A note on the sequential version of Π^1_2 statements

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Abstract. In connection with uniform computability and intuitionistic provability, the strength of the sequential version of Π^1_2 theorems has been investigated in reverse mathematics. In some examples, we illustrate that it occasionally depends on the way of formalizing the Π^1_2 statement, so the investigation of sequential strength demands careful attention to the formalization. Moreover our results suggest the optimality of Dorais's uniformization theorems.

Keywords: Reverse mathematics, Sequential version, Uniformity, Marriage theorem, Bounded König's lemma, Weak weak König's lemma.

1 Introduction

Definition 1 (Sequential version). *The sequential version of a* Π^1_2 *statement having the form:*

$$(\spadesuit) \qquad \forall X\,(\varphi(X) \to \exists Y\,\psi(X,Y))$$

is the statement

$$\forall \langle X_n\rangle_{n\in\mathbb{N}}\,(\forall n\ \varphi(X_n) \to \exists\langle Y_n\rangle_{n\in\mathbb{N}}\forall n\ \psi(X_n,Y_n))\,,$$

where X *is possibly a tuple of set (or function) variables. Throughout this paper, we denote the sequential version of a statement* T *having a form* ($\spadesuit$) *as* $\mathrm{Seq}(\mathrm{T})$.

Many mathematical statements have the form ($\spadesuit$), and their sequential forms have been investigated in order to reveal the lack of uniformity of their proof in classical subsystems of second-order arithmetic (e.g. [2], [3], [6]). For instance,

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the intermediate value theorem is provable in $\mathsf{RCA_0}$, but the sequential version of it is equivalent to WKL (weak König's lemma), and so, not provable in $\mathsf{RCA_0}$. This is of course caused by the necessity of non-uniformity in the proof in $\mathsf{RCA_0}$. However, the strength of the sequential version may increase for another reason. In this paper, we concentrate our attention on Π^1_2 statements having the following syntactical form:

$$(\natural') \qquad \forall X\,(\exists Z\theta(X,Z) \to \exists Y\psi(X,Y))\,,$$

where $\theta(X,Z)$ is arithmetical. Despite the fact that $(\natural')$ is, even in intuitionistic predicate logic, equivalent to the following statement:

$$(\natural) \qquad \forall X, Z\,(\theta(X,Z) \to \exists Y\psi(X,Y))\,,$$

the sequential version of $(\natural')$ is occasionally stronger than that of $(\natural)$ even if $\theta(X,Z)$ has a very weak complexity such as Π^0_1. This is caused by the difficulty of obtaining the sequence of Z in $(\natural')$. Using the finite marriage theorem and the bounded König's lemma, we illustrate this phenomenon. On the other hand, the sequential version of a statement of the form $(\natural')$ is not always stronger than that of $(\natural)$ as we see in the case of the weak weak König's lemma. The important point is that the sequential form of $(\natural')$ captures the difficulty of obtaining a solution Y from X alone while that of $(\natural)$ captures the difficulty of obtaining a solution Y using both X and Z. That is to say, whenever we consider the sequential version of a Π^1_2 statement, we must pay attention to the formalization and what information can be used to obtain a solution.

In addition, it has been recently established in [7] and [1] that the intuitionistic provability of Π^1_2 statements of some syntactical form guarantees its classical sequential provability. Such kind of results are called "uniformization theorems". Our results can be used to show that Dorais's uniformization theorems in [1] are the best possible for the syntactical classes involved.

Throughout this paper, we use the standard notation and terminology in reverse mathematics (cf. [9]). In addition, $\mathbf{x}\subset_{\text{fin}}X$ denotes that $\mathbf{x}$ is a finite subset of X, and $\mathbb{Q}^+$ denotes the set of positive rational numbers. We recall that $\mathsf{WKL_0} = \mathsf{RCA_0} + \text{WKL}$ and $\mathsf{ACA_0} = \mathsf{RCA_0} + \text{ACA}$ (arithmetical comprehension).

