An Extension of Context Model for Representing Vague Knowledge

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Abstract: In this paper, a framework for representing vague knowledge based on the notion of context model introduced by Gebhardt and Kruse (1993) is discussed. From a concept analysis point of view, it has been shown that the context model can be semantically considered as a data model for fuzzy concept analysis (Huynh *et al.*, 2004). From a decision analysis point of view, in order to deal with the problem of synthesis of vague evidence linguistically provided by experts in some situations of decision analysis, the notions of context-dependent vague characteristics and fuzzy context model will be introduced. It is shown that each context-dependent vague characteristic within fuzzy context model directly induces a uncertainty measure of type 2 interpreted as "vague" belief function, which is inferred from vague evidence expressed linguistically.

Keywords: Context model; Dempster-Shafer theory; fuzzy context model; vague knowledge; uncertainty measure of type 2.

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1 INTRODUCTION

The notion of context model was introduced by Gebhardt and Kruse (1993) as an integrating model of vagueness and uncertainty. The motivation for the context model arises from the intention to develop a common formal framework that supports a better understanding and comparison of existing models of partial ignorance to reduce the rivalry between well-known approaches. Particularly, the authors presented basic ideas keyed to the interpretation of Bayes theory and the Dempster-Shafer theory within the context model. Furthermore, a direct comparison between these two approaches based on the well-known decision-making problems within the context model were also examined in the paper.

In (Huynh and Nakamori, 2001), an approach to the problem of mathematical modeling of fuzzy concepts was introduced based on the theory of formal concept analysis (Ganter and Wille, 1999) and the notion of context model (Gebhardt and Kruse, 1993; Kruse *et al.*, 1993). In particular, we introduced the notion of fuzzy formal concepts within a context model and the membership functions associated with these fuzzy concepts. It is shown that fuzzy formal concepts can be interpreted exactly as the collections of α-cuts of their membership functions. Furthermore, based on the meta-theory developed by Resconi *et al.* (1992) and Resconi *et al.* (1993), in (Huynh *et al.*, 2002) the authors consider a model of modal logic for fuzzy concept analysis from a context model. By this approach, one can integrate context models by using a model of modal logic, and then develop a method of calculating the expression for the membership functions of composed and/or complex fuzzy concepts based on values $\{0, 1\}$ corresponding to the truth values $\{F, T\}$ assigned to a given sentence as the response of a context considered as a possible world. It is of interest that fuzzy intersection and fuzzy union operators by this model form a well-known dual pair of *Product t-norm* T_P and *Probabilistic Sum t-conorm* S_P.

As such, the notion of context model can be used as a framework for modeling fuzziness in concept formation as well as uncertainty in decision analysis situations. In addition, one can also extend the notion of context model to the so-called fuzzy context model to deal with situations of data analysis where both vagueness and conflict co-exist. To clarify the motivation for such an extension, let us recall briefly the interpretation of data and the kinds of imperfectness within the context model (Gebhardt and Kruse, 1993).

According to Gebhardt and Kruse (1993), data characterizes the state of an object (obj) with respect to underlying relevant frame conditions ($cond$). In this sense, we assume that it is possible to characterize obj by an element $\mathsf{state}(obj, cond)$ of a well-defined set $\mathsf{dom}(obj)$ of distinguishable object states. $\mathsf{dom}(obj)$ is usually called the *universe of discourse* (or *frame of discernment*) of obj with respect to $cond$. Then we are interested in the problem that the original characterization of $\mathsf{state}(obj, cond)$ is not available due to a lack of information about obj and $cond$. Generally, $cond$ merely permits us to use statements like "$\mathsf{state}(obj, cond) \in \mathsf{char}(obj, cond)$", where $\mathsf{char}(obj, cond) \subseteq \mathsf{dom}(obj)$ and called an *imprecise characterization of obj* with respect to $cond$. For example, You were seeing a robbery taking place in front of Your eyes, and You also noticed a suspicious character (Suspect) who was running away. Assume that police then asked You about, say, the *height* of Suspect. Obviously, in this case You could not determine precisely the actual height of Suspect (obj), but by observation You might stated that Suspect is between 1.65 meter and 1.75 meter tall. Fornally, we may then define $\mathsf{dom}(obj) = [0, 3]$ as the domain of the *height* variable, and by taking into account the frame conditions $cond$ (Your location, distance between You and Suspect, and time of Your observation), Your statement is formulated as $\mathsf{state}(obj, cond) \in [1.65, 1.75]$, i.e., the corresponding characterization is defined by $\mathsf{char}(obj, cond) = [1.65, 1.75]$.

The second kind of imperfect knowledge in context model is *conflict* (Gebhardt and Kruse, 1993). This kind of imperfectness is induced by information about preferences between the elements of $\mathsf{char}(obj, cond)$ that interprets for the existence of contexts. The combined occurrence of imprecision and conflict in data reflects vagueness in the context model, and $\mathsf{state}(obj, cond)$ is described by the so-called *vague characteristic* of obj with respect to $cond$.

Although information about preferences between the elements of $\mathsf{char}(obj, cond)$ is modelled by contexts, this also means they have the same possibility or chance to be the unknown original value of $\mathsf{state}(obj, cond)$ in each context. However, in many practical situations, even in the same context elements of $\mathsf{char}(obj, cond)$ may have different degrees of possibility to be the unknown original value of $\mathsf{state}(obj, cond)$. Especially in the situations where $cond$ only permits us to express in the form of verbal statements like "$\mathsf{state}(obj, cond)$ is A", where A is a linguistic value represented by a fuzzy set in $\mathsf{dom}(obj)$.

