Propagation of partial randomness

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Abstract

Let f be a computable function from finite sequences of 0's and 1's to real numbers. We prove that strong f-randomness implies strong f-randomness relative to a PA-degree. We also prove: if X is strongly f-random and Turing reducible to Y where Y is Martin-Löf random relative to Z, then X is strongly f-random relative to Z. In addition, we prove analogous propagation results for other notions of partial randomness, including non-K-triviality and autocomplexity. We prove that f-randomness relative to a PA-degree implies strong f-randomness, hence f-randomness does not imply f-randomness relative to a PA-degree.

Keywords: partial randomness, effective Hausdorff dimension, Martin-Löf randomness, Kolmogorov complexity, models of arithmetic.

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Contents

1 Introduction

We begin by recalling two known results concerning Martin-Löf randomness relative to a Turing oracle. Let $\mathbb{N}$ denote the set of positive integers. Let $\{0,1\}^{\mathbb{N}}$ denote the Cantor space, i.e., the set of infinite sequences of 0's and 1's.

Theorem 1.1. Let $X \in \{0,1\}^{\mathbb{N}}$ be Martin-Löf random. Suppose X is Turing reducible to Y where Y is Martin-Löf random relative to Z. Then X is Martin-Löf random relative to Z.

Theorem 1.2. Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random, then X is Martin-Löf random relative to some $Z \in Q$.

Recall from [18] that a PA-degree is defined to be the Turing degree of a complete consistent theory extending first-order Peano arithmetic. It is well known that, via Gödel numbering, the set of complete consistent theories extending first-order Peano arithmetic may be viewed as a Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. Moreover [18, 29, 32, 33], this particular Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$ is *universal* in the following sense: a Turing oracle Z is of PA-degree if and only if every nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$ contains an element which is Turing reducible to Z. Consequently, Theorem 1.2 may be restated as follows:

Theorem 1.3. If $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random, then X is Martin-Löf random relative to some PA-degree.

Theorem 1.1, which we call the *XYZ Theorem*, is due to Miller/Yu [22, Theorem 4.3]. Theorems 1.2 and 1.3 are due independently to several groups of researchers: Downey et al [10, Proposition 7.4], Reimann/Slaman [27, Theorem 4.5] (also cited in [10]), and Simpson/Yokoyama [37, Lemma 3.3].

Theorems 1.2 and 1.3 have been very useful in the study of randomness. Reimann/Slaman [27] used Theorem 1.2 to prove that any noncomputable $X \in \{0,1\}^{\mathbb{N}}$ is nonatomically random with respect to some probability measure on $\{0,1\}^{\mathbb{N}}$. Simpson/Yokoyama [37] used a generalization of Theorem 1.2 to study the reverse mathematics of Loeb measure. Recently Brattka/Miller/Nies [5] used Theorem 1.2 to prove that $x \in [0,1]$ is random if and only if every computable continuous function of bounded variation is differentiable at x.

The theme of Theorems 1.1–1.3 is what might be called "propagation of Martin-Löf randomness." Namely, all of these theorems assert that if X is Martin-Löf random then X is Martin-Löf random relative to certain Turing oracles.

The purpose of this paper is to present some new results which are generalizations of Theorems 1.1–1.3. The theme of our new results might be called "propagation of partial randomness." Here "partial randomness" refers to certain properties which are in the same vein as Martin-Löf randomness. Recent studies of partial randomness include [6, 16, 19, 21, 25, 28, 39]. Our main new results involve a specific notion of partial randomness known as *strong f-randomness* where f is an arbitrary computable function from finite sequences of 0's and 1's to real numbers. Along the way we present some old and new characterizations of strong f-randomness. We also consider other notions of partial randomness including *complexity* [4, 15, 19], *autocomplexity* [19], and *non-K-triviality* [11, 23].

The plan of this paper is as follows. In §2 we define *f-randomness* and *strong f-randomness* and characterize these notions in terms of Kolmogorov complexity. In §3 we prove that $(f + 2\log_2 f)$-randomness implies strong f-randomness. Note that §2 and §3 and §8 are largely expository. In §4 we present our main results concerning propagation of strong f-randomness. Namely, we prove appropriate generalizations of Theorems 1.1–1.3 with Martin-Löf randomness replaced by strong f-randomness. In §5 and §6 we prove analogous results concerning propagation of non-K-triviality and propagation of diagonal nonrecursiveness, respectively. In §7 we prove analogous results concerning propagation of autocomplexity, and we characterize autocomplexity in terms of f-randomness and strong f-randomness. In §8 we define *vehement f-randomness* and prove that it is equivalent to strong f-randomness, provided f is *convex*. In §9 we prove a version of Theorem 1.2 with Martin-Löf randomness replaced by vehement f-randomness. In §10 we present two new characterizations of strong f-randomness. In §11 we show that our results concerning propagation of strong f-randomness fail for f-randomness.

2 f-randomness and strong f-randomness

Let $f : \{0,1\}^* \to [-\infty, \infty]$ be an arbitrary computable function from finite sequences of 0's and 1's to the extended[1] real numbers. In this section we define what it means for $X \in \{0,1\}^{\mathbb{N}}$ to be f-random, strongly f-random, f-complex, and strongly f-complex.

Recall that according to Schnorr's Theorem (see [11, Theorem 6.2.3] or [23, Theorem 3.2.9] or [34, Theorem 10.7]), X is Martin-Löf random if and only if for all n the prefix-free Kolmogorov complexity of the first n bits of X is at least n modulo an additive constant. In this section we prove generalizations of Schnorr's Theorem, replacing Martin-Löf randomness by f-randomness and strong f-randomness. Our proofs are modeled on one of the standard proofs [34, Theorem 10.7] of Schnorr's Theorem.

This section is mostly expository. For the history of the concepts and results in this section, see Calude/Staiger/Terwijn [6] and Tadaki [39].

Definition 2.1. For $X \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$ we write $X{\upharpoonright}n = X{\upharpoonright}\{1, \ldots, n\} =$ the first n bits of X. Given $f : \{0,1\}^* \to [-\infty, \infty]$ we define X to be f-complex or strongly f-complex if

$$\exists c \, \forall n \, (\mathrm{KP}(X{\upharpoonright}n) \geq f(X{\upharpoonright}n) - c)$$

or

$$\exists c \, \forall n \, (\mathrm{KA}(X{\upharpoonright}n) \geq f(X{\upharpoonright}n) - c)$$

respectively. Here KP and KA denote prefix-free complexity (see [11, §3.5] or [23, §2.2] or [34, §10]) and a priori complexity (see [11, §3.16] or [40]) respectively.

Definition 2.2. Given $f : \{0,1\}^* \to [-\infty, \infty]$, the f-weight of $\sigma \in \{0,1\}^*$ is defined as $\mathrm{wt}_f(\sigma) = 2^{-f(\sigma)}$. The direct f-weight of $A \subseteq \{0,1\}^*$ is defined as $\mathrm{dwt}_f(A) = \sum_{\sigma \in A} \mathrm{wt}_f(\sigma)$. A set $P \subseteq \{0,1\}^*$ is said to be prefix-free if no element of P is a proper initial segment of an element of P. The prefix-free f-weight of A is defined as

$$\mathrm{pwt}_f(A) = \sup\{\mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\}.$$

Definition 2.3. For $\sigma \in \{0,1\}^*$ we write $[\![\sigma]\!] = \{X \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset X\}$. For $A \subseteq \{0,1\}^*$ we write $[\![A]\!] = \bigcup_{\sigma \in A} [\![\sigma]\!]$ and

$$\widehat{A} = \{\sigma \in A \mid \nexists \rho \, (\rho \subset \sigma \text{ and } \rho \in A)\} = \text{the set of minimal elements of } A.$$

Note that $\widehat{A}$ is prefix-free and $[\![\widehat{A}]\!] = [\![A]\!]$.

We write r.e. as an abbreviation for *recursively enumerable*. A sequence of sets $A_i \subseteq \{0,1\}^*$, $i \in \mathbb{N}$ is said to be uniformly r.e. if $\{(\sigma, i) \mid \sigma \in A_i\}$ is r.e.

[1] We define $f : \{0,1\}^* \to [-\infty, \infty]$ to be computable if $f/(|f| + 1) : \{0,1\}^* \to [-1, 1]$ is computable.

Definition 2.4. Assume that $f : \{0,1\}^* \to [-\infty, \infty]$ is computable. We define $X \in \{0,1\}^{\mathbb{N}}$ to be <u>f-random</u> or <u>strongly f-random</u> if $X \notin \bigcap_i [\![A_i]\!]$ whenever A_i is uniformly r.e. with $\mathrm{dwt}_f(A_i) \le 2^{-i}$ or $\mathrm{pwt}_f(A_i) \le 2^{-i}$ respectively.

Remark 2.5. Since $\mathrm{pwt}_f(A) \le \mathrm{dwt}_f(A)$ for all A, it is clear that strong f-randomness implies f-randomness. Similarly, since $\exists c \, \forall \tau \, (\mathrm{KA}(\tau) \le \mathrm{KP}(\tau) + c)$, it is clear that strong f-complexity implies f-complexity. Note also that wt_f is a *premeasure* in the sense of [26, Definition 1].

The next theorem is a straightforward generalization of Tadaki [39, Theorem 3.1].

Theorem 2.6. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. Then f-randomness is equivalent to f-complexity.

Proof. Suppose X is f-random. Let $S_i = \{\tau \mid \mathrm{KP}(\tau) < f(\tau) - i\}$. Clearly S_i is uniformly r.e., and by Kraft's Inequality [34, Theorem 10.3] we have

$$\mathrm{dwt}_f(S_i) = \sum_{\tau \in S_i} 2^{-f(\tau)} \le \sum_{\tau \in S_i} 2^{-\mathrm{KP}(\tau) - i} = 2^{-i} \sum_{\tau \in S_i} 2^{-\mathrm{KP}(\tau)} < 2^{-i}$$

so S_i is a test for f-randomness. Since X is f-random it follows that $X \notin \bigcap_i [\![S_i]\!]$, i.e., $\exists i \, \forall n \, (\mathrm{KP}(X{\upharpoonright}n) \ge f(X{\upharpoonright}n) - i)$, i.e., X is f-complex.