2 The Finite Marriage Theorem

The so-called marriage theorem for finite graphs states that *a finite binary graph* $(B,G;R)$ *satisfying the Hall condition:*

$$\forall \mathbf{x}\subset_{\text{fin}}B\exists\mathbf{y}\subset_{\text{fin}}G\,(|\mathbf{x}|\leq|\mathbf{y}|\wedge\forall g\in\mathbf{y}\exists b\in\mathbf{x}\,((b,g)\in R))\,,$$

has an injection $M\subseteq R$ *from* B *to* G. It is well-known that there is a uniform algorithm to construct an injection from a given finite bipartite graph satisfying the Hall condition, which suggests that the sequential version of the finite marriage theorem is provable in $\mathsf{RCA_0}$. However, it depends on the formalization.

We provide the following two formalizations of the finite marriage theorem.

FMT :

$$\forall B, G, R, k \left(\left(\begin{array}{c} (B,G;R) \text{ is a bipartite graph} \\ \text{which satisfies the Hall condition} \\ \text{and } k \text{ bounds } B \cup G \end{array} \right) \to \exists M \left(\begin{array}{c} M \subseteq R \\ \text{is injective} \end{array} \right) \right),$$

F′MT :

$$\forall B, G, R \left(\exists k \left(\begin{array}{c} (B,G;R) \text{ is a bipartite graph} \\ \text{which satisfies the Hall condition} \\ \text{and } k \text{ bounds } B \cup G \end{array} \right) \to \exists M \left(\begin{array}{c} M \subseteq R \\ \text{is injective} \end{array} \right) \right),$$

where "k bounds $B \cup G$" denotes that for all $v \in B \cup G$, $v < k$. Note that the premise of $(\ldots \to \ldots)$ in FMT is purely universal. Throughout this paper, we use a little odd notation (e.g. F′MT) to indicate which assumption of uniformity is dropped by sequentializing.

Proposition 2.

1. $\mathsf{RCA_0} \vdash \mathrm{Seq(FMT)}$.
2. $\mathsf{RCA_0} \vdash \mathrm{Seq(F'MT)} \leftrightarrow \mathrm{ACA}$.

Proof. (1) A slight recasting of the proof of the finite marriage theorem in $\mathsf{RCA_0}$ ([4, Theorem 2.1]).

(2) $\mathsf{ACA_0} \vdash \mathrm{Seq(F'MT)}$ follows from the fact that the infinite marriage theorem is provable in $\mathsf{ACA_0}$ ([5, Theorem 2.2]). For the reverse direction, it suffices to find the range of an injection $f : \mathbb{N} \to \mathbb{N}$ ([9, Lemma III.1.3]). The basic idea is to construct, simultaneously in $\mathsf{RCA_0}$, infinite numbers of finite bipartite graphs such that the solution of the i-th graph indicates whether i is in $Rng(f)$ or not. By Σ^0_0 comprehension, take $\langle B_n \rangle_{n\in\mathbb{N}}$ and $\langle G_n \rangle_{n\in\mathbb{N}}$ as

$$b \in B_n \Leftrightarrow b = 0 \lor f\left(\frac{b-2}{2}\right) = n,$$
$$g \in G_n \Leftrightarrow g = 1 \lor f\left(\frac{g-3}{2}\right) = n,$$

which means that in addition to the underlying sequence $\{0,1\}_{n\in\mathbb{N}}$ of vertices, the odd numbers are divided into $\{B_n\}_{n\in\mathbb{N}}$ and the even numbers are divided into $\{G_n\}_{n\in\mathbb{N}}$ according to f, and take $\langle R_n \rangle_{n\in\mathbb{N}}$ as

$$(b,g) \in R_n \Leftrightarrow (b,g) = (0,1) \lor \left(b = 0 \land f\left(\frac{g-3}{2}\right) = n\right) \lor \left(g = 1 \land f\left(\frac{b-2}{2}\right) = n\right).$$

Then it is easy to see that $(B_n, G_n; R_n)$ satisfies the Hall condition for each $n \in \mathbb{N}$. Moreover if n is in the range of f via j, $B_n \cup G_n$ is bounded by $2j+4$, and otherwise, $B_n \cup G_n$ is bounded by 2. Thus, by Seq(F′MT), there exists a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ of injections. Put $S := \{n : M_n(0) \neq 1\}$, then S is the range of f by the above construction. □

The previous proposition indicates that ACA is not needed to construct an injection from a finite bipartite graph satisfying the Hall condition, and only used to take a sequence of bounds. In fact, the next proposition follows from the previous proposition immediately. (One can even prove it directly.)