The rest of this paper is organized as follows. In Section 2, basic concepts of context model are briefly presented. Section 3 provides a review of fuzzy concept analysis within the framework of context model. In Section 4, after introducing basic notions of Dempster-Shafer theory in relation to the context model, the concept of fuzzy context model is presented. Then a uncertainty measure of type 2 called vague beliefs induced from the fuzzy context model is discussed in Section 5. Finally, Section 6 presents some concluding remarks.

2 BASIC CONCEPTS OF CONTEXT MODEL

Formally, a context model is defined as a triple $\langle D, C, A_C(D)\rangle$, where D is a nonempty *universe of discourse*, C is a nonempty *finite set of contexts*, and the set $A_C(D) = \{a | a : C \to 2^D\}$ which is called the set of all vague characteristics of D with respect to C. Let $a \in A_C(D)$, a is said to be *contradictory* (respectively, *consistent*) if and only if $\exists c \in C, a(c) = \emptyset$ (respectively, $\bigcap_{c \in C} a(c) \neq \emptyset$). For $a_1, a_2 \in A_C(D)$, then a_1 is said to be *more specific* than a_2 iff $(\forall c \in C)(a_1(c) \subseteq a_2(c))$. In this paper we confine ourselves to only vague characteristics that are not contradictory in the context model.

If there is a finite measure $P_C : 2^C \to \mathbb{R}^+$ that fulfills $(\forall c \in C)(P_C(\{c\}) > 0)$, then $a \in A_C(D)$ is called a *valuated vague characteristic* of D with respect to P_C. Then we call a quadruple

$$\mathcal{C} = \langle D, C, A_C(D), P_C \rangle$$

a valuated context model. Mathematically, if $P_C(C) = 1$ the mapping $a : C \to 2^D$ is a random set but obviously with a different interpretation within the context model.

Let a be a vague characteristic in the valuated context model $\mathcal{C}$. For each $X \in 2^D$, we define the acceptance degree $\mathsf{Acc}_a(X)$ that evaluates the proposition "$\mathsf{state}(obj, cond) \in X$" is true. Due to inherent imprecision of a, it does not allow us to uniquely determine acceptance degrees $\mathsf{Acc}_a(X)$, $X \in 2^D$. However, as shown in (Gebhardt and Kruse, 1993), we can calculate lower and upper bounds for them as follows:

$$\underline{\mathsf{Acc}}_a(X) = P_C(\{c \in C | \emptyset \neq a(c) \subseteq X\}) \quad (1)$$

$$\overline{\mathsf{Acc}}_a(X) = P_C(\{c \in C | a(c) \cap X \neq \emptyset\}) \quad (2)$$

More details on the context model and its applications can be found in (Gebhardt and Kruse, 1993, 1998; Gebhardt, 2000; Kruse *et al.*, 1993).

3 FUZZY CONCEPT ANALYSIS

3.1 Fuzzy Concepts by Context Model

Fuzzy set was originally introduced as a mathematical modeling of vague concepts in natural language. Obviously, the usefulness of a fuzzy set for modeling a linguistic label depends on the appropriateness of its membership function. Therefore, the practical determination of an accurate and justifiable function for any particular situation is of major concern.

As noted in (Resconi and Turksen, 2001), the specific meaning of a vague concept in a proposition is usually evaluated in different ways for different assessments of an entity by different agents, contexts, etc. This observation has been also implicitly accepted in, e.g. (Klir, 1994).

Let us consider a context model $\mathcal{C} = \langle D, C, A_C(D) \rangle$, where D is a domain of an attribute at which is applied to objects of concern, C is a non-empty finite set of contexts, and $A_C(D)$ is a set of *linguistic terms* associated with the domain D considered now as vague characteristics in the context model. For example, consider $D = [0, 3m]$ which is interpreted as the domain of the attribute height for people, C is a set of contexts such as Japanese, American, Swede, etc., and $A_C(D) = \{$*very short, short, medium, tall, more or less tall,* $\ldots\}$. Each context determines a subset of D given as being compatible with a given linguistic term. Formally, each linguistic term can be considered as a mapping from C to 2^D. Furthermore, we can also associate with the context model a weighting function or a probability distribution Ω defined on C. As such we obtain a valuated context model

$$\mathcal{C} = \langle D, C, A_C(D), \Omega \rangle$$

By this context model, each linguistic term $a \in A_C(D)$ may be semantically represented by the fuzzy set A whose membership function, μ_A, is defined for all $x \in D$ as follows

$$\mu_A(x) = \sum_{c \in C} \Omega(c) \lambda_{a(c)}(x)$$

where $\lambda_{a(c)}$ denotes the characteristic function of $a(c)$. Intuitively, while each subset $a(c)$, for $c \in C$, represents the c's view of the vague concept a, the fuzzy set A is the result of a weighted combination view of the vague concept. At this juncture, we can formulate further for the set-theoretic operations on fuzzy sets by a straightforward manner in this model (Huynh *et al.*, 2002, 2004).