Now suppose X is not f-random, say $X \in \bigcap_i [\![A_i]\!]$ where A_i is uniformly r.e. and $\mathrm{dwt}_f(A_i) \le 2^{-i}$. Then

$$\sum_i \sum_{\tau \in A_{2i}} 2^{-f(\tau)+i} = \sum_i 2^i \mathrm{dwt}_f(A_{2i}) \le \sum_i 2^i 2^{-2i} = \sum_i 2^{-i} = 1$$

so by the Kraft/Chaitin Lemma (see [34, Corollary 10.6]) we have

$$\exists c \, \forall i \, \forall \tau \, (\tau \in A_{2i} \Rightarrow \mathrm{KP}(\tau) \le f(\tau) - i + c).$$

Since $X \in \bigcap_i [\![A_{2i}]\!]$ it follows that $\exists c \, \forall i \, \exists n \, (\mathrm{KP}(X{\upharpoonright}n) \le f(X{\upharpoonright}n) - i + c)$. In other words, X is not f-complex. This completes the proof. $\square$

Corollary 2.7. The sets $S_i = \{\tau \mid \mathrm{KP}(\tau) < f(\tau) - i\}$ form a universal test for f-randomness.

Proof. Paraphrasing Theorem 2.6 we see that X is f-random if and only if $X \notin \bigcap_i [\![S_i]\!]$. It remains to prove that $\mathrm{dwt}_f(S_i) \le 2^{-i}$, but we have already seen this as part of the proof of Theorem 2.6. $\square$

The next theorem is a straightforward generalization of Calude/Staiger/Terwijn [6, Corollary 4.10].

Theorem 2.8. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. Then strong f-randomness is equivalent to strong f-complexity.

Proof. Recall that $\mathrm{KA}(\tau) = -\log_2 m(\tau)$ where $m : \{0,1\}^* \to [0,1]$ is a universal left-r.e. semimeasure. See for instance [11, §3.16] or [40].

Suppose X is strongly f-random. Let $S_i = \{\tau \mid \mathrm{KA}(\tau) < f(\tau) - i\}$. Clearly S_i is uniformly r.e. We claim that $\mathrm{pwt}_f(S_i) \leq 2^{-i}$. To see this, let $P \subseteq S_i$ be prefix-free. Then

$$\mathrm{dwt}_f(P) = \sum_{\tau \in P} 2^{-f(\tau)} \leq \sum_{\tau \in P} 2^{-i-\mathrm{KA}(\tau)} = 2^{-i} \sum_{\tau \in P} m(\tau) \leq 2^{-i}$$

since m is a semimeasure. This proves our claim. Thus S_i is a test for strong f-randomness. Since X is strongly f-random, we have $X \notin \bigcap_i [\![S_i]\!]$, i.e., $\exists i\, \forall n\, (\mathrm{KA}(X{\upharpoonright}n) \geq f(X{\upharpoonright}n) - i)$, i.e., X is strongly f-complex.

Now suppose X is not strongly f-random, say $X \in \bigcap_i [\![A_i]\!]$ where A_i is uniformly r.e. and $\mathrm{pwt}_f(A_i) \leq 2^{-i}$. For each i let m_i be the uniformly left-r.e. semimeasure given by $m_i(\sigma) = \mathrm{pwt}_f(\{\tau \in A_i \mid \tau \supseteq \sigma\})$. Note that $m_i(\tau) \geq \mathrm{wt}_f(\tau)$ whenever $\tau \in A_i$. For each i we have $m_i(\langle\rangle) = \mathrm{pwt}_f(A_i) \leq 2^{-i}$, hence $2^i m_{2i}(\langle\rangle) \leq 2^i 2^{-2i} = 2^{-i}$, so consider the left-r.e. semimeasure $\overline{m}(\sigma) = \sum_i 2^i m_{2i}(\sigma)$. Since m is a universal left-r.e. semimeasure, let c be such that $\overline{m}(\sigma) \leq 2^c m(\sigma)$ for all σ. Then for all $\tau \in A_{2i}$ we have $2^{i-f(\tau)} = 2^i \mathrm{wt}_f(\tau) \leq 2^i m_{2i}(\tau) \leq \overline{m}(\tau) \leq 2^c m(\tau) = 2^{c-\mathrm{KA}(\tau)}$, hence $\mathrm{KA}(\tau) \leq f(\tau) - i + c$. Since $X \in \bigcap_i [\![A_{2i}]\!]$ it follows that $\forall i\, \exists n\, (\mathrm{KA}(X{\upharpoonright}n) \leq f(X{\upharpoonright}n) - i + c)$. In other words, X is not strongly f-complex. This completes the proof. ☐

Corollary 2.9. The sets $S_i = \{\tau \mid \mathrm{KA}(\tau) < f(\tau) - i\}$ form a universal test for strong f-randomness.

Proof. Paraphrasing Theorem 2.8 we see that X is strongly f-random if and only if $X \notin \bigcap_i [\![S_i]\!]$. It remains to prove that $\mathrm{pwt}_f(S_i) \leq 2^{-i}$, but we have already seen this as part of the proof of Theorem 2.8. ☐

Remark 2.10. As a special case, consider the functions $f_s : \{0,1\}^* \to [0, \infty)$ given by $f_s(\sigma) = s|\sigma|$ where s is rational and $0 < s \leq 1$. Here we are writing $|\sigma|$ = the length of σ. Define $X \in \{0,1\}^{\mathbb{N}}$ to be <u>s-random</u> if it is f_s-random, and <u>strongly s-random</u> if it is strongly f_s-random. Note that Martin-Löf randomness is equivalent to 1-randomness and to strong 1-randomness. The <u>effective Hausdorff dimension</u> of X is

$$\mathrm{effdim}(X) = \sup\{s \mid X \text{ is } s\text{-random}\} = \sup\{s \mid X \text{ is strongly } s\text{-random}\}$$

and this notion has been studied in [21, 25, 39] and many other publications.

Remark 2.11. Given a computable function $f : \{0,1\}^* \to [-\infty, \infty]$, it is easy to see that $\{X \mid X \text{ is } f\text{-random}\}$ and $\{X \mid X \text{ is strongly } f\text{-random}\}$ are Σ^0_2 subsets of $\{0,1\}^{\mathbb{N}}$. Conversely, given a Σ^0_2 set $S \subseteq \{0,1\}^{\mathbb{N}}$, we can easily construct a computable function $f : \{0,1\}^* \to \mathbb{N}$ such that

$$S = \{X \mid X \text{ is } f\text{-random}\} = \{X \mid X \text{ is strongly } f\text{-random}\}.$$

Namely, if $S = \bigcup_i \{\text{paths through } T_i\}$ where $T_i \subseteq \{0,1\}^*$, $i \in \mathbb{N}$ is a computable sequence of computable trees, let

$$f(\tau) = \begin{cases} 1 & \text{if } h(\tau{\restriction}(|\tau|-1)) = h(\tau), \\ 2|\tau| & \text{otherwise,} \end{cases}$$

where $h(\tau) =$ the least i such that $i = |\tau|$ or $\tau \in T_i$. We mention these examples in order to suggest how our concepts of f-randomness and strong f-randomness may apply to a wide variety of situations. See also Theorem 7.3 below.

3 g-randomness implies strong f-randomness

Suppose we have two computable functions $f, g : \{0,1\}^* \to [-\infty, \infty]$. Clearly g-randomness implies f-randomness provided $\forall\sigma\, (f(\sigma) \leq g(\sigma))$. We now prove that g-randomness implies strong f-randomness provided g grows significantly faster than f. Our result here is a slight refinement of known results due to Calude/Staiger/Terwijn [6] and Reimann/Stephan [28]. See also Uspensky/Shen [40, §4.2].

Definition 3.1. The increasing set of $f : \{0,1\}^* \to [-\infty, \infty]$ is

$$I(f) = \{\sigma \mid (\forall\rho \subset \sigma)\, (f(\rho) < f(\sigma))\}.$$

Lemma 3.2. Given a computable function $f : \{0,1\}^* \to [-\infty, \infty]$, we can effectively find a computable function $\overline{f} : \{0,1\}^* \to \mathbb{N}$ such that for all σ,

$$f_0(\sigma) < \overline{f}(\sigma) < f_0(\sigma) + 2 \tag{1}$$

where $f_0(\sigma) = \min(\max(f(\sigma), 0), 2|\sigma|)$. It then follows that f-randomness is equivalent to $\overline{f}$-randomness, and strong f-randomness is equivalent to strong $\overline{f}$-randomness.

Proof. Given $\sigma \in \{0,1\}^*$ we can effectively approximate $f_0(\sigma)$ to find $\overline{f}(\sigma) \in \mathbb{N}$ such that (1) holds. In this way we obtain a computable function $\overline{f} : \{0,1\}^* \to \mathbb{N}$. Using the fact that $\exists c\, \forall\sigma\, (0 < \mathrm{KP}(\sigma) < 2|\sigma| + c$ and $0 < \mathrm{KA}(\sigma) \leq 2|\sigma| + c)$, we can easily see that (strong) f-complexity is equivalent to (strong) $\overline{f}$-complexity. The desired conclusions then follow in view of Theorems 2.6 and 2.8. ☐

Lemma 3.3. Let $f : \{0,1\}^* \to \mathbb{N}$ be computable. Given an r.e. set $A \subseteq \{0,1\}^*$ we can effectively find an r.e. set $\overline{A} \subseteq I(f)$ such that $[\![A]\!] \subseteq [\![\overline{A}]\!]$ and $\mathrm{dwt}_f(\overline{A}) \leq \mathrm{dwt}_f(A)$ and $\mathrm{pwt}_f(\overline{A}) \leq \mathrm{pwt}_f(A)$.

Proof. Let $\overline{A} = \{\overline{\sigma} \mid \sigma \in A\}$ where $\overline{\sigma} = \min\{\rho \subseteq \sigma \mid f(\rho) \geq f(\sigma)\}$. It is straightforward to verify that this $\overline{A}$ has the desired properties. ☐

Remark 3.4. Because of Lemmas 3.2 and 3.3, we are often safe in assuming that $f : \{0,1\}^* \to \mathbb{N}$ and that $A \subseteq I(f)$.

Theorem 3.5. Let $f, g : \{0,1\}^* \to [-\infty, \infty]$ be computable with g of the form $g(\sigma) = f(\sigma) + h(f(\sigma))$ where h is nondecreasing and $\sum_{n=1}^{\infty} 2^{-h(n)} < \infty$. If X is g-random, then X is strongly f-random.

