalently, a graph G is a permutation graph if there exists a permutation π such that G is an intersection model of the permutation diagram of π. We say that π (or, sometimes, the permutation diagram of π) represents G. If a permutation graph G is bipartite, we call G a *bipartite permutation graph*.

There is a good characterization for bipartite permutation graphs.

Theorem 2.1 (P. Hell and J. Huang [8]). *A graph G is a bipartite permutation graph if and only if G has neither the graphs in Fig. 2, nor K_3, nor cycles of length more than four as an induced subgraph.*

It is known that a connected bipartite permutation graph has at most four representing permutation diagrams. If a permutation diagram P is representing a connected bipartite permutation graph G, the other representing permutation diagram of G must be one of P^H, P^V, and P^R [14]. Thus a permutation diagram representing a connected bipartite graph is essentially unique. Note that a disconnected bipartite permutation graph may have more than four representing permutation diagrams. Together with the fact that cards in a deck of a connected graph can be disconnected, this is the reason why our proof is not very simple.
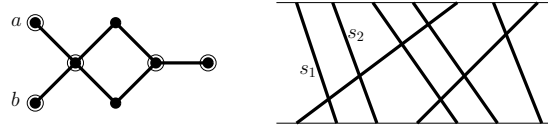
Fig. 3. An bipartite permutation graph and its representation. The polar vertices are circled. Vertices a and b are isomorphic, and can correspond to both the segments s_1 and s_2. Thus, both a and b are polar vertices.

Let P be a permutation diagram representing a connected bipartite permutation graph G. There are two left-most segments in P, and there are two right-most segments in P. Here, we say that a segment is the *left-most* (*right-most*) if it is the left-most (right-most) among the segments not intersecting with it. We call the vertices that can correspond to the left-most or right-most segments *polar vertices*. the number of polar vertices can be less than four, since the left-most segment in P can also be the right-most segment when the segment intersecting all the segments belonging to the other partite. The number of polar vertices may be more than four, since there may exist some isomorphic polar vertices.[d] See Fig. 3 for an example.

By repeatedly removing degree one polar vertices from a connected bipartite permutation graph G, we obtain a connected bipartite permutation graph G'. We call the graph G' *trunk* of G, and we denote the trunk by $\text{Tr}(G)$. The vertex in $\text{Tr}(G)$ nearest from a degree one polar vertex v of G is called the *root* of v. The path in G whose ends are v and v's root is called a *limb*.

It is clear that every card G' of a bipartite permutation graph G is a bipartite permutation graph, since we can obtain a representing permutation diagram of G' by removing a line segment from a representing permutation diagram of G.

3. Main Proof

The main idea of our proof is simple. However, if there is a degree one polar vertex, there are many exceptional cases, and the proof gets complex. Therefore, we first show the simple case, and then prove the exceptional cases.

3.1. *No degree one polar vertex case*

We show an algorithm which reconstructs G from its deck. The algorithm directly shows the uniqueness of the preimage. However, the proof of the uniqueness uses a bunch of bipartite permutation graph specific properties. We are afraid that checking the properties one by one makes the readers lose the way in the main-line of the proof. Therefore, we leave some of the proofs in Section 4.

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[d]Vertices v and u are isomorphic, if the neighbors of them are identical.

We need the two lemmas below to keep the main proof simple.

Lemma 3.1. *All the preimages of the deck of a bipartite graph G are bipartite.*

Proof. Immediate from the fact that the chromatic number of G is reconstructible [16]. □

Lemma 3.2. *All the preimages of a deck of a bipartite permutation graph are bipartite permutation graphs.*

Proof. Immediate from Lemma 3.1 and the fact that permutation graphs are recognizable [13]. □

We can easily check the following lemma.

Lemma 3.3. *A card obtained from a connected bipartite graph G by removing its polar vertex is connected, if every polar vertex of G has degree more than one.*

Proof. Let L be a set of segments representing G. Let l_1 be the left-most segment in L, and l_2 be the second left-most one. Note that there must exist l_2, since the right-most polar vertex has degree more than one. Since G is connected, $N_L(l_1) \subset N_L(l_2)$ holds, where $N_L(l)$ is the set of segments in L intersecting l. Therefore, for a polar vertex v corresponding to l_1, there exist a vertex whose neighbors contain the neighbors of v. Thus the graph obtained from G by removing v is connected. □

Our main proof assumes that a bipartite permutation graph $G = (X, Y, E)$ does not have a polar vertex in X whose degree is $|Y|$. Therefore, we need the following lemma.