3.2 Fuzzy Set Theoretic Operations by Context Model

To deal with the general case of composed fuzzy sets which represent linguistic combinations of linguistic terms of several context models, let us consider a pair of variables x and y which may be interpreted as the values of two attributes at_1 and at_2 for objects of concern, ranging on domains D_1 and D_2, respectively. Let $\mathcal{C}_t = \langle D_t, C_t, A_{C_t}(D_t), \Omega_t \rangle$, for $t = 1, 2$ be context models defined on D_1 and D_2, respectively. Recall that each element in $A_{C_t}(D_t)$ is a linguistic term understood as a mapping from $C_t \to 2^{D_t}$.

We now define a unified Kripke model as follows

$$M = \langle W, R, V, \Omega \rangle$$

where $W = C_1 \times C_2$, R is the identity relation on W, and

$$\Omega : \quad C_1 \times C_2 \to [0, 1]$$
$$w_{(i,j)} \mapsto \omega_{ij} = \omega_i \omega_j$$

where the simplified notations $w_{(i,j)} = (c_i^1, c_j^2)$, $\Omega(w_{(i,j)}) = \omega_{ij}$, $\Omega_1(c_i^1) = \omega_i$, $\Omega_2(c_j^2) = \omega_j$ are used.

For $a_t \in A_{C_t}(D_t)$, for $t = 1, 2$, we now formulate composed fuzzy sets, which represent combined linguistic terms like *"a_1 and a_2"* and *"a_1 or a_2"* within model M.

For simplicity of notation, let us denote O a set of objects of concern which we may apply for two attributes $\mathsf{at}_1, \mathsf{at}_2$ those values range on domains D_1 and D_2, respectively. Then instead of considering fuzzy sets defined on different domains, we can consider fuzzy sets defined only on a universal set, the set of objects O. As such, we now consider atomic propositions of the form

a_o : "An object o is in relation to a linguistic term a"

where $a \in A_{C_1}(D_1) \cup A_{C_2}(D_2)$ or a is a linguistic combination of linguistic terms in $A_{C_1}(D_1) \cup A_{C_2}(D_2)$.

Notice that this constructive formulation of composed fuzzy sets is comparable with the notion of the translation of a proposition a_o into a *relational assignment equation* (Zadeh, 1975).

Case 1: a is a single term.

Firstly we consider the case where $a \in A_{C_1}(D_1)$. For this case, we define the valuation function V in M for atomic propositions a_o by

$$V(w_{(i,j)}, a_o) = \begin{cases} T & \text{if } \mathsf{at}_1(o) \in a(c_i^1) \\ F & \text{otherwise} \end{cases}$$

here $\mathsf{at}_1(o) \in D_1$ denotes the value at attribute at_1 of object o. Let us denote

$$^{(i,j)}a_o = \begin{cases} 1 & \text{if } V(w_{(i,j)}, a_o) = T \\ 0 & \text{if } V(w_{(i,j)}, a_o) = F \end{cases}$$

Then the fuzzy set A which represents the meaning of the linguistic term a is defined in the model M as follows

$$\mu_A^M(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} \, ^{(i,j)}a_o$$

Then, we have

$$\mu_A^M(o) = \mu_A^{M_1}(o)$$

where $\mu_A^{M_1}(o)$ is represented by $\mu_A^{M_1}(\mathsf{at}_1(o))$ in the model M_1 derived from $\mathcal{C}_1$.

Similar for the case where $a \in A_{C_2}(D_2)$, we define the valuation function V in M for atomic propositions a_o by

$$V(w_{(i,j)}, a_o) = \begin{cases} T & \text{if } \mathsf{at}_2(o) \in a(c_j^2) \\ F & \text{otherwise} \end{cases}$$

Obviously, we also have $\mu_A^M(o) = \mu_A^{M_2}(o)$.

Case 2: a is a composed linguistic term

We now consider for the case where a is a composed linguistic term which is of the form like *"a_1 and a_2"* and *"a_1 or a_2"*, where $a_i \in A_{C_i}(D_i)$, for $i = 1, 2$. To formulate the composed fuzzy set A corresponding to the term a in the model M, we need to define the valuation function V for propositions a_o. It is natural to express a_o by

$$a_o = \begin{cases} a_{1,o} \vee a_{2,o} & \text{if } a \text{ is "}a_1 \text{ or } a_2\text{"} \\ a_{1,o} \wedge a_{2,o} & \text{if } a \text{ is "}a_1 \text{ and } a_2\text{"} \end{cases}$$

where $a_{t,o}$, for $t = 1, 2$, are propositions of the form

$a_{t,o}$: "An object o is in relation to linguistic term a_t."

Consider the case where a is "a_1 *or* a_2". Then, the valuation function V for propositions a_o is defined as follows

$$V(w_{(i,j)}, a_{1,o} \vee a_{2,o}) = \begin{cases} T & \text{if "}\mathsf{at}_1(o) \in a_1(c_i^1) \text{ or} \\ & \mathsf{at}_2(o) \in a_2(c_j^2)\text{''} \\ F & \text{otherwise} \end{cases}$$

With this notation, we are now ready to define the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 *or* a_2" in the model M by

$$\mu_A(o) = \mu_{A_1 \cup A_2}(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} \, ^{(i,j)}(a_{1,o} \vee a_{2,o}) \qquad (3)$$

where A_1, A_2 denote fuzzy sets which represent component linguistic terms a_1, a_2, respectively.