Proof. Because h is nondecreasing, we may safely apply Lemma 3.2 to assume that $f : \{0,1\}^* \to \mathbb{N}$. Fix c such that $\sum_n 2^{-h(n)} \leq 2^c < \infty$. Suppose X is not strongly f-random, say $X \in \bigcap_i [\![A_i]\!]$ where A_i is uniformly r.e. and $\mathrm{pwt}_f(A_i) \leq 2^{-i}$. By Lemma 3.3 we may safely assume that $A_i \subseteq I(f)$ for all i. Let $P_{in} = \{\sigma \in A_i \mid f(\sigma) = n\}$. Clearly $A_i = \bigcup_n P_{in}$ and P_{in} is prefix-free. Thus $\mathrm{dwt}_f(P_{in}) \leq \mathrm{pwt}_f(A_i)$ and

$$
\begin{aligned}
\mathrm{dwt}_g(A_i) &= \textstyle\sum_{\sigma \in A_i} 2^{-g(\sigma)} \\
&= \textstyle\sum_{\sigma \in A_i} 2^{-h(f(\sigma))} 2^{-f(\sigma)} \\
&= \textstyle\sum_n \sum_{\sigma \in P_{in}} 2^{-h(n)} 2^{-f(\sigma)} \\
&= \textstyle\sum_n 2^{-h(n)} \sum_{\sigma \in P_{in}} 2^{-f(\sigma)} \\
&= \textstyle\sum_n 2^{-h(n)} \mathrm{dwt}_f(P_{in}) \\
&\leq 2^c \mathrm{pwt}_f(A_i) \\
&\leq 2^{c-i}.
\end{aligned}
$$

Since $X \in \bigcap_i [\![A_i]\!]$ it follows that X is not g-random, Q.E.D. □

Theorem 3.6. Let $f : \{0,1\}^* \to (0, \infty]$ be computable. Suppose X is $(f + (1 + \epsilon) \log_2 f)$-random for some $\epsilon > 0$. Then X is strongly f-random.

Proof. We may safely assume that ϵ is rational. In this case it suffices to apply Theorem 3.5 with $h(x) = (1 + \epsilon) \log_2 x$. □

Remark 3.7. Consider the computable function $f = f_s$ where $s = 1/2$, i.e., $f(\sigma) = |\sigma|/2$ for all σ. (More generally, let f be computable and satisfy certain other conditions which we shall not specify here.) Reimann/Stephan [28] have constructed an X which is f-random but not strongly f-random. Hudelson [16] has constructed an X which is strongly f-random but such that no Y Turing reducible to X is $(f + (1 + \epsilon) \log_2 f)$-random for any $\epsilon > 0$. We conjecture that there exists an X which is f-random but such that no Y Turing reducible to X is strongly f-random.

Remark 3.8. In Theorem 3.6 and Remark 3.7 we may replace $f + (1+\epsilon) \log_2 f$ by $f + \log_2 f + (1 + \epsilon) \log_2 \log_2 f$, etc., as in [40, §4.2].

4 Propagation of strong f-randomness

The purpose of this section is to prove generalizations of Theorems 1.1–1.3 in which Martin-Löf randomness is replaced by strong f-randomness. These generalizations are perhaps the most important new results of this paper. Let μ be the fair-coin probability measure on $\{0,1\}^{\mathbb{N}}$ given by $\mu([\![\sigma]\!]) = 2^{-|\sigma|}$.

Definition 4.1. A Levin system is an indexed family of sets $V_\sigma \subseteq \{0,1\}^{\mathbb{N}}$, $\sigma \in \{0,1\}^*$, such that

1. V_σ is Σ^0_1 uniformly in σ,
2. $V_\sigma \supseteq V_{\sigma^\frown\langle 0\rangle} \cup V_{\sigma^\frown\langle 1\rangle}$ for all σ,
3. $V_{\sigma^\frown\langle 0\rangle} \cap V_{\sigma^\frown\langle 1\rangle} = \emptyset$ for all σ.

These properties easily imply

4. $V_\rho \supseteq V_\sigma$ whenever $\rho \subseteq \sigma$,
5. $V_\sigma \cap V_\tau = \emptyset$ whenever σ and τ are incompatible.

Lemma 4.2. Let V_σ be a Levin system, and let f be computable. If X is strongly f-random, then $\exists c \forall n\, (\mu(V_{X\restriction n}) \leq 2^{c-f(X\restriction n)})$.

Proof. Let $A_i = \{\sigma \mid \mu(V_\sigma) > 2^{i-f(\sigma)}\}$. Clearly A_i is uniformly r.e. We claim that $\mathrm{pwt}_f(A_i) \leq 2^{-i}$. To see this, let $P \subseteq A_i$ be prefix-free. By part 5 of Definition 4.1 we have $1 \geq \mu(\bigcup_{\sigma\in P} V_\sigma) = \sum_{\sigma\in P} \mu(V_\sigma) \geq \sum_{\sigma\in P} 2^{i-f(\sigma)} = 2^i \mathrm{dwt}_f(P)$, so $\mathrm{dwt}_f(P) \leq 2^{-i}$. This proves our claim. Thus A_i is a test for strong f-randomness. Since X is strongly f-random, it follows that $X \notin [\![A_i]\!]$ for some i. In other words, $\mu(V_{X\restriction n}) \leq 2^{i-f(X\restriction n)}$ for all n, Q.E.D. □

Remark 4.3. Our idea of using strong f-randomness in Lemma 4.2 was inspired by Reimann's use of strong f-randomness in [26, Theorem 14].

Lemma 4.4. Let r_σ, $\sigma \in \{0,1\}^*$, be a uniformly left-r.e. system of real numbers. Given a Levin system V_σ, we can effectively find a Levin system $\widetilde{V}_\sigma$ such that

1. $\widetilde{V}_\sigma \subseteq V_\sigma$ for all σ,
2. $\mu(\widetilde{V}_\sigma) \leq r_\sigma$ for all σ,
3. $\widetilde{V}_\sigma = V_\sigma$ whenever σ is such that $\mu(V_\rho) < r_\rho$ for all $\rho \subseteq \sigma$.

Proof. The proof is awkward but straightforward. □

Theorem 4.5. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. Suppose X is strongly f-random and Turing reducible to Y where Y is Martin-Löf random relative to Z. Then X is strongly f-random relative to Z.

Proof. Let Φ be a partial recursive functional such that $X = \Phi^Y$. Consider the Levin system $V_\sigma = \{\overline{Y} \mid \Phi^{\overline{Y}} \supseteq \sigma\}$. By Lemma 4.2 let c be such that $\mu(V_{X\restriction n}) < 2^{c-f(X\restriction n)}$ for all n. Applying Lemma 4.4 with $r_\sigma = 2^{c-f(\sigma)}$ we obtain a Levin system $\widetilde{V}_\sigma$ such that $\mu(\widetilde{V}_\sigma) \leq 2^{c-f(\sigma)}$ for all σ, and $Y \in V_{X\restriction n} = \widetilde{V}_{X\restriction n}$ for all n. Now suppose X is not strongly f-random relative to Z, say $X \in \bigcap_i [\![A_i^Z]\!]$ where A_i^Z is uniformly Z-r.e. and $\mathrm{pwt}_f(A_i^Z) \leq 2^{-i}$. Let $W_i^Z = \bigcup_{\sigma\in A_i^Z} \widetilde{V}_\sigma$. Clearly W_i^Z is uniformly $\Sigma^{0,Z}_1$. Because $X \in \bigcap_i [\![A_i^Z]\!]$ and $Y \in \bigcap_n V_{X\restriction n} = \bigcap_n \widetilde{V}_{X\restriction n}$, we have $Y \in \bigcap_i W_i^Z$. Let $P_i = \widehat{A}_i^Z = \{\text{minimal elements of } A_i^Z\}$. Because $\widetilde{V}_\sigma$ is a Levin system, we have $W_i^Z = \bigcup_{\sigma \in P_i} \widetilde{V}_\sigma$ and hence

$$\mu(W_i^Z) = \sum_{\sigma \in P_i} \mu(\widetilde{V}_\sigma) \leq \sum_{\sigma \in P_i} 2^{c-f(\sigma)} = 2^c \text{dwt}_f(P_i) \leq 2^c \text{pwt}_f(A_i^Z) \leq 2^{c-i}$$

since P_i is a prefix-free subset of A_i^Z. Thus Y is not Martin-Löf random relative to Z, Q.E.D. □

Remark 4.6. In Theorem 4.5 the assumption "Y is Martin-Löf random relative to Z" cannot be weakened to "Y is strongly f-random relative to Z." For example, define $Z(n) = Y(2n)$ where Y is Martin-Löf random. Then Z is strongly 1/2-random (indeed Martin-Löf random) and Turing reducible to Y, and Y is strongly 1/2-random relative to Z, but of course Z is not strongly 1/2-random relative to Z.

Theorem 4.7. For each $i \in \mathbb{N}$ let $f_i : \{0,1\}^* \to [-\infty, \infty]$ be computable and let $X_i \in \{0,1\}^{\mathbb{N}}$. Suppose $\forall i\,(X_i$ is strongly f_i-random). Then, we can find Z of PA-degree such that $\forall i\,(X_i$ is strongly f_i-random relative to Z).

Proof. By the Kučera/Gács Theorem (see [11, Theorem 8.3.2] or [23, §3.3] or [34, Theorem 3.8]), let Y be Martin-Löf random such that $\forall i\,(X_i$ is Turing reducible to Y). By Theorem 1.3 let Z be of PA-degree such that Y is Martin-Löf random relative to Z. If $\forall i\,(X_i$ is strongly f_i-random), it follows by Theorem 4.5 that $\forall i\,(X_i$ is strongly f_i-random relative to Z). □

Corollary 4.8. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. If X is strongly f-random, then X is strongly f-random relative to some PA-degree.

Proof. Apply Theorem 4.7 with $X_i = X$ and $f_i = f$ for all i. □

Even the following corollary appears to be new.

Corollary 4.9. Suppose $(\forall i \in \mathbb{N})\,(X_i$ is Martin-Löf random). Then, we can find Z of PA-degree such that $(\forall i \in \mathbb{N})\,(X_i$ is Martin-Löf random relative to Z).

