Notes on the first-order part of Ramsey's theorem for pairs

Keita Yokoyama*
(Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology)

Abstract

We give the Π^0_2-part, the Π^0_3-part and the Π^0_4-part of RT^2_2 and related combinatorial principles.

1 Introduction

Determinating the first-order part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$ and other important combinatorial principles is a one of the crucial topics in the study of Reverse Mathematics (see, e.g., [2, 4]). The usual approach for these questions is using forcing arguments to construct a second-order part for the target combinatorial principle. On the other hand, there is a traditional way to study the strength of combinatorial principles by using indicator functions. (For the details of indicator functions, see [6].) In [1], Bovykin and Weiermann gave the Π^0_2-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$ by means of an indicator function defined by a density notion, using the idea of Paris [7] and Paris/Kirby [8]. Using similar arguments, we can show that the Π^0_2-part of $\mathsf{WKL}^*_0 + \mathrm{RT}^2_2$ is equivalent to Elementary Function Arithmetic (see [9]). In this paper, we give the Π^0_3-part and the Π^0_4-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$ based on [1]. We will also give several density notions to characterize the Π^0_2-part, the Π^0_3-part and the Π^0_4-part of $\mathrm{RT}^2_{<\infty}$, SRT^2_2, $\mathrm{SRT}^2_{<\infty}$ and EM.

2 The Π^0_2-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$

This section is essentially due to Bovykin/Weiermann[1].

Definition 2.1 (within $\mathrm{I}\Sigma_1$)**.** For a finite set X, we define the notion of *n-density* as follows.

- A finite set X is said to be 0-*dense* if $|X| > \min X$.
- A finite set X is said to be $n+1$-*dense* if for any (coloring) function $P : [X]^2 \to 2$, there exists a subset $Y \subseteq X$ such that Y is n-dense and Y is P-homogeneous, *i.e.*, P is constant on $[Y]^2$.

*2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, JAPAN. e-mail: yokoyama.k.ai@m.titech.ac.jp

Note that "X is m-dense" can be expressed by a Σ_0-formula.

Definition 2.2. $n\mathrm{PH}^2_2$ asserts that for any a there exists an n-dense set X such that $\min X > a$.

Define $T_0 := \{k\mathrm{PH}^2_2 \mid k \in \omega\} \cup \mathrm{I}\Sigma_1$.

Lemma 2.1.

- $\mathsf{WKL_0} + \mathrm{RT}^2_2 \vdash n\mathrm{PH}^2_2$ *for any* $n \in \omega$.
- $\mathrm{I}\Sigma_1 \vdash m\mathrm{PH}^2_2 \to \mathrm{PH}^2_{m+1}$.

Proof. Easy. □

Lemma 2.2 (Bovykin/Weiermann[1]). *Let M be a countable model of* $\mathrm{I}\Sigma_1$*, and let $X \subseteq M$ is a (M-)finite set which is k-dense for any $k \in \omega$. Then, there exists a cut $I \subseteq M$ such that* $\min X \in I < \max X$*, $X \cap I$ is unbounded in I and* $(I, \mathrm{Cod}(I/M)) \models \mathsf{WKL_0} + \mathrm{RT}^2_2$.

Proof. See [1]. □

Theorem 2.3 (Bovykin/Weiermann[1]). *A Π^0_2 sentence ψ is provable in* $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *if and only if it is provable in T_0.*

Proof. See [1]. □

In fact, we can generalize this theorem as follows.

Theorem 2.4. *A Π^0_2 formula ψ (ψ may contain set parameters) is provable in* $\mathsf{WKL_0}+\mathrm{RT}^2_2$ *if and only if it is provable in* $\mathrm{I}\Sigma^0_1 \cup \{k\mathrm{PH}^2_2 \mid k \in \omega\}$*. (Here,* $\mathrm{I}\Sigma^0_1$ *is a system of second-order arithmetic which contains basic axioms and induction axioms for Σ^0_1-formulas with set parameters.)*

3 The Π^0_3-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$

Definition 3.1. Let $\theta(a, x, y)$ be a Σ_0-formula. We say that a finite set $X = \{a_i \mid i \leq l\}$ dominates $\theta(a, \cdot, \cdot)$ if $\forall i < l\ \forall x \leq a_i\ \exists y \leq a_{i+1} \theta(a, x, y)$ holds. We define several variations of PH^2_2 as follows:

- $\theta\text{-}n\mathrm{PH}^2_2 :\equiv \forall a(\forall x \exists y \theta(a, x, y) \to \exists X$ (X is finite, n-dense, and dominates $\theta(a, \cdot, \cdot)$)),
- $n\widetilde{\mathrm{PH}^2_2} :\equiv \forall X(\forall x \exists y \geq x\ y \in X \to \exists Y$ (Y is finite, n-dense, and $Y \subseteq X$)).