Similar for the case where a is "a_1 *and* a_2". The valuation function V for propositions a_o is then defined as follows

$$V(w_{(i,j)}, a_{1,o} \wedge a_{2,o}) = \begin{cases} T & \text{if "}\mathsf{at}_1(o) \in a_1(c_i^1) \text{ and} \\ & \mathsf{at}_2(o) \in a_2(c_j^2)\text{''} \\ F & \text{otherwise} \end{cases}$$

and the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 *and* a_2" in the model M is defined by

$$\mu_A(o) = \mu_{A_1 \cap A_2}(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} \, ^{(i,j)}(a_{1,o} \wedge a_{2,o}) \qquad (4)$$

Notice that in the case without the weighting function Ω in the model M, the membership expressions of composed fuzzy sets defined in (3) and (4) are comparable with those given in (Resconi and Turksen, 2001).

With this formulation of fuzzy set theoretic operators, we have

$$\mu_{A_1 \cap A_2}(o) = \mu_{A_1}(o)\mu_{A_2}(o) \qquad (5)$$

$$\mu_{A_1 \cup A_2}(o) = \mu_{A_1}(o) + \mu_{A_2}(o) - \mu_{A_1}(o)\mu_{A_2}(o) \qquad (6)$$

Expressions (5) and (6) show that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, moreover, they form a well-known dual pair of *Product t-norm* T_P and *Probabilistic Sum t-conorm* S_P (Klement, 1997). This justifies for the situation when linguistic terms belong to different universes of discourse, for example *tall* and *high income*, there is no constraint of semantic consistency between them, and reflecting such independence, the product-sum rule is appropriate in applications.

4 FUZZY CONTEXT MODEL

We first recall in this section necessary notions from the Dempster-Shafer theory of evidence (DS theory, for short). The theory aims at providing a mechanism for representing and reasoning with uncertain, imprecise and incomplete information.

4.1 Dempster-Shafer Theory within the Context Model

DS theory originated from the work by Dempster (Dempster, 1967) on the modeling of uncertainty in terms of upper and lower probabilities induced by a multivalued mapping.

A multivalued mapping F from space Q into space S associates to each element q of Q a subset $F(q)$ of S. The domain of F, denoted by $\mathsf{Dom}(F)$, is defined by

$$\mathsf{Dom}(F) = \{q \in Q | F(q) \neq \emptyset\}$$

From a multivalued mapping F, a probability measure P on Q can be propagated to S in such a way that for any subset T of S the lower and upper bounds of probabilities of T are defined as

$$P_*(T) = \frac{P(F^-(T))}{P(F^-(S))} \quad (7)$$

$$P^*(T) = \frac{P(F^+(T))}{P(F^+(S))} \quad (8)$$

where

$$\begin{aligned} F^-(T) &= \{q \in Q | q \in \mathsf{Dom}(F) \wedge F(q) \subseteq T\} \\ F^+(T) &= \{q \in Q | F(q) \cap T \neq \emptyset\} \end{aligned}$$

Clearly, $F^+(S) = F^-(S) = \mathsf{Dom}(F)$, and P_*, P^* are well defined only when $P(\mathsf{Dom}(F)) \neq 0$. Furthermore, Dempster also observed that, in the case that S is finite, these lower and upper probabilities are completely determined by the quantities

$$P(F^{-1}(T)), \text{ for } T \in 2^S$$

where for each $T \in 2^S$,

$$F^{-1}(T) = \{q \in Q | F(q) = T\}$$

As such Dempster implicitly gave the prototype of a mass function also called *basic probability assignment*. Shafer's contribution has been to explicitly define the basic probability assignment and to use it to represent evidence directly. Simultaneously, Shafer has reinterpreted Dempster's lower and upper probabilities as degrees of belief and plausibility respectively, and abandoned the idea that they arise as lower and upper bounds over classes of Bayesian probabilities (Shafer, 1976).

As we already observed above, both DS theory and the context model are closely related to the theory of multivalued mappings. In fact, each vague characteristic in the context model is formally a multivalued mapping from the set of contexts into the universe of discourse.

Let $\mathcal{C} = \langle D, C, A_C(D), P_C\rangle$ be a valuated context model. Here, for the sake of discussing essential remarks regarding the interpretation of the Dempster-Shafer theory within the context model, we assume that P_C is a probability measure on C. Let a be a vague characteristic in $\mathcal{C}$ considering now as a multivalued mapping from C into D. Then a induces lower and upper probabilities, in the sense of Dempster, on 2^D as respectively defined in (7) and (8). Namely, for any $X \in 2^D$,

$$P(a)_*(X) = \frac{P_C(a^-(X))}{P_C(a^-(D))}$$

$$P(a)^*(X) = \frac{P_C(a^+(X))}{P_C(a^+(D))}$$

In the case where a is non-contradictory, we have $\mathsf{Dom}(a) = C$. Then, these probabilities coincide with lower and upper acceptance degrees as defined in (1) and (2) respectively. That is, for any $X \in 2^D$,

$$P(a)_*(X) = \underline{\mathsf{Acc}}_a(X)$$

$$P(a)^*(X) = \overline{\mathsf{Acc}}_a(X)$$

Furthermore, Gebhardt and Kruse also defined the so-called *mass distribution* m_a of a as follows

$$m_a(X) = P_C(a^{-1}(X)), \text{ for any } X \in 2^D$$

Then, for any $X \in 2^D$, we have

$$\underline{\mathsf{Acc}}_a(X) = \sum_{A \in a(C): \emptyset \neq A \subseteq X} m_a(A)$$

$$\overline{\mathsf{Acc}}_a(X) = \sum_{A \in a(C): A \cap X \neq \emptyset} m_a(A)$$

As such the mass distribution m_a induced from a in the context model $\mathcal{C}$ can be considered as the counterpart of a basic probability assignment in the DS theory.