Proof. Consider $f : \{0,1\}^* \to [0, \infty)$ where $f(\sigma) = |\sigma|$ for all σ. By Remark 2.10 X_i is Martin-Löf random if and only if X_i is strongly f-random, and similarly X_i is Martin-Löf random relative to Z if and only if X_i is strongly f-random relative to Z. Apply Theorem 4.7 with $f_i = f$ for all i. □

We end this section by presenting a kind of Borel/Cantelli Lemma for strong f-randomness. Let us say that X is <u>strongly BC-f-random</u> if $\{i \mid X \in [\![A_i]\!]\}$ is finite whenever A_i is uniformly r.e. and $\sum_i \text{pwt}_f(A_i) < \infty$. This notion resembles a generalization of Tadaki's earlier notion of Solovay D-randomness [39, Definition 3.8].

Theorem 4.10. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. Suppose X is strongly f-random and Turing reducible to Y where Y is Martin-Löf random relative to Z. Then X is strongly BC-f-random relative to Z.

Proof. Suppose X is not strongly BC-f-random relative to Z. Let A_i^Z be uniformly Z-r.e. such that $\sum_i \mathrm{pwt}_f(A_i^Z) < \infty$ and $X \in [\![A_i^Z]\!]$ for infinitely many i. Let V_σ, c, $\widetilde{V}_\sigma$, W_i^Z, P_i be as in the proof of Theorem 4.5. For all i we have $\mu(W_i^Z) \leq 2^c \mathrm{pwt}_f(A_i^Z)$, hence $\sum_i \mu(W_i^Z) \leq 2^c \sum_i \mathrm{pwt}_f(A_i^Z) < \infty$. On the other hand, for all i such that $X \in [\![A_i^Z]\!]$ we have $Y \in W_i^Z$, so $Y \in W_i^Z$ for infinitely many i. Relativizing Solovay's Lemma [34, Lemma 3.5] to Z, we see that Y is not Martin-Löf random relative to Z, Q.E.D. □

Theorem 4.11. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. If X is strongly f-random, then X is strongly BC-f-random relative to some PA-degree.

Proof. By the Kučera/Gács Theorem, let Y be Martin-Löf random such that X is Turing reducible to Y. By Theorem 1.3 let Z be of PA-degree such that Y is Martin-Löf random relative to Z. If X is strongly f-random, Theorem 4.10 tells us that X is strongly BC-f-random relative to Z, Q.E.D. □

Corollary 4.12. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. Then strong f-randomness is equivalent to strong BC-f-randomness.

Proof. Trivially strong BC-f-randomness implies strong f-randomness. The converse is immediate from Theorem 4.11. □

Remark 4.13. It is possible to give a direct proof of Corollary 4.12 resembling the standard proof of Solovay's Lemma [34, Lemma 3.5].

5 Propagation of non-K-triviality

Recall from [11, 23] that X is <u>LR-reducible</u> to Z, abbreviated $X \leq_{\mathrm{LR}} Z$, if $\forall Y$ ((Y Martin-Löf random relative to Z) $\Rightarrow$ (Y Martin-Löf random relative to X)). The concept of LR-recucibility has been very useful [20, 34, 35] in the reverse mathematics of measure-theoretic regularity. It is also known (see [11, Chapter 11] or [23, Chapter 5]) that LR-reducibility can be used to characterize <u>K-triviality</u>. Namely, X is K-trivial if and only if $X \leq_{\mathrm{LR}} 0$.

From our point of view in this paper, it seems reasonable to view non-K-triviality as a kind of partial randomness notion. Accordingly, we now present appropriate analogs of our main propagation results, Theorems 4.5 and 4.7. Our results in this section are easy consequences of previously known characterizatons of K-triviality.

Theorem 5.1. Suppose X is Turing reducible to Y where Y is Martin-Löf random relative to Z. Then $X \not\leq_{\mathrm{LR}} 0$ implies $X \not\leq_{\mathrm{LR}} Z$.

Proof. Since $X \not\leq_{\mathrm{LR}} 0$, it follows by [11, Chapter 11] or [23, Chapter 5] that X is not a *base for Martin-Löf randomness*. In particular, since X is Turing reducible to Y, Y is not Martin-Löf random relative to X. But then, since Y is Martin-Löf random relative to Z, we have $X \not\leq_{\mathrm{LR}} Z$, Q.E.D. □

Theorem 5.2. Suppose $X_i \not\leq_{\mathrm{LR}} 0$ for all $i \in \mathbb{N}$. Then, we can find Z of PA-degree such that $X_i \not\leq_{\mathrm{LR}} Z$ for all $i \in \mathbb{N}$.

Proof. For each i let Y_i be Martin-Löf random but not Martin-Löf random relative to X_i. By Corollary 4.9 let Z be of PA-degree such that $\forall i\,(Y_i$ is Martin-Löf random relative to $Z)$. It follows that $\forall i\,(X_i \not\leq_{\mathrm{LR}} Z)$, Q.E.D. ☐

Remark 5.3. In Theorems 5.1 and 5.2, the conclusion $X \not\leq_{\mathrm{LR}} Z$ implies that $X \oplus Z \not\leq_{\mathrm{LR}} Z$, i.e., X is not K-trivial relative to Z. On the other hand, results such as Theorems 1.3 and 4.7 and 5.2 bear an obvious resemblance to the well known *GKT Theorem* (see Gandy/Kreisel/Tait [13] or [18, Theorem 2.5] or [31, Theorem VIII.2.24]). Indeed, Theorem 5.2 is just the GKT Theorem with Turing reducibility replaced by LR-reducibility.

6 Propagation of diagonal nonrecursiveness

Let $\{n\}$ denote the partial recursive functional with index n. Let DNR be the set of functions $f : \mathbb{N} \to \mathbb{N}$ which are diagonally nonrecursive, i.e., $f(n) \neq \{n\}(n)$ for all n. Known results concerning diagonal nonrecursiveness may be found in [1, 17, 19, 32]. We also consider relative DNR-ness: $\mathrm{DNR}^Z = \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n\,(f(n) \neq \{n\}^Z(n))\}$. The purpose of this section is to obtain propagation results for diagonal nonrecursiveness.

Theorem 6.1. Suppose there exists a DNR function which is Turing reducible to X. Suppose also that X is Turing reducible to Y where Y is Martin-Löf random relative to Z. Then there exists a DNR^Z function which is Turing reducible to X.

In order to prove Theorem 6.1 we need the following lemma, which is a variant of the Parametrized Recursion Theorem. In stating and proving our lemma, we shall use standard recursion-theoretic notation. In particular, for any expression E we write $E{\downarrow}$ to mean that E is defined, and $E{\uparrow}$ to mean that E is undefined. We also write $E_1 \simeq E_2$ to mean that either ($E_1{\downarrow}$ and $E_2{\downarrow}$ and $E_1 = E_2$) or ($E_1{\uparrow}$ and $E_2{\uparrow}$). Via Gödel numbering, we identify finite sequences of positive integers with positive integers. We write $f \leq_{\mathrm{T}} X$ to mean that f is Turing reducible to X.

Lemma 6.2. Let $\Theta(n, j, \sigma, -)$ be a partial recursive functional. Then, we can find a primitive recursive function $p(n, j)$ such that

$$\{p(n, j)\}(-) \;\simeq\; \Theta(n, j, \langle p(n, i) \mid i \leq j\rangle, -)$$

for all $n, j, -$.

Proof. By the Parametrized Recursion Theorem, let q be a primitive recursive function such that $\{q(n, j, \sigma)\}(-) \simeq \Theta(n, j, \sigma^\frown\langle q(n, j, \sigma)\rangle, -)$ for all $n, j, \sigma, -$.

Define p primitive recursively by letting $p(n,j) = q(n,i,\langle p(n,i) \mid i < j\rangle)$ for all n,j. We then have

$$\begin{aligned}
\{p(n,j)\}(-) &\simeq \{q(n,j,\langle p(n,i) \mid i<j\rangle)\}(-) \\
&\simeq \Theta(n,j,\langle p(n,i) \mid i<j\rangle^\frown\langle q(n,j,\langle p(n,i) \mid i<j\rangle)\rangle,-) \\
&\simeq \Theta(n,j,\langle p(n,i) \mid i<j\rangle^\frown\langle p(n,j)\rangle,-) \\
&\simeq \Theta(n,j,\langle p(n,i) \mid i\le j\rangle,-)
\end{aligned}$$

and this proves our lemma. □

We now prove Theorem 6.1.

Proof of Theorem 6.1. Let $f \in \mathbb{N}^{\mathbb{N}}$ be DNR and $\le_{\mathrm{T}} X$. Then $f \le_{\mathrm{T}} Y$ so let Φ be a partial recursive functional such that $f = \Phi^Y$, i.e., $f(n) = \Phi(Y,n)$ for all n. As in §4 let μ be the fair-coin probability measure on $\{0,1\}^{\mathbb{N}}$. Define a partial recursive function $\theta(n,j,\sigma) \simeq$ some m such that $\mu(\{\overline{Y} \mid \Phi^{\overline{Y}}(\sigma(j))\downarrow = m$ and $(\forall i<j)\,(\Phi^{\overline{Y}}(\sigma(i))\downarrow \neq \{\sigma(i)\}(\sigma(i))\downarrow)\}) > 2^{-n}$. Apply Lemma 6.2 to obtain a primitive recursive function $p(n,j)$ such that $\{p(n,j)\}(p(n,j)) \simeq$ some m such that $\mu(V_{n,j,m}) > 2^{-n}$ where $V_{n,j,m} = \{\overline{Y} \mid \Phi^{\overline{Y}}(p(n,j))\downarrow = m$ and $(\forall i<j)\,(\Phi^{\overline{Y}}(p(n,i))\downarrow \neq \{p(n,i)\}(p(n,i))\downarrow)\}$. Thus $\{p(n,j)\}(p(n,j))\downarrow$ implies $\mu(V_{n,j}) > 2^{-n}$ where $V_{n,j} = V_{n,j,\{p(n,j)\}(p(n,j))}$. On the other hand, $i \neq j$ implies $V_{n,i} \cap V_{n,j} = \emptyset$ so for each n there is at least one $j \le 2^n$ such that $\{p(n,j)\}(p(n,j))\uparrow$.

Let Ψ be a partial recursive functional defined by

$$\Psi^{\overline{Y}}(n) \simeq \langle \Phi^{\overline{Y}}(p(n,i)) \mid i \le 2^n\rangle$$

for all n. In particular we have $g \in \mathbb{N}^{\mathbb{N}}$ defined by

$$g(n) = \Psi^Y(n) = \langle f(p(n,i)) \mid i \le 2^n\rangle$$

for all n. Clearly $g \le_{\mathrm{T}} f \le_{\mathrm{T}} X$, so it will suffice to prove that $g(n) \neq \{n\}^Z(n)$ for all but finitely many n.

