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CONNECTIVITY OF METRIC SPACES IN CONSTRUCTIVE MATHEMATICS

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Chapter 1

Introduction

Connectivity of metric spaces is the notion that it cannot be the union of its disjoint open (or closed) subsets. For example, differentiable function and integrable one in complex analysis are defined on such a space, so that it is important in such a theory.

In this paper, we will define three connectivities and consider their properties respectively and the relation on them.

Now, we show constructive mathematics in this chapter,

and will consider real numbers theory, a metric space and connectivity of metric spaces from the next chapter.

Classical mathematics, which is called mathematics by the majority mathematicians, is formalized by *classical logic*, and correspondingly *constructive mathematics* is done by *intuitionistic logic* (see [14]). Actually, the character can be showed by the following *BHK* $(Brouwer - Heyting - Kolmogorov) - interpretation$, which is that of logical operators by Brouwer, Heyting and Kolmogorov (see [14] and [11]).

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting a proof of A or B.
- A proof of $A \Rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.
- Absurdity $\perp$ $(contbradiction)$ has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical poof of A into a proof of a contradiction.
- A proof of $\forall x A(x)$ is a construction which transforms any $d \in D$ (D the intended range of the variable x) into a proof of $A(d)$.
- A proof of $\exists x A(x)$ is given by presenting a $d \in D$ and a proof of $A(d)$.

This interpretation is restricted than that of classical mathematics. Actually, for a proof of $A \vee B$, though it is enough to show that $\neg A \wedge \neg B$ details a contradiction in classical mathematics, it is at least necessary in constructive mathematics either to give a proof of A or to give a proof of B. For a proof of $\exists x A(x)$, we can regard it classically as showing that $\forall x \neg A(x)$ is impossible, but constructively we must present explicitly d with $A(d)$. Therefore, it can be thought that constructive mathematics classifies rules, translations and existences in mathematics under computability.

Now, there are some propositions not to be provable in constructive mathematics but to be provable in classical mathematics. For example, *the Principle of Excluded Middle* $A \vee \neg A$ is unprovable since for a open problem A, we cannot give a proof of A or a proof of $\neg A$ under BHK-interpretation. Then, *Axiom of Choice* cannot be a part of constructive mathematics since this axiom implies the Principle of Excluded Middle, where Axiom of Choice is is as follows.

$$\forall S \subset A \times B[\forall x \in A \exists y \in B((x, y) \in S) \Rightarrow \exists f : A \to B \forall x \in A((x, f(x)) \in S).]$$

by implying the Principle of Excluded Middle as follows (see [7]):

(*Proof*) Let P be a proposition, and let $A := \{s, t\}$ with $s = t$ if and only if P holds. Let $B := \{0, 1\}$. Let $S := \{(s, 0), (t, 1)\}$. Then, by Axiom of Choice, there exists a choice function $f : A \to B$ such either that $f(s) = 1$, that $f(t) = 0$ or that $f(s) = 0$ and $f(t) = 1$. That is, either $s = t$ or $\neg(s = t)$. Therefore, either P or $\neg P$ holds. ∎

But, *Axiom of Countable Choice* is acceptable in constructive mathematics, where it is that of replacing A with the set of natural numbers $\mathbf{N}$ in the above Axiom of Choice.

Now, in constructive mathematics, there are the three main schools, which are *Bishop's constructive mathematics*, *Brouwer's intuitionistic mathematics* and *Markov's constructive mathematics*. Bishop's constructive mathematics is the mathematics accepting Axiom of countable choice under the BHK-interpretation, and Brouwer's intuitionistic mathematics can be regarded as Bishop's one added Brouwer's characteristic axioms. Markov's constructive mathematics can be also think of Bishop's one added Church's thesis i.e. "all sequences of natural numbers are recursive" and Markov's principle

MP $\quad \forall(\alpha_n) \in \{0, 1\}^{\mathbf{N}}[\neg\neg\exists n(\alpha_n = 1) \Rightarrow \exists n(\alpha_n = 1)]$.

That is, two other theories are extended from Bishop's constructive mathematics. Still, classical one is also the extension from Bishop's one since classical mathematics is regarded as a system that is added the Principle of Excluded Middle to Bishop's constructive mathematics syntactically. Then, Brouwer' intuitionistic mathematics and Markov's constructive mathematics are inconsistent with classical mathematics respectively, but Bishop's constructive mathematics is not so. Therefore, in this paper, the author considers in Bishop's constructive mathematics. From now on, "constructive mathematics" means "Bishop's consructive mathematics" in this paper.

Well, the following propositions are unprovable in constructive mathematics.

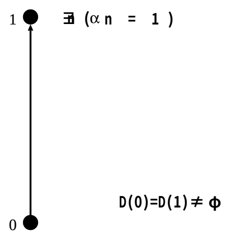
LPO(the Least Principle of Omniscience)

$\quad \forall(\alpha_n) \in \{0, 1\}^{\mathbf{N}}[\exists n(\alpha_n = 1) \vee \neg\exists n(\alpha_n = 1)]$.

LLPO(the Lesser Principle of Omniscience)

$$\forall(\alpha_n),(\beta_n)\in\{0,1\}^{\mathbf{N}}[\neg(\exists n(\alpha_n=1)\wedge\exists n(\beta_n=1))\Rightarrow\neg\exists n(\alpha_n=1)\vee\neg\exists n(\beta_n=1)].$$

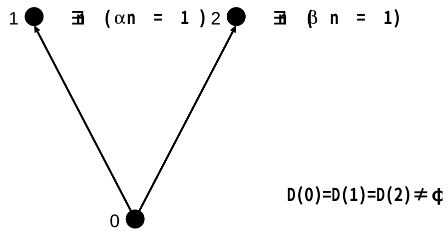
Actually, we cannot give a proof by the soundness for intuitionistic logic since there exist some models in such that they are false semantically (see [14]), and MP is also unprovable constructively in the same way. For example, there is the following Kripke model for LPO(see [14]):



LPO is false in this model since $0 \not\models \exists n(\alpha_n = 1)$ and $0 \not\models \neg\exists n(\alpha_n = 1)$.

MP is also false in this model since $0 \models \neg\neg\exists n(\alpha_n = 1)$ and $0 \not\models \exists n(\alpha_n = 1)$.

LLPO is also false in the following model



since $0 \models \neg(\exists n(\alpha_n = 1 \wedge \exists n(\beta_n = 1))$ and $0 \not\models \neg\exists n(\alpha_n = 1) \vee \neg\exists n(\beta_n = 1)$.

On the other hand, we can show it by another method.

For example, the unprovability of LPO is showed as follows (see [11]): Let (α_n) be a sequence such that

$$\alpha_n = \begin{cases} 0, & \text{if } 2n+4=p+q \text{ and } 2n+9=r+s+t \text{ for some pure numbers } p,q,r,s \text{ and } t \\ 1, & \text{otherwise} \end{cases}$$

Then, under the BHK-interpretation, $\exists n(\alpha_n = 1) \vee \neg\exists n(\alpha_n = 1)$ holds only when either *Goldbach's problem* "for all nonnegative integer n, there exist some pure numbers p, q, r, s and t such that $2n + 4 = p + q$ and $2n + 9 = r + s + t$" or its negation have proved. But, we have not known weather or not we can give the proof of either Goldbach's problem or its negation yet. Therefore, we now cannot prove LPO under the BHK-interpretation.

But, LPO, LLPO and MP hold in classical mathematics.

In constructive mathematics, "A given position does not hold" means that we can prove its negation or that it implies LPO, LLPO, MP and so on.

Now, in chapter 2, we will define real numbers, where we assume that the readers know rational numbers and its theory, and will consider metric spaces and its connectivity in chapter 3 and 4 respectively.

Chapter 2

Real Numbers

2.1 Real Number System

A *ratinal number* is an expression of the form p/q, where p and q are integers with $q\sharp 0$ ($q\sharp 0 \Leftrightarrow \neg(q=0)$ for all rational number q). Two rational numbers p/q and p'/q' are *equal* if $pq' = p'q$. The integer n is identified with the rational number $n/1$.

The contents of this chapter are based to [2] and [7].

Definition 2.1.1 Let (x_n) be a sequence of rational numbers. Then, (x_n) is a *real number* if (x_n) is a *regular* sequence i.e. (x_n) satisfies

$$|x_m - x_n| \leq m^{-1} + n^{-1}$$

for all positive integer m and n. The set of real numbers is denoted by $\mathbf{R}$.

Definition 2.1.2 Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers. x and y are *equal* if

$$|x_n - y_n| \leq 2n^{-1}$$

for all positive integer n. What x and y are equal is described by $x = y$.

Proposition 2.1.3 *Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers. Then, $x = y$ if and only if for each positive integer k, there exists a positive integer N such that $|x_n - y_n| \leq k^{-1}$ for all positive integer n with $n \geq N$.*

(*Proof*) Let k be a positive integer. Define $N := 2k$. Then, for all natural number n with $n \geq N$, we have $|x_n - y_n| \leq 2n^{-1} \leq 2N^{-1} < 2(2k)^{-1} = k^{-1}$. That is, $|x_n - y_n| \leq 2k^{-1}$.

Conversely, let n and k be positive integers. Then, by the assumption, there exists a positive integer N' such that $|x_n - y_n| \leq k^{-1}$ for all positive integer n with $n \geq N$. Then, define $N := \max\{k, N'\}$, and we have $|x_n - y_n| \leq |x_n - x_m| + |x_m - y_m| + |y_m - y_n| \leq 2n^{-1} + 2m^{-1} + k^{-1} \leq 2n^{-1} + 3k^{-1}$ for all positive integer m with $m \geq N$. Thus, $\forall n \forall k(|x_n - y_n| \leq 2n^{-1} + 3k^{-1})$ since $\forall k(q \leq k^{-1})$ if and only if $q \leq 0$ for a rational number q. ∎

Proposition 2.1.4 *" =" is equivalent relation on* $\mathbf{R}$.

(*Proof*) Reflexivity and symmetry are trivial.
Transitivity:
Let $x \equiv (x_n), y \equiv (y_n)$ and $z \equiv (z_n)$ be real numbers, assume that $x = y$ and $y = z$, and let k be a positive number. Then, there exist positive integers N_1 and N_2 such that $\forall n \geq N_1(|x_n - y_n| \leq (2k)^{-1})$ and $\forall n \geq N_2(|y_n - z_n| \leq (2k)^{-1})$ by Proposition 2.1.3. Then, define $N := \max\{N_1, N_2\}$, and we have $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \leq (2k)^{-1} + (2k)^{-1} = k^{-1}$ for all positive integer n with $n \geq N$. Hence, $x = z$ by Proposition 2.1.3. ∎

Definition 2.1.5 A sequence of rational numbers (x_n) is a *Cauchy sequence* if for all positive integer k, there exists a positive integer N such that $|x_m - x_n| < k^{-1}$ for all positive integers m and n with $m, n \geq N$.

Cauchy sequences of rational numbers have upper bound i.e. there exists a positive integer K such that $|x_n| < K$ for all positive integer n: there exists a positive integer N such that $|x_m - x_n| \leq 1$ for all positive integers n and m with $m, n \geq N$, hence we can take a natural number K such that $K \geq \max\{|x_1|, ..., |x_{N-1}|, |x_N| + 1\}$. The K is called *canonical bound*.

Then, a regular sequence of rational numbers is a Cauchy sequence since for all positive integer k, $|x_m - x_n| \leq m^{-1} + n^{-1} \leq (2k)^{-1} + (2k)^{-1} = k^{-1}$ for all m and n with $m, n \geq 2k$. Hence, any regular sequence has canonical bound.

Definition 2.1.6 Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers and be α be a rational number, and define $K := \max\{K_x, K_y\}$, where K_x and K_y is canonical bounds of x and y. Then,

(1) $x + y := (x_{2n} + y_{2n})$

(2) $xy := (x_{2Kn} y_{2Kn})$

(3) $\max\{x, y\} := (\max\{x_n, y_n\})$

(4) $-x := (-x_n)$

(5) $\alpha^* := (\alpha, \alpha, \alpha, ...)$

(6) $|x| := \max\{x, -x\}$

(7) $\min\{x, y\} := -\max\{-x, -y\}$.

Trivially, $-(-x) = x, \max\{x, y\} = \max\{y, x\}, |x| = (|x_n|), \min\{x, y\} = (\min\{x_n, y_n\})$ and $|-x| = |x|$.

Lemma 2.1.7 *Let p, p', q and q' be rational numbers. Then,*

$$|\max\{p, q\} - \max\{p', q'\}| \leq \max\{|p - p'|, |q - q'|\}.$$

(*Proof*) In the case that $\max\{p,q\} = p$ *and* $\max\{p',q'\} = q'$, $\max\{p,q\} - \max\{p',q'\} \leq p-p' \leq \max\{|p-p'|, |q-q'|\}$, and $\max\{p',q'\} - \max\{p,q\} \leq q'-q \leq \max\{|p-p'|, |q-q'|\}$. Thus, $|\max\{p,q\} - \max\{p',q'\}| \leq \max\{|p-p'|, |q-q'|\}$.

In the case that $\max\{p,q\} = q$ *and* $\max\{p',q'\} = p'$, it is similar.

In the case that $\max\{p,q\} = p$ *and* $\max\{p',q'\} = p'$, $|\max\{p,q\} - \max\{p',q'\}| = |p-p| \leq \max\{|p-p'|, |q-q'|\}$.

In the case that $\max\{p,q\} = p$ *and* $\max\{p',q'\} = p'$, it is similar. ∎

Proposition 2.1.8 *The sequences* $x+y, xy, \max\{x,y\}, -x, \alpha^*, |x|$ *and* $min\{x,y\}$ *of Definition* 2.1.1 *are real numbers.*

(*Proof*) Let m and n be positive integers.

$x+y$: $|(x_{2m}+y_{2m}) - (x_{2n}+y_{2n})| \leq |x_{2m}-x_{2n}| + |y_{2m}-y_{2n}| \leq ((2m)^{-1}+(2n)^{-1}) + ((2m)^{-1}+(2n)^{-1}) = m^{-1}+n^{-1}$. Thus, $(x_{2n}+y_{2n})$ is regular, hence $x+y$ is a real number.

xy : $|(x_{2km}y_{2km}) - (x_{2kn}y_{2km})| \leq |y_{2km}||x_{2km}-x_{2kn}| + |y_{2kn}||y_{2km}-y_{2kn}| \leq k((2km)^{-1} + (2kn)^{-1}) + k((2km)^{-1}+(2kn)^{-1}) \leq m^{-1}+n^{-1}$.

$\max\{x,y\}$: $|\max\{x_m,y_m\} - \max\{x_n,y_n\}| \leq \max\{|x_m-x_n|, |y_m-y_n|\} \leq m^{-1}+n^{-1}$ by Lemma 2.1.7.

$-x$ and α^* : trivial.