4.2 Fuzzy Context Model

Although the context model can be considered as an autonomous approach to the handling of imperfect knowledge, it in its standard form does not allow us to directly model situations where *cond* only permits us to express $\mathsf{state}(obj, cond)$ in each context in the form of verbal statements like "$\mathsf{state}(obj, cond)$ is A", where A is a linguistic value represented by a fuzzy set in $\mathsf{dom}(obj)$. Let us consider the following example.

Example 1. *Assume that we want to forecast the temperature of the next day. Let $D = \{-40, \ldots, 40\}$ be the frame of discernment (temperatures measured in °C). We are told by expert E_1 that tomorrow's temperature will be* very high*, whereas another expert E_2 asserts that it will be* medium*. Assuming that we have degree of confidence of 0.4 in expert E_1 and of 0.6 in expert E_2, what is our belief about some predicted intervals of tomorrow's temperature?*

This example is inspired by Denœux (2000). However, Denœux proposed a principled approach to the representation and manipulation of imprecise degrees of belief within the framework of DS theory (Denœux, 2000, 1999). In the sequel we introduce an extension of the context model for dealing with both vagueness and partial conflict in such a situation. Let D be a nonempty universe of discourse, and denote $\overline{\mathcal{F}}(D)$ the set of all normal fuzzy subsets of D. Now a fuzzy context model is defined as a quadruple

$$\mathcal{FC} = \langle D, C, \Gamma_C(D), P_C\rangle,$$

where D, C, P_C are previously defined as in Section 2, and

$$\Gamma_C(D) = \{a | a : C \rightarrow \overline{\mathcal{F}}(D)\}$$

and each $a \in \Gamma_C(D)$ is called a context-dependent vague characteristic. As such, we also restrict to consider only context-dependent vague characteristics that are not contradictory in the fuzzy context model.

It is of interest to note that each context-dependent vague characteristic a in the fuzzy context model $\mathcal{FC}$ is formally equivalent to a type-2 fuzzy set on C with respect to D in the sense of Zadeh (1975). Furthermore, there is a very close interrelation between context-dependent vague characteristics within the fuzzy context model and the notion of context-dependent fuzzy sets introduced by Thiele (2001). Indeed, given a context-dependent vague characteristic $a \in \Gamma_C(D)$, we then obtain a context-dependent fuzzy set f defined as follows

$$\begin{array}{rccl} f: & D \times C & \to & [0,1] \\ & (d,c) & \mapsto & f(d,c) =_{def} a(c)(d) \end{array}$$

So the notion of fuzzy context model can also provide a constructive approach to fuzzy sets of type 2. However, we do not consider this issue in the present paper. While the notion of context-dependent fuzzy sets has been introduced as an interpretation for vague concepts in connection with linguistic variables, context-dependent vague characteristics are motivated by situations of decision analysis with linguistic information as illustrated above.

5 MODELING VAGUE BELIEFS

5.1 Vague Beliefs by Fuzzy Context Model

Now from a point of view of decision analysis, we also intend to evaluate the acceptance degree $\mathsf{Acc}_a(X)$, for $X \in 2^D$, that the proposition "$\mathsf{state}(obj, cond) \in X$" is true. Obviously, vagueness and partial conflict due to contexts in a do not allow us to uniquely determine an interval of acceptance degrees as in the original form of context model, but a fuzzy quantity in the set of non-negative real numbers $\mathbb{R}^+$. This can be done in terms of the α-cuts of fuzzy sets $a(c), c \in C$, as follows.

Let α be any real number in $(0, 1]$, and ${}^\alpha a(c)$, for any $c \in C$, the α-cut of $a(c)$. Then by (1) and (2) we obtain

$${}^\alpha\underline{\mathsf{Acc}}_a(X) = P_C(\{c \in C | \ \emptyset \neq \ {}^\alpha a(c) \subseteq X\}) \quad (9)$$

$${}^\alpha\overline{\mathsf{Acc}}_a(X) = P_C(\{c \in C | \ {}^\alpha a(c) \cap X \neq \emptyset\}) \quad (10)$$

For any $\alpha, \beta \in (0, 1]$ and $\alpha \leq \beta$, we have ${}^\beta a(c) \subseteq \ {}^\alpha a(c)$. It directly follows by (9) and (10) that

$${}^\alpha\underline{\mathsf{Acc}}_a(X) \leq \ {}^\beta\underline{\mathsf{Acc}}_a(X)$$

and

$${}^\beta\overline{\mathsf{Acc}}_a(X) \leq \ {}^\alpha\overline{\mathsf{Acc}}_a(X)$$

Equivalently, we have

$$[{}^\beta\underline{\mathsf{Acc}}_a(X), \ {}^\beta\overline{\mathsf{Acc}}_a(X)] \subseteq [{}^\alpha\underline{\mathsf{Acc}}_a(X), \ {}^\alpha\overline{\mathsf{Acc}}_a(X)]$$