Define $T_1 := \{\theta\text{-}k\mathrm{PH}^2_2 \mid k \in \omega, \theta \in \Sigma_0\} \cup \mathrm{I}\Sigma_1$ and $\widetilde{T_1} := \{k\widetilde{\mathrm{PH}^2_2} \mid k \in \omega\} \cup \mathsf{RCA_0}$. Note that T_1 is a Π^0_3-theory, *i.e.*, T_1 is a set of Π^0_3-sentences.

Lemma 3.1. *Let $\theta(a, x, y)$ be a Σ_0-formula, and let $n \in \omega$. Then,* $\mathsf{WKL_0}+\mathrm{RT}^2_2 \vdash \theta\text{-}n\mathrm{PH}^2_2$, *and* $\mathsf{WKL_0} + \mathrm{RT}^2_2 \vdash n\widetilde{\mathrm{PH}^2_2}$.

Proof. Easy. □

Theorem 3.2. *A* Π^0_3 *sentence* ψ *is provable in* $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *if and only if it is provable in* T_1*. Thus,* T_1 *is the* Π^0_3*-part of* $\mathsf{WKL_0} + \mathrm{RT}^2_2$*.*

Proof. We show that $T_1 \nvdash \psi$ implies $\mathsf{WKL_0} + \mathrm{RT}^2_2 \nvdash \psi$ for any Π^0_3-sentence ψ. Assume that $\psi \equiv \forall a \exists x \forall y \theta(a, x, y)$ is not provable from T_1. Then, there exists a nonstandard countable model $M \models T_1$ such that $M \models \forall x \exists y \neg\theta(a, x, y)$ for some $a \in M$. By $(\neg\theta)$-$k\mathrm{PH}^2_2$ and overspill, there exists an m-dense set X which dominates $\neg\theta(a, \cdot, \cdot)$ for some $m \in M \setminus \omega$. By Lemma 2.2, there exists an initial segment $I \subseteq_e M$ such that $(I, \mathrm{Cod}(I/M)) \models \mathsf{WKL_0} + \mathrm{RT}^2_2$ and $I \cap X$ is unbounded in I. Since X dominates $\neg\theta$, for any $x \in I$ there exists $y \in I$ such that $I \models \neg\theta(a, x, y)$. Thus, we have $(I, \mathrm{Cod}(I/M)) \models \neg\psi$, which means that $\mathsf{WKL_0} + \mathrm{RT}^2_2 \nvdash \psi$. □

Theorem 3.3. *A* Π^0_3 *formula* ψ *is provable in* $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *if and only if it is provable in* $\widetilde{T_1}$*.*

Proof. Similar to the proof of Theorem 3.2. □

Note that $\widetilde{T_1}$ is equivalent to $\mathrm{I}\Sigma^0_1 \cup \{\forall A \forall a (\forall x \exists y \theta(A, a, x, y) \to \exists X$ (X is finite, n-dense, and dominates $\theta(A, a, \cdot, \cdot))) \mid n \in \omega, \theta \in \Sigma^0_0\}$ with respect to Π^1_1-sentences.

4 The Π^0_4-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$

Definition 4.1 (within $\mathrm{I}\Sigma_1$)**.** Let $\theta(a, x, y, z)$ be a Σ_0-formula. Then, we define the notion of *weakly domination* as follows.

- A 0-dense set X *weakly dominates* $\theta(a, \cdot, \cdot, \cdot)$.
- An $n+1$-dense set X *weakly dominates* $\theta(a, \cdot, \cdot, \cdot)$ if for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y \subseteq X$ such that $\forall x < \min X \exists y < \min Y \forall z < \max Y \theta(a, x, y, z)$, Y is n-dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$.