Lemma 3.4. *Let $G = (X, Y, E)$ be a connected bipartite permutation graph such that a polar vertex in X has degree equal to $|Y|$. Then G is reconstructible.*

We leave the proof in Section 4. In the rest of this subsection, we consider a connected bipartite permutation graph $G = (X, Y, E)$ satisfying the condition below.

Condition 1. *There is no polar vertex of degree $|Y|$ in X, and there is no polar vertex of degree $|X|$ in Y.*

Note that, under this condition, a polar vertex on the left-end cannot be adjacent to any polar vertex on the right-end. And, $|X|, |Y| \geq 2$ holds, since if $|X| = 1$, then $x \in X$ must be a polar vertex and adjacent to every vertex in Y.

Now, we explain the main idea. Let $G = (X, Y, E)$ be a connected bipartite permutation graph satisfying Condition 1.

We can prove that $|X|$ and $|Y|$ are reconstructible. Since the proof of this fact becomes a bit long, we leave it in Section 4.

We first consider the case that $|X| \neq |Y|$. We assume without loss of generality that $|X| > |Y|$. There are the left- and the right-polar vertices x_l and x_r in X, such that x_l corresponds to the left-most line segment, and x_r corresponds to the right-most line segment in a permutation diagram representing G. In a similar fashion, we define vertices y_l and y_r as the left- and the right-polar vertices in Y. We assume without loss of generality that $\deg_G(x_l) \leq \deg_G(x_r)$ holds.

Let $G_l = (X_l, Y_l, R_l)$ and $G_r = (X_r, Y_r, R_r)$ be cards of G obtained by removing y_l and y_r from G, respectively. By Lemma 3.3, G_l and G_r are connected. We denote by D_Y the set of G's connected cards that are obtained by removing a vertex belonging to Y. Clearly, G_l and G_r are in D_Y.

Consider a connected bipartite permutation graph $G' = (X', Y', E')$ in D_Y. We assume without loss of generality that $|X'| \geq |Y'|$ holds. Then, since $|X| > |Y|$ holds, $|X'| = |X|$ and $|Y'| = |Y| - 1$ hold. On the other hand, with a connected graph $G'' = (X'', Y'', E'')$ obtained by removing a vertex in X from G, $|X''| = |X| - 1$, $|Y''| = |Y|$ hold. Therefore, we can choose all the cards that belong to D_Y from the deck of G, and we can determine which partite of each card corresponds to X.

Now, consider the degrees of the polar vertices in X'. If G' is G_l, the degrees of the polar vertices in X' are $\{\deg_G(x_l) - 1, \deg_G(x_r)\}$. If G' is G_r, the degrees of the polar vertices in X' are $\{\deg_G(x_l), \deg_G(x_r) - 1\}$. Otherwise, the degrees of the polar vertices in X' are either $\{\deg_G(x_l) - 1, \deg_G(x_r)\}$, $\{\deg_G(x_l), \deg_G(x_r) - 1\}$, $\{\deg_G(x_l), \deg_G(x_r)\}$, or $\{\deg_G(x_l) - 1, \deg_G(x_r) - 1\}$. We call G' *good*, if the degrees of the polar vertices in X' are $\{\deg_G(x_l) - 1, \deg_G(x_r)\}$. Note that we can chose good cards from D_Y, since $\deg_G(x_l) - 1$ is the minimum degree of the polar vertices, and $\deg_G(x_r)$ is the maximum degree of the polar vertices. And, the key point is that, in a good card, the degrees of the left- and the right-polar vertices in X differ.