Under such a condition of monotonicity, now we can define $\mathsf{Acc}_a(X)$ as a fuzzy set of $\mathbb{R}^+$ whose membership function $\mu_{\mathsf{Acc}_a(X)}$ is defined by

$$\mu_{\mathsf{Acc}_a(X)}(r) = \sup_\alpha\{\alpha | \ r \in [{}^\alpha\underline{\mathsf{Acc}}_a(X), \ {}^\alpha\overline{\mathsf{Acc}}_a(X)]\} \quad (11)$$

Note that if $a(c)$ is crisp for any $c \in C$, then $\mathsf{Acc}_a(X)$ reduces to an interval of $\mathbb{R}^+$, where lower and upper bounds of the interval are determined by (1) and (2), respectively.

In the case where P_C is a probability measure over C, the function Acc_a is formally equivalent to a type-2 fuzzy set of 2^D, i.e. that a fuzzy set with fuzzy membership values (Zadeh, 1975). Then, as the interpretation established in the previous section, $\mathsf{Acc}_a(X)$ could be considered now as the *degree of belief*, which is directly inferred from "vague" evidence expressed linguistically, in the proposition "$\mathsf{state}(obj, cond) \in X$".

Example 2. *This example models Example 1 by using the notion of fuzzy context model. Let $D = \{-40, \ldots, 40\}$ be the domain of temperature, and $C = \{E_1, E_2\}$ be the set of contexts. Assume that linguistic values* very high *and* medium *are represented by normal fuzzy sets in D whose membership functions are denoted by μ_{VH} and μ_M, respectively. Then* tomorrow's temperature *(temp) is considered as a context-dependent vague characteristic, that is*

$$\begin{array}{rccl} temp: & C & \longrightarrow & \overline{\mathcal{F}}(D) \\ & E_1 & \longmapsto & \mu_{VH} \\ & E_2 & \longmapsto & \mu_M \end{array}$$

The measure P_C that reflects degrees of confidence in Experts is defined by $P_C : 2^C \longrightarrow \mathbb{R}^+$ so that $P_C(\emptyset) = 0, P_C(\{E_1\}) = 0.4, P_C(\{E_2\}) = 0.6,$ and $P_C(C) = 1$.

Assuming that we have to decide a forecasted interval for tomorrow's temperature from some predicted intervals of temperature available, say TI_1, TI_2, TI_3. By the procedure specified above, we can calculate $\mathsf{Acc}_{temp}(TI_i)$ for $TI_i, i = 1, 2, 3$. The next step in the decision process may consist in comparison of the obtained fuzzy quantities. In the following we will provide a procedure for ranking vague beliefs based on their alpha-cut representation and comparison probabilities of interval values as proposed in (Huynh *et al.*, 2008).

5.2 Ranking Vague Beliefs

Let $X_i \in 2^D$, $i = 1, \ldots, n$, be a collection of subsets of D available for evaluations of the form "$\mathsf{state}(obj, cond) \in X_i$", given a context-dependent vague characteristic a. As discussed previously, for each X_i, we obtain $\mathsf{Acc}_a(X_i)$ as a quantification of our belief about the proposition "$\mathsf{state}(obj, cond) \in X_i$". In order to decide which one of X_i $(i = 1, \ldots, n)$, say, for a description of $\mathsf{state}(obj, cond)$, we might want to define a preference order on the set $\{X_i\}_{i=1}^n$ induced by a ranking of vague beliefs $\mathsf{Acc}_a(X_i)$, for $i = 1, \ldots, n$. This can be done as follows.

1. Based on the method proposed in (Huynh *et al.*, 2008), for any pair X_i and X_j, we first calculate:

$$S(X_i, X_j) \triangleq P(\mathsf{Acc}_a(X_i) \succeq \mathsf{Acc}_a(X_j)) = \int_0^1 P({}^\alpha\mathsf{Acc}_a(X_i) \succeq {}^\alpha\mathsf{Acc}_a(X_j))d\alpha \quad (12)$$

   where ${}^\alpha\mathsf{Acc}_a(X_i) = [{}^\alpha\underline{\mathsf{Acc}}_a(X_i), {}^\alpha\overline{\mathsf{Acc}}_a(X_i)]$, and $S(X_i, X_j)$ is interpreted as the strength of preference of X_i over X_j defined by the expected probability of $\mathsf{Acc}_a(X_i)$ dominating $\mathsf{Acc}_a(X_j)$.

2. Then, for each X_i, its score can be defined by

$$T(X_i) = \sum_{j=1}^{n} S(X_i, X_j) \quad (13)$$

   as proposed in (Yager *et al.*, 2001).

3. Finally, these scores can be used to obtain an ordering over $\{X_i\}_{i=1}^n$.

Note that in case of subnormal fuzzy quantities $\mathsf{Acc}_a(X_i)$ and $\mathsf{Acc}_a(X_j)$ in the expression (12) above, we define

$$S(X_i, X_j) \triangleq P(\mathsf{Acc}_a(X_i) \succeq \mathsf{Acc}_a(X_j)) = \int_0^\beta P({}^\alpha\mathsf{Acc}_a(X_i) \succeq {}^\alpha\mathsf{Acc}_a(X_j))d\alpha \quad (14)$$

where $\beta = \min(\text{hgt}(\mathsf{Acc}_a(X_i)), \text{hgt}(\mathsf{Acc}_a(X_j)))$, and $\text{hgt}(F)$ denotes the height of fuzzy set F. More details on this ranking procedure as well as its properties can be referred to Huynh *et al.* (2008); Yager *et al.* (2001).