Note that "X is m-dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$" can be expressed by a Σ_0 formula.

Definition 4.2. Let $\theta(a, x, y, z)$ be a Σ_0-formula. Then, the assertion θ^*-$n\mathrm{PH}^2_2$ is the following

$$\forall a \forall b (\forall x \exists y \forall z \theta(a, x, y, z) \to \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(a, \cdot, \cdot, \cdot) \text{ and } \min X > b).$$

Define $T_2 := \{\theta^*\text{-}n\mathrm{PH}^2_2 \mid n \in \omega, \theta(a, x, y, z) \in \Sigma_0\} \cup \mathrm{I}\Sigma_1$. Note that T_2 is a Π^0_4-theory.

Lemma 4.1. *Let* $\theta(a, x, y, z)$ *be a* Σ_0*-formula, and let* $n \in \omega$*. Then,* $\mathsf{WKL_0} + \mathrm{RT}^2_2 \vdash \theta^*$*-*$n\mathrm{PH}^2_2$*.*

Proof. Easy. □

Theorem 4.2. *A Π^0_4 sentence ψ is provable in $\mathsf{WKL_0} + \mathrm{RT}^2_2$ if and only if it is provable in T_2. Thus, T_2 is the Π^0_4-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$.*

Proof. We show that $T_2 \nvdash \psi$ implies $\mathsf{WKL_0} + \mathrm{RT}^2_2 \nvdash \psi$ for any Π^0_4-sentence ψ. Assume that $\psi \equiv \forall a \exists x \forall y \forall z \theta(a, x, y, z)$ is not provable from T_2. Then, there exists a nonstandard countable model $M \models T_2$ such that $M \models \forall x \exists y \neg\theta(a, x, y, z)$ for some $a \in M$. By $(k, \neg\theta)\mathrm{PH}^2_2$ and overspill, there exists an $(m, \theta(a, \cdot, \cdot, \cdot))$-dense set X such that $\min X > a$ for some $m \in M \setminus \omega$. As the proof of Theorem 1 of [1], we can construct a descending sequence $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ which satisfies the following:

- $I = \sup\{\min X_i \mid i \in \omega\} \subseteq_e M$,
- $(I, \mathrm{Cod}(I/M)) \models \mathsf{WKL_0} + \mathrm{RT}^2_2$,
- $I \cap X$ is unbounded in I,
- $\forall x \le \min X_i \ \exists y \le \min X_{i+1} \ \forall z \le \max X_{i+1} \neg\theta(a, x, y, z)$ for any $i \in \omega$.

Since $\min X_i < \min X_{i+1} < I < \max X_{i+1}$ for any $i \in \omega$, we have $I \models \forall x \exists y \forall z \neg\theta(a, x, y, z)$, *i.e.*, $(I, \mathrm{Cod}(I/M)) \models \neg\psi$. This means that $\mathsf{WKL_0} + \mathrm{RT}^2_2 \nvdash \psi$. □

Remark 4.3. Adding set parameters, we can easily show the following: a Π^0_4 formula ψ is provable in $\mathsf{WKL_0} + \mathrm{RT}^2_2$ if and only if it is provable in

$$\mathrm{I}\Sigma^0_1 \cup \{\forall A \forall a \forall b (\forall x \exists y \forall z \theta(A, a, x, y, z) \to \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(A, a, \cdot, \cdot, \cdot) \text{ and } \min X > b) \mid n \in \omega, \theta \in \Sigma^0_0\}.$$

5 PH^2_2 with stronger largeness notion

In this section, we compare $n\mathrm{PH}^2_2$ with PH^2_2 plus "stronger largeness".

Definition 5.1 (within $\mathrm{I}\Sigma_1$)**.**

- A finite set X is said to be 0-*large* if $X \neq \emptyset$.
- A finite set X is said to be $r+1$-*large* if there is a partition $X = \bigsqcup_{i \le \min X} Y_i$ such that $\max Y_i < \min Y_{i+1}$ for any $i < \min X$ and each Y_i is r-large.

Remark 5.1.