$|x|$ and $\min\{x,y\}$: By (1) and (3). ∎

Let $x \equiv (x_n), x' \equiv (x'_n), y \equiv (y_n)$ and $y' \equiv (y'_n)$ be real numbers. Then , if $x = x'$ and $y = y'$, then $x - x' = y - y'$: for each positive integer k, there exist positive integers N_1 and N_2 such that $|x_m - x'_m|, |y_n - y'_n| \leq (2k)^{-1}$ for all m and n with $m \geq N_1$ and $n \geq N_2$. Then, define $N := \max\{N_1, N_2\}$, and $|(x_{2n}-x'_{2n}) - (y_{2n}-y'_{2n})| \leq |x_{2n}-x'_{2n}| + |y_{2n}-y'_{2n}| \leq (2k)^{-1} + (2k)^{-1} = k^{-1}$ for all n with $n \geq N$.

Lemma 2.1.9 *Let* (x_n) *and* (y_n) *be Cauchy sequences of rational numbers. Then,* $(x_n + y_n)$ *and* $(x_n y_n)$ *are Cauchy sequences of rational numbers.*

(*Proof*) Define $K := \max\{K_x, K_y\}$, where K_x and K_y is a canonical bound of x and y respectively.

(x_n+y_n): Let k a positive integer. Then, there exist positive numbers N_1 and N_2 such that $|x_m - x_n| \leq (2k)^{-1}$for all m and n with $m,n \geq N_1$ and that $|y_m - y_n| \leq (2k)^{-1}$ for all m and n with $m,n \geq N_2$. Then $|(x_m+y_m) - (x_n+y_n)| \leq |x_m-x_n| + |y_m-y_n| \leq k^{-1}$. That is, $\forall k \exists N \forall m,n \geq N(|(x_m+y_m) - (x_n+y_n)| \leq k^{-1})$. Hence, $(x_n y_n)$ is a Cauchy sequence.

$(x_n y_n)$: Let k be a positive integer. Then, there exist positive integers N_x and N_y such that $|x_m - x_n| \leq (2Kk)^{-1}$ for all m and n with $m,n \geq N_x$ and $|y_m - y_n| \leq (2Kk)^{-1}$ for all m and n with $m,n \geq N_y$, and define $N := \max\{N_x, N_y\}$. Then $|x_m y_m - x_n y_n| \leq |x_n||y_m - y_n| + |y_m||x_m - x_n| \leq K(2Kk)^{-1} + K(2Kk)^{-1} = k^{-1}$ for all m and n with $m,n \geq N$. ∎

Proposition 2.1.10 *Let* x, y *and* z *be real numbers and* α *and* β *be rational numbers. Then,*

(1) *If x=y, then* $x + z = y + z, xz = yz$ *and* $|x| = |y|$.

(2) $x + y = y + x,\ xy = yx$.

(3) $(x + y) + z = x + (y + z), x(yz) = (xy)z$.

(4) $x(y + z) = xy + xz$.

(5) $0^* \pm x = x,\ 1^*x = x$.

(6) $x - x = 0^*$.

(7) $|xy| = |x||y|$.

(8) $(\alpha + \beta)^* = \alpha^* + \beta^*, (\alpha\beta)^* = \alpha^*\beta^*$.

(9) $(-\alpha)^* = -\alpha^*$.

(*Proof*) $x \equiv (x_n), y \equiv (y_n)$ *and* $z \equiv (z_n)$. Let K_x, K_y and K_z be canonical bound of x, y and z respectively.

(1): Suppose that $x = y$. Then, $\forall k \exists N \forall n \geq N(|x_n - y_n| \leq k^{-1})$ by Proposition 2.1.3.

$x + z = x + y$: For each positive integer k, there exists a positive integer N such that $|(x_{2n} + z_{2n}) - (y_{2n} + z_{2n})| = |x_{2n} - y_{2n}| \leq n^{-1} < 2n^{-1}$ for all n with $n \geq N$. Thus, $x + z = y + z$. Similarly, $|(x_{2n} - z_{2n}) - (y_{2n} - z_{2n})| \leq 2n^{-1}$.

$xz = yz$: Define $K_1 := \max\{K_x, K_z\}$ and $K_2 := \max\{K_y, K_z\}$, define $K := \min\{K_1, K_2\}$ and let k be a positive number. Then, there exists a positive number N' such that $|x_n - y_n| \leq (2K_2k)^{-1}$ for all n with $n \geq N'.Then,$let $N := \max\{2k_2k, N'\}$.

Now, for all n with $n \geq N$,
$|x_{2K_1n}z_{2K_1n} - y_{2K_2n}z_{2K_2n}|$
$\leq |z_{2k_1n}||x_{2K_1n} - y_{2K_2n}| + |y_{2K_2n}||z_{2K_1n} - z_{2K_2n}|$
$\leq K_2(2K_2k)^{-1} + K_2((2K_1n)^{-1} + (2K_2n)^{-1})$
$\leq (2k)^{-1} + K_2((2Kn)^{-1} + (2Kn)^{-1})$
$\leq (2k)^{-1} + K_2(K(2K_2k))^{-1}$
$= (2k)^{-1} + (2Kk)^{-1} \leq (2k)^{-1} + (2k)^{-1} = k^{-1}$.
Thus, $\forall k \exists N \forall n \geq N(|x_{2K_1n}z_{2L_1n} - y_{2K_2n}z_{2K_2n}| \leq k^{-1})$. Hence, $xz = yz$.

$|x| = |y|$: Since $||x_n| - |y_n|| \leq |x_n - y_n|$ for all n.

(3): $|((x_{4n}+y_{4n})+z_{2n})-(x_{2n}+(y_{4n}+z_{4n})) \leq |x_{4n}-x_{2n}|+|z_{4n}-z_{2n}| \leq 3(2n)^{-1} \leq 2n^{-1}$. Hence, $(x + y) + z = x + (y + z)$.

Next, $(x_ny_nz_n)$ is a Cauchy sequence by Lemma 2.1.9, and for any k, there exist some N such that $|x_my_mz_m - x_ny_nz_n| \leq k^{-1}$ for all m and n with $m, n \geq N$. Thus, for $x(yz) \equiv (x_{2Kn}y_{4KK_{yz}n}z_{4KK_{yz}})$ and $(xy)z \equiv (x_{4K_{xy}n}y_{4KK_{xy}n}z_{2Kn})$ and any positive integer k, where $K_{yz} := K_yK_z$, $K_{xy} := K_xK_y$ and $K := K_xk_yK_z$, there exists a positive integer N such that $|x_{2Kn}y_{4KK_{yz}n}z_{4KK_{yz}} - x_{4K_{xy}n}y_{4KK_{xy}n}z_{2Kn}| \leq k^{-1}$ for all n with $n \geq N$. Hence, $x(yz) = (zy)z$.

(4): Define $K_{xy} := \max\{K_x, K_y\}, K_{xz} := \max\{K_x, K_y\}, K := \max\{K_{xy}, K_{xz}\}$ and $K' := \max\{K_x, K_{y+x}\}$, where K_{y+z} is one of $x+y$, and define $x(y+z) := (x_{2K'n}(y_{4K'K_{y+z}n}+$

$z_{4K'K_{y+z}n}))$ and $xy + yz := (x_{4K_{xy}n}y_{4K_{xy}n} + x_{4K_{xz}n}z_{4K_{xz}n})$. Then, $|x_{2K'n}(y_{4K'K_{y+z}n} + z_{4K'K_{y+z}n}) - (x_{4K_{xy}n}y_{4K_{xy}n} + x_{4K_{xz}n}z_{4K_{xz}n})| \leq |x_{2K'n}y_{4K'K_{y+z}n} - x_{4K_{xy}n}y_{4K_{xy}n}| + |x_{4k_{xy}n}z_{4K'K_{y+z}n} - x_{4K_{xz}n}z_{4K_{xz}n}|$.

Now, (x_ny_n) and (x_nz_n) are Cauchy sequence of rational numbers by Lemma 2.1.9. Thus, for any k, there exist positive integers N_1 and N_2 such that $|x_my_m - x_ny_n| \leq (2k)^{-1}$ for all positive integers m and n with $m, n \geq N_1$ and that $|x_mz - x_nz_n| \leq (2k)^{-1}$ for all positive integers m and n with $m, n \geq N_2$.

Here, define $N := \max\{N_1, N_2\}$. Then, $|x_{2K'n}y_{4K'K_{y+z}n} - x_{4K_{xy}n}y_{4K_{xy}n}| + |x_{4K_{xy}n}z_{4K'K_{y+z}n} - x_{4K_{xz}n}z_{4K_{xz}n}| \leq k^{-1}$ for n with $n \geq N$.

(5): Since (x_n) is a Cauchy sequence, $\forall k \exists N \forall n \geq N(|x_{2n} - x_n| \leq k^{-1}.)$

(7): Since $|xy| = (|x_{2K_{xy}n}y_{2K_{xy}n}|), |x||y| \equiv (|x_{2K_{xy}n}||x_{2K_{xy}n}|)$.

(2), (6), (8) and (9) are trivial. ∎

Next, we show another relations on $\mathbf{R}$.

Definition 2.1.11 Let $x \equiv (x_n)$ be a real number. x is *positive* if there exists a positive integer N such that $x_N > N^{-1}$. x is also *nonnegative* if $x_n \geq -n^{-1}$ for all positive integers n.

Lemma 2.1.12 *Let p and q be rational numbers. Then $p \leq q$ if and only if $p \leq q + k^{-1}$ for any positive integer k.*

(*Proof*) It is trivial that if $p \leq q$, then $p \leq q + k^{-1}$ for all positive integer k. Conversely, suppose that $p > q$. Then, there exists a positive integer N such that $p > q + N^{-1}$. On th other hand, $p \leq q + N^{-1}$ by the assumption. Thus, $N < N$. It is contradictory. Hence, $p \leq q$ since $\alpha \leq 0$ or $\alpha > 0$ for all rational number α. ∎

Proposition 2.1.13 *Let $x \equiv (x_n)$ be a real number. Then x is positive if and only if there exists a positive integer N such that $x_n \geq N^{-1}$ for all positive integer n with $n \geq N$.*

x is also nonnegative if and only if for any positive integer k, there exists a positive integer N such that $x_n \geq -k^{-1}$ for all positive integer n with $n \geq N$.

(*Proof*) At first, we show $x > 0^*$ if and only if $\exists N \forall n \geq N(x_n \geq N^{-1})$.

Let N' be a positive integer with $x'_N \geq N'^{-1}$, and let N be a positive integer with $x'_N - N'^{-1} > 2N^{-1} > 0$. Then, $|x_n - x_{N'}| \leq n^{-1} + N'^{-1}$ for all positive numbers n with $n \geq N^{-1}$. Thus, $x_n \geq -n^{-1} + (x_n - N'^{-1}) > -n^{-1} + 2N^{-1} \geq -N^{-1} + 2N^{-1} = N^{-1}$.

Conversely, let N' be a positive integer with $x_n \geq N'$ for all n with $n \geq N'$, and define $N := N' + 1$. Then $x_N = x_{N'+1} \geq N'^{-1} > (N+1)^{-1} = N^{-1} > 0$.

Next, we show $x \geq 0^*$ if and only if $\forall k \exists N \forall n \geq N(x_n \geq -k^{-1})$.

Let k be a positive integer. Then, $x_n \geq -n^{-1} \geq -k^{-1}$ for all positive integers n with $n \geq k$.

Conversely, let k a positive integer, and let N be a positive integer with $x_n \geq -k^{-1}$ for all $n \geq N$. Then, $|x_m - x_n| \leq m^{-1} + n^{-1}$ for all n with $n \geq N$. Therefore, $x_m \geq x_n - n^{-1} - m^{-1} \geq -k^{-1} - n^{-1} - m^{-1}$. Thus, $x_m \geq -m^{-1}$ by Lemma 2.1.12 since k and n are positive integers. ∎

Proposition 2.1.14 *Let x and y be real numbers.*

(1) *Suppose $x = y$. Then $y > 0^*$ whenever $x > 0^*$, and $y \geq 0^*$ whenever $x \geq 0^*$.*

(2) *$x + y \geq 0^*$ and $xy \geq 0^*$ whenever $x \geq 0^*$ and $y \geq 0^*$.*

(3) *$x + y > 0^*$ and $xy > 0^*$ whenever $x > 0^*$ and $y > 0^*$.*

(4) *$x + y > 0^*$ whenever $x > 0^*$ and $y \geq 0^*$.*

(5) *$|x| > 0^*$ if and only if $x > 0^*$ or $x < 0^*$.*

(*Proof*) Let $x \equiv (x_n)$ and $y \equiv (y_n)$ be real numbers, and let $K := \max\{K_x, K_y\}$, where K_x and K_y are canonical bound of x and y respectively.

(1): In the case $x > 0^*$, let N_1 be a positive integer such that $x_n \geq {N_1}^{-1}$ for all n with $n \geq N_1$, and let N_2 be a positive number with $|x_n - y_n| \leq (2N_1)^{-1}$ for all n with $n \geq N_2$. Then, define $N := \max\{2N_1, N_2\}$, and for all n with $n \geq N$, $y_n \geq x_n - (2N_1)^{-1} \leq N_1 - (2N_1)^{-1} = (2N_1)^{-1} = N^{-1}$. Thus, $\exists N \forall n \geq N(y_n \geq N^{-1})$.

In the case $x \geq 0^*$, let k be a positive integer, let N_3 be a positive integer with $x_n \geq -(2k)^{-1}$ for all n with $n \geq N_3$ and let N_4 be a positive integer such that $|x_n - y_n| \leq (2k)^{-1}$ for all n with $n \geq N_4$. Define $N' := \max\{N_3, N_4\}$. Then for all n with $n \geq N'$, $y_n \geq x_n - (2k)^{-1} \geq -(2k)^{-1} - (2k)^{-1} = -k^{-1}$. Thus, $\forall k \exists N' \forall n \geq N'(y_n \geq -k^{-1})$.

(2): Let k be a positive integer.

In the case that $x + y \geq 0^*$, let N_1 and N_2 be positive integers such that $x_m, y_n \geq -(2k)^{-1}$ for all m and n with $m \geq N_1$ and $n \geq N_2$, and define $N := \max\{N_1, N_2\}$. Then, for all n with $n \geq N$, $x_{2n} + y_{2n} \geq -(2k)^{-1} + -(2k)^{-1} = k^{-1}$.

In the case that $xy \geq 0^*$, let N_3 and N_4 be a positive integer such that $x_m, y_n \geq -k^{-2}$ for all m and n with $m \geq N_3$ and $n \geq N_4$, and define $N' := \max\{N_3, N_4\}$. Then, for all n with $n \geq N'$, $x_{2Kn} y_{2Kn} \geq -k^{-2} \geq -k^{-1}$.

(3): Let N_x be a positive integer such that $x_n \geq {N_x}^{-1}$ for all n with $n \geq N_x$, and let N_y be a positive integer such that $y_n \geq {N_y}^{-1}$ for all n with $n \geq N_y$.

In the case that $x + y > 0^*$, define $N_1 := N_x + N_y$. Then, for all n with $n \geq N_1$, $x_{2n} + y_{2n} \geq {N_x}^{-1} + {N_y}^{-1} \geq (N_x + N_y)^{-1} = N_1^{-1}$.