Let $U_n^Z = \{\overline{Y} \mid \Psi^{\overline{Y}}(n)\downarrow = \{n\}^Z(n)\downarrow\}$. Clearly U_n^Z is uniformly $\Sigma_1^{0,Z}$. Given a rational number r, let $U_n^Z[r]$ be U_n^Z enumerated so long as its μ-measure is $\le r$. Thus $U_n^Z[r]$ is uniformly $\Sigma_1^{0,Z}$ and $\mu(U_n^Z[r]) \le r$. Moreover, $U_n^Z[r] = U_n^Z$ if and only if $\mu(U_n^Z) \le r$. Since Y is Martin-Löf random, it follows by Solovay's Lemma [34, Lemma 3.5] that $Y \notin U_n^Z[2^{-n}]$ for all but finitely many n. Therefore, it will suffice to prove $g(n) \neq \{n\}^Z(n)$ for all such n.

Supposing otherwise, we would have $\Psi^Y(n) = g(n) = \{n\}^Z(n)$, hence $Y \in U_n^Z$, hence $\mu(U_n^Z) > 2^{-n}$. Moreover, for all $\overline{Y} \in U_n^Z$ we would have $\Psi^{\overline{Y}}(n) = \{n\}^Z(n) = g(n)$, hence $\Phi^{\overline{Y}}(p(n,i)) = f(p(n,i)) \neq \{p(n,i)\}(p(n,i))$ for all $i \le 2^n$. Let $j \le 2^n$ be such that $(\forall i<j)\,(\{p(n,i)\}(p(n,i))\downarrow)$. Then $U_n^Z \subseteq V_{n,j,f(p(n,j))}$, hence $\mu(V_{n,j,f(p(n,j))}) \ge \mu(U_n^Z) > 2^{-n}$, hence $\{p(n,j)\}(p(n,j))\downarrow$, so by induction on j we see that $\{p(n,j)\}(p(n,j))\downarrow$ holds for all $j \le 2^n$. This contradiction completes the proof. □

Theorem 6.3. Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If $(\forall i \in \mathbb{N})\,(\exists f \in \mathrm{DNR})\,(f \leq_{\mathrm{T}} X_i)$, then $(\exists Z \in Q)\,(\forall i \in \mathbb{N})\,(\exists g \in \mathrm{DNR}^Z)\,(g \leq_{\mathrm{T}} X_i)$.

Proof. By the Kučera/Gács Theorem (see [11, Theorem 8.3.2] or [23, §3.3] or [34, Theorem 3.8]), let Y be Martin-Löf random such that $\forall i\,(X_i \leq_{\mathrm{T}} Y)$. By Theorem 1.2 let $Z \in Q$ be such that Y is Martin-Löf random relative to Z. If $\forall i\,\exists f\,(f \in \mathrm{DNR}$ and $f \leq_{\mathrm{T}} X_i)$, it follows by Theorem 6.1 that $\forall i\,\exists g\,(g \in \mathrm{DNR}^Z$ and $g \leq_{\mathrm{T}} X_i)$. □

Corollary 6.4. Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If there exists a DNR function which is Turing reducible to X, then for some $Z \in Q$ there exists a DNR^Z function which is Turing reducible to X.

Proof. This is the special case of Theorem 6.3 with $X_i = X$ for all $i \in \mathbb{N}$. □

Theorem 6.5. Suppose $(\forall i \in \mathbb{N})\,(\exists f \in \mathrm{DNR})\,(f \leq_{\mathrm{T}} X_i)$. Then, there exists Z of PA-degree such that $(\forall i \in \mathbb{N})\,(\exists g \in \mathrm{DNR}^Z)\,(g \leq_{\mathrm{T}} X_i)$.

Proof. In Theorem 6.3 let Q be the Π^0_1 set consisting of all completions of first-order Peano arithmetic. □

Corollary 6.6. If there exists a DNR function which is Turing reducible to X, then for some Z of PA-degree there exists a DNR^Z function which is Turing reducible to X.

Proof. In Corollary 6.4 let Q be the Π^0_1 set consisting of all completions of first-order Peano arithmetic. □

Remark 6.7. As in [32, §10] and [36, §2.2], let C be a "nice" class of recursive functions. For example, C could be the class of *all* recursive functions, or the class of primitive recursive functions, or the class of recursive functions up to level α of the transfinite Ackermann hierarchy for some limit ordinal $\alpha \leq \varepsilon_0$. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be <u>C-bounded</u> if $(\exists F \in C)\,\forall n\,(f(n) < F(n))$. In particular, f is <u>recursively bounded</u> if it is C-bounded where $C =$ the class of all recursive functions. Our proofs above show that Theorems 6.1 and 6.3 and 6.5 and Corollaries 6.4 and 6.6 also hold with "DNR" replaced by "C-bounded DNR." It suffices to note that, in our proof of Theorem 6.1, if f is C-bounded then so is g. See also the refinements mentioned in Remarks 7.8 and 7.9 below.

We end this section by presenting an alternative proof of Corollary 6.4.

Alternative proof of Corollary 6.4. Let $\mathbb{N}^*$ be the set of finite sequences of positive integers. For each $\sigma \in \mathbb{N}^*$ let

$$Q_\sigma = \{Z \in Q \mid (\forall n < |\sigma|)\,(\sigma(n) \neq \{n\}^Z(n))\}$$

where $|\sigma| =$ the length of σ. Clearly $\sigma \subseteq \tau$ implies $Q_\sigma \supseteq Q_\tau$. By the Parametrization or S-m-n Theorem, let $p(n, \sigma)$ be a primitive recursive function such that for all m, $\{p(n,\sigma)\}(p(n,\sigma)) = m$ if and only if $\{n\}^Z(n) = m$ for all $Z \in Q_\sigma$. Let $f \leq_T X$ be a DNR function. Define $g \leq_T X$ recursively by letting $g(n) = f(p(n, \langle g(i) \mid i < n\rangle))$ for all n. We are going to show that g is DNR relative to some $Z \in Q$.

We claim that $Q_{\langle g(i)|i<n\rangle} \neq \emptyset$ for all n. To begin with, we have $Q_{\langle\rangle} = Q \neq \emptyset$. Assume inductively that $Q_{\langle g(i)|i<n\rangle} \neq \emptyset$. We shall prove that $Q_{\langle g(i)|i\leq n\rangle} \neq \emptyset$. There are two cases. If $\{p(n, \langle g(i) \mid i < n\rangle)\}(p(n, \langle g(i) \mid i < n\rangle)) = m$, we have $\{n\}^Z(n) = m$ for all $Z \in Q_{\langle g(i)|i<n\rangle}$, but $g(n) = f(p(n, \langle g(i) \mid i < n\rangle)) \neq m$ since f is DNR. Thus $Q_{\langle g(i)|i\leq n\rangle} = Q_{\langle g(i)|i<n\rangle} \neq \emptyset$. If $\{p(n, \langle g(i) \mid i < n\rangle)\}(p(n, \langle g(i) \mid i < n\rangle))$ is undefined, there exists $Z \in Q_{\langle g(i)|i<n\rangle}$ such that $\{n\}^Z(n) \neq g(n)$, and then Z belongs to $Q_{\langle g(i)|i\leq n\rangle}$. This proves our claim.

By compactness, our claim implies that $\bigcap_{n=0}^{\infty} Q_{\langle g(i)|i<n\rangle} \neq \emptyset$. Moreover, from the definition of $Q_{\langle g(i)|i<n\rangle}$ we see that g is DNR relative to any $Z \in \bigcap_{n=0}^{\infty} Q_{\langle g(i)|i<n\rangle}$. This completes the proof. ☐

Remark 6.8. Our alternative proof of Corollary 6.4 is more constructive than the previous proof via Theorem 6.1 and the Kučera/Gács Theorem. In particular, the alternative proof can be formalized in WKL_0 (see [31]) while the previous proof cannot.

There are some issues here which are interesting from the viewpoint of reverse mathematics [31]. For example, consider the following statement.

> Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If X_1 and X_2 are Martin-Löf random, there exists $Z \in Q$ such that X_1 and X_2 are Martin-Löf random relative to Z.

By Corollary 4.9 this statement is true, and from the truth of the statement it follows easily that the statement is true in all ω-models of WKL_0. Moreover, we conjecture that the statement is provable in WKL_0. On the other hand, by [2, Theorem 2.1] together with [38], the following special case of the Kučera/Gács Theorem is false in all ω-models of WKL_0 except those which contain $0^{(1)}$ = the Turing jump of 0.

> If X_1 and X_2 are Martin-Löf random, there exists a Martin-Löf random Y such that $X_1 \leq_{\mathrm{T}} Y$ and $X_2 \leq_{\mathrm{T}} Y$.

7 Propagation of autocomplexity

In this section we prove propagation results for *autocomplexity* and *complexity*. We also obtain a characterization of autocomplexity in terms of f-randomness and strong f-randomness. With this characterization plus [19, Theorem 2.3], we see considerable overlap between the propagation results of this section and those of §4 and §6.

Definition 7.1.

1. Following [19] we define $X \in \{0,1\}^{\mathbb{N}}$ to be <u>autocomplex</u> if there exists an unbounded function $h : \mathbb{N} \to \mathbb{N}$ such that $h \leq_{\mathrm{T}} X$ and $h(n) \leq \mathrm{KS}(X\restriction n)$ for all n. Here KS denotes <u>simple Kolmogorov complexity</u> [40], also known as <u>plain complexity</u> [11, 23].

2. Following [4] and [15] and [19], we define $X \in \{0,1\}^{\mathbb{N}}$ to be complex if there exists an unbounded computable function $h : \mathbb{N} \to \mathbb{N}$ such that $h(n) \le \mathrm{KS}(X{\restriction}n)$ for all n.

Remark 7.2. By [40, §4.3.1] there exist constants c_1 and c_2 such that $\mathrm{KS}(\sigma) \le \mathrm{KP}(\sigma) + c_1 \le \mathrm{KS}(\sigma) + 3\log_2|\sigma| + c_2$ and $\mathrm{KA}(\sigma) \le \mathrm{KP}(\sigma) + c_1 \le \mathrm{KA}(\sigma) + 3\log_2|\sigma| + c_2$ for all $\sigma \in \{0,1\}^*$. These inequalities imply that the distinctions among KS and KP and KA are immaterial for some purposes. In particular, we can replace KS in Definition 7.1 by KP or KA.

We begin with autocomplexity.