1. For any $r \in \omega$, $\mathrm{I}\Sigma_1$ proves that for any a, there exists a finite set X such that $\min X > a$ and X is r-large.
2. $Q(a, b) := \max\{r \mid [a, b] \text{ is } r\text{-large}\}$ is an indicator function for $\mathsf{WKL_0}$.
3. More generally, if M is a model of $\mathrm{I}\Sigma_1$ and $X \subseteq M$ is r-large for some $r \in M \setminus \omega$, then there exists a cut $I \subseteq_e M$ such that $(I, \mathrm{Cod}(I/M)) \models \mathsf{WKL_0}$ and $X \cap I$ is unbounded in I.

Definition 5.2.

1. $\mathrm{PH}^2_{2,r}$ asserts that for any a, there exists a finite set X such that $\min X > a$ and for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y \subseteq X$ which is r-large.

2. $\widetilde{\mathrm{PH}^2_{2,r}}$ asserts that for any infinite set A, there exists a finite set X such that $X \subseteq A$ and for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y \subseteq X$ which is r-large.

3. In general, $\widetilde{n\mathrm{PH}^2_{2,r}}$ asserts that for any infinite set A, there exists a finite set X such that $X \subseteq A$ and X is (n, r)-dense, where the notion of (n, r)-density is defined as follows:

   - A finite set X is said to be $(0, r)$-*dense* if X is r-large.
   - A finite set X is said to be $(n + 1, r)$-*dense* if for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y \subseteq X$ which is (n, r)-dense.

Proposition 5.2. $\mathrm{I}\Sigma_1 \vdash n\mathrm{PH}^2_2 \to \mathrm{PH}^2_{2,n}$.

Proof. Easy. □

The strength of $\mathrm{PH}^2_{2,r}$ is related to the strength of $n\mathrm{PH}^2_2$ in the following meaning.

Proposition 5.3. *Assume that* $\mathsf{WKL_0} \vdash \widetilde{\mathrm{PH}^2_{2,r}}$ *for all* $r \in \omega$*, then we have* $\mathsf{WKL_0} \vdash \widetilde{n\mathrm{PH}^2_2}$ *for all* $n \in \omega$*.*

Proof. Our assumption is $\mathsf{WKL_0} \vdash \widetilde{1\mathrm{PH}^2_{2,r}}$ for any $r \in \omega$. We will show by induction on n that $\mathsf{WKL_0} \vdash \widetilde{n\mathrm{PH}^2_{2,r}}$ for any $r \in \omega$ and for any $n \in \omega$. Let $\mathsf{WKL_0} \vdash \widetilde{n\mathrm{PH}^2_{2,r}}$ for any $r \in \omega$. Assume for the sake of contradiction that $\mathsf{WKL_0} \nvdash \widetilde{(n+1)\mathrm{PH}^2_{2,r}}$ for some $r \in \omega$. Then, there exists a model $(M, S) \models \mathsf{WKL_0}$ and $A \in S$ such that $M \not\cong \omega$, A is unbounded in M and any (M-)finite subset of A is not $(n+1, r)$-dense. By the assumption, there exists an (n, s)-dense subset of A for any $s \in \omega$. Thus, by overspill, for some $m \in M \setminus \omega$, we can take an (n, m)-dense subset $X \subseteq A$. We will show that this X is in fact $(n+1, r)$-dense, which leads to a contradiction. By the definition of (n, m)-density, for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y_1 \subseteq X$ which is $(n - 1, m)$-dense, and we can repeat this process n-times then the result set Y_n is m-large. By Remark 5.1.3, there exists a cut $I \subseteq_e M$ such that $(I, \mathrm{Cod}(I/M)) \models \mathsf{WKL_0}$ and $Y_n \cap I$ is unbounded in I. Thus, there exists a finite subset of $Y_n \cap I$ which is $(1, r)$-dense. This means that Y_n is $(1, r)$-dense, and hence X is $(n + 1, r)$-dense. □

Thus, if $\mathsf{WKL_0} \vdash \widetilde{\mathrm{PH}^2_{2,r}}$, then $\mathsf{WKL_0} + \mathrm{RT}^2_2$ is a Π^0_2-conservative extension of $\mathsf{WKL_0}$. This may give a new approach to study the proof-theoretic strength of $\mathsf{WKL_0} + \mathrm{RT}^2_2$.

Question 5.3. Is $\mathrm{I}\Sigma_1 \cup \{n\mathrm{PH}^2_2 \mid n \in \omega\}$ equivalent to $\mathrm{I}\Sigma_1 \cup \{\mathrm{PH}^2_{2,r} \mid r \in \omega\}$?