In the case that $xy > 0^*$, Let $N_2 := N_x N_y$. Then, for all n with $n \geq N_2$, $x_{2Kn} y_{2Kn} \geq (N_x N_y)^{-1} = N_2^{-1}$.

(4): Let N' be a positive integer with $x_n \geq N'^{-1}$, let N'' be a positive number such that $y_n \geq -(2N')^{-1}$ for all n with $n \geq N''$ and let $N := \max\{2N', N''\}$. Then, for all n with $n \geq N$, $x_{2n} + y_{2n} \geq N'^{-1} - (2N')^{-1} = (2N')^{-1} \leq N^{-1}$.

(5): Let $x \equiv (x_n)$ be a real number with $|x| > 0^*$. Then, for some positive integer N, $|x_N| > N^{-1}$ i.e. $x_n > N^{-1}$ or $-x_N > 0$. Thus, $x_N > N^{-1}$ for some N or $-x_N > N^{-1}$ for some N. Therefore, $x > 0^*$ or $x < 0^*$.

The converse is trivial. ∎

Definition 2.1.15 Let x and y be real numbers. Then we write $x < y$ (or $y > x$) if and only if $y - x$ is positive.

We also write $x \leq y$ (or $y \geq x$) if and only if $y - x$ is nonnegative.

By Proposition 2.1.10 and 2.1.14 (1), we find that x is positive if and only if $x > 0^*$, and that x is nonnegative if and only if $x \geq 0^*$ for all real number x.

Let x, x', y and y' be real numbers. Then, if $x = x'$ and $y = y'$, then $x < y$ i.e $y - x > 0^*$ i.e. $y' - x' > 0^*$ i.e. $y' > x'$ by $y - x = y' - x'$ and Proposition 2.1.14(1). Hence, $x < y$ if and only if $x' < y'$, and $x \leq y$ if and only if $x' \leq y'$.

Proposition 2.1.16 *Let x, x', y and y' be real numbers.*

(1) $x < y$ *whenever either* $x < x'$ *and* $x' \leq y$ *or* $x \leq x'$ *and* $x' < y$.

(2) $x \leq y$ *whenever* $x \leq x'$ *and* $x' \leq y$.

(3) $x + y \leq x' + y'$ *whenever* $x \leq x'$ *and* $y \leq y'$, *and,*
$x + y < x' + y'$ *whenever* $x \leq x'$ *and* $y < y'$.

(4) $xy \leq x'y$ *whenever* $x \leq x'$ *and* $y \geq 0^*$, *and*
$xy < x'y$ *whenever* $x < x'$ *and* $y > 0^*$.

(5) *If* $x < y$, *then* $-x > -y$, *and, if* $x \leq y$, *then* $-x \geq -y$.

(6) $x \leq y$ *and* $x \geq y$ *if and only if* $x = y$.

(7) $\max\{x, y\} \geq x, \quad \min\{x, y\} \leq x$.

(8) *If* $x < y$ *and* $x' < y$, *then* $\max\{x, x'\} < y$, *and,*
if $x < y$ *and* $x < y'$, *then* $x < \min\{y, y'\}$.

(9) *If* $x \leq y$ *and* $x' \leq y'$, *then* $\max\{x, x'\} \leq y$,*and,*
if $x \leq y$ *and* $x' \leq y'$, *then* $x \leq \min\{y, y'\}$.

(10) *If* $x \leq y$, *then* $\max\{x, y\} = y$ *and* $\min\{x, y\} = x$.

(11) $|x| \geq 0^*$.

(12) *If* $x < y$ *and* $-x < y$, *then* $|x| < y$.

(13) *If* $x \leq y$ *and* $-x \leq y$, *then* $|x| \leq y$.

(14) $|x + y| \leq |x| + |y|$.

(15) $||x| - |y|| \leq |x - y|$.

(16) $\neg(x < x)$.

(*Proof*) (1): In the case $x < x'$ and $x' \le y$, $y - x = (y - x') + (x' - x) > 0^*$ by Proposition 2.1.14 (4) since $y - x' \ge 0^*$ and $x' - x > 0^*$.

(2): By Proposition 2.1.14 (2), it is similar to the proof of (1).

(3): By Proposition 2.1.14 (2), $(y' + x') - (x + y) = (y' - y) + (x' - x) \ge 0^*$. Similarly, $(x' + y') - (x + y) > 0^*$ by Proposition 2.1.14(4).

(4): $x'y - xy = (x' - x)y \ge 0^*$ by Proposition 2.1.14 (2). Similarly, $x'y - xy \ge 0^*$ by Proposition 2.1.14 (2).

(5): If $x < y$, then $x - y < 0^*$ i.e. $-y + x < 0^*$ i e. $-y - (-x) < 0^*$. Thus, $-x > -y$. Also, if $-x \ge -y$, then $-x - (-y) \le 0^*$ i.e. $-x + y \le 0^*$ i.e. $y - x \le 0^*$. Thus, $x \le y$.

(6): For any n, $y_n - x_n \ge -n^{-1}$ and $x_n - y_n \ge -n^{-1}$ i.e. $|x_n - y_n| \le n^{-1} \le 2n^{-1}$. Thus, $x = y$.

Conversely, for any k, there exists N such that $x_n - y_n \ge -|x_n - y_n| \ge -k^{-1}$ and $y_n - y_n \ge -|x_n - y_n| \ge -k^{-1}$ for all n with $n \ge N$. Hence, $x \ge y$ and $x \le y$.

(7): $\max\{x_n, y_n\} - x_n \ge 0 \ge -k^{-1}$ and $x_n - \min\{x_n, y_n\} \ge 0 \ge -k^{-1}$ for all k and n.

(8): By (1), $y - \max\{x, x'\} \le y - x < 0^*$. That is, $\max\{x, x'\} < y$.

Similarly, $\min\{y, y'\} - x \ge y - x > 0^*$.

(9): By the same way as (8).

(10): $y - \max\{x, y\} \le y - y = 0^*$. Thus, $y = \max\{x, y\}$ by (6) and (7).

Similarly, $x = \min\{x, y\}$.

(11): Since $|x_{2n}| \ge 0$ for all n.

(12): $|x| = \max\{x, -x\} < y$ by (8).

(13): It's similar to (13).

(14): For any n, $(|x_{4n}| + |y_{4n}|) - |x_{4n} + y_{4n}| \ge 0 \ge -n^{-1}$.

(15): For any n, $|x_{4n} - y_{4n}| - ||x_{4n}| - |y_{4n}|| \ge 0 \ge -n^{-1}$.

(16): Since $\neg(0^* > 0^*)$, $x - x$ is not positive. ∎

Proposition 2.1.17 *Let x and y be real numbers. Then , if $x < y$ is contradictory, then $x \ge y$.*

(*Proof*) For $x \equiv (x_n)$ and $y \equiv (y_n)$, if $\neg\exists n(y_n - x_n > n^{-1})$, then $\forall n(x_n - y_n \ge -n^{-1})$. Hence, $y \ge x$. ∎

Now, for all real numbers x and y, $x \le y$ whenever $x < y$ or $x = y$. But the converse does not hold. See Proposition 2.3.1.

Definition 2.1.18 For real numbers x and y , we write $x \sharp y$ if and only if $x < y$ or $x > y$.

Proposition 2.1.19 *Let x and y be real numbers. Then $x \sharp y$ is contradictory if and only if x=y.*

(*Proof*) $x \sharp y$ is contradictory if and only if $x < y$ or $x > y$ are contradictory i.e. $x \ge y$ and $x \le y$. That is, $x = y$ by Proposition 2.1.16(7) and Proposition 2.1.17. ∎

If $x\sharp y$, then $\neg(x = y)$. But the converse doesn't holds since $\neg(x = y) \Rightarrow x\sharp y$ if and only if MP. See Proposition 2.3.1.

Next proposition means the existence of the inverse element in $\mathbf{R} - 0^*$ for the product. Hence, $\mathbf{R}$ is a field by Proposition 2.1.10.

Definition 2.1.20 Let $x \equiv (x_n)$ be a real number with $x\sharp 0^*$, and let N be a positive integer such that $x_n \geq N^{-1}$ for all n with $n \geq N$ or that $-x_n \geq N^{-1}$ for all n with $n \geq N$. Let $y \equiv (y_n)$ be a rational sequence with

$$y_n = \begin{cases} (x_{N^3})^{-1}, & \text{if } n < N \\ (x_{nN^2})^{-1}, & \text{if } n \geq N. \end{cases}$$

Define that $x^{-1} := (y_n)$.

Proposition 2.1.21 *Let x and y be real numbers with $x, y\sharp 0^*$. Let α be a rational number with $\alpha\sharp 0^*$. Then*

(1) x^{-1} *is a real number.*

(2) $xx^{-1} = x^{-1}x = 1^*$.

(3) $x^{-1} > 0^*$ *whenever* $x > 0^*$, *and* $x^{-1} < 0^*$ *whenever* $x < 0^*$.

(4) *If* $xt = 1$, *then* $t = x^{-1}$.

(5) $(x^{-1})^{-1} = x$.

(6) $(xy)^{-1} = y^{-1}x^{-1}$.

(7) $|x^{-1}| = |x|^{-1}$.

(8) $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

(*Proof*) Let N be a positive integer such that $x_n \geq N^{-1}$ for all n with $n \geq N$ or that $-x_n \geq N^{-1}$ for all n with $n \geq N$, and let K be a upper bound of canonical bounds of (x_n) and (y_n) with $K \geq N$, where $x^{-1} := (y_n)$.

(1): In case $m \geq N$ and $n < N$, $|y_m - y_n| \leq |x_{mN^2}{}^{-1} - x_{N^3}{}^{-1}| \leq N^2(N^{-3} + m^{-1}N^{-2}) = N^{-1} + m^{-1} \leq n^{-1} + m^{-1}$.

(2): Let k be a positive integer, and define $N' := \min\{k, N\}$. Then, for all positive integer n with $n \geq N'$, $|x_{2Kn}y_{2Kn} - 1| = |x_{2Kn}x_{2KN^2n}{}^{-1} - 1| \leq N(2Kn)^{-1} + (2KNn)^{-1} \leq (2k)^{-1} + (2N^2k)^{-1} = (2k)^{-1} + (2N^2k)^{-1} \leq (2k)^{-1} + (2k)^{-1} = k^{-1}$. Thus, $\forall k \exists N' \forall n \geq N'(|x_{2Kn}y_{2Kn} - 1^*| \leq k^{-1})$. Hence, $xx^{-1} = 1^*$. Thus, $xx^{-1} = x^{-1}x = 1^*$ by Proposition 2.1.3.

(3): In case $x > 0^*$, there exists N such that $x_n \geq N^{-1}$ for all n with $n \geq N$. Then, for all n with $n \geq N$, $(K_x)^{-1} < (x_{N^2n})^{-1}$, where K_x is a canonical bound of x. Here, define $N' := \max\{N, K_x\}$. Then, $(x_{N^2n})^{-1} \geq N'^{-1}$ for all n with $n \geq N'$. Thus, $x^{-1} > 0^*$. Similarly, $x^{-1} < 0^*$ whenever $x < 0^*$.

(4): Let t be a real number with $xt = 1^*$. Then $x^{-1}xt = x^{-1}1^*$ by Proposition 2.1.10 (1).

(5): By (4).

(6): $(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}y = 1^*$. Thus, by (4), $(xy)^{-1} = y^{-1}x^{-1}$.

(7): $|x^{-1}||x| = |x^{-1}x| = 1$. Thus, $|x|^{-1} = |x^{-1}|$ by (4).

(8): It's trivial. ∎

By Proposition 2.1.10 and 2.1.21, $(\alpha\beta)^* = \alpha^*\beta^*$, $(\alpha+\beta) = \alpha^* + \beta^*$, $(-\alpha)^* = -\alpha^*$, $|\alpha|^* = |\alpha^*|$ and $(\alpha^{-1})^* = (\alpha^*)^{-1}$ for all rational numbers α and β. It is also that $\alpha\Delta\beta$ if and only if $\alpha^*\Delta\beta^*$, where δ stands for any of the relations $=, <, >, \leq, \geq$ and $\sharp$.

Thus the map $* : \mathbf{Q} \to \mathbf{R}$ is order isomorphic, hence we can identify a rational number α with a real number α^*.

Proposition 2.1.22 *Let $x \equiv (x_n)$ be a real number. Then $\forall n(|x - {x_n}^*| \leq n^{-1*})$.*

(*Proof*) Let n be fixed. Then, we may show $n^{-1} - |x_{4m} - x_n| \geq -m^{-1}$ for all positive integer m.

For all m, $n^{-1} - |x_{4m} - x_n| \geq n^{-1} - ((4m)^{-1} + n^{-1}) = -(4m)^{-1} \geq m^{-1}$. ∎

Proposition 2.1.23 *Let x and y be real number with $x < y$. Then, there exists a rational number α such that $x < \alpha < y$.*

(*Proof*) Let $x \equiv (x_n)$ and $y \equiv (y_n)$, let N_0 be a positive integer with $y_{4n} - x_{2n} \geq N_0^{-1}$ for all n with $n \geq N_0$, define let $N := 2N_0$, and define $\alpha := 2^{-1}(x_{2N} + y_{2N})$. Then, $y_{2N} - \alpha = y_{2N} - x_{2N} \geq (2N_0)^{-1} = N^{-1}$. Thus, $y - \alpha > 0$. Similarly, $\alpha - x > 0$. ∎

Corollary 2.1.24 *Let x be a real number. Then for any $\epsilon > 0$, there exists a rational number α such that $|x - \alpha| < \epsilon$.*

(*Proof*) For all real number x, there exists a rational number α such that $-\epsilon + x < \alpha < \epsilon + x$ for all $\epsilon > 0$ by Proposition 2.1.23. Thus, $|x - \alpha| < \epsilon$. ∎

Next, we show the proposition for a certain judgment on $\mathbf{R}$.

Lemma 2.1.25 *Let $x_1, ..., x_n$ be real numbers with $x_1 + ... + x_n > 0$. Then, there exists a positive integer i such that $0 \leq i \leq n$ and $x_i > 0$.*

(*Proof*) There exists a rational number α such that $x_1 + ... + x_n > \alpha > 0$ by Proposition 2.1.23, and there exists a rational finite sequence (a_i) such that $|x_i - a_i| < \alpha n^{-1}$ for all i with $i \leq n$ by Proposition 2.1.24. Then, $\sum_{i=1}^n a_i - \sum_{i=1}^n x_i = \sum_{i=1}^n (a_i - x_i) > \sum_{i=1}^n -\alpha n^{-1} = -\alpha$. Thus, $\sum_{i=1}^n a_i > \sum_{i=1}^n x_i - \alpha > 0$ by the assumption. Thus, there exists i such that $a_i > 0$ and that $i \leq n$.