Proposition 3. *The following assertion* SeqB *is equivalent to* ACA *over* RCA_0.

(SeqB) *For any sequence of sets* $\langle X_n \rangle_{n\in\mathbb{N}}$*, if* X_n *is finite for all* n*, then there exists a function* $g : \mathbb{N} \to \mathbb{N}$ *such that* $g(n)$ *bounds* X_n.

Proof. $\mathsf{ACA}_0 \vdash$ SeqB is straightforward. For the reverse direction, it suffices to show Seq(F′MT) from SeqB over RCA_0. Let $\langle (B_n, G_n; R_n) \rangle_{n\in\mathbb{N}}$ be a sequence of finite bipartite graphs satisfying the Hall condition. Using SeqB, we have a function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n)$ bounds $B_n \cup G_n$ for all $n \in \mathbb{N}$. Then the existence of a sequence of injections follows from Seq(FMT). □

3 The Bounded König's lemma

It is known that the bounded König's lemma, which states that *an infinite tree having a bounding function has an infinite path*, is equivalent to WKL [9, Lemma IV.1.4]. As in the previous section, we provide the two formalizations of it.

$$\mathrm{BKL} : \forall T, g \left(\begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

$$\mathrm{B'KL} : \forall T \left(\exists g \begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

where "g bounds T" denotes that for all $\sigma \in T$ and $i < \mathrm{lh}(\sigma)$, $\sigma(i) < g(i)$. In addition, we now treat a weaker version of the bounded König's lemma in which a tree in question is bounded by a constant.

$$\mathrm{B_cKL} : \forall T, k \left(\begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } k \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

$$\mathrm{B'_cKL} : \forall T \left(\exists k \begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } k \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

where "k bounds T" denotes that for all $\sigma \in T$ and $i < \mathrm{lh}(\sigma)$, $\sigma(i) < k$. Note that the premise of $(\ldots \to \ldots)$ in $\mathrm{B_cKL}$ is purely universal.

Proposition 4.

1. $\mathsf{RCA}_0 \vdash \mathrm{Seq(BKL)} \leftrightarrow \mathrm{Seq(B_cKL)} \leftrightarrow \mathrm{WKL}$.
2. $\mathsf{RCA}_0 \vdash \mathrm{Seq(B'KL)} \leftrightarrow \mathrm{Seq(B'_cKL)} \leftrightarrow \mathrm{ACA}$.

Proof. We reason in RCA_0.

(1) WKL implies Seq(WKL) ([6, Lemma 5]), and Seq(WKL) implies Seq(BKL) by imitating the proof of BKL in WKL_0 ([9, Lemma IV.1.4]). The implication from Seq(BKL) to Seq($\mathrm{B_cKL}$) is obvious. That from Seq($\mathrm{B_cKL}$) to WKL follows immediately from the fact that binary trees are bounded by 2.

(2) It is straightforward that ACA implies Seq($\mathrm{B'KL}$) by imitating the proof of König's lemma in ACA_0 ([9, Lemma III.7.2]). Seq($\mathrm{B'KL}$) implies Seq($\mathrm{B'_cKL}$). The implication from Seq($\mathrm{B'_cKL}$) to ACA follows from Lemma 11 below. □

In the reverse mathematics of analysis, the bounded König's lemma corresponds to the Heine/Borel compactness of effectively totally bounded complete separable metric spaces. Thus, to consider the strength of a sequential version of a mathematical statement which is related to Heine/Borel compactness, it is important to check which version of bounded König's lemma is needed. Here, we will consider the maximum principle of continuous functions as an example. The following statement is equivalent to WKL over RCA_0. (See [9, Section IV].)

(MP) For any f, if f is a continuous function from $[-1,1]$ to $\mathbb{R}$, then there exists $a \in [-1,1]$ such that

$$\max\{f(x) : x \in [-1,1]\} = f(a).$$

By an easy consideration, we can see that MP is equivalent to the following.

(MP^+) For any f, if f is a continuous function from $(-1,1)$ to $\mathbb{R}$ such that $f(0) > 0$ and $\lim_{x\to\pm 1} f(x) = 0$, then there exists $a \in (-1,1)$ such that

$$\max\{f(x) : x \in (-1,1)\} = f(a).$$

For the sequential version of MP, the following is well-known, actually, it is an easy consequence of $\mathsf{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{Seq(WKL)}$ ([6, Lemma 5]).