Let $\{G'_1, \ldots, G'_k\}$ be the set of good graphs in D_Y. Let v_i be a vertex such that G'_i is obtained by removing v_i from G. We can determine $\deg_G(v_i)$, since the degree sequence is reconstructible. We can prove that $\deg_G(y_l) = \min_{i=1,\ldots,k} \deg_G(v_i)$ by Lemma 4.2 which we will state in Section 4. Thus, we can uniquely reconstruct the preimage from the deck. (We only have to add a degree $\deg_G(y_l)$ polar vertex in Y' adjacent to the polar vertex of degree $\deg_G(x_l) - 1$ in X'. This can be done deterministically on the permutation diagram by Lemma 4.3 which we will prove in Section 4.)

Now we consider the case that $|X| = |Y|$. In this case, every connected card is in the form $G_i = (X_i, Y_i, E)$ such that $|X_i| = |Y_i| - 1$. Thus we cannot determine which partite of G' corresponds to which partite of G. However, we know that G_i is obtained by removing a vertex in Y_i's partite. Therefore, polar vertices of G_i in X_i are also polar vertices of a preimage. Thus, the minimum degree of polar vertices in X_i among all the connected cards is equal to $p' - 1$, where p' is the minimum degree of polar vertices of a preimage. Moreover, a card that has a polar vertex of degree $p' - 1$ in X_i is obtained by removing a vertex adjacent to a polar vertex of degree p' from a preimage. Hence, we can uniquely reconstruct a preimage in the

same fashion above (using Lemmas 4.2, 4.3).

Therefore, we have the theorem below.

Theorem 3.5. *A connected bipartite permutation graph $G = (X, Y, E)$ satisfying Condition 1 is reconstructible, if every polar vertex of G has degree more than one.*

3.2. Polar vertices with degree one

We can determine if a preimage G has a polar vertex of degree one, by Lemma 4.4 in Section 4. In this subsection, we consider the case that G has a polar vertex of degree one.

First, we show the fundamental lemmas.

Lemma 3.6. *Let P be a permutation diagram of a connected bipartite permutation graph G having at least one cycle. Any two limbs of the same side (the left-side or the right-side) of P have the same root.*

Proof. If not, G cannot be connected.   □

Lemma 3.7. *If a connected bipartite permutation graph G has a polar vertex of degree one, $Tr(G)$ is reconstructible.*

Proof. Let G' be a card obtained by removing a polar vertex of degree one from G. Then, $Tr(G')$ and $Tr(G)$ are clearly isomorphic. Let G'' be a connected card obtained by removing a vertex that is not a polar vertex of degree one. Then, $|V(Tr(G''))| < |V(Tr(G))|$ holds. Thus, we can reconstruct $Tr(G)$ by choosing the $Tr(G')$ whose number of vertices is the maximum.   □

Now we prove the reconstructivity, one by one.

Lemma 3.8. *A connected bipartite permutation graph $G = (X, Y, E)$ with a limb whose length is more than one is reconstructible.*

Proof. Let L be a permutation diagram of G. Let $\{G'_1, \ldots, G'_k\}$ be the multi-set of connected cards of G that satisfy $Tr(G'_i) = Tr(G)$. If there are more than one limbs having the same root, and both of them have the lengths more than one, G contains the left forbidden graph in Fig 2. Thus, only one limb can have the length more than one among limbs of the same root. We concentrate the limbs of the maximum length, on the both sides.

First we consider the case that there are two limbs, one is on the left-side of L having the maximum length among limbs on the left-side, and the other is on the right-side having the maximum length among limbs on the right-side. If both the two limbs have the lengths more than one, we can easily reconstruct G, since we can determine the lengths p, q of the two limbs from $\{G'_1, \ldots, G'_k\}$. Note that each G'_i has limbs of lengths $p - 1, q$, or p, q, or $p, q - 1$. Hence, we consider the case that the

left-side limb has the length exactly one. The right-side limb has the length q more than one. Even in this case, we can determine that the maximum lengths of limbs on the both sides of any preimage are one and q. The remaining problem is how to reconstruct G from $\{G'_1, \ldots, G'_k\}$. If q is more than two, the reconstruction is easy. Only find the limb of length $q - 1$, and add a degree one vertex to it. Thus, we consider the case that q is equal to two. In this case, there is a card in $\{G'_1, \ldots, G'_k\}$ that has length one limbs on the both side. Thus, we can determine if the roots of the two limbs of a preimage belongs to the same partite. And, there is a card G' in $\{G'_1, \ldots, G'_k\}$ that has a limb l of the length of two. Thus, we can reconstruct G uniquely by adding a degree one vertex to the opposite side to l.