Theorem 7.3. The following are pairwise equivalent.

1. X is autocomplex.

2. X is f-random for some computable $f : \{0,1\}^* \to \mathbb{N}$ such that $\{f(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded.

3. X is strongly f-random for some computable $f : \{0,1\}^* \to \mathbb{N}$ such that $\{f(X{\restriction}n) \mid n \in \mathbb{N}\}$ is unbounded.

Proof. The equivalence $2 \Leftrightarrow 3$ is clear in view of Remark 7.2.

To prove $2 \Rightarrow 1$, suppose 2 holds via f. By Theorem 2.6 X is f-complex, so let $c \in \mathbb{N}$ be such that $\mathrm{KP}(X{\restriction}n) \ge f(X{\restriction}n) - c$ for all n. Then for all n we have $\mathrm{KP}(X{\restriction}n) \ge h(n)$ where $h(n) = \max(1, f(X{\restriction}n) - c)$. Clearly $h \le_{\mathrm{T}} X$ (in fact h is Lipschitz computable from X) and h is unbounded, so it follows by Remark 7.2 that X is autocomplex, i.e., 1 holds.

It remains to prove $1 \Rightarrow 2$. Suppose X is autocomplex. By Remark 7.2 let $h : \mathbb{N} \to \mathbb{N}$ be unbounded such that $h \le_{\mathrm{T}} X$ and $h(n) \le \mathrm{KP}(X{\restriction}n)$ for all n. Let Φ be a partial recursive functional such that $h = \Phi^X$. Consider the primitive recursive function $f : \{0,1\}^* \to \mathbb{N}$ defined by $f(\sigma) = \max\{p(\sigma,n) \mid n \le |\sigma|\}$ where $p(\sigma,n) = \Phi^\sigma_{|\sigma|}(n)$ if $\Phi^\sigma_{|\sigma|}(n)\!\downarrow$, otherwise $p(\sigma,n) = 1$. Then for all n and all sufficiently large $m \ge n$ we have $h(n) = p(X{\restriction}m, n) \le f(X{\restriction}m)$. Since $\{h(n) \mid n \in \mathbb{N}\}$ is unbounded, it follows that $\{f(X{\restriction}m) \mid m \in \mathbb{N}\}$ is unbounded. Consider the primitive recursive function $q(\sigma) =$ the least $n \le |\sigma|$ such that $f(\sigma) = p(\sigma,n)$. Let c be a constant such that $\mathrm{KP}(\sigma{\restriction}q(\sigma)) \le \mathrm{KP}(\sigma) + c$ for all σ. Then for all m we have $\mathrm{KP}(X{\restriction}m) + c \ge \mathrm{KP}(X{\restriction}q(X{\restriction}m)) \ge h(X{\restriction}q(X{\restriction}m)) \ge p(X{\restriction}m, q(X{\restriction}m)) = f(X{\restriction}m)$ so X is f-complex. It follows by Theorem 2.6 that X is f-random. This completes the proof. ∎

Theorem 7.4.

1. If X is autocomplex and $\le_{\mathrm{T}} Y$ where Y is Martin-Löf random relative to Z, then X is autocomplex relative to Z.

2. If $(\forall i \in \mathbb{N})\,(X_i$ is autocomplex$)$, there exists Z of PA-degree such that $(\forall i \in \mathbb{N})\,(X_i$ is autocomplex relative to $Z)$.

First proof. Part 1 is immediate from Theorems 4.5 and 7.3. Part 2 is immediate from Theorems 4.7 and 7.3. □

Second proof. By Kjos-Hanssen/Merkle/Stephan [19, Theorem 2.3] we know that X is autocomplex if and only if there exists a DNR function which is Turing reducible to X. Modulo this result, parts 1 and 2 are equivalent to Theorems 6.1 and 6.5 respectively. □

Remark 7.5. Yet another proof of Theorem 7.4 was obtained independently by Bienvenu [3] who had seen it conjectured in an earlier draft of the present paper. The earlier draft included Theorems 4.5 and 4.7, as well as Corollary 6.4 with our alternative proof, but it did not include Theorem 6.1 or 6.5 or 7.3.

We now turn to propagation results for complexity. Let us define $f : \{0,1\}^* \to [-\infty, \infty]$ to be length-invariant if $\forall\sigma\,\forall\tau\,(|\sigma| = |\tau| \Rightarrow f(\sigma) = f(\tau))$.

Theorem 7.6. The following are pairwise equivalent.

1. X is complex.
2. X is f-random for some computable $f : \{0,1\}^* \to \mathbb{N}$ which is unbounded and length-invariant.
3. X is strongly f-random for some computable $f : \{0,1\}^* \to \mathbb{N}$ which is unbounded and length-invariant.

Proof. This is immediate from Theorems 2.6 and 2.8 and Remark 7.2. □

Theorem 7.7.

1. If X is complex and $\leq_{\mathrm{T}} Y$ where Y is Martin-Löf random relative to Z, then X is complex relative to Z.
2. If $(\forall i \in \mathbb{N})\,(X_i$ is complex), there exists Z of PA-degree such that $(\forall i \in \mathbb{N})\,(X_i$ is complex relative to $Z)$.

Proof. Part 1 is immediate from Theorems 4.5 and 7.6. Part 2 is immediate from Theorems 4.7 and 7.6. □

Remark 7.8. By [19, Theorem 2.3] we know that X is complex if and only if some DNR function is truth-table reducible to X. Consequently, the Turing degrees of complex X's are the same as the Turing degrees of recursively bounded DNR functions. And of course, the Turing degrees of autocomplex X's are the same as the Turing degrees of DNR functions. Thus Theorems 4.5 and 4.7 may be viewed as far-reaching refinements not only of Theorems 7.4 and 7.7 but also of Theorems 6.1–6.5 and Remark 6.7.

Remark 7.9. By [1, Theorem 1.8] there exists an autocomplex X such that no complex Y is Turing reducible to X. Within the class of complex X's, much more refined results of the same kind have been obtained by Hudelson [16] generalizing the main result of Miller [21, Theorem 4.1]. See also Remarks 3.7 and 3.8 above, as well as [36, §§2.1–2.3, Figure 1].

8 Vehement f-randomness

In this section we define vehement f-randomness and discuss its relationship with strong f-randomness. The notion of vehement f-randomness was originally introduced by Kjos-Hanssen (unpublished, but see [26]). We prove that, under a convexity hypothesis on f, vehement f-randomness is equivalent to strong f-randomness. Our result is a generalization of known results due to Reimann [26, Corollary 21] and Miller [21, Lemma 3.3].

Definition 8.1. Given $f : \{0,1\}^* \to [-\infty, \infty]$, the vehement f-weight of $A \subseteq \{0,1\}^*$ is defined as $\mathrm{vwt}_f(A) = \inf\{\mathrm{dwt}_f(S) \mid \llbracket A \rrbracket \subseteq \llbracket S \rrbracket\}$.

Remark 8.2. Note that $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ implies $\mathrm{vwt}_f(A) \leq \mathrm{vwt}_f(B)$. In particular, $\mathrm{vwt}_f(A)$ depends only on $\llbracket A \rrbracket$.

Lemma 8.3. For all A we have $\mathrm{vwt}_f(A) \leq \mathrm{dwt}_f(\widehat{A}) \leq \mathrm{pwt}_f(A)$.

Proof. The first inequality holds because $\llbracket A \rrbracket \subseteq \llbracket \widehat{A} \rrbracket$. The second inequality holds because $\widehat{A}$ is a prefix-free subset of A. □

Definition 8.4. Fix $f : \{0,1\}^* \to [-\infty, \infty]$. A good cover of A is a set B such that $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ and $\mathrm{pwt}_f(B) \leq \mathrm{vwt}_f(A)$. It follows by Remark 8.2 and Lemma 8.3 that $\mathrm{vwt}_f(A) = \mathrm{vwt}_f(B) = \mathrm{dwt}_f(\widehat{B}) = \mathrm{pwt}_f(B)$.

Lemma 8.5. Suppose B is a good cover of A. Given $F \subseteq \widehat{B}$ let us write $A_F = \{\sigma \in A \mid \llbracket F \rrbracket \supseteq \llbracket \sigma \rrbracket\}$ and $B_F = \{\tau \in B \mid \llbracket F \rrbracket \supseteq \llbracket \tau \rrbracket\}$. Then B_F is a good cover of A_F.

Proof. Clearly $\widehat{B_F} = F$, hence $\llbracket A_F \rrbracket \subseteq \llbracket F \rrbracket = \llbracket \widehat{B_F} \rrbracket = \llbracket B_F \rrbracket$. In order to show that B_F is a good cover of A_F, it remains to show that $\mathrm{dwt}_f(P) \leq \mathrm{dwt}_f(S)$ whenever $P \subseteq B_F$ is prefix-free and $\llbracket A_F \rrbracket \subseteq \llbracket S \rrbracket$. Letting $G = \widehat{B} \setminus F$ we see that $P \cap G = \emptyset$ and $P \cup G$ is a prefix-free subset of B and $\llbracket A \rrbracket = \llbracket A_F \rrbracket \cup \llbracket A_G \rrbracket \subseteq \llbracket S \rrbracket \cup \llbracket G \rrbracket = \llbracket S \cup G \rrbracket$. Thus $\mathrm{dwt}_f(P) + \mathrm{dwt}_f(G) = \mathrm{dwt}_f(P \cup G) \leq \mathrm{pwt}_f(B) \leq \mathrm{vwt}_f(A) \leq \mathrm{dwt}_f(S \cup G) \leq \mathrm{dwt}_f(S) + \mathrm{dwt}_f(G)$, hence $\mathrm{dwt}_f(P) \leq \mathrm{dwt}_f(S)$, Q.E.D. □

Remark 8.6. Let B be a good cover of A, and suppose τ is such that $\llbracket B \rrbracket \not\supseteq \llbracket \tau \rrbracket$. Then obviously no initial segment of τ belongs to B. In other words, $\tau \in \widehat{B \cup \{\tau\}}$. Letting $F = \widehat{B \cup \{\tau\}} \setminus \{\tau\}$ and applying Lemma 8.5, we see that $\widehat{B_F} = F$ and B_F is a good cover of A_F.