Proposition 5. Seq(MP) *is equivalent to* WKL *over* RCA_0.

However, the sequential version of MP^+ is strictly stronger than that of MP. (In general, ACA is required to extend a continuous function $f : (-1,1) \to \mathbb{R}$ with $\lim_{x\to\pm 1} f(x) = 0$ into a continuous function from $[-1,1]$ to $\mathbb{R}$.)

Proposition 6. *The following are equivalent over* RCA_0.

1. ACA.
2. *The sequential version of the following statement: for any f, if f is a bounded support continuous function from $\mathbb{R}$ to $\mathbb{R}$, then there exists $a \in \mathbb{R}$ such that* $\max\{f(x) : x \in \mathbb{R}\} = f(a)$. (*Here, f is said to have bounded support if there exists $k \in \mathbb{N}$ such that the closure of* $\{x \in \mathbb{R} : f(x) \neq 0\}$ *is a subset of* $[-k,k]$.)
3. *The sequential version of the following statement: for any f, if f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) > 0$ and* $\lim_{x\to\pm\infty} f(x) = 0$, *then there exists $a \in \mathbb{R}$ such that* $\max\{f(x) : x \in \mathbb{R}\} = f(a)$.

4. Seq(MP^+).

Proof. By modifying the proof of MP $\leftrightarrow$ WKL, we can easily see that 2 is equivalent to the sequential version of the following statement: if $T \subseteq \mathbb{N}^{\mathbb{N}}$ is an infinite tree such that $T \subseteq 2k \times 2^{<\mathbb{N}}$ for some k, then T has an infinite path. Note that this is a weaker version of Seq(B'_{c}KL), and still is equivalent to ACA as in the proof of Lemma 11 below. For a given continuous function f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) > 0$ and $\lim_{|x|\to\infty} f(x) = 0$, define a continuous function g as $g(x) = \max\{0, f(x) - f(0)/2\}$. Then, g has bounded support and $\max\{g(x) : x \in \mathbb{R}\} + f(0)/2 = \max\{f(x) : x \in \mathbb{R}\}$, hence we have 2 $\leftrightarrow$ 3. By an easy rescaling, we have 3 $\leftrightarrow$ 4. Thus, they are all equivalent to ACA. □

4 The Weak Weak König's Lemma

The weak weak König's lemma, which states that *a binary tree with positive measure has an infinite path*, has an intermediate strength between RCA_0 and WKL_0 ([9, Remark X.1.8]). In this case, both of sequential versions are stronger than the instancewise version and actually equivalent to WKL.

$$\mathrm{WWKL} : \forall T, m \left(\begin{pmatrix} T \subseteq 2^{<\mathbb{N}} \text{ is a tree and} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W}_2) \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

$$\mathrm{W}'\mathrm{WKL} : \forall T \left(\exists m \begin{pmatrix} T \subseteq 2^{<\mathbb{N}} \text{ is a tree and} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W}_2) \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right),$$

where (W_2) denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{2^n} \geq m.$$

Proposition 7.

1. $\mathsf{RCA}_0 \vdash$ Seq(WWKL) $\leftrightarrow$ WKL. ([2, Theorem 4.1.(2)])
2. $\mathsf{RCA}_0 \vdash$ Seq(W$'$WKL) $\leftrightarrow$ WKL.

Proof (of 2). It is easy to show within RCA_0 that for binary tree T, if there exists $m \in \mathbb{Q}^+$ such that $\lim_{n\to\infty} \frac{|\{\sigma \in T :\ \mathrm{lh}(\sigma) = n\}|}{2^n} \geq m$, then T is infinite. Therefore $\mathsf{WKL}_0 \vdash$ Seq(W$'$WKL) immediately follows from $\mathsf{WKL}_0 \vdash$ Seq(WKL) ([6, Lemma 5]). For the reverse direction, Seq(W$'$WKL) obviously implies Seq(WWKL), which is equivalent to WKL over RCA_0 from (1). □

Remark 8. *Note that the previous proposition does not suggest that the sequential strength of a mathematical statement equivalent to* WWKL *is* WKL *in general. Here, we will consider Riemann integrability for bounded functions as an example. The following statement is equivalent to* WWKL *over* RCA_0. (*See [8].*)