Next, we consider the case that G has limbs only on the left-side. It is easy to reconstruct G in this case, since finding the connected card that has limbs most, and adding a degree one vertex to the longest limb (the other limbs have length one), we have G. □

Lemma 3.9. *A bipartite permutation graph $G = (X, Y, E)$ with two limbs of different roots is reconstructible.*

Proof. From Lemma 3.8, we only have to prove the case that every limb has length exactly one. In this case, we can determine if the two roots belong to the same vertex set, since we can reconstruct $\{|X|, |Y|\}$. Thus we can reconstruct G uniquely. □

Lemma 3.10. *A bipartite permutation graph $G = (X, Y, E)$ whose limbs have the same root is reconstructible.*

Proof. If there are more than one limbs, it is easy to reconstruct G. Let G' be a connected card satisfying $\mathrm{Tr}(G') = \mathrm{Tr}(G)$. Find a limb in G' and add a degree one vertex to its root. Therefore, we consider the case that G has only one limb, and the length is one. Assume that the limb is on the left-side of L, where L is a permutation diagram of G. Then two polar vertices on the right-side have degrees p, q larger than one. If both of p and q are larger than two, we can reconstruct G, since connected cards of G with degree one polar vertex on the one side of their permutation diagram have polar vertices of degree p, q, or $p - 1, q$, or $p, q - 1$, on the opposite side.

Now we consider the case that p is exactly two, and q is also equal to two. There is a connected card of G whose polar vertices on the same side have degrees one and two. Since the limb of G has length exactly one, the polar vertices of the same side having degree one and two cannot be the degree one vertex of G. Therefore we can reconstruct G uniquely.

Lastly, we consider the case that p is exactly two, and q is larger than two. Let $\{G'_1, \ldots, G'_k\}$ be connected cards of G obtained by removing a vertex whose degree is larger than one. Looking $\{G'_1, \ldots, G'_k\}$, we can determine the value p and q. Hence we can reconstruct G uniquely. □

From above lemmas, we have the theorem below.

Theorem 3.11. *A connected bipartite permutation graph with a polar vertex of degree one is reconstructible.*

Combining Lemma 3.4, Theorems 3.5 and 3.11, we have the main theorem. Note that since disconnected graphs are reconstructible, disconnected bipartite permutation graphs are of course reconstructible.

Theorem 3.12. *Bipartite permutation graphs are reconstructible.*

4. Miscellaneous Proofs

We prove Lemmas not proved yet, in this section.

Lemma 4.1. *The numbers of vertices in X and Y are reconstructible for a connected bipartite permutation graph $G = (X, Y, E)$.*

Proof. Let D be the deck of G. There are at least two connected cards in a deck of a connected graph with more than two vertices (Consider removing a vertices which are leaves of a spanning tree of G.) Let $G_1 = (X_1, Y_1, E_1), G_2 = (X_2, Y_2, E_2), \ldots, G_k = (X_k, Y_k, E_k)$ be connected bipartite permutation graphs in D. The following cases can occur.

(1) $\{|X_i|, |Y_i|\} = \{p_1, q_1\}$ for some $i \in \{1, \ldots, k\}$, and $\{|X_i|, |Y_i|\} = \{p_2, q_2\}$ for other $i \in \{1, \ldots, k\}$, where $\{p_1, q_1\} \neq \{p_2, q_2\}$ holds.
(2) $\{|X_i|, |Y_i|\}$ is the identical set $\{p, q\}$ for every $i \in \{1, \ldots, k\}$.

First we consider the case 1. Clearly, $\max\{p_1, p_2, q_1, q_2\}$ is equal to $\max\{|X|, |Y|\}$, and $\min\{p_1, q_1, p_2, q_2\}$ is equal to $\min\{|X|, |Y|\} - 1$. Therefore, we can uniquely determine $\{|X|, |Y|\}$, in this case.