Definition 8.7. We define $f : \{0,1\}^* \to [-\infty, \infty]$ to be convex if $\mathrm{wt}_f(\sigma) \leq \mathrm{wt}_f(\sigma^\frown\langle 0\rangle) + \mathrm{wt}_f(\sigma^\frown\langle 1\rangle)$ for all $\sigma \in \{0,1\}^*$. Equivalently, $\mathrm{wt}_f(\sigma) \leq \mathrm{dwt}_f(S)$ for all $\sigma \in \{0,1\}^*$ and all $S \subseteq \{0,1\}^*$ such that $\llbracket S \rrbracket = \llbracket \sigma \rrbracket$.

Lemma 8.8. Assume that f is convex. Suppose B is a good cover of A but not of $A' = A \cup \{\sigma\}$. Choose $\tau \subseteq \sigma$ so as to minimize $\mathrm{dwt}_f(\widehat{B \cup \{\tau\}})$. Then $B' = B \cup \{\tau\}$ is a good cover of A'.

Proof. Obviously $[\![B']\!] \supseteq [\![A']\!]$ so it remains to prove that $\mathrm{dwt}_f(P') \le \mathrm{dwt}_f(S')$ whenever $P' \subseteq B'$ is prefix-free and $[\![A']\!] \subseteq [\![S']\!]$.

Since $[\![\sigma]\!] \subseteq [\![A']\!] \subseteq [\![S']\!]$, let $\tau^* \subseteq \sigma$ be as short as possible such that $[\![\tau^*]\!] \subseteq [\![S']\!]$. Obviously S' contains no proper initial segment of τ^*. Hence $[\![\tau^*]\!] = [\![S^*]\!]$ for some $S^* \subseteq S'$. It follows by Definition 8.7 that $\mathrm{wt}_f(\tau^*) \le \mathrm{dwt}_f(S^*)$. Therefore, replacing S' by $(S' \setminus S^*) \cup \{\tau^*\}$, we may safely assume that $\tau^* \in S'$.

Since $[\![\sigma]\!] \not\subseteq [\![B]\!]$ and $\tau \subseteq \sigma$ and $\tau^* \subseteq \sigma$, we obviously have $[\![\tau]\!] \not\subseteq [\![B]\!]$ and $[\![\tau^*]\!] \not\subseteq [\![B]\!]$. Applying Remark 8.6 to τ and to τ^*, we obtain sets $F = \widehat{B \cup \{\tau\}} \setminus \{\tau\}$ and $F^* = \widehat{B \cup \{\tau^*\}} \setminus \{\tau^*\}$. In particular, since $[\![A]\!] \subseteq [\![S']\!]$ we have $[\![A_{F^*}]\!] = [\![A]\!] \setminus [\![\tau^*]\!] \subseteq [\![S']\!] \setminus [\![\tau^*]\!] \subseteq [\![S' \setminus \{\tau^*\}]\!]$. Moreover, by our choice of τ we have $\mathrm{dwt}_f(\widehat{B \cup \{\tau\}}) \le \mathrm{dwt}_f(\widehat{B \cup \{\tau^*\}})$.

We are now ready to complete the proof of Lemma 8.8. If $\tau \notin P'$ we have $P' = P \subseteq B$, hence $\mathrm{dwt}_f(P) \le \mathrm{pwt}_f(B) \le \mathrm{vwt}_f(A) \le \mathrm{dwt}_f(S')$ and we are done. Suppose now that $\tau \in P'$. Then $P' = P \cup \{\tau\}$ where $P \subseteq B_F$. Thus we have

$$\begin{aligned}
\mathrm{dwt}_f(P') &= \mathrm{dwt}_f(P) + \mathrm{wt}_f(\tau)\\
&\le \mathrm{pwt}_f(B_F) + \mathrm{wt}_f(\tau)\\
&= \mathrm{dwt}_f(F) + \mathrm{wt}_f(\tau)\\
&= \mathrm{dwt}_f(\widehat{B \cup \{\tau\}})\\
&\le \mathrm{dwt}_f(\widehat{B \cup \{\tau^*\}})\\
&= \mathrm{dwt}_f(F^*) + \mathrm{wt}_f(\tau^*)\\
&= \mathrm{vwt}_f(A_{F^*}) + \mathrm{wt}_f(\tau^*)\\
&\le \mathrm{dwt}_f(S' \setminus \{\tau^*\}) + \mathrm{wt}_f(\tau^*)\\
&= \mathrm{dwt}_f(S')
\end{aligned}$$

and again we are done. □

Definition 8.9. Given $f : \{0,1\}^* \to [-\infty, \infty]$ define

$$L_f : \{(P_1, P_2) \mid P_1, P_2 \text{ are finite and prefix-free}\} \to \{0,1\}$$

by

$$L_f(P_1, P_2) = \begin{cases} 1 & \text{if } \mathrm{dwt}_f(P_1) < \mathrm{dwt}_f(P_2),\\ 0 & \text{otherwise.} \end{cases}$$

We say that f is <u>strongly computable</u> if both f and L_f are computable. This is often the case, e.g., if f is computable and integer-valued as in Lemma 3.2. Note also that Lemma 3.3 depends only on strong computability.

Lemma 8.10. Let f be strongly computable and convex. If A is r.e., we can effecively find an r.e. set B such that B is a good cover of A.

Proof. For $n = 0, 1, 2, \ldots$ let A_n consist of the first n elements in some fixed recursive enumeration of A. Assume inductively that we have found a finite set B_n which is a good cover of A_n. Let $A_{n+1} = A_n \cup \{\sigma_n\}$. If $[\![\sigma_n]\!] \subseteq [\![B_n]\!]$ let $B_{n+1} = B_n$. Otherwise, use strong computability to effectively choose $\tau_n \subseteq \sigma_n$ which minimizes $\mathrm{dwt}_f(\widehat{B_n \cup \{\tau_n\}})$. Lemma 8.8 then implies that $B_{n+1} = B_n \cup \{\tau_n\}$ is a good cover of A_{n+1}. Finally let $B = \bigcup_{n=1}^{\infty} B_n$. Clearly B is r.e. and $[\![A]\!] \subseteq [\![B]\!]$, so it remains to prove that $\mathrm{dwt}_f(P) \leq \mathrm{vwt}_f(A)$ for all prefix-free sets $P \subseteq B$. But clearly $\mathrm{dwt}_f(P) = \sup\{\mathrm{dwt}_f(P_0) \mid P_0 \text{ is a finite subset of } P\}$, so it suffices to consider finite prefix-free sets. If $P \subseteq B$ is finite and prefix-free, let n be such that $P \subseteq B_n$. Then $\mathrm{dwt}_f(P) \leq \mathrm{pwt}_f(B_n) \leq \mathrm{vwt}_f(A_n) \leq \mathrm{vwt}_f(A)$, Q.E.D. □

Lemma 8.11. Let f be strongly computable and convex. If A is r.e., we can effecively find an r.e. set B such that $[\![A]\!] \subseteq [\![B]\!]$ and $\mathrm{pwt}_f(B) \leq \mathrm{vwt}_f(A)$.

Proof. This is a restatement of Lemma 8.10. □

Definition 8.12. Assume that $f : \{0,1\}^* \to [-\infty, \infty]$ is computable. We define $X \in \{0,1\}^*$ to be <u>vehemently f-random</u> if $X \notin \bigcap_i [\![A_i]\!]$ whenever A_i is uniformly r.e. such that $\mathrm{vwt}_f(A_i) \leq 2^{-i}$.

Theorem 8.13. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be strongly computable and convex. Then vehement f-randomness is equivalent to strong f-randomness.

Proof. Suppose X is not strongly f-random, say $X \in \bigcap_i [\![A_i]\!]$ where A_i is uniformly r.e. and $\mathrm{pwt}_f(A_i) \leq 2^{-i}$. By Lemma 8.3 we have $\mathrm{vwt}_f(A_i) \leq \mathrm{pwt}_f(A_i) \leq 2^{-i}$ so X is not vehemently f-random.

Now suppose X is not vehemently f-random, say $X \in \bigcap_i [\![A_i]\!]$ where A_i is uniformly r.e. and $\mathrm{vwt}_f(A_i) \leq 2^{-i}$. By Lemma 8.11 we can find uniformly r.e. B_i such that $[\![A_i]\!] \subseteq [\![B_i]\!]$ and $\mathrm{pwt}_f(B_i) \leq \mathrm{vwt}_f(A_i) \leq 2^{-i}$. Clearly $X \in \bigcap_i [\![B_i]\!]$, so X is not strongly f-random. □

We now sketch how to replace "strongly computable" by "computable."

Lemma 8.14. Let f be computable and convex. Given $\epsilon > 0$ we can effectively find an $\overline{f}$ which is strongly computable and convex and such that $f(\sigma) < \overline{f}(\sigma) < f(\sigma) + \epsilon$ for all σ.

Proof. Let $\mathbb{Q}$ be the set of rational numbers. By a straightforward but awkward construction, we can find $\overline{f} : \{0,1\}^* \to \mathbb{Q}$ which is computable and convex and such that $f(\sigma) < \overline{f}(\sigma) < f(\sigma) + \epsilon$ for all σ. From the $\mathbb{Q}$-valuedness of $\overline{f}$ it follows easily that $\overline{f}$ is strongly computable. □

Lemma 8.15. Let f be computable and convex. Given $\delta > 0$ and an r.e. set A, we can effectively find an r.e. set B such that $[\![A]\!] \subseteq [\![B]\!]$ and $\mathrm{pwt}_f(B) \leq (1 + \delta) \cdot \mathrm{vwt}_f(A)$.

Proof. Let $\overline{f}$ be as in Lemma 8.14 with $\epsilon = \log_2(1 + \delta)$. If A is r.e., apply Lemma 8.11 to find an r.e. set B such that $[\![A]\!] \subseteq [\![B]\!]$ and $\mathrm{pwt}_{\overline{f}}(B) \leq \mathrm{vwt}_{\overline{f}}(A)$. It is then easy to check that $\mathrm{pwt}_f(B) \leq (1 + \delta) \cdot \mathrm{vwt}_f(A)$. □

Theorem 8.16. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable and convex. Then vehement f-randomness is equivalent to strong f-randomness.

Proof. Proceed as in the proof of Theorem 8.13 but instead of Lemma 8.11 use Lemma 8.15 with $\delta = 1$. □

9 Propagation of vehement f-randomness

In this section we present an alternative proof of one of our main results concerning propagation of strong f-randomness, Corollary 4.8. Our alternative proof proceeds via vehement f-randomness and depends heavily on Remark 8.2. Our alternative proof has the advantage of being a direct generalization of one of the known proofs (see [10, Proposition 7.4]) of the corresponding result for Martin-Löf randomness, Theorem 1.2.