Now, define $M := \min\{{a_i}^{-1}, n\alpha^{-1}\}$, where i satisfies that $i \leq n$ and that $a_i > 0$. Then, $M > 0$, therefore $a_i - N^{-1} > 0$. Thus, $x_i > a_i - \alpha n^{-1} \geq a_i - N^{-1} \geq 0$. Hence, $x_i > 0$. ∎

Corollary 2.1.26 *Let x, y and z be real numbers with $x < y$. Then $x < z$ or $z < y$.*

(*Proof*) Since $y-x>0$, $(z-x)+(y-z)=y-x>0$. Thus, by Lemma 2.1.25, $z-x>0$ or $y-z>0$ i.e. $x<z$ or $z<y$. ∎

Here, we show uncountability for **R**.

Theorem 2.1.27 *Let (a_n) be a sequence of real numbers. Let x_0 and y_0 be real number with $x_0<y_0$. Then there exists a real number x such that $x_0 \le x \le y_0$ and that $a_n \sharp x$ for all positive integer n.*

(*Proof*) We define sequence of rational numbers (x_n) and (y_n) such that

(1) $x_0 \le x_n \le x_m < y_m \le y_n \le y_0$ for all positive integers m and n with $m \ge n$

(2) $a_n < x_n$ or $y_n < a_n$ for positive integer n

(3) $y_n - x_n < n^{-1}$ for all positive integer n

as follows inductively.

Assume that for n, $x_0 \le x_n < y_n \le y_0$, that $a_n < x_n$ or $y_n < a_n$ and that $y_n - x_n < n^{-1}$. Then, $x_n < a_{n+1}$ or $a_{n+1} < y_n$ by Corollary 2.1.26. In the case $x_n < a_{n+1}$, we can take a rational number x_{n+1} with $x_n < x_{n+1} < \min\{a_{n+1}, y_n\}$, and let y_n be one with $x_{n+1} < y_{n+1} < \min\{a_{n+1}, y_n, x_{n+1} + (n+1)^{-1}\}$ by Proposition 2.1.23 and 2.1.16. In the case $a_{n+1} < y_n$, we can take a rational number y_{n+1} with $\max\{a_{n+1}, x_n\} < y_{n+1} < y_n$, and let x_{n+1} be one with $\max\{a_{n+1}, x_n, y_{n+1} - (n+1)^{-1}\} < x_{n+1} < y_{n+1}$. In the both case, $x_0 \le x_n < x_{n+1} < y_{n+1} < y_{n+1} \le y_0$, that $a_{n+1} < x_{n+1}$ or $y_{n+1} < a_{n+1}$ and that $y_{n+1} - x_{n+1} < (n+1)^{-1}$. Thus, (x_n) and (y_n) satisfy (1), (2) and (3) by induction on n (since $x_0 < a_1$ or $a_1 < y_0$ for x_0 and y_0 with $x_0 < y_0$ by Proposition 2.1.26, x_1 and y_1 are defined as the above.)

Now, by (1), (2) and (3), $|x_m - x_n| = x_m - x_n < y_m - x_n < y_n - x_n < n^{-1} < m^{-1} + n^{-1}$ for all positive integer m and n . Thus, (x_n) is a real number. Similarly, (y_n) is a real number. Therefore, $(x_n) = (y_n)$ by (3).

Next, let $x = (x_n)$. Then, $x_n \le x$ for all n and $x \le y_n$ for all n by (1). Thus, $a_n < x$ or $x < a_n$ i.e. $a_n \sharp x$ for all n. ∎

Next, we show completeness for **R**.

Definition 2.1.28 A sequence of real numbers (x_n) *conveges* to a real number x if for all $\epsilon > 0$, there exists positive integer N such that $|x_n - x| < k^{-1}$ for all n with $n \ge N$. Then, we write $x_n \to x$ and call x a limit of (x_n).

Here, A *Cauchy seqence* of real numbers is defined by replacing "rational"in Definition 2.1.5 with "real", and it has also canonical bounds : for a Cauchy sequence (x_n) of real numbers, there exists N such that $|x_m - x_n| \le 1$ for all m and n with $m, n \ge N$. Then, $|x_n| \le \max\{|x_1|, ..., |x_{N-1}|, |x_N| + 1\}$ for all n. Thus, we can take a positive integer K with $K \ge \max\{K_1, ..., K_{N_1}, K_N + 1\}$ as canonical bound of (x_n) since $x_n < {K_n}^*$ for all n, where K_n $(1 \le n \le N)$ is a canonical bound of a regular sequence of rational numbers x_n $(1 \le n \le N)$.

Theorem 2.1.29 *A real number sequence (x_n) is a Cauchy sequence if and only if (x_n) converges.*

(*Proof*) Let a real number sequence (x_n) be a Cauchy sequence.

Let (N_k) be a sequence of positive integers such that $|x_m - x_n| \le (2k)^{-1}$ for all m and n with $m, n \ge N_k$, and for each positive integer k, let (y_k) be a sequence of rational numbers with $|x_{N_k} - y_k| \le (2k)^{-1}$ (by Corollary 2.1.24).

We will show that $y \equiv (y_n)$ is a real number and that y is a limit of (x_n).

Now, define $N_{m,n} := \max\{N_m, N_n\}$ for each m and n. Then, $|y_m - y_n| \le |y_m - x_{N_m}| + |x_{N_m} - x_{N_{m,n}}| + |x_{N_{m,n}} - x_{N_n}| + |x_{N_n} - y_n| \le (2m)^{-1} + (2m)^{-1} + (2n)^{-1} + (2n)^{-1} = m^{-1} + n^{-1}$. Thus, $|y_m - y_n| \le m^{-1} + n^{-1}$. Hence, y is a real number.

Next, for any k, we have $|x_n - y| \le |x_n - x_{N_{2k}}| + |x_{N_{2k}} - y_{2k}| + |y_{2k} - y| \le (4k)^{-1} + (4k)^{-1} + (2k)^{-1} = k^{-1}$ for all n with $n \ge N_{2k}$ by Proposition 2.1.22. Thus, $|x_n - y| \le k^{-1}$. Hence, $x_n \to x$.

Conversely, let x be the limit of (x_n).

Let k be a positive integer, and let N be a positive integer with $|x_n - x| < (2k)^{-1}$ for all n with $n \ge N$. Then, $|x_m - x_n| \le |x_m - x| + |x - x_n| < (2k)^{-1} + (2k)^{-1} = k^{-1}$ for all m and $n$with $n \ge N$. Thus, $|x_m - x_n| < k^{-1}$. Hence, (x_n) is a Cauchy sequence. ∎

2.2 Sequences of Real Numbers

Now, we show some properties of converging sequence.

Lemma 2.2.1 *Let x and y be real numbers. Then, $x \le y$ if and only if $x \le y + k^{-1}$ for all positive integer k.*

(*Proof*) It is easy that if $x \le y$, then $x \le y + k^{-1}$.

Conversely, if $x > y$, then there exists positive integer N such that $x - y > N^{-1} > 0$, therefore $y + N^{-1} < x \le y$. It's contradictory. Thus, $x \le y$ by Proposition 2.1.17. ∎

If x_n has limits x and x', then $x = x'$:
let k be a positive number, and let N_1 and N_2 be positive numbers such that $|x_m - x| < (2k)^{-1}$ and $|x_n - x'| < (2k)^{-1}$ for all m and n with $m \ge N_1$ and $n \ge N_2$. Then, $|x - x'| \le |x - x_n| + |x_n - x'| \le (2k)^{-1} + (2k)^{-1} \le k^{-1}$. Thus, $|x - x'| \le 0$ by the above lemma. Thus, $x = x'$.

Lemma 2.2.2 *Let $x \equiv (x_n), x' \equiv (x'_n), y \equiv (y_n)$ and $y' \equiv (y'_n)$ be real numbers. Then,*

$$|\max\{x, x'\} - \max\{y, y'\} \le \max\{|x - x'|, |y - y'|\}.$$

(*Proof*) Since $|\max\{x_{4n}, x'_{4n}\} - \max\{y_{4n}, y'_{4n}\}| \le \max\{|x_{4n} - x'_{4n}|, |y_{4n} - y'_{4n}|\}$ for all n by Lemma 2.1.7. ∎

Proposition 2.2.3 *Let (x_n) and (y_n) be sequences of real numbers such that (x_n) converges to a real number x and that (y_n) converges to a real number y. Then*

(1) $(x_n + y_n) \rightarrow (x+y)$.

(2) $(x_n y_n) \rightarrow (xy)$.

(3) $\max\{x_n, y_n\} \rightarrow \max\{x, y\}$.

(4) *If $x_n = c$ for all n, then $x = c$.*

(5) *If $x, x_n \sharp 0$ for all n, then $x_n^{-1} \rightarrow x^{-1}$.*

(6) *If $x_n \leq y_n$ for all n, then $x \leq y$.*

(7) *Suppose $x = y$. Then for a sequence of real numbers (z_n), if $x_n \leq z_n \leq y_n$ for all n, then $z_n \rightarrow x$.*

(*Proof*) (1) and (2): They are showed by the same way as Lemma 2.1.9 respectively.

(3): Let k a positive integer, and let N_1 and N_2 be positive integers with $|x_m - x|, |y_n - y| < k^{-1}$ for all m and n with $m \geq N_1$ and $n \geq N_2$. Then, $|\max\{x_n, y_n\} - \max\{x, y\}| \leq \max\{|x_n - x|, |y_n - y|\} \leq k^{-1}$ by Lemma 2.2.2.

(4): $x_n \rightarrow c$ since $\forall k, n(|x_n - c| < k^{-1})$. Thus, $x = c$.

(5): Let k be a positive integer, and let N_1 be a positive number such that $|x_n - x| < (2k)^{-1}|x|^2$ for all n $n \geq N_2$. Then, $|x| < (2k)^{-1}|x|^2$ for all n with $n \geq N_1$. Also, let N_2 a positive number such that $|x_n - x| < 2^{-1}|x|$ for all n with $n \geq N_2$. Then, $|x_n| > 2^{-1}|x|$ for all n with $n \geq N_2$. Here, let $N := \max\{N_1, N_2\}$. Then, $|x_n^{-1} - x^{-1}| = |x_n|^{-1}|x|^{-1}|x_n - x| < 2|x|^{-1}|x|^{-1}(2k)^{-1}|x|^2 = k^{-1}$ for all n with $n \geq N$.

(6): Assume that $x > y$. Then, there exits N_0 such that $x - y > N^{-1}$ by Proposition 2.1.23, and there exist N_1 and N_2 such that $|x_m - x|, |y_n - y| < (2N_0)^{-1}$ for all m and n with $m \geq N_1$ and $n \geq N_2$. Let $N := \max\{N_1, N_2\}$. Then, $N_0 < x - y = (x - x_n) + (x_n - y_n) + (y_n - y) \leq |x - x_n| + 0 + |y_n - y| \leq (2N_0)^{-1} + (2N_0)^{-1} = N_0{}^{-1}$ for all n with $n \geq N$. It is contradictory. Then, $x \leq y$ by Proposition 2.1.17.

(7): Let k a positive integer, let N_1 and N_2 be positive integers such that $|x_m - x| < k^{-1}$ and $|y_n - x| < k^{-1}$ for all m and n with $m \geq N_1$ and $n \geq N_2$ and let $N := \max\{N_1, N_2\}$. Then, $z_n - x \leq y_n - x < k^{-1}$ and $x - z_n \leq x - x_n \leq k^{-1}$ for all n with $n \geq N$. Thus, $|z_n - x| < k^{-1}$ by Proposition 2.1.16 (12). ∎

2.3 On Omniscience Principles

Here, we show some propositions are equivalent to LPO, LLPO and MP respectively.

Proposition 2.3.1

(1) *LPO if and only if* $\forall r \in \mathbf{R}(r > 0 \vee r < 0 \vee r = 0)$.

(2) *LLPO if and only if* $\forall r \in \mathbf{R}(r \geq 0 \vee r \leq 0)$.

(3) *MP if and only if* $\forall r \in (\neg(r = 0) \Rightarrow r > 0)$.

(*Proof*) (1): At first, we show that if *LPO* holds, then $\forall r \in \mathbf{R}(r > 0 \vee r < 0 \vee r = 0)$.

Let $r \equiv (r_n)$ be a real number. Let (α_n) be a sequence of integers such that

$$\alpha_n = \begin{cases} 0, & \text{if } |r_n| \leq n^{-1} \\ 1, & \text{if } |r_n| > n^{-1}. \end{cases}$$

Then, $\alpha_n = 1$ for some n if and only if $|r| > 0$ i.e. $r > 0$ or $r < 0$, and $\alpha_n = 0$ for all n if and only if $r = 0$. Thus, $r > 0$, $r < 0$ or $r = 0$ by the assumption.

Conversely, let (α_n) be a sequence of integers taking 0 or 1, and let $r \equiv (r_n)$ be a sequence of rational numbers with $r_n = \sum_{i=1}^{n} 2^{-i}\alpha_i$. Then, r is a real number since $|r_m - r_n| = |\sum_{i=1}^{m} 2^{-i}\alpha_i - \sum_{i=1}^{n} 2^{-i}\alpha_i| \leq \sum_{i=n+1}^{m} |2^{-i}\alpha_i| \leq \sum_{i=1}^{n} 2^{-i} = 2^{-n} - 2^{-m} \leq 2^{-n} \leq n^{-1} < n^{-1} + m^{-1}$ for all m and n with $m \geq n$. Then, $|r| > 0$ if and only if $\exists n(\alpha_n = 1)$ since $\sum_{i=1}^{n} 2^{-i}\alpha_i \geq n^{-1}$ for some n if and only if $\alpha_n = 1$ for some n, and $|r| = 0$ if and only if $\forall n(\alpha_n = 0)$ since $|\sum_{i=1}^{n} 2^{-i}\alpha_i| \leq n^{-1}$ for all n if and only if there exist no n such that $\alpha_n = 1$.

Thus, $\exists n(\alpha_n = 1)$ or $\neg\exists n(\alpha_n = 1)$ by the assumption.

(2): We show that if LLPO holds then $\forall r \in \mathbf{R}(r \geq 0 \vee r \leq 0)$.

Let r be a real number, and let (α_n) and (β_n) sequences of integers such that

$$\alpha_n = \begin{cases} 0, & \text{if } r > -n^{-1} \\ 1, & \text{if } r < -(n+1)^{-1}, \end{cases}$$

and

$$\beta_n = \begin{cases} 0, & \text{if } r < n^{-1} \\ 1, & \text{if } r > (n+1)^{-1}. \end{cases}$$

Then, if there exist m and n such that $\alpha_m = 1$ and $\beta_n = 1$, then $r < -(m+1)^{-1}$ and $r > (n+1)^{-1}$. It's contradictory. Thus, $\neg(\exists n(\alpha_n = 1) \wedge \exists n(\beta_n = 1))$. Therefore, $\neg\exists n(\alpha_n = 1) \vee \neg\exists n(\beta_n = 1)$ by the assumption. Thus, $\alpha_n = 0$ for all n or $\beta_n = 0$ for all n. Hence, $r \geq 0$ or $r \leq 0$ by Lemma 2.2.1.