Now, we consider the case 2. There are two more detailed cases. One case is that $|X| = |Y|$ (case 2a), and the other case is that every connected card is obtained by removing a vertex from one partite (case 2b).

In the case 2a, $\max\{p, q\} = \min\{p, q\} + 1 = |X| = |Y|$ holds. Thus, we can determine $\{|X|, |Y|\}$, if we can realize that the case is 2a, not 2b. We explain how to distinguish the case 2a from the case 2b later.

In the case 2b, let T be a spanning tree of G. Since a graph obtained from G by removing a leaf of T is connected, all the leaves of T belong to the same partite. We can assume without loss of generality that the partite is X. Then apparently $|X| > |Y|$ holds. Thus $\{|X|, |Y|\}$ is $\{\max\{p, q\} + 1, \min\{p, q\}\}$.

The remaining problem is how to distinguish the case 2a from the case 2b. In the case 2a, $|p - q| = 1$ always holds. Therefore, we consider the case that $|p - q| = 1$ holds in the case 2b. In this case, $|X| = |Y| + 2$ must hold.

Let L be a permutation diagram of G. Let x_l and x_r be polar vertices in X that correspond to the left-most segment, and the right-most segment of L, respectively.

Let P be the shortest path in G from x_l to x_r. Let y_l be the vertex adjacent to x_l in P, and let y_r be the vertex adjacent to x_r in P. Since every vertex $y \in Y$ is a cut-vertex of G, every path from x_l to x_r passes y. Therefore, all the vertices in Y are in P. Hence, there exist $|Y| + 1$ X-vertices in P. Note that the graph induced from G by these $|Y| + 1$ X-vertices and all the vertices in Y is exactly P, since otherwise some vertex in Y is not a cut-vertex of G. Since we here consider the case that $|X| = |Y| + 2$ holds, there is only one vertex $v \in X$ remaining not in P. There are four possibilities,

 (i) The vertex v is adjacent to y_l, and not adjacent to y_r.
 (ii) The vertex v is adjacent to y_r, and not adjacent to y_l.
 (iii) The vertex v is adjacent to both y_l and y_r.
 (iv) The vertex v is not adjacent to neither y_l nor y_r.

In the case i, iii, and iv, by removing a vertex y_r, we obtain a isolated vertex x_r and a connected component of remaining vertices. In the case ii, by removing a vertex y_l, we obtain a isolated vertex x_l and a connected component of remaining vertices. In both the cases, the connected components are bipartite, and the difference of the numbers of vertices in the two partite is exactly two. On the other hand, if there is a card consists of an isolated vertex and a connected component in the case 2a, the size of each partite of the connected component must be the same. Therefore, we can distinguish the two cases. □

Proof of Lemma 3.4. If $\min\{|X|, |Y|\} = 1$, G is a tree, and is thus reconstructible. Therefore we assume that $\min\{|X|, |Y|\} \geq 2$.

Let x be a polar vertex adjacent to every vertex in Y. Let L be a permutation diagram representing G. Assume without loss of generality that x corresponds to a line segment s in L whose lower-end is the left-most among all the lower-ends of the segments in L. Then each vertex in $X \setminus \{x\}$ corresponds to each segment that lays on the right-side of s in L. On the other hand, a segment s' corresponding to a vertex in Y must intersect with s. Therefore, the upper-end of s' must be on the left-side of that of s. Consider the segment s'' whose upper-end is the right-most among segments corresponding to vertices in Y. Then lower-end of s'' must be the right-most, since otherwise G cannot be connected. Hence, the vertex corresponding to s'' is adjacent to all the vertices in X. This means that if a bipartite permutation graph $G = (X, Y, E)$ has a polar vertex $x \in X$ satisfying $\deg(x) = |Y|$, then G also has a polar vertex $y \in Y$ satisfying $\deg(y) = |X|$. See Fig. 4 for an illustration.