(Int) *For any f, if f is a continuous function from $[0,1]$ to $[0,1]$, then there exists $r \in \mathbb{R}$ such that*

$$\int_0^1 f(x)\,dx = r.$$

However, Seq(Int) *does not imply* WKL. *This is because* Seq(Int) *follows from the following sequential contrapositive of* W′WKL*:*

$$(\star) \qquad \forall T \left(\forall n \begin{pmatrix} T_n \subseteq 2^{<\mathbb{N}} \textit{ is a tree} \\ \textit{which has no path} \end{pmatrix} \to \forall n \lim_{k\to\infty} \frac{|\{\sigma \in T_n : \mathrm{lh}(\sigma) = k\}|}{2^k} = 0 \right).$$

The contrapositive of W′WKL *does not have the form* (♠) *from Definition 1 any more and* ($\star$) *is trivially equivalent to* WWKL*. Therefore* Seq(Int) *is actually equivalent to* WWKL*. In fact, for many sequential versions of mathematical statements which are equivalent to* WWKL*, we do not need* Seq(WWKL) *or* Seq(W′WKL) *but* ($\star$)*.*

Next, we will investigate the effect of uniformity for positive measure more precisely. For this, we shall consider some more variants, namely, bounded König's lemmas with respect to measure.

– $\mathrm{WBKL} : \forall T, m, g \left(\begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_g}), \\ g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right)$,

  where $(\mathrm{W_g})$ denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} \geq m.$$

– $\mathrm{WB_cKL} : \forall T, m, k \left(\begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_k}), \\ k \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right)$,

  where $(\mathrm{W_k})$ denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{k^n} \geq m.$$

Proposition 9. WBKL *and* $\mathrm{WB_cKL}$ *are equivalent to* WWKL *over* RCA_0*.*

Proof. We reason in RCA_0. WBKL to $\mathrm{WB_cKL}$ to WWKL is trivial. We will show WBKL from WWKL. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree bounded by $g : \mathbb{N} \to \mathbb{N}$ such that for some $q \in \mathbb{Q}^+$,

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} \geq q.$$

For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define $l_g(\sigma)$ and $r_g(\sigma)$ as follows:

$$l_g(\sigma) = \sum_{k<\mathrm{lh}(\sigma)} \frac{\sigma(k)}{\prod_{i\leq k} g(i)}, \qquad r_g(\sigma) = l_g(\sigma) + \frac{1}{\prod_{i<\mathrm{lh}(\sigma)} g(i)}.$$

Similarly, for $\sigma \in 2^{<\mathbb{N}}$, define $l_2(\sigma)$ and $r_2(\sigma)$ as follows:

$$l_2(\sigma) = \sum_{k<\mathrm{lh}(\sigma)} \sigma(k)2^{-k-1}, \qquad r_2(\sigma) = l_2(\sigma) + 2^{-\mathrm{lh}(\sigma)}.$$

Note that $\bigcup_{\sigma\in T,\mathrm{lh}(\sigma)=m}[l_g(\sigma), r_g(\sigma)]$ are disjoint intervals in $[0,1]$ whose lengths sum to the measure of level m of T and these intervals can be approximated arbitrarily well from within by intervals with dyadic rational endpoints. That is, for $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$\frac{\left|\left\{\sigma \in 2^{<\mathbb{N}} : \begin{array}{l}\mathrm{lh}(\sigma) = N \wedge \\ \exists\tau \in T(\mathrm{lh}(\tau) = m \wedge l_g(\tau) \le l_2(\sigma) \wedge r_2(\sigma) \le r_g(\tau))\end{array}\right\}\right|}{2^N}$$
$$> \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = m\}|}{\prod_{i<m} g(i)} - \frac{q}{2^{m+2}}.$$

We define $h(m)$ as the least such N.