Conversely, let (α_n) and (β_n) be sequences of integers taking either 0 or 1, and let $r \equiv (r_n)$ be a sequence of rational numbers with $r_n = \sum_{i=1}^{n} 2^{-i}(\beta_i - \alpha_i)$. Then, r is a real number since $|r_m - r_n| \leq \sum_{i=n+1}^{m} |2^{-i}(\beta_i - \alpha_i)| \leq \sum_{i=n+1}^{m} 2^{-i} \leq m^{-1} + n^{-1}$ for all m and n with $m \geq n$.

Now, suppose that $\neg(\exists n(\alpha_n = 1) \wedge \exists n(\beta_n = 1))$. Then, we can classify by the assumption as follows: in the case that $r \geq 0$, suppose that $\alpha_n = 1$ for some n. Then, there exist no i such that $\beta_i = 1$ by the supposition. That is, $\beta_i = 0$ for all i. Therefore, for some n with $\alpha_n = 1$, $-2^{-(n+1)} \leq r_{2^{n+1}} = \sum_{i=1}^{2^{n+1}} 2^{-i}(\beta_i - \alpha_i) = \sum_{i=1}^{2^{n+1}} 2^{-i} - \alpha_i \leq -2^{-n}$ since $r \geq 0$ i.e. $r_n \geq -n^{-1}$ for all n and $1 \leq n \leq 2^{n+1}$. It is contradictory. Thus, $\neg\exists n(\alpha_n = 1)$. Hence, $\neg\exists n(\alpha_n = 1) \vee \neg\exists n(\beta_n = 1)$.

In case the case that $r \leq 0$, suppose that $\beta_n = 1$ for some n. Then, $\alpha_i = 0$ for all i. Therefore, for some n with $\beta_n = 1$, $2^{-(n+1)} \geq r_{2^{n+1}} = \sum_{i=1}^{2^{n+1}} 2^{-i}(\beta_i - \alpha_i) = \sum_{i=1}^{2^{n+1}} 2^{-i}\beta_i \geq 2^{-n}$ since $r \leq 0$ i.e. $r_n \leq -n^{-1}$ for all n and $1 \leq n \leq 2^{n+1}$. It is contradictory. Thus, $\neg\exists n(\alpha_n = 1)$. Hence, $\neg\exists n(\alpha_n = 1) \vee \neg\exists n(\beta_n = 1)$.

(3): We show that if MP, then $\forall r \in (\neg(r = 0) \rightarrow r > 0)$.

Let $r \equiv (r_n)$ be a real number, and let (α_n) be a sequence of integers such that

$$\alpha_n = \begin{cases} 0, & \text{if } |r_n| \leq n^{-1} \\ 1, & \text{if } |r_n| > n^{-1}. \end{cases}$$

Then, $\alpha_n = 1$ for some n if and only if $|r| > 0$, and $\alpha_n = 0$ for all n if and only if $r = 0$. Here, suppose that $\neg(r = 0)$. Then $\neg\forall n(\alpha_n = 0)$. Thus, $\neg\neg\exists n(\alpha_n = 1)$ by $\neg\exists n(\alpha_n = 1) \Rightarrow \forall n(\alpha_n = 0)$. Therefore, $\exists n(\alpha_n = 1)$ by MP. Hence, $|r| > 0$ i.e. $r\sharp 0$.

Conversely, let (α_n) be a sequence of integers taking 0 or 1, and let $r \equiv (r_n)$ be a sequence of rational numbers with $r_n = \sum_{i=1}^{n} 2^{-i}\alpha_i$. Then, r is a real number by the same way as (1). $|r| > 0$ if and only if $\alpha_n = 1$ for some n, and $r = 0$ if and only if $\alpha_n = 0$ for all n.

Now, suppose that $\neg\neg\exists n(\alpha_n = 1)$ i.e. $\neg\neg(|r| > 0)$. Then, $r = 0$. Thus, $r\sharp 0$ by the assumption. That is, $|r| > 0$. Therefore, $\exists n(\alpha_n = 1)$. ∎

Then, $\forall x \in \mathbf{R}(r \geq 0 \Rightarrow r > 0 \vee r = 0)$ if and only if LPO as follows:

Assume that for all real number r, if $r \leq 0$, then $r = 0$ or $r > 0$. Then, $|r| \geq 0$ for any real number r. Thus, $|r| = 0$ or $|r| > 0$. That is, $r = 0$, $r > 0$ or $r < 0$. Hence, LPO holds by the above proposition.

The converse is trivial by the above proposition.

Chapter 3

Metric Spaces

Here, "A set X is inhabited" means that X has at least one element. Actually, in constructive mathematics, "X is nonempty" is not equivalent to "X is inhabited" since "If X is nonempty, then X is inhabited" implies MP (by $X := \{n \in \mathbf{N} | \alpha_n = 1\}$ for a given sequence (α_n) taking 0 or 1).

3.1 Metric Spaces

Definition 3.1.1 Let X be a set, and let d be a map by X to $\mathbf{R^{0+}}$ with the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$

(2) $d(x, y) = d(y, x)$

(3) $d(x, y) \leq d(x, z) + d(z, y)$,

where $\mathbf{R^{0+}}$ is the set of nonnegative real numbers. Then, this d is called *metric* or *distance function*, and (X, d) is called *metric space*. We sometimes omit the metric d.

In a metric space (X, d), the subset A of X is a metric space for the following map d_A:

$$d_A(x, y) := d(x, y) \ (x, y \in A).$$

This (A, d_A) is called a *metric subspace* of (X, d).

Example 3.1.2

(1) Euclid space $\mathbf{R}$, where $d(x, y) := |x - y| \ (x, \ y \in \mathbf{R})$.

(2) $(\prod_{i=1}^{n} X_i, d)$, where (X_i, d_i) $(1 \leq i \leq n)$ are metric spaces and $d(x, y) := \sum_{i=1}^{n} d_i(x_i, y_i)$ $(x \equiv (x_1, ..., x_n), \ y \equiv (y_1, ..., y_n) \in \prod_{i=1}^{n} X_i \equiv X_1 \times ... \times X_n)$.

Definition 3.1.3 A *open ball of radius $r > 0$* (or r-neighborhood) for a point x in metric space (X, d) is the subset $B(x, r) \equiv \{y \in X | d(x, y) < r\}$.

The open ball in the subspace A of X is as follows:

$$\begin{aligned} B_A(x,r) &\equiv \{y \in A | d_A(x,y) < r\} \\ &= \{y \in X | d(x,y) < r\} \cap A. \end{aligned}$$

Definition 3.1.4 Let (X,d) be a metric space, and let V be a subset of X. Then,

- $V^i := \{x \in V | \exists r > 0 (B(x,r) \subset V)\}$.
- $V^- := \{x \in X | \forall r > 0 (B(x,r) \cap V \textit{is inhabited})\}$.

V^i is called *the interior of* V, and V^- is called *the closure of* V.

Then, V is *open* if $V = V^i$, and V is *closed* if $V = V^-$.

Proposition 3.1.5 *Let X be a metric space, and let A and B be subsets of X. Then*

(1) $(A \cap B)^i = A^i \cap B^i$.

(2) $A^- \cup B^- \subset (A \cup B)^-$.

(*Proof*) (1): Let $x \in (A \cap B)$. Then, $B(x,r) \subset A \cap B$ for some $r > 0$. That is, $B(x,r) \subset A$ and $B(x,r) \subset B$. Hence, $x \in A^i \cap B^i$.

Conversely, let $x \in A^i \cap B^i$. Then, $N(x,r_1) \subset A$ and $N(x,r_2) \subset B$ for some $r_1 > 0$ and $r_2 > 0$. Here, let $r := \min\{r_1, r_2\}$. Then, $B(x,r) \subset A \cap B$ since $B(x,r) \subset N(x,r_1) \cap N(x,r_2)$. Hence, $x \in A \cap B$.

(2): Let $x \in (A \cup B)^-$. Then, $B(x,r) \cap (A \cap B)$ is inhabited for all $r > 0$. Thus, $N(x,r_1) \cap A$ is inhabited for all $r_1 > 0$, and $N(x,r_2) \cap B$ is inhabited for all $r_2 > 0$. That is, $x \in A^- \cap B^-$. ∎

About the above (2), " $(A \cup B)^- \supset A^- \cup B^-$" imply LLPO. See Example 3.1.12.

Lemma 3.1.6 *Let X be a metric space, and let A be a subset of X. Then,*

(1) *A^i is the maximal open set contained in A.*

(2) *A^- is the minimal closed set containing A.*

(*Proof*) (1): Let O be a open set contained in A and x be any element of O. Then, $B(x,r)$ is contained in O for some $r > 0$. Thus, $B(x,r)$ is contained in A. Therefore, x belongs to A^i.

(2): Let F be a closed set containing A, and let x be any element of A^-. Then, $B(x,r) \cap A$ is inhabited for all $r > 0$. Thus, $B(x,r) \cap F$ is inhabited, hence x belongs to F. ∎

Example 3.1.7 Let a and b be a real number with $a < b$.

(1) $[a,b]$ is closed in Euclid space $\mathbf{R}$.

(2) (a, b) is open in Euclid space $\mathbf{R}$.

(3) $\{0\}$ and $\{1\}$ is open and closed in subspace $\{0, 1\}$ of Euclid space $\mathbf{R}$.

(4) In Example 3.1.2 (2), O_i is open in X_i for each $i = 1, ..., n$ if and only if $O_1 \times ... \times O_n$ is open in $\prod_{i=1}^n X_i$.

(*Proof*) (1): We may show $[a, b]^- \subset [a, b]$.

Let $x \in [a, b]^-$. Then, $B(x, r) \cap [a, b]$ is inhabited for all $r > 0$. Here, $x \leq b$ or $x \geq a$ by Corollary 2.1.26.

In the case $x \leq b$, suppose that $a < x$. Then, $x < q < a$ for some rational number q by Corollary 2.1.24 and $N(x, |q - x|) \cap [a, b] = \phi$. It is contradictory. Thus, $x \leq a$ by Proposition 2.1.17. Therefore, $x \in [a, b]$.

In the case $x \geq a$, $x \in [a, b]$ similarly. Hence, $x \in [a, b]$.

(2):We may show $(a, b) \subset (a, b)^i$.

Let $x \in (a, b)$, and let $r := 2^{-1} \min\{|x - a|, |b - x|\}$. Then, if $y \in B(x, r)$, then $|x - y| < |x - a|$ and $|x - y| < |b - x|$. Thus, $a < y < b$ i.e. $y \in (a, b)$ by $a < x < b$. Hence, $x \in (a, b)^i$.

(3): Since $B(0, 2^{-1}) = \{0\}$, $0 \in \{0\}^i$. That is, $\{0\} = \{0\}^i$. Therefore, if $x \in \{0\}^-$, then $B(x, r) \cap \{0\} = \{0\}$ for all $r > 0$. Thus, $x \in \{0\}$. Hence, $\{0\} = \{0\}^-$.

About $\{1\}$, it is similar to $\{0\}$.

(4): Let $O_1, ..., O_n$ be open sets in $X_1, ..., X_n$ respectively, and let $x \equiv (x_1, ..., x_n)$ be in $O \equiv O_1 \times ... \times O_n$. Then, there exist positive real numbers $r_1, ..., r_n$ such that $B_i(x_i, r_i) \subset O_i$ for any $i = 1, ..., n$, where $B_i(x_i, r_i)$ is a open ball in X_i. Here, let $r := \min\{r_1, ..., r_n\}$, and let $y \equiv (y_1, ..., y_n)$ be in $\prod_{i=1}^n X_i$ with $d(x, y) \equiv \sum_{i=1}^n d_i(x_i, y_i) < r$. Then, $d_i(x_i, y_i) < r \leq r_i$ for all $i = 1, ..., n$ i.e. $y_i \in B_i(x_i, r)$ for all i. Thus, $y_i \in O_i$ for all i i.e. $y \in O$. Therefore, $B(x, r) \subset O$. Hence O is open, where $B(x, r)$ is a open ball in $\prod_{i=1}^n X_i$.

Conversely, let $O_1 \times ... \times O_n$ be a open set in $\prod_{i=1}^n X_i$. Then, there exists $r > 0$ such that $B(x, r) \subset O_1 \times ... \times O_n$. Now, let $y \equiv (y_1, ..., y_n)$ be a element of $\prod_{i=1}^n X_i$ with $y_i \in B_i(x_i, n^{-1}r)$. Then, $d(x, y) := \sum_{i=1}^n d_i(x_i, y_i) < \sum_{i=1}^n n^{-1}r = r$. Thus, $y \in O$ i.e. $y_i \in O_i$ for all $i = 1, ..., n$. Hence, O_i is open for all for all $i = 1, ..., n$. ∎

Lemma 3.1.8 *Let A a subspace of X. Then*

(1) for all open set O in X, $O \cap A$ is open in A

(2) F is closed in A if and only if $F = F' \cap A$ for some closed set F' in X.

(*Proof*) For a set M, let M^{i_A} be the interior of M in A and M^{-_A} be the closure of M in A.

(1): We may show $O \cap A \subset (O \cap A)^{i_A}$.

Let $x \in O \cap A$. Then, $B(x, r) \subset O$ for some $r > 0$. Thus, $B(x, r) \cap A \subset O \cap A$ i.e. $B_A(x, r) \subset O \cap A$. Therefore, $x \in (O \cap A)^{i_A}$.

(2): Let F be a closed set in A, and let $F' := F^-$. We show $F = F' \cap A$. $F \subset F' \cap A$ is trivial.

we show that $F \supset F' \cap A$. Let $x \in F' \cap A$ i.e. $x \in F^- \cap A$. Then, $B(x,r) \cap F$ is inhabited for all $r > 0$, and $x \in A$. That is, $B(x,r) \cap A \cap F$ is inhabited for all $r > 0$. Therefore, $x \in F^{-_A} = F$, hence $F' \cap A \subset F$.

Conversely, let $x \in (F' \cap A)^{-_A}$. Then, $B(x,r)_A \cap (A \cap F')$ is inhabited for all $r > 0$. Thus, $B(x,r) \cap F'$ is inhabited for all $r > 0$. That is, $x \in F'$. On the other hand, $x \in A$ since $(F' \cap A)^{-_A}$. Hence, $x \in F' \cap A$. ∎

The author hasn't known weather or not "$M^{i_A} = M^i \cap A$" holds constructively.

Definition 3.1.9 Let (x_n) be a sequence in X, and let x be in X. (x_n) *converges* to x if for all positive real number ϵ, there exist a positive integer N such that $d(x_n, x) < \epsilon$ for all positive integer n with $n \geq N$.

Then, we write $x_n \to x$.

Lemma 3.1.10 *A subset A of X is closed if and only if there exists a sequence converging to x for all x in A.*

(*Proof*) Let x be a element of A^-. Then, $B(x, k^{-1}) \cap A$ is inhabited for all positive integer k. Thus, there exists a sequence (x_k) of A such that $x_k \in B(x, k^{-1})$ for all k. That is , (x_k) converges to x.