Now, we know that there are two special polar vertices x and y in G. There must be two other polar vertices. One corresponds to the segment in L whose upper-end is the left-most, and the other corresponds to the segment whose upper-end is the right-most. We denote the vertices by v and w, respectively. We assume without loss of generality $|X| \geq |Y|$. Removing $v(\in Y)$ results in a connected bipartite graph G'. The size of the vertex sets of G is $|X|$ and $|Y| - 1$. Thus, there is at least one connected card whose vertex sets have sizes $|X|$ and $|Y| - 1$. Moreover, since
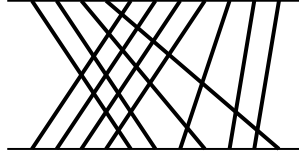
Fig. 4. An example of permutation diagram of a connected bipartite graph $G = (X, Y, E)$ with a polar vertex $x \in X$ whose degree is $|Y|$.

$|X| > |Y| - 1$, we can find such a card from the deck of G. Let G'' be a card of G whose vertex sets have sizes $|X|$ and $|Y| - 1$. Since the degree sequence of G is reconstructible, we can determine the degree of the vertex z, where G'' is obtained by removing z from G. Hence, we can determine the preimage uniquely. □

Lemma 4.2. *Given a connected bipartite permutation graph $G = (X, Y, E)$ satisfying Condition 1, let x be a polar vertex in X, and let Y' be the set of vertices adjacent to x. A vertex $y \in Y'$ is a polar vertex of G if and only if y's degree is the minimum in Y'.*

Proof. Let L be a permutation diagram representing G. Since x is a polar vertex of G, we can assume without loss of generality that the line segment s in L whose upper-end is the left-most corresponds to x. Then, all the line segments corresponding to the vertices in $X \setminus \{x\}$ are at the right-side of s.

   Let y^* be a polar vertex in Y'. Since G satisfies Condition 1, a line segment s' in L corresponding to y^* is the left-most in those corresponding to vertices in Y'. Therefore, the degree of y^* is the minimum among the vertices in Y'. □

Lemma 4.3. *Let $G' = (X', Y', E')$ be a connected bipartite permutation graph. Let x be a polar vertex in X'. Then, graph $G = (X, Y, E)$ that is obtained by adding a degree $k \in \{1, \ldots, |X'|\}$ vertex y to Y' is uniquely determined, under the conditions that G is a bipartite permutation graph, y is a polar vertex of G, and y is adjacent to x in G.*

Proof. Let L' be a permutation diagram representing G', and let L be a permutation diagram representing G. It is clear that L can be obtained by adding to L' a line segment s_y corresponding to y.

   Since x is a polar vertex of G', we can assume without loss of generality that the line segment s_x in L' and L corresponding to x is the left-most among those corresponding to vertices in X. We can assume without loss of generality that the upper-end of s_x is the left-most among the upper-ends of all the segments in L. Since y is a polar vertex in G, s_y in L corresponding to y is the left-most among those corresponding to vertices in Y. That is, the lower-end of the s_y is the left-most among the lower-ends of all the segments in L. Then, we can determine the position

of the upper-end of s_y uniquely, since s_y must intersect to exactly k segments in X. □

Lemma 4.4. *The number of polar vertices whose degree is one is reconstructible for a connected bipartite permutation graph $G = (X, Y, E)$ satisfying Condition 1.*

Proof. If a polar vertex v of G has degree one, the polar vertex adjacent to v has degree more than one, since otherwise G is disconnected.

When we remove a polar vertex that is adjacent to another polar vertex of degree one from G, we obtain a graph consisting of some isolated vertices and a connected component. Conversely, if there is a graph consisting of some isolated vertices and a connected component in the deck of G, this graph must be obtained from some preimage by removing a polar vertex adjacent to an other polar vertex of degree one. Otherwise, there must be at least two connected components. Thus, the number of polar vertices whose degree is one is equal to the number of cards consisting of some isolated vertices and a connected component. □

5. Concluding Remarks

Proving that permutation graphs are reconstructible is a challenging problem. Since a permutation diagram of a permutation graph is not unique, it seems not to be easy. Recently, we developed an algorithm which enumerates all the preimages of a deck that consists of permutation graphs in the polynomial time [10]. The algorithm shows that the number of preimages for a deck of permutation graphs is at most $O(n^3)$.

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