Note that an ordering over $\{X_i\}_{i=1}^n$ can be also induced, for example, on the basis of a partial order such as $\mathsf{Acc}_a(X_1) \leq \mathsf{Acc}_a(X_2)$ if and only if

$${}^\alpha\underline{\mathsf{Acc}}_a(X_1) \leq {}^\alpha\underline{\mathsf{Acc}}_a(X_2) \quad (15)$$

and

$${}^\alpha\overline{\mathsf{Acc}}_a(X_1) \leq {}^\alpha\overline{\mathsf{Acc}}_a(X_2) \quad (16)$$

for any $\alpha \in (0, 1]$. However, in this case we have to admit indeterminacy when two fuzzy degrees of belief are incomparable.

Let us consider the context-dependent vague characteristic *temp* given in Example 2 again, and assume that membership functions of linguistic temperature values are defined as graphically depicted in Fig. 1. Assume further that we have two predicted intervals of tomorrow's temperature, say [20,27.5] and [25,32.5], available. Then, refer to (11), we obtain

$$\mu_{\mathsf{Acc}_{temp}([20,27.5])}(r) = \begin{cases} 0.333, & \text{if } 0 \leq r \leq 0.6 \\ 0.167, & \text{if } 0.6 < r \leq 1 \end{cases}$$

and

$$\mu_{\mathsf{Acc}_{temp}([25,32.5])}(r) = \begin{cases} 0.433, & \text{if } 0 \leq r \leq 0.4 \\ 0, & \text{otherwise} \end{cases}$$

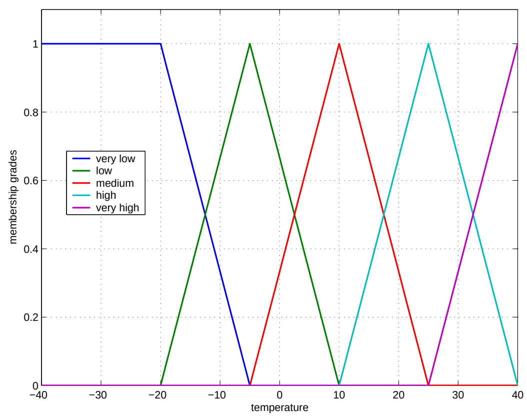

Figure 1: Linguistic temperature values

Clearly, both $\mathsf{Acc}_{temp}([20, 27.5])$ and $\mathsf{Acc}_{temp}([25, 32.5])$ are subnormal fuzzy sets and, by applying (14), we get

$$S(\mathsf{Acc}_{temp}([20, 27.5]), \mathsf{Acc}_{temp}([25, 32.5])) = 0.836$$

which means the prediction of $[20, 27.5]$ is preferred to the prediction of $[25, 32.5]$ when taking experts' opinion into account. Instead, if we use the partial order defined by (15) and (16) for comparing the fuzzy quantities, we then have $\mathsf{Acc}_{temp}([20, 27.5])$ and $\mathsf{Acc}_{temp}([25, 32.5])$ are incomparable.

Some Remarks

- The manipulation of fuzzy quantities may be considerably simplified by restricting the consideration on fuzzy numbers with the LL parameterization introduced in (Dubois and Prade, 1987). Then many methods for total ordering of fuzzy numbers that have been suggested in the literature can be used in the comparison of fuzzy degrees of acceptance. It should be noticed that in the spirit of previous applications of fuzzy set theory to decision analysis, e.g. (Dubois and Prade, 1982; Freeling, 1980; Watson *et al.*, 1979), the utilities were often described in terms of fuzzy numbers.

- For any $X \in 2^D$, as an alternative representation of $\mathsf{Acc}_a(X)$, we also define the so-called *fuzzy mass distribution* m_a of a via α-cuts as follows.

$${}^\alpha m_a(X) = P_C(\{c \in C | \; {}^\alpha a(c) = X\})$$

  Due to the additivity property of P_C, we have

$${}^\alpha\underline{\mathsf{Acc}}_a(X) = \sum_{A \in {}^\alpha a(C): A \subseteq X} {}^\alpha m_a(A)$$

$${}^\alpha\overline{\mathsf{Acc}}_a(X) = \sum_{A \in {}^\alpha a(C): A \cap X \neq \emptyset} {}^\alpha m_a(A)$$

6 CONCLUSIONS

The notion of context model can be viewed as a framework for modeling and handling vagueness and uncertainty. Furthermore, in order to deal with the problem of synthesis of vague evidence linguistically provided by the experts in some situations of decision analysis, the notions of context-dependent vague characteristics and fuzzy context model have been introduced. Each context-dependent vague characteristic within a fuzzy context model directly induces a uncertainty measure of type 2 interpreted as "vague" belief function, which is inferred from vague evidence expressed linguistically. The notion of fuzzy context model may allow us to model some situations where heterogeneous data from a variety of sources considered as contexts have to be taken into account. Especially, the possibility to describe states of belief employing verbal statements is also expected to be useful in situations involving the elicitation of degrees of belief from experts, such as encountered in the development of decision support systems.