Conversely, suppose that for each x in A, there exists a sequence (x_k) of elements of A converging to x. Then, for any k, there exists a positive integer N such that $d(x_n, x) < k^{-1}$ for all n with $n \geq N$. That is, for each positive integer k, there exists n such that $x_n \in B(x, k^{-1})$. Hence, $B(x, k^{-1}) \cap A$ is inhabited for all k by $(x_k) \subset A$. ∎

For the above proof, we reason (x_n) converges to x if and only if for each positive integer k, there exists a positive integer N such that $d(x_n, x) < k^{-1}$ for all n with $n \geq N$. The reason is that if the later holds, then for each ϵ there exists k such that $0 < k^{-1} < \epsilon$ and that if the former holds, particularly we can take a positive rational number k^{-1} as any $\epsilon > 0$.

Example 3.1.11 In Example 3.1.2(2), $A_1{}^- \times ... \times A_n{}^- = (A_1 \times ... \times A_n)^-$.

(*Proof*) Let $x \equiv (x_1, ..., x_n)$ be a element of $A_1{}^- \times ... \times A_n{}^-$. Then, for any i, there exists a sequence $(x^i{}_k)$ of elements of A_i converging to x_i by Lemma 3.1.10. Thus, a sequence $((x_k^1, ..., x_k^n))$ is a sequence of elements of $A_1 \times ... \times A_n$ converges to x. Hence, $x \in (A_1 \times ... \times A_n)^-$ by Lemma 3.1.10.

Conversely, let $x \equiv (x_1, ..., x_n)$ be a elements of $(A_1 \times ... \times A_n)^-$. Then, there exists a sequence $(x_k) \equiv ((x_k^1, ..., x_k^n))$ of elements of $A_1 \times ... \times A_n$ converging to x . Thus, for any i, a sequence (x_k^i) of elements of A_i converges to x_i. Hence, x belongs to $A_i{}^- \times ... \times A_i{}^-$ by Lemma 3.1.10. ∎

Example 3.1.12 $([-1, 0] \cup [0, 1])^- \subset [-1, 0] \cup [0, 1]$ implies LLPO. Thus, the converse of Proposition 3.1.5(2) doesn't holds.

(*Proof*) Let (α_n) and (β_n) be binary sequences, which are a sequence taking 0 or 1 for each n, and assume that $\neg(\exists n(\alpha_n = 1) \wedge \exists n(\beta_n = 1))$. Let $r_n := \sum_{i=1}^{n} 2^{-1}(\beta_i - \alpha_i)$, and let $r := (r_n)$. Then, r is a real number in $[-1, 1]$ by the same way as Proposition 2.3.1(2), and (r_n) converges to r by Proposition 2.1.22.

Now, suppose that a positive integer n is given. Then, in case that $\alpha_i = 0$ and $\beta_i = 0$ for all i with $i \leq n$, $r_n \in [-1, 0] \cap [0, 1]$.

In case that $\alpha_i = 1$ for some $i \leq n$ and that $\beta_i = 0$ for all $i \leq n$, $\sum_{i=1}^{n} 2^{-1}(\beta_i - \alpha_i) = \sum_{i=1}^{n} 2^{-1}(-\alpha_i) \leq 0$. Thus, $r_n \in [-1, 0]$.

In case that $\alpha_i = 0$ for all $i \leq n$, and $\beta_i = 1$ for some $i \leq n$, $\sum_{i=1}^{n} 2^{-1}(\beta_i - \alpha_i) = \sum_{i=1}^{n} 2^{-1}\beta_i \geq 0$. Thus, $r_n \in [0, 1]$.

In the case that $\alpha_i = 1$ and $\beta_j = 1$ for some i and j, it is opposed to the assumption.

Therefore, (r_n) is in $[-1, 0] \cup [0, 1]$. Thus, $r \in [-1, 0]$ or $r \in [0.1]$ by Lemma 3.1.10. Hence, $\neg\exists n(\alpha_n = 1)$ or $\neg\exists(\beta_n = 1)$ by the same way as Proposition 2.3.1. ∎

By Proposition 2.1.22, there exists a rational sequence (q_n) converging to r for any real number r. Thus, $\mathbf{Q}^- = \mathbf{R}$.

3.2 Continuity

Let X and Y be metric spaces.

Definition 3.2.1 A map f from X to Y is *continuus* if for any x in X and any positive real number ϵ, there exist a positive real number δ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$.

Proposition 3.2.2 *Let f and g be continuous maps from X to $\mathbf{R}$. Then,* $\max\{f, g\}$, $f + g$ *and fg are continuous.*

(*Proof*) Let ϵ be any positive real number and x be any element of X.

($\max\{f, g\}$): there exist positive real numbers δ_1 and δ_2 such that for all y and y' in X, $|f(x) - f(y)| < \epsilon$ and $|g(x) - g(y')| < \epsilon$ whenever $d(x, y) < \delta_1$ and $d(x, y') < \delta_2$. Then, let $\delta_{\max} := \min\{\delta_1, \delta_2\}$.

Now, $|\max\{f(x), g(x)\} - \max\{f(y), g(y)\}| \leq \max\{|f(x) - f(y)|, |g(x) - g(y)|\} < \epsilon$ for any $y \in X$ with $d(x, y) < \delta_{\max}$, by Lemma 2.2.2.

$(f + g)$: For $\epsilon > 0$ and x, there exist positive real numbers δ_3 and δ_4 such that for all y and y' in X, $|f(x) - f(y)| < 2^{-1}\epsilon$ and $|g(x) - g(y')| < 2^{-1}\epsilon$ whenever $d(x, y) < \delta_3$ and $d(x, y') < \delta_4$. Then, let $\delta_+ := \min\{\delta_1, \delta_2\}$.

Now, for any $y \in X$ with $d(x, y) < \delta_+$, $|(f(x) + g(x)) - (f(y) + g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)| < 2^{-1}\epsilon + 2^{-1}\epsilon = \epsilon$.

(fg) : First, let δ_5 be a positive real number such that for any $y \in X$, if $d(x, y) < \delta_5$, then $|f(x) - f(y)| < 1$, and let δ_6 be a positive real number such that for all $y \in X$, if $d(x, y) < \delta_6$, then $|g(x) - g(y)| < 1$. Then, $|f(y)| < |f(x)| + 1, |g(y)| < |g(x)| + 1$ for all $y \in X$.

Next, there exist positive real numbers δ_7 and δ_8 such that for any y and any y' in X, if $d(x,y) < \delta_7$ and $d(x,y') < \delta_8$, then $|f(x) - f(y)| < (2(|g(x)+1))^{-1}\epsilon$ and $|g(x) - g(y')| < (2(|f(x)+1))^{-1}\epsilon$. Then, let $\delta_* := \min\{\delta_5, \delta_6, \delta_7, \delta_8\}$.

Now, for any $y \in X$ with $d(x,y) < \delta_*$, $|f(x)g(x) - f(y)g(y)| \leq |g(y)||f(x) - f(y)| + |f(x)||g(x) - g(y)| < (|g(x)| + 1)|f(x) - f(y)| + (|f(x)| + 1)|g(x) - g(y)| < (|g(x)| + 1)(2(|g(x)+1))^{-1}\epsilon + (|f(x)| + 1)(2(|f(x)+1))^{-1}\epsilon = \epsilon$. ∎

Proposition 3.2.3 *Let f be a continuous map from X to* $\mathbf{R}$ *such that there exists a positive real number c such that $|f(x)| \geq c$ for all x. Then, $(f)^{-1}$ is continuous.*

(*Proof*) For any $x \in X$ and any positive real number ϵ, there exists a positive real number δ such that for all $y \in X$, if $d(x,y) < \delta$, then $|f(x) - f(y)| < c^2\epsilon$.

Now, for any $y \in X$ with $d(x,y) < \delta$, $|f(x)^{-1} - f(y)^{-1}| = |f(x)|^{-1}|f(y)|^{-1}|f(x) - f(y)| < c^{-2}c^2\epsilon = \epsilon$. ∎

Proposition 3.2.4 *Let X, Y and Z be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$. Then, if f and g are continuous, then $g \circ f$ is continuous.*

(*Proof*) (1): $\forall \epsilon > 0 \exists \delta, \delta' > 0 (B(x,\delta) \subset f^{-1}(B(f(x),\delta')) \subset f^{-1}(g^{-1}(B(g(f(x)),\epsilon)))$. ∎

Definition 3.2.5 A map from X to Y is *seqentially continuous* if for all x in X and all sequences (x_n) in X, $f(x_n) \to f(x)$ whenever $x_n \to x$.

Propositions replacing "continuous" with " sequentially continuous" on Proposition 3.2.2, 3.2.3 and 3.2.4 are also shown by the same way.

Definition 3.2.6 A map from X to Y is *nondiscontinuous* if for all x in X and all sequences (x_n) in X, what (x_n) converges to x and what there exists a real number δ such that $d(f(x_n), f(x)) \geq \delta$ for all n imply that $\delta \leq 0$.

Proposition 3.2.7 *Let f be a map from X to Y.*

(1) f is continuous if and only if $f^{-1}(O)$ is open in X for all open subset O of Y.

(2) f is continuous $\Rightarrow$
f is sequentially continuous $\Rightarrow$
$f^{-1}(F)$ is closed in X for all closed subset F of Y $\Rightarrow$
$f(A^-)$ is contained in $(f(A))^-$ for all subset F of X $\Rightarrow$
f is nondiscontinuous.

(*Proof*) (1): Let O e be an open subset of Y.

Take any x in $f^{-1}(O)$. Then $f(x)$ belongs to O, so that $B(f(x),\epsilon)$ is contained in O for some $\epsilon > 0$. That is, $f^{-1}(B(f(x),\epsilon)$ is contained in $f^{-1}(O)$. Then, there exists a positive real number δ such that $B(x,\delta)$ is contained in $f^{-1}(B(f(x),\epsilon))$ by f is continuous. Thus, $B(x,\delta)$ is contained in $f^{-1}(O)$ since $f^{-1}(B(f(x),\epsilon)$ is contained in $f^{-1}(O)$.

Conversely, let ϵ be a positive real number, and let x be a elements of X. Then, $f^{-1}(B(f(x),\epsilon))$ is open in X by the assumption since $B(f(x),\epsilon)$ is open in Y. On the other hand, x belongs to $f^{-1}(B(f(x),\epsilon))$. Thus, there exists a positive real number δ such that $B(x,\delta)$ is contained in $f^{-1}(B(f(x),r))$.

(2): f is continuous $\Rightarrow$ f is sequentially continuous:

Let x be any element of X, and let (x_n) be a sequence of elements of X. Suppose that (x_n) converges to x, and let ϵ be a positive real number. Then, there exists a positive real number δ such that $B(x,\delta)$ is contained in $f^{-1}(B(f(x),\epsilon))$ since f is continuous, and there exists a positive integer N such that x_n belongs to $B(x,\delta)$ for all n with $n \geq N$. Therefore, $f(x_n)$ belongs to $B(f(x),\epsilon)$. That is, $f(x_n) \to f(x)$.

f is sequentially continuous $\Rightarrow$ $f^{-1}(F)$ is closed in X for all closed subset F of Y:

Let F be any closed subset of Y. Then, we may show $f^{-1}(F)^{-} \subset f^{-1}(F)$.

Let x be a element of $f^{-1}(F)^{-}$. Then, there exists a sequence (x_n) of elements of $f^{-1}(F)$ conversing to x. Thus, the sequence $(f(x_n))$ converges to $f(x)$ since f is sequentially continuous. Therefore, $f(x)$ belongs to $F$by Lemma 3.1.10 since $(f(x_n))$ is a sequence of elements of F. Hence, x belongs to $f^{-1}(F)$.

$f^{-1}(F)$ is closed in X for all closed subset F of Y $\Rightarrow$ $f(A^{-})$ is contained in $f(A)^{-}$ for all subset F of X:

Let A be a subset of X. Then, since A is a subset of A^{-}, A is contained in $f^{-1}(f(A)^{-})$. On the other hand, $f^{-1}(f(A)^{-})$ is closed by the assumption. Thus, A^{-} is a subset of $f^{-1}(f(A)^{-})$ by Lemma 3.1.6. That is, $f(A^{-})$ is contained in $f(A)^{-}$.

$f(A^{-})$ is contained in $f(A)^{-}$ for all subset F of X $\Rightarrow$ f is nondiscontinuous:

Let x be a element of X and (x_n) be a sequence of elements of X. Let δ be a real number.

Suppose that (x_n) converges to x and $d(f(x_n),f(x)) \geq \delta$ for all n. Then, $f(x) \in f((x_n)^{-})$ since $x \in (x_n)^{-}$. Then, $f(x)$ belongs to $(f(x_n))^{-}$ by the assumption. Therefore, there exists a subsequence $(f(x_{n_k}))$ converging to $f(x)$ by Lemma 3.1.6. Here, if $\delta > 0$, then there exists N such that $d(f(x_{n_k}),f(x)) < \delta$ for all n with $n \geq N$. It is opposed to the supposition. Therefore, $\delta > 0$. Hence, $\delta \leq 0$. ∎

Example 3.2.8 Constant function and identical function etc. are continuous. Moreover, about the product space $(\prod_{i=1}^{n} X_i, d)$ in Example 3.1.2, the projection p_i from $\prod_{i=1}^{n} X_i$ to X_i $(1 \leq i \leq n)$ is also one.

(*Proof*) For all open subset O_i in X_i, $p_i^{-1}(O_i) = X_1 \times ... \times X_{i-1} \times O_i \times X_{i+1} \times ... \times X_n$, thus $p_i^{-1}(O_i)$ is open in $\prod_{i=1}^{n} X_i$. Therefore, p_i is continuous by Proposition 3.2.7(1). ∎

Now, Ishihara showed in [9] "Ever sequentially continuous map on a metric space is continuous" doesn't hold in constructive mathematics. Therefore, "$f^{-1}(F)$ is closed in X for all closed subset F of Y" , "$f(A^{-})$ is contained in $f(A)^{-}$ for all subset F of X" and "f is continuous" are classically equivalent to each other (see [1],[13] or [16]), but aren't so constructively.

However, the author hasn't known for the following:

- f is nondiscontinuous $\Rightarrow$
  $f^{-1}(F)$ is closed in X for all closed subset F of Y.

- $f(A^-)$ is contained in $f(A)^-$ for all subset F of X $\Rightarrow$
  $f^{-1}(F)$ is closed in X for all closed subset F of Y.

- $f^{-1}(F)$ is closed in X for all closed subset F of Y $\Rightarrow$
  f is sequentially continuous.

But, they are showed partly. Ishihara showed in [8]

- "Every nondiscontinuous map from a complete metric space to a metric space is sequentially continuous."

- "Every map from a complete metric space to a metric space is strongly extensional."

- **WMP**:"Every pseudopositive real number is positive."

are equivalent to each other. Here

Definition 3.2.9 Let (x_n) be a sequence in X.Then, (x_n) is a *Cauchy sequence* if for any positive number r, there exist some positive integer N such that $d(x_m, x_n) < r$ for all m and n with $m, n \geq N$. A metric space X is *complete* if any Cauchy sequence in X is converges.