Now we define $T^* \subseteq 2^{\mathbb{N}}$ as

$$\sigma \in T^* \Leftrightarrow$$
$$\forall m < \mathrm{lh}(\sigma) \begin{pmatrix} h(m) \le \mathrm{lh}(\sigma) \to \exists\tau \in T(\mathrm{lh}(\tau) = m \wedge \\ l_g(\tau) \le l_2(\sigma \restriction h(m)) \wedge r_2(\sigma \restriction h(m)) \le r_g(\tau)) \end{pmatrix}.$$

Then, T^* is a tree such that for all $n \in \mathbb{N}$,

$$\frac{|\{\sigma \in T^* : \mathrm{lh}(\sigma) = n\}|}{2^n} > \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} - \sum_{m<n} \frac{q}{2^{m+2}} \ge \frac{q}{2}.$$

Thus, by WWKL, there exists a path P^* through T^*. For any $m \in \mathbb{N}$, there exists a unique $\tau_m \in T$ such that $\mathrm{lh}(\tau_m) = m$ and $l_g(\tau_m) \le l_2(P \restriction h(m)) \wedge r_2(P \restriction h(m)) \le r_g(\tau_m)$. Then, $P = \bigcup_{m\in\mathbb{N}} \tau_m$ is a path through T. □

Next we investigate the sequential strength of the statements in question. The following proposition means that the uniformity for positive-measure does not help to weaken the sequential strength of the bounded König's lemma.

Proposition 10.

1. Seq(W′BKL), Seq(WBKL), Seq(W′B$_\mathrm{c}$KL) *and* Seq(WB$_\mathrm{c}$KL) *are equivalent to* WKL *over* RCA_0.
2. Seq(W′B′KL), Seq(WB′KL), Seq(W′B′$_\mathrm{c}$KL) *and* Seq(WB′$_\mathrm{c}$KL) *are equivalent to* ACA *over* RCA_0.

Here WB′KL, W′BKL, W′B′KL, WB′$_\mathrm{c}$KL, W′B$_\mathrm{c}$KL, *and* W′B′$_\mathrm{c}$KL *are defined in the same manner as before, that is,* W′ *(resp.* B′, B′$_\mathrm{c}$*) means that the universal quantifier* $\forall m$ *(resp.* $\forall g$, $\forall k$*) is moved into* $(\ldots \to \ldots)$ *as the existential quantifier* $\exists m$ *(resp.* $\exists g$, $\exists k$*).*

To show the previous proposition, we first show the following lemma.

Lemma 11. $\mathsf{RCA_0} \vdash \mathrm{Seq}(\mathrm{WB'_cKL}) \to \mathrm{ACA}$*, that is, the following statement implies* ACA *over* $\mathsf{RCA_0}$:

$$\forall \langle T_n \rangle_{n\in\mathbb{N}}, \langle m_n \rangle_{n\in\mathbb{N}} \left(\forall n \exists k \begin{pmatrix} T_n \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m_n \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_k}) \text{ for } T_n, \\ k \text{ bounds } T_n \end{pmatrix} \right.$$
$$\left. \longrightarrow \exists \langle P_n \rangle_{n\in\mathbb{N}} \forall n \, (P_n \text{ is an infinite path of } T_n) \right).$$

Proof. As in the proof of Proposition 2.(2), we will find the range of an injection $f : \mathbb{N} \to \mathbb{N}$ ([9, Lemma Ⅲ.1.3]). By Σ^0_0 comprehension, we take a sequence $\langle T_n \rangle_{n\in\mathbb{N}}$ of trees from the given injection f as

$$\sigma \in T_n \;\Leftrightarrow\; \begin{array}{l} \forall i < \mathrm{lh}(\sigma)\Big(\sigma(0) = 0 \wedge \sigma(i+1) \leq 1 \wedge f(i) \neq n\Big) \vee \\ \exists j < \sigma(0)\Big(\forall i < \mathrm{lh}(\sigma)\,(\sigma(i) \leq 2j+1) \wedge f(j) = n\Big). \end{array}$$

Then, each $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ clearly forms a tree. Put $m_n :\equiv 1/2$. We need to find a required bound k for each n. For given n, if there exists j such that $f(j) = n$, then put $k := 2j + 2$, and otherwise, put $k := 2$. In either case, we can check that k bounds T_n and $m_n (= 1/2)$ satisfies $(\mathrm{W_k})$ for T_n. Thus, by $\mathrm{Seq}(\mathrm{WB'_cKL})$, there exists a sequence $\langle P_n \rangle_{n\in\mathbb{N}}$ of paths. Put $S := \{n : P_n(0) \neq 0\}$. It is easy to see that $P_n(0) \neq 0 \leftrightarrow \exists j\,(f(j) = n)$, namely, S is the range of f. □

Proof (of Proposition 10). We reason in $\mathsf{RCA_0}$.