6 Other combinatorial principles

In this section, we give several density notions for SRT^2_2, $\mathrm{RT}^2_{<\infty}$, $\mathrm{SRT}^2_{<\infty}$, EM and ADS. (For the definitions of these combinatorial principles, see [2, 5, 1].) Using these notions, we can characterize Π^0_2, Π^0_3 or Π^0_4 part of the target combinatorial principle as in Sections 2,3 and 4.

We reason within $\mathrm{I}\Sigma_1$.

Proposition 6.1. *The Π^0_2-part, Π^0_3-part and the Π^0_4-part of $\mathsf{WKL_0}+\mathrm{SRT}^2_2$ is characterized by the following density notion.*

A finite set X is said to be

- *0-dense if $|X|>\min X$, and*
- *$m+1$-dense if for any $P:[X]^2\to 2$,*
  - *there exists a P-homogeneous subset $Y\subseteq X$ which is m-dense, or,*
  - *there exists $Y=\{y_0<y_1<\cdots<y_l\}\subseteq X$ such that $P(y_0,y_i)\neq P(y_0,y_{i+1})$ for any $0<i<l$ and Y is m-dense.*

For the strength of SRT^2_2, see also Chong/Slaman/Yang [3].

Proposition 6.2. *The Π^0_2-part, Π^0_3-part and the Π^0_4-part of $\mathsf{WKL_0}+\mathrm{RT}^2_{<\infty}$ is characterized by the following density notion.*

A finite set X is said to be

- *0-dense if $|X|>\min X$, and*
- *$m+1$-dense if for any coloring $P:[X]^2\to k$ such that $k<\min X$, there exists a P-homogeneous subset $Y\subseteq X$ which is m-dense.*

Proposition 6.3. *The Π^0_2-part, Π^0_3-part and the Π^0_4-part of $\mathsf{WKL_0}+\mathrm{SRT}^2_{<\infty}$ is characterized by the following density notion.*

A finite set X is said to be

- *0-dense if $|X|>\min X$, and*
- *$m+1$-dense if for any coloring $P:[X]^2\to k$ such that $k<\min X$,*
  - *there exists a P-homogeneous subset $Y\subseteq X$ which is m-dense, or,*
  - *there exists $Y=\{y_0<y_1<\cdots<y_l\}\subseteq X$ such that $P(y_0,y_i)\neq P(y_0,y_{i+1})$ for any $0<i<l$ and Y is m-dense,*

Proposition 6.4. *The Π^0_2-part, Π^0_3-part and the Π^0_4-part of $\mathsf{WKL_0}+\mathrm{EM}$ is characterized by the following density notion.*

A finite set X is said to be

- 0-*dense if* $|X| > \min X$, *and*
- $m+1$-*dense if*
  - *for any coloring* $P : [X]^2 \to 2$, *there exists* $Y \subseteq X$ *such that* P *is transitive on* Y *and* Y *is* m-*dense, and,*
  - *there is a partition* $X = \bigsqcup_{i \le \min X} Y_i$ *such that* $\max Y_i < \min Y_{i+1}$ *for any* $i < \min X$ *and each* Y_i *is* m-*dense.*

Here, a coloring P *is said to be transitive if* $P(a,b) = P(b,c) \Rightarrow P(a,b) = P(a,c)$.

Proposition 6.5. *The* Π^0_2-*part,* Π^0_3-*part and the* Π^0_4-*part of* $\mathsf{WKL_0} + \mathrm{ADS}$ *is characterized by the following density notion.*

A finite set X *is said to be*

- 0-*dense if* $|X| > \min X$, *and*
- $m+1$-*dense if for any transitive coloring* $P : [X]^2 \to 2$, *there exists a* P-*homogeneous subset* $Y \subseteq X$ *which is* m-*dense.*

In fact, Slaman/Chong/Yang[4] showed that $\mathsf{WKL_0} + \mathrm{ADS}$ is a Π^1_1-conservative extension of $\mathrm{B}\Sigma^0_2$. Thus, for any $n \in \omega$, $\mathsf{WKL_0}$ actually proves for any a, there exists a finite set X such that $\min X > a$ and X is n-dense for ADS.

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