Definition 3.2.10 A map by (X, d) to (Y, d') is *strongly extentinal* if $d'(f(x), f(y)) > 0$ implies $d(x, y) > 0$ for all x and y in X.

Definition 3.2.11 A real number r is *pseudopsitive* if

$$\neg\neg(0 < x) \vee \neg\neg(x < r)$$

for all real number x.

(WMP(weak Markov's principle) is a omniscience principle implied by MP.)

Here, all maps are strongly extensional in classical mathematics , but it doesn't holds by the above in constructive mathematics [8]. Actually, the next holds.

Proposition 3.2.12 *Any map from X to Y is strongly continuous if and only if MP.*

(*Proof*) Let r be a real number. Suppose that $\neg\neg(|r| > 0)$, and let $X := \{rx | x \in \mathbf{R}\}$ and $Y := \mathbf{R}$. Then, X and Y are metric space for normal metric. And, let f be a correspondence rx in X with x in Y. Then, f is a map since $rx = ry$ implies $x = y$, so that f is strongly extensional.

Now, $|r| > 0$ i.e. $r > 0$ or $r < 0$ since $f(r) = f(r1) = 1$ and $f(0) = f(r0) = 0$ imply $|f(r) - f(0)| > 0$. Thus, by the supposition, $r > 0$.

Conversely, let f be a map from X to Y. Suppose that $d(f(x), f(y)) > 0$ for x and y in X. Then, $\neg(d(f(x), f(y)) = 0)$ i.e. $\neg(f(x) = f(y))$, so that $\neg(x = y)$ i.e.$\neg(d(x, y) = 0)$ i.e.$\neg\neg(d(x, y) > 0)$. Thus, $d(x, y) > 0$ by MP. ∎

Also, Ishihara showed in [9]

- a map f from a complete metric space to a metrics space is sequentially continuous if and only if f is strongly extensional and nondiscontinuous.

Chapter 4

Connectivity

First, we will consider under Definition 4.1.1 which appear in [14], and second under another definitions which classically is equivalent to each other (see [1]). Finally, we will consider the relation on these definitions and *the intermediate value theorem.*

Let X and Y be metric spaces.

4.1 Connectivity

Definition 4.1.1 *[connected]* A metric space X is *connected* if for any inhabited and open set V and W with $V \cup W = X$, $V \cap W$ is inhabited.

A subset A of X is *connected* if subspace A of X is connected.

Proposition 4.1.2 *Let A be a connected set. Then, for all subset M of X, if $A \subset M \subset A^-$, then M is connected.*

(*Proof*) Let V and W be inhabited and open set in M with $V \cup W = M$. Then, $V \cap A$ and $W \cap A$ are open in A by Lemma 3.1.8, and $(V \cap A) \cup (W \cap A) = A$. Thus, we may show that $V \cap A$ and $W \cap A$ are inhabited respectively.

Let x be a element of V, and let δ be a positive real number with $B(x, \delta)_M \subset V$ i.e. $B(x, \delta) \cap M \subset V$. Now, since $x \in A^-$, $B(x, \delta) \cap A$ is inhabited, and $B(x, \delta) \cap A \subset B(x, \delta) \cap M$. Thus, there exists y with $y \in B(x, \delta) \cap A$, therefore $y \in V \cap A$.

Similarly, it is shown that $W \cap A$. ∎

Proposition 4.1.3 *Let $\{X_n \subset X | n \in \mathbf{N}\}$ be a class of connected subsets of X such that $\bigcap\{X_n \subset X | n \in \mathbf{N}\}$ is inhabited. Then, $\bigcup\{X_n \subset X | n \in \mathbf{N}\}$ is connected.*

(*Proof*) Let $x \in \bigcap\{X_n \subset X | n \in \mathbf{N}\}$, and let V and W be inhabited and open in subspace $\bigcup\{X_n \subset X | n \in \mathbf{N}\}$ with $V \cup W = \bigcup\{X_n \subset X | n \in \mathbf{N}\}$.

In case $x \in W$, let $y \in V$ since V is inhabited.

Here, in case $y \in X_n$, $V \cap X_n$ and $W \cap X_n$ are inhabited and open in X_n, and $(V \cup X_n) \cup (W \cap X_n) = X_n$. Thus, since X_n is connected, $V \cap X_n \cap W \cap X_n$ is inhabited. Therefore, $V \cap W$ is inhabited.

In case $x \in V$, it is similarly shown that $V \cap W$ is inhabited. ∎

In classical mathematics, the proposition that is replaced "**N**" with "Λ" in the above proposition holds by Axiom of Choice, where Λ is any index set. But, since we assume only Axiom of Countable Choice in constructive mathematics, such rewritten proposition holds only under the assumption that there exist some choice functions for $\{X_\lambda \subset X | \lambda \in \Lambda\}$.

Theorem 4.1.4 *Let X be a connected set, and let f be a continuous map from X to Y. Then, $f(X)$ is connected.*

(*Proof*) Let V and W be open and inhabited subsets of X with $V \cup W = f(X)$. Then, since $f^{-1}(V) \cup f^{-1}(W) = f^{-1}(V \cup W)$ and $f^{-1}(V \cup W) = X$, $f^{-1}(V)$, and $f^{-1}(W)$ are open and inhabited in X and $f^{-1}(V) \cup f^{-1}(W) = X$. Thus, there exists x in $f^{-1}(V) \cap f^{-1}(W)$. Therefore, $f(x) \in V \cap W$ since $f^{-1}(V) \cap f^{-1}(W) = f^{-1}(V \cap W)$. ∎

Proposition 4.1.5 *Every metric space (X_i, d_i) $(1 \leq i \leq n)$ is connected if and only if $(\prod_{i=1}^n X_i, d)$ is connected, where $d(x, y) := \sum_{i=1}^n d_i(x_i, y_i)$ $(x = (x_1, ..., x_n), y := (y_1, ..., y_n) \in \prod_{i=1}^n X_i)$.*

(*Proof*) Let $V \equiv V_1 \times ... \times V_n$ and $W \equiv W_1 \times ... \times W_n$ be inhabited and open set with $V \cup W = \prod_{i=1}^n X_i$. Then, $V_i \cup W_i = X_i$, and V_i and W_i are open in X_i for all $i = 1, ..., n$ by Example 3.1.2. Thus, for all i, $V_i \cap W_i$ is inhabited since X_i is connected. Hence, $V \cap W$ is inhabited.

Conversely, since the projection $p_i : \prod_{i=1}^n X_i \to X_i$ is continuous and surjective for any i , any X_i is connected by Theorem 4.1.4. ∎

Definition 4.1.6 (path-connectivity) A metric space X is $path-connected$ if all a and b are *joined by an arc f* i.e. there exists a continuous map from $[0, 1]$ to X with $f(0) = a$ and $f(1) = b$.

Theorem 4.1.7 *A path-connected set is connected.*

(*Proof*) Let X be path- connected, and let V and W be open and inhabited subsets in X with $V \cup W = X$. Then, we can take x_0 in V and y_0 in W, and there exists a continuous map f such that $f(0) = x_0, f(1) = y_0$. Here, let $a_0 := 0, b_0 := 1$. Then, we construct sequences (x_n) and (y_n) on X and (a_n) and (b_n) on $[0, 1]$ as follows:
let x_n, y_n, a_n and b_n be given.

In the case that $f(2^{-1}(a_n + b_n)) \in V$, let $x_{n+1} := f(2^{-1}(a_n + b_n))$, $y_{n+1} := y_n$, $a_{n+1} := 2^{-1}(a_n + b_n)$ and $b_{n+1} := b_n$.

In the case that $f(2^{-1}(a_n + b_n)) \in W$, let $x_{n+1} := x_n$, $y_{n+1} := f(2^{-1}(a_n + b_n))$, $a_{n+1} := a_n$ and $b_{n+1} := 2^{-1}(a_n + b_n)$.

Then,

(1) (a_n) is increasing, and (b_n) is decreasing.

(2) $a_n \leq b_n$ for all n.

(3) $|b_n - a_n| \leq 2^{-n}$.

(1) and (2) is trivial by induction on n. About (3), if $|b_n - a_n| \leq 2^{-n}$, then $|b_{n+1} - a_{n+1}| = |2^{-1}(a_n + b_n) - a_n| \leq 2^{-1}2^{-n} = 2^{-(n+1)}$, so that it holds by induction on n.

Thus, $|a_m - a_n| = a_m - a_n \leq b_m - a_n \leq b_n - a_n \leq 2^{-n}$ for all m and n, therefore (a_n) is a Cauchy sequence. Thus (a_n) converges by Theorem 2.1.29. Here, let a be a limit of (a_n) , and let $x := f(a)$. Then, (b_n) converges to a since $|b_n - a| \leq |b_n - a_n| + |a_n - a|$ for all n.

Therefore, (x_n) and (y_n) converge x since f is also sequentially continuous by Proposition 3.2.7 (2).

Here, in the case that $x \in W$, let δ with $B(x, \delta) \subset W$, then $x_n \in V \cap W$ for some n since there exists N such that $x_n \in B(x, \delta)$ for all n with $n \geq N$.

In the case that $x \in V$, $y_n \in V \cap W$ for some n since there exists N' such that $y_n \in B(x, \delta)$ for all n with $n \geq N'$.

Hence, $V \cap W$ is inhabited. ∎

Example 4.1.8 In Euclid space $\mathbf{R}$, a interval is path-connected, where a *interval* is one in the next:

$(a,b) := \{x \in \mathbf{R} | a < x < b\}$, $\quad [a,b] := \{x \in \mathbf{R} | a \leq x \leq b\}$,
$[a,b) := \{x \in \mathbf{R} | a \leq x < b\}$, $\quad (a,b] := \{x \in \mathbf{R} | a < x \leq b\}$,
$(-\infty,b) := \{x \in \mathbf{R} | x < b\}$, $\quad (-\infty,b] := \{x \in \mathbf{R} | x \leq b\}$,
$(a,\infty) := \{x \in \mathbf{R} | a < x\}$, $\quad [a,\infty) := \{x \in \mathbf{R} | a \leq x\}$,
$(-\infty,\infty) := \mathbf{R}$,
where $a < b$.

Actually, $f(t) = at + (1-t)b$ $(t \in [0,1])$ is continuous for every a and b by Proposition 3.2.2, hence the f can be take as the arc.

In classical mathematics, there exists a space such that it is not path-connected but connected (see [1], [16] or [13]). However, the author has not known weather or not the metric space is connected in constructive mathematics.

Proposition 4.1.9 *Let X be path-connected, and let f be a continuous map from X to Y. Then, $f(X)$ is path-connected.*

(*Proof*) By Theorem 4.1.4 and Proposition 3.2.4. ∎

Proposition 4.1.10 *The path-connected set defined by replacing "continuous" with "sequentially continuous" in Definition* 4.1.6 *is connected.*

(*Proof*) It is trivial by the proof of Theorem 4.1.7. ∎

Theorem 4.1.11 *Let X be a path-connected set in Proposition 4.1.10, and let f be a sequentially continuous map from X to Y. Then, $f(X)$ is path-connected in Theorem 4.1.10.*

(*Proof*) By Proposition 3.2.4, it is shown similarly to the proof of Theorem 4.1.7. ∎

The author has not known weather or not the proposition obtained by replacing "continuous" with "sequentially continuous" in Theorem 4.1.4 holds constructively.

4.2 C-connectivity

Definition 4.2.1 *[C-connectivity]* A metric space X is *$C-connected$* if for any inhabited and closed set V and W with $V \cup W = X$, $V \cap W$ is inhabited.

Proposition 4.2.2 *Let $\{X_n \subset X | n \in \mathbf{N}\}$ be a class of C-connected subsets of X such that $\bigcap\{X_n \subset X | n \in \mathbf{N}\}$ is inhabited . Then, $\bigcup\{X_n \subset X | n \in \mathbf{N}\}$ is C-connected.*

(*Proof*) By the same way as Theorem 4.1.3. ∎

Theorem 4.2.3 *Let X be C-connected, and let $f : X \to Y$ such that $f^{-1}(F)$ is closed for all closed subset F of X. Then, $f(X)$ is C-connected.*

(*Proof*) By the same way as Theorem 4.1.4. ∎

Proposition 4.2.4 *Every metric space (X_i, d_i) $(1 \leq i \leq n)$ is C-connected if and only if $(\prod_{i=1}^n X_i, d)$ is C-connected, where $d(x,y) := \sum_{i=1}^n d_i(x_i, y_i)$ $\quad (x = (x_1, ..., x_n), y := (y_1, ..., y_n) \in \prod_{i=1}^n X_i)$.*

(*Proof*) we show by the same way as Proposition 4.1.5 that what (X_i, d_i) $(1 \leq i \leq n)$ are C-connected spaces implies what $(\prod_{i=1}^n X_i, d)$ is C-connected.

Conversely, the projection $p_i : \prod_{i=1}^n X_i \to X_i$ is continuous for any i, and p_i is surjective and satisfies that $p_i^{-1}(F)$ is closed in $\prod_{i=1}^n X_i$ for all closed subset F in X_i by Proposition 3.2.7. Thus, any X_i is C-connected by Theorem 4.2.3. ∎

Theorem 4.2.5 *A path-connected set is C-connected.*

(*Proof*) Let X be path-connected. Let V and W be closed and inhabited subsets in X with $V \cup W = X$. Then, we can take x_0 in V and y_0 in W, and there exists a continuous map f such that $f(0) = x_0, f(1) = y_0$. Let $a_0 := 0, b_0 := 1$. Then, we construct sequences (x_n) and (y_n) on X and (a_n) and (b_n) on $[0,1]$ by the same way as Theorem 4.1.7. Then, (a_n) and (b_n) satisfy

- (a_n) is increasing, and (b_n) is decreasing
- $a_n \leq b_n$ for all n

- $|b_n - a_n| \leq 2^{-n}$.

Thus, (a_n) and (b_n) have the same limit respectively. Here, let a be a limit of (a_n) , and let $x := f(a)$. Then, (x_n) and (y_n) converge to x since f is also sequentially continuous by Proposition 3.2.7. Hence, $x \in V \cap W$ by Lemma 3.1.10. ∎

Proposition 4.2.6 *The path-connected set defined by replacing "continuous" with "sequentially continuous" in Definition* 4.1.6 *is C-connected.*

(*Proof*) It is trivial by the proof of Theorem 4.2.5. ∎

The author has not known weather or not for a C-connected set A, if $A \subset M \subset A^-$ for all subset M of A, then M is C-connected.

4.3 Strong Connectivity

Definition 4.3.1 *[strong connectivity]* A metric space X is *strongly connected* if for any inhabited set V with $V \cap W = X$, $V^- \cap W$ or $V \cap W^-$ is inhabited.

Proposition 4.3.2 *Let $\{X_n \subset X | n \in \mathbf{N}\}$ be a class of strongly connected subsets of X such that $\bigcap\{X_n \subset X | n \in \mathbf{N}\}$ is inhabited. Then, $\bigcup\{X_n \subset X | n \in \mathbf{N}\}$ is strongly connected.*

(*Proof*) Let $x \in \bigcap\{X_n \subset X | n \in \mathbf{N}\}$, and let V and W be inhabited sets with $V \cup W = \bigcup\{X_n \subset X | n \in \mathbf{N}\}$.