(1) Each of Seq(W′BKL), Seq(WBKL), Seq(W′B$_\mathrm{c}$KL), Seq(WB$_\mathrm{c}$KL) follows from Seq(BKL), then also from WKL by Proposition 4.(1). On the other hand, each of them implies Seq(WWKL) which is equivalent to WKL.

(2) Each of Seq(W′B′KL), Seq(WB′KL), Seq(W′B′$_\mathrm{c}$KL), Seq(WB′$_\mathrm{c}$KL) follows from Seq(B′KL), then also from ACA by Proposition 4.(2). On the other hand, each of Seq(W′B′KL), Seq(WB′KL), Seq(W′B′$_\mathrm{c}$KL) implies Seq(WB′$_\mathrm{c}$KL) and Seq(WB′$_\mathrm{c}$KL) implies ACA by Lemma 11. □

5 The Best Possibility of Dorais's Uniformization Results

The first uniformization theorems are established in [7], which can be applied for Π^1_2 statements of the form (♠) (from Definition 1) with purely universal φ. Dorais has recently shown other uniformization theorems in second-order setting with function-based language, which can be applied for more Π^1_2 statements.

Proposition 12 (Dorais [1]).

1. *For any* $\mathrm{T} : \forall f\,(\varphi(f) \to \exists g \psi(f,g))$ *such that* $\varphi(f)$ *is in* N_K *and* $\psi(f,g)$ *is in* Γ_K*, if* $\mathsf{EL} + \mathrm{GC} + \mathrm{CN} \vdash \mathrm{T}$*, then* $\mathsf{RCA} \vdash \mathrm{Seq}(\mathrm{T})$.
2. *For any* $\mathrm{T} : \forall f\,(\varphi(f) \to \exists g \psi(f,g))$ *such that* $\varphi(f)$ *is in* N_L *and* $\psi(f,g)$ *is in* Γ_L*, if* $\mathsf{EL} + \mathrm{WKL} + \mathrm{GC}_L + \mathrm{CN}_L \vdash \mathrm{T}$*, then* $\mathsf{RCA} + \mathrm{WKL} \vdash \mathrm{Seq}(\mathrm{T})$.

We refer the readers to see [1] for precise definitions of each of the symbols in the previous proposition. In fact, the restriction of ψ to Γ_K and Γ_L is not tight and the interest is only in the possibility of extending $\mathrm{N_K}$ and $\mathrm{N_L}$. All purely existential and purely universal formulas are included in N_K. In addition, all formulas of the form $\exists x \leq t \forall z A_{qf}$ are included in N_L. Here we show that $\mathrm{N_K}$ and $\mathrm{N_L}$ cannot be extended to the class including all formulas of the form $\exists x \forall z A_{qf}$ in Proposition 12.

Suppose that in Proposition 12.(1), N_K can be extended to such a class. Since each purely universal formula in set-based language is translated as a purely universal formula in function-based language by identifying sets with their characteristic functions, the premise of $(\ldots \to \ldots)$ in function-based $\mathrm{F'MT}$ (intuitionistically equivalent to function-based FMT) has the form $\exists x \forall z A_{qf}$. Then Proposition 2.(2) derives that function-based $\mathrm{F'MT}$ is not provable in $\mathsf{EL} + \mathrm{GC} + \mathrm{CN}$. However, it is provable in $\mathsf{EL_0}$ by transforming the proof of the finite marriage theorem in $\mathsf{RCA_0}$ ([4, Theorem 2.1]).

Next we suppose in Proposition 12.(2), N_L can be extended to such a class. As in the previous paragraph, Proposition 4.(2) derives that function-based $\mathrm{B'_cKL}$ (intuitionistically equivalent to function-based $\mathrm{B_cKL}$) is not provable in $\mathsf{EL} + \mathrm{WKL} + \mathrm{GC}_L + \mathrm{CN}_L$. However, it is provable in $\mathsf{EL_0} + \mathrm{WKL}$ by transforming the proof of the bounded König's lemma in $\mathsf{WKL_0}$ ([9, Lemma IV.1.4]).

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