Up to now we have considered the representation and manipulation of vague knowledge by the fuzzy context model. To justify how to come to decision-making aspects by the fuzzy context model, we may need to explore further important operations such as conditioning, data revision, combination on context-dependent vague characteristics similar to those in (Gebhardt and Kruse, 1993). Along with context-dependent vague characteristics, these operations within the fuzzy context model may constitute a flexible framework allowing to express, and reason with vaguely specified degrees of belief. These problems are being the subject of our further work.

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REFERENCES

Dempster, A. P. Upper and lower probabilities induced by a multivalued mapping *Annals of Mathematics and Statistics* 38: 325–339, 1967.

Denœux, T. Modeling vague beliefs using fuzzy-valued belief structures. *Fuzzy Sets and Systems* 116: 167–199, 2000.

Denœux, T. Reasoning with imprecise belief structures. *International Journal of Approximate Reasoning* 20: 79–111, 1999.

Dubois, D. and H. Prade. *Possibility Theory – An Approach to Computerized Processing of Uncertainty.* Plenum Press, New York, 1987.

Dubois, D., H. Prade. The use of fuzzy numbers in decision analysis. In: M. M. Gupta, E. Sanchez (Eds.), *Fuzzy Information and Decision Processes*, pp. 309–321. North-Holland, 1982.

Freeling, A. N. S. Fuzzy sets and decision analysis. *IEEE Transactions on Systems, Man and Cybernetics* 10(7): 341–354, 1980.

Ganter, B., R. Wille. *Formal Concept Analysis: Mathematical Foundations.* Springer, 1999.

Gebhardt, J., R. Kruse. The context model: An integrating view of vagueness and uncertainty. *International Journal of Approximate Reasoning* 9: 283–314, 1993.

Gebhardt, J., R. Kruse. Parallel combination of information sources. In D. M. Gabbay, P. Smets (Eds.), *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, vol. 3, pp. 393–439. Kluwer, 1998.

Gebhardt, J. Learning from data – possibilistic graphical models. In D. M. Gabbay, P. Smets (Eds.), *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, vol 4, pp. 314–389. Kluwer, 2000.

Huynh, V. N., Y. Nakamori, J. Lawry. A probability-based approach to comparison of fuzzy numbers and applications to target oriented decision making. *IEEE Transactions on Fuzzy Systems* 16: 371–387, 2008.

Huynh, V. N., Y. Nakamori. Fuzzy concept formation based on context model. In N. Baba, R. J. Howlett, L. C. Jain (Eds.), *Knowledge-Based Intelligent Information Engineering Systems & Allied Technologies*, pp. 687–691. IOS Press, 2001.

Huynh, V. N., Y. Nakamori, T. B. Ho, G. Resconi. A context model for constructing membership functions of fuzzy concepts based on modal logic. In T. Eiter, K.-D. Schewe (Eds.), *Foundations of Information and Knowledge Systems*, LNCS 2284, pp. 93–104. Springer-Verlag, 2002.

Huynh, V. N., Y. Nakamori, T. B. Ho, G. Resconi, A context model for fuzzy concept analysis based upon modal logic, *Information Sciences* 160: 111–129, 2004.

Klement, E. P. Some mathematical aspects of fuzzy sets: triangular norms, fuzzy logics, and generalized measures. *Fuzzy Sets and Systems* 90: 133–140, 1997.

Klir, R. Multi-valued logic versus modal logic: Alternate framework for uncertainty modelling. In P. P. Wang, editor, *Advances in Fuzzy Theory and Technology*, vol. II, pp. 3–47. Duke University Press, 1994.

Kruse, R., J. Gebhardt, F. Klawonn. Numerical and logical approaches to fuzzy set theory by the context model. In R. Lowen and M. Roubens, editors, *Fuzzy Logic: State of the Art*, pp. 365–376. Kluwer, 1993.

Resconi, G., G. Klir, U. St. Clair. Hierarchically uncertainty metatheory based upon modal logic. *International Journal of General Systems* 21: 23–50, 1992.

Resconi, G., G. Klir, U. St. Clair, D. Harmanec. On the integration of uncertainty theories. *Inter. J. Uncertainty, Fuzziness and Knowledge-Based Systems* 1: 1–18, 1993.

Resconi, G., I. B. Turksen. Canonical forms of fuzzy truthhoods by meta-theory based upon modal logic. *Information Sciences* 131: 157–194, 2001.

Shafer, G. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, 1976.

Watson, S. R., J. J. Weiss, M. L. Donnell. Fuzzy decision analysis. *IEEE Transactions on Systems, Man and Cybernetics* 9(1): 1–9, 1979.

Thiele, H. A new approach to type–2 fuzzy sets. In J. M. Abe, J. I. da Silva Filho (Eds.) *Logic, Artificial Intelligence and Robotics*, pp. 255–262. IOS Press, 2001.

Yager, R. R., M. Detyniecki, B. Bouchon-Meunier. A context-dependent method for ordering fuzzy numbers using probabilities. *Information Sciences* 138: 237–255, 2001.

Zadeh, L. A. The concept of linguistic variable and its application to approximate reasoning. *Information Sciences*, part I, 8: 199–249, 1975; part II, 8: 310–357, 1975.

Zadeh, L. A. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1: 3–38, 1978.