Theorem 9.1. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable and convex. Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If X is strongly f-random, then X is strongly f-random relative to some $Z \in Q$.

Proof. Relativizing Corollary 2.9 let S^Z_i be a universal test for strong f-randomness relative to Z. In other words, S^Z_i is uniformly r.e. relative to Z and $\mathrm{pwt}_f(S^Z_i) \leq 2^{-i}$ and $\forall X\, \forall Z\, (X \notin \bigcap_i [\![S^Z_i]\!] \Leftrightarrow X$ is strongly f-random relative to $Z)$. By Lemma 8.3 we have $\mathrm{vwt}_f(S^Z_i) \leq \mathrm{pwt}_f(S^Z_i) \leq 2^{-i}$ so by Theorem 8.16 S^Z_i is also a universal test for vehement f-randomness relative to Z. Thus, letting $U^Z_i = [\![S^Z_i]\!]$, we have

$$\forall X\, \forall Z\, (X \notin \textstyle\bigcap_i U^Z_i \Leftrightarrow X \text{ is vehemently } f\text{-random relative to } Z)$$

and U^Z_i is uniformly Σ^0_1 relative to Z.

Let $\widetilde{U}_i = \bigcap_{Z \in Q} U^Z_i$. Since Q is Π^0_1, it follows by compactness that $\widetilde{U}_i$ is uniformly Σ^0_1. Therefore, let $\widetilde{S}_i$ be uniformly r.e. such that $[\![\widetilde{S}_i]\!] = \widetilde{U}_i$. For any $Z \in Q$ we have $\widetilde{U}_i \subseteq U^Z_i$, i.e., $[\![\widetilde{S}_i]\!] \subseteq [\![S^Z_i]\!]$, so $\mathrm{vwt}_f(\widetilde{S}_i) \leq \mathrm{vwt}_f(S^Z_i) \leq 2^{-i}$ by Remark 8.2. Thus $\widetilde{S}_i$ is a test for vehement f-randomness. In particular we have $\forall X\, (X$ vehemently f-random $\Rightarrow X \notin \bigcap_i \widetilde{U}_i)$.

Suppose now that X is strongly f-random. By Theorem 8.16 X is vehemently f-random, so

$$X \notin \bigcap_i \widetilde{U}_i = \bigcap_i \bigcap_{Z \in Q} U^Z_i = \bigcap_{Z \in Q} \bigcap_i U^Z_i.$$

Let $Z \in Q$ be such that $X \notin \bigcap_i U^Z_i$. Then X is vehemently f-random relative to Z, so by Theorem 8.16 X is strongly f-random relative to Z, Q.E.D. □

Theorem 9.2. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable and convex. Let Q be a nonempty Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. If X is vehemently f-random, then X is vehemently f-random relative to some $Z \in Q$.

Proof. This is immediate from Theorems 8.16 and 9.1. □

10 Other characterizations of strong f-randomness

In this section we present two new characterizations of strong f-randomness. One of our new characterizations is in terms of f-randomness relative to a PA-degree. The other is in terms of what we call *provable noncomplexity*.

Theorem 10.1. Let $f : \{0,1\}^* \to [-\infty, \infty]$ be computable. The following are pairwise equivalent.

1. X is strongly f-random.
2. X is strongly f-random relative to some PA-degree.
3. X is f-random relative to some PA-degree.

Proof. The implication $1 \Rightarrow 2$ follows from Theorem 4.7. The implication $2 \Rightarrow 3$ is trivial. It remains to prove $3 \Rightarrow 1$. Assume that 1 fails, i.e., X is not strongly f-random. Let A_i, $i \in \mathbb{N}$ be uniformly r.e. such that $\mathrm{pwt}_f(A_i) \leq 2^{-i}$ and $X \in \bigcap_i [\![A_i]\!]$. For each i let $\widehat{A}_i$ be the set of minimal elements of A_i. Let Q be the set of sequences Z_i, $i \in \mathbb{N}$ such that $Z_i \subseteq \{0,1\}^*$ and $\mathrm{dwt}_f(Z_i) \leq 2^{-i}$ and $\forall \sigma\,(\sigma \in A_i \Rightarrow \exists \rho\,(\rho \subseteq \sigma \text{ and } \rho \in Z_i))$. The sequence $\widehat{A}_i$, $i \in \mathbb{N}$ belongs to Q, so Q is nonempty. Moreover, Q may be viewed as a Π^0_1 set in the Cantor space. Therefore, given Z of PA-degree, we can find a sequence B_i, $i \in \mathbb{N}$ which is Turing reducible to Z and belongs to Q. From the definition of Q it follows that $\mathrm{dwt}_f(B_i) \leq 2^{-i}$ and $X \in \bigcap_i [\![B_i]\!]$. Thus X is not f-random relative to Z. This holds for all PA-degrees, so 3 fails, Q.E.D. $\square$

For our second characterization, let PA denote first-order Peano arithmetic. Within PA we define prefix-free complexity $\mathrm{KP} : \{0,1\}^* \to \mathbb{N}$ and a priori complexity $\mathrm{KA} : \{0,1\}^* \to (0, \infty)$ as usual. Also within PA we define $\mathrm{KP}^{(j)} =$ prefix-free complexity relative to $0^{(j)}$, and $\mathrm{KA}^{(j)} =$ a priori complexity relative to $0^{(j)}$, where $0^{(j)} =$ the jth Turing jump of 0. Let $f : \{0,1\}^* \to \mathbb{N}$ and $X \in \{0,1\}^{\mathbb{N}}$ be arbitrary.

Definition 10.2. Let K stand for KP or KA, and let Z be a Turing oracle. We define X to be <u>K^Z-f-complex</u> if $\exists c\, \forall n\, (K^Z(X{\upharpoonright}n) > f(X{\upharpoonright}n) - c))$.

Definition 10.3. Let M be a nonstandard model of PA. We define X to be <u>M-f-complex</u> if $\forall r\, \exists c\, \forall n\, (\mathrm{KP}_r(X{\upharpoonright}n) > f(X{\upharpoonright}n) - c)$. Here r ranges over M-finite functions r with prefix-free domain, and $\mathrm{KP}_r(\tau) = \min\{|\sigma| \mid r(\sigma) = \tau\}$.

Definition 10.4. Let K stand for KP or KA or $\mathrm{KP}^{(j)}$ or $\mathrm{KA}^{(j)}$. Let T be a consistent theory extending PA. We define X to be <u>provably K-f-noncomplex in T</u> if $\forall c\, \exists n\, (T \vdash (\exists m < n)\, (K(X{\upharpoonright}m) < f(X{\upharpoonright}m) - c))$.

Theorem 10.5. Let K stand for KP or KA. Let T be a recursively axiomatizable, consistent theory extending PA. Let $f : \{0,1\}^* \to \mathbb{N}$ and $X \in \{0,1\}^{\mathbb{N}}$ be arbitrary. For each $j \in \mathbb{N}$ the following are pairwise equivalent.

1. X is not K^Z-f-complex for any Z of PA-degree.

2. X is not M-f-complex for any nonstandard $M \models T$.

3. X is provably K-f-noncomplex in some recursively axiomatizable, consistent theory extending T.

4. X is provably $K^{(j)}$-f-noncomplex in some recursively axiomatizable, consistent theory extending T.

Proof. This is a special case of [41, Theorems 4.1 and 4.4]. □

Theorem 10.6. Let $f : \{0,1\}^* \to \mathbb{N}$ be computable. Let T be a recursively axiomatizable, consistent theory extending PA. For all $X \in \{0,1\}^{\mathbb{N}}$ the following are pairwise equivalent.

1. X is strongly f-random.

2. X is M-f-complex for some nonstandard $M \models T$.

3. X is not provably KP-f-noncomplex in any recursively axiomatizable, consistent theory extending T.

Proof. By Theorems 2.8 and 10.1 X is strongly-f-random if and only if X is KA^Z-f-complex for all Z of PA-degree. The equivalences $1 \Leftrightarrow 2$ and $1 \Leftrightarrow 3$ then follow by Theorem 10.5. □

Define K^Z-length-complexity to mean K^Z-f-complexity where $f(\sigma) =$ the length of σ, and similarly for M-length-complexity and provable KP-length-noncomplexity.

Theorem 10.7. Let T be a recursively axiomatizable, consistent theory extending PA. For all $X \in \{0,1\}^{\mathbb{N}}$ the following are pairwise equivalent.

1. X is Martin-Löf random.

2. X is M-length-complex for some nonstandard $M \models T$.

3. X is not provably KP-length-noncomplex in any recursively axiomatizable, consistent theory extending T.

Proof. This is the special case of Theorem 10.6 with $f(\sigma) = |\sigma|$ for all σ. □

Remark 10.8. By Theorem 10.5 our notion of provable length-noncomplexity is stable under relativation to a strong oracle. Thus, letting X be Martin-Löf random and Turing reducible to $0^{(1)}$, we see that X is not $\mathrm{KP}^{(1)}$-length-complex but not provably $\mathrm{KP}^{(1)}$-noncomplex in any recursively axiomatizable, consistent extension T of PA. It follows that for any such T there exist $\tau \in \{0,1\}^*$ and $n \in \mathbb{N}$ such that $\mathrm{KP}^{(1)}(\tau) < n$ but $T \nvdash \mathrm{KP}^{(1)}(\tau) < n$. Comparing this to the celebrated Chaitin Incompleteness Theorem [7, 24], we now have a somewhat different example of a statement which is true but not provable in T.

11 Non-propagation of f-randomness

In this section we show that Theorems 4.5 and 4.7 and 10.6 fail if strong f-randomness is replaced by f-randomness.

Theorem 11.1. For many f's, e.g., $f(\sigma) = |\sigma|/2$, we can find an X which is f-random but not f-random relative to any PA-degree.

Proof. By Reimann/Stephan [28] let X be f-random but not strongly f-random. By Theorem 10.1 X is not f-random relative to any PA-degree. □

Corollary 11.2. For many f's, e.g., $f(\sigma) = |\sigma|/2$, we can find an X which is f-random but provably KP-f-noncomplex in some recursively axiomatizable, consistent extension of PA. Indeed, X is provably KP-f-noncomplex in some recursively axiomatizable, consistent extension of any recursively axiomatizable, consistent extension of PA.

Proof. By Theorem 11.1 let X be f-random but not f-random relative to any PA-degree. The desired conclusion follows by Theorem 10.5. □

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