In the case $x \in W$. Let y a elements of V.

In the case $y \in X_n$, $V \cap X_n$ and $W \cap X_n$ is inhabited and $W \cup X_n = X_n$. Since X_n is strongly connected, $(V \cap X_n)^{-X_n} \cap (W \cap X_n)$ is inhabited or $(V \cap X_n) \cap (W \cap X_n)^{-X_n}$ is inhabited.

If there exists x in $(V \cap X_n)^{-X_n} \cap (W \cap X_n)$, then $x \in (V \cap X_n)^{-X_n} \cap (W \cap X_n) \subset V^- \cap W$ since $(V \cap X_n)^{-X_n} \subset V^{-X_n} \cap X_n \subset V^- \cap X_n$ by Lemma 3.1.8

If there exists x in $V \cap X_n \cap (W \cap X_n)^{-X_n} \subset V^- \cap W$, x belongs to $V \cap W^-$ by the same way.

In the case $x \in V$, it is similar. ∎

Theorem 4.3.3 *Let X be strongly connected, and let $f : X \to Y$ such that $f(A^-)$ is contained in $f(A)^-$ for all subset A in Y. Then, $f(X)$ is strongly connected.*

(*Proof*) Let V and W be inhabited sets with $V \cup W = X$.

Now, $f^{-1}(V), f^{-1}(W)$ is inhabited, and $f^{-1}(V) \cup f^{-1}(W) = f^{-1}(V \cup W) = X$. Thus, $f^{-1}(V)^- \cap f^{-1}(W)$ or $f^{-1}(V) \cap f^{-1}(W)^-$ is inhabited.

In the case $\exists x \in f^{-1}(V)^- \cap f^{-1}(W)$, $f(f^{-1}(V)^-) \subset (ff^{-1}(V))^- = V$ by the assumption. Therefore, $f(x) \in f(f^{-1}(V)^-) \cap f(f^{-1}(W)) \subset V^- \cap W$.

In the case $\exists x \in f^{-1}(V) \cap f^{-1}(W)$, $f(x) \in V \cap W^-$ by the same way. ∎

Proposition 4.3.4 *Every* (X_i, d_i) $(1 \leq i \leq n)$ *is strongly connected if and only if* $(\prod_{i=1}^n X_i, d)$ *is strongly connected, where* $d(x,y) := \sum_{i=1}^n d_i(x_i, y_i)$ $(x = (x_1, ..., x_n), y := (y_1, ..., y_n) \in \prod_{i=1}^n X_i)$.

(*Proof*) Let $V \equiv V_1 \times ... \times V_n$ and $W \equiv W_1 \times ... \times W_n$ be inhabited sets with $\prod_{i=1}^n X_i = V \cup W$. Then, for any i, $V_i \cup W_i = X_i$, therefore $V_i^- \cap W_i$ or $V_i \cup W_i^-$ is inhabited since X_i is strongly connected. Thus, $(V_1 \times ... \times V_n)^- \cap W_1 \times ... \times W_n$ or $V_1 \times ... \times V_n \cap (W_1 \times ... \times W_n)^-$ is inhabited by Example 3.1.11.

Conversely, the projection $p_i : \prod_{i=1}^n X_i \to X_i$ is continuous for any i, therefore p_i satisfies that $f(A^-) \subset f(A)^-$ for all inhabited set A in X by Proposition 3.2.7, and any p_i is surjective. Thus, any X_i is strongly connected by Theorem 4.3.3. ∎

Theorem 4.3.5 *A path-connected set is strongly connected.*

(*Proof*) Let X be path-connected. Let V and W be closed and inhabited subsets in X with $V \cup W = X$. Then, we can take x_0 in V and y_0 in W, and there exists a continuous map f such that $f(0) = x_0, f(1) = y_0$. Here, let $a_0 := 0, b_0 := 1$, then we construct sequences (x_n) and (y_n) on X and (a_n) and (b_n) on $[0,1]$ by the same way as Theorem 4.1.7. Then, (a_n) and (b_n) satisfy

(1) (a_n) is increasing, and (b_n) is decreasing

(2) $a_n \leq b_n$ for all n

(3) $|b_n - a_n| \leq 2^{-n}$.

Thus, (a_n) and (b_n) have the same limit. Here, let a be a limit of (a_n) , and define $x := f(a)$. Then, (x_n) and (y_n) converge to x since f is also sequentially continuous by Proposition 3.2.7.

Now, in the case that $x \in W$, $x \in V^-$ since (x_n) converges to x. Therefore $x \in V^- \cap W$. In the case $x \in V$, $x \in V \cap W^-$ since (y_n) converges to x. ∎

Theorem 4.3.6 *The path-connected set defined by replacing "continuous" with "sequentially continuous" in Definition* 4.1.6 *is strongly connected.*

(*Proof*) It is trivial by the proof of Theorem 4.3.5. ∎

The author has not known weather or not for a strongly connected set A, if $A \subset M \subset A^-$ for all subset M of A, then M is strongly connected.

Next, we consider the relation on three definitions for connectivity.

Proposition 4.3.7

(1) If X is strongly connected, then X is connected

(2) If X is strongly connected, then X is C-connected.

(*Proof*) (1): Let V and W be inhabited and open sets in X with $V \cup W = X$, so that $V^- \cap W$ or $V \cap W^-$ is inhabited by the assumption. Then, in the case that $V^- \cap W$ is inhabited, let x be a element of $V^- \cap W$ and δ be a positive number with $N(x, \delta) \subset W$. Then, there exists a sequence (x_n) on V such that (x_n) converges to x. Thus, there exist Some N such that $x_n \in V \cap W$ for all n with $n \geq N$ since $x_n \in N(x, \delta)$.

In the case that $V \cap W^-$ is inhabited, it is similar.

(2): Let V and W be inhabited and closed sets in X with $V \cup W = X$, so that $V^- \cap W$ or $V \cap W^-$ is inhabited by the assumption . In both cases, $V \cap W$ is inhabited. ∎

The author has not known the converses of (1) and (2) in the above and the relation between connectivity and C-connectivity hold.

4.4 The Intermediate Value Theorem

The intermediate value theorem is one of the conclusion by the argument of connectivity in classical mathematics, where the theorem is as the next:
"Let X be connected and f be a continuous map from X to $\mathbf{R}$, let a and b be a element of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for each $\epsilon > 0$, there exists c in X with $f(c) = \gamma$."

In section, we show that *the intermediate value theorem* implies LLPO, so that doesn't hold in constructive mathematics, but a certain weak theorem holds.

Proposition 4.4.1 *The intermediate value theorem implies LLPO*

(*Proof*) Let a be any real number.

Let f be a map from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = \min\{x-1, 0\} + \max\{0, x-2\}$ $(x \in [0,3])$, and let $g(x) := f(x) + a \quad (x \in [0,3])$.

Now, $[0,3]$ is connected, therefore g is continuous by Proposition 3.2.2 and $g(0) = -1 + a < 1 + a = g(3)$. Thus, by the intermediate value theorem, we can take x in $[0,3]$ such that $g(x) = 0$ for a suitable a.

a in $[0,3]$ such that $g(x) = 0$ is decided as follows: since $x < 2$ or $1 < x$, in the case that $x < 2$, $\max\{0, x-2\} = 0$ and $\min\{x-1, 0\} \leq 0$. Thus, $0 = g(x) = f(x) + a = min\{x-1, 0\} + a \leq a$. That is, $0 \leq a$.

In the case $1 < x$, $\max\{0, x-2\} \geq 0$ and $min\{x-1, 0\} = 0$. Thus, $0 = g(x) = f(x) + a = \max\{0, x-2\} + a \geq a$. That is, $0 \geq a$.

Therefore, $0 \leq a \vee 0 \geq a$. That is, $\forall a \in \mathbf{R}(0 \leq a \vee 0 \geq a)$. Hence, LLPO is implied by Proposition 2.3.1. ∎

Theorem 4.4.2 *Let X be connected, and let x and y be elements of X. Let f be a continuous map from X to $\mathbf{R}$, let a and b be elements of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for any ϵ, there exists c in X with $|f(c) - \gamma| < \epsilon$.*

(*Proof*) Let ε be a positive number. Let $V := \{c \in X | f(c) < \gamma + \epsilon\}$ and $W := \{c \in X | \gamma - \epsilon < f(c)\}$. Then, V and W are inhabited sets by $f(c) \leq \gamma \leq f(c)$, and $V \cup W = X$

by Corollary 2.1.26. Therefore, V and W are open in X by f's continuity and Proposition 3.2.7, since $f(V)$ and $f(W)$ are open in $\mathbf{R}$. Thus, $V \cap W$ is inhabited since X is connected. That is, for any $\epsilon > 0$, there exists c in X with $\gamma - \epsilon < f(c) < \gamma + \epsilon$ i.e. $|f(c) - \gamma| < \epsilon$. ∎

Theorem 4.4.3 *Let X be C-connected, and let f be a map from X to $\mathbf{R}$ such that $f^{-1}(F)$ is closed in X for all closed subset F of $\mathbf{R}$. Let a and b be elements of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in X with $|f(c) - \gamma| < \epsilon$.*

(*Proof*) Let ε be a positive number. Let $V := \{c \in X | f(c) \leq \gamma + \epsilon\}$ and $W := \{c \in X | \gamma - \epsilon \leq f(c)\}$. Then, V and W are inhabited sets, and $V \cup W = X$. Therefore, V and W are closed by the assumption for f since $f(V)$ and $f(W)$ are closed. Thus, $V \cup W$ is inhabited since X is C-connected. That is, for any $\epsilon > 0$, there exists c in X with $\gamma - \epsilon < f(c) < \gamma + \epsilon$ i.e. $|f(c) - \gamma| < \epsilon$. ∎

Theorem 4.4.4 *Let X be strongly connected, and let f be a map from X to $\mathbf{R}$ such that $f(A^-)$ is contained in $f(A)^-$ for all subset A of Y. Let a and b be the elements of X with $f(a) < f(b)$, and let γ be a positive number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in X with $|f(c) - \gamma| < \epsilon$.*

(*Proof*) Let ε be a positive number. Let $V := \{c \in X | f(c) \leq \gamma + \epsilon\}$ and $W := \{c \in X | \gamma - \epsilon \leq f(c)\}$. Then, V and W are inhabited sets, and $V \cup W = X$, and $V^- \cap W$ or $V \cap W^-$ is inhabited since X is strongly connected. Thus, $f(V)^- \cap f(W)$ or $f(V) \cap f(W)^-$ is inhabited since $f(V^- \cap W) = f(V^-) \cap f(W) \subset f(V)^- \cap f(W)$ and $f(V \cap W^-) = f(V) \cap f(W^-) \subset f(V) \cap f(W)^-$ by the assumption for f. Therefore, in the both cases, for any $\epsilon > 0$, there exists c in X with $\gamma - \epsilon < f(c) < \gamma + \epsilon$ i.e. $|f(c) - \gamma| < \epsilon$. ∎

Theorem 4.4.3 and 4.4.4 imply the next.

Corollary 4.4.5 *Let f be a map from $[a, b]$ to $\mathbf{R}$ such that $f(A^-)$ is contained in $f(A)^-$ for all subset A of Y and that $f(a) < f(b)$. Let γ be a positive number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in $[a, b]$ with $|f(c) - \gamma| < \epsilon$.*

Chapter 5

Conclusion

5.1 Conclusion

Here, we arrange what we showed in this paper.

First, we showed that for any map f from a metric space X to a metric space Y, f is continuous $\Rightarrow$ f is sequentially continuous $\Rightarrow$ $f^{-1}(F)$ is closed in X for all closed subset F of Y $\Rightarrow$ $f(A^-)$ is *contained* in $f(A)^-$ for all subset A of X.

Next, we defined connectivity, C-connectivity and strongly connectivity of metric spaces and showed that a image of a connected space by a continuous map is connected, that a image of a C-connected space by the map such that $f^{-1}(F)$ is closed in X for all closed subset F of Y is C-connected and that a image of a strongly connected space by the map such that $g(A^-)$ is contained in $g(A)^-$ for all subset A of Y is strongly connected.

We also defined path-connectivity and showed that a path-connected space is connected, C-connected and strongly-connected.

Then, we showed that a strongly connected space is connected and C-connected.

Finally, we showed that the intermediate value theorem does not hold in constructive mathematics, and did the weak versions of it for connected, C-connected and strongly connected metric spaces respectively and found that the weak version hold for intervals on Euclid space $\mathbf{R}$ as the corollary.

5.2 Notes

In classical mathematics, any connected set in Euclid space $\mathbf{R}$ is a interval (see [13]), but it implies the intermediate value theorem. Actually, assume that any connected set in Euclid space $\mathbf{R}$ is a interval, and let S be connected in $\mathbf{R}$. Then, $[a, b] \subset S$ for all a and b in S. Now, let X be connected, let $f : X \to \mathbf{R}$ be continuous, let a and b be elements of X with $f(a) < f(b)$ and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, $f(X)$ is connected by Theorem 4.1.4, therefore $[f(a), f(b)] \subset f(X)$ by the above. Hence, $f(c) = \gamma$ for some c in X.

Now, it is known that all of propositions in Proposition 3.2.7(2) are classically equivalent to each other (see [1], [13] or [16]), but they are open in constructive mathematics. Also, Definition 4.1.1, 4.2.1 and 4.3.1 are also classically equivalent (see [1]), but it has not be clear constructively except Proposition 4.3.7. What they are solved means that the relation on the three notions of connectivity as a topological property that is conserved by continuous maps in constructive mathematics becomes clear.

Here, if X is connected, there exist no continuous and surjective map from X to $\{0,1\}$: (*Proof*) Assume that $f : X \to \{0,1\}$ is continuous and surjective. Then, $f^{-1}(0)$ and $f^{-1}(1)$ are inhabited, open and closed by Example 3.1.7 and Proposition 3.2.7 and $X = f^{-1}(0) \cup f^{-1}(1)$. Since X is connected, $\exists x \in f^{-1}(0) \cap f^{-1}(1)$. Then, $f(x) = 0$ and $f(x) = 0$. It is opposed to the definition of a map. ∎

"there exist no continuous and surjective map from X to $\{0,1\}$" is classically equivalent to " X is connected," hence is classically equivalent to "X is C-connected" and " X is strongly connected" respectively (see [1]). But, in constructive mathematics, the author has not known weather or not it holds. Also, "If A is open and closed, then $A = X$ or $A = \phi$" is equivalent to them in classical mathematics (see [1]), but the author knows nothing about the relation on them.

Finally, in [15], Troelstra considered the connectivity in the sense that it cannot be the union of finite , inhabited, disjoint and closed sets in *intuitionistic topology*, and Bridges do under the original definition in [3], [4], [5] and [6]. Mandelker showed that intervals on the set of real numbers couldn't be the union of two inhabited disjoint open subsets, and Bishop and Bridges gave connected sets in [2], where they are defined as $path-connected\ sets$ in this paper.

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