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A Gametheoretic Approach to Second-Order Definability

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Contents

Chapter 0

Introduction

Our interest in this paper is to study the expressive power of second-order logic in finite structures, using the methodology of *finite model theory*, a branch of model theory.

Model theory or the theory of models, first named by Tarski in 1954, is the part of the semantics of formalized languages that is concerned with the relationships between syntactic constructions of axiom systems and (mainly algebraic) properties of mathematical structures ("models"). In classical model theory, the expressive power of first-order language has been studied extensively already, for it obeys some fundamental principles such as the compactness theorem, which says that if each finite subset of a set Σ of sentences has a model then the whole set Σ has a model. Another typical classical result is Löwenheim's theorem: if a sentence has an infinite model, then it has a countable model. These results help us to observe some limitations of the expressive power of first-order logic; Löwenheim's theorem shows that no consistent set of sentences can imply that a model is uncountable, and the compactness theorem has been used to show that many mathematical properties cannot be expressed by a set of first-order sentences – for instance, there is no set of sentences whose models are precisely all the finite models. These two theorems we have stated are proved using classical methods of model constructions, which rest essentially upon the realm of infinite structures.

However, principal theorems of first-order logic fail and important methods become useless when we restrict ourselves to finite structures. The first landmark is Trakhtenbrot's Theorem (1950) which implies that first-order logic, when restricted to the finite, does not admit a complete proof calculus. Later, for the last twenty years we really come to ask questions of a modeltheoretic flavor with the restriction to finite structures, and it turned out that the questions are deeply connected to computational aspects, as the proof of Trakhtenbrot's theorem is based on the undecidability of the halting problem for Turing machines. The following facts of great importance have been shown in the field, which show evidence of the close relation between finite model theory and complexity theory.

Theorem Let K be a class of ordered finite structures.

$K \in$ LOGSPACE	iff	K is axiomatizable in FO(DTC)
$K \in$ NLOGSPACE	iff	K is axiomatizable in FO(TC)
$K \in$ PTIME	iff	K is axiomatizable in FO(IFP)

$$K \in \text{NPTIME} \quad \text{iff} \quad K \text{ is axiomatizable in } \Sigma^1_1$$
$$K \in \text{PSPACE} \quad \text{iff} \quad K \text{ is axiomatizable in FO(PFP)}$$

(Σ^1_1 denotes the fragment of second-order logic consisting of the sentences of the form $\exists X_1 \cdots \exists X_n \psi$, where ψ is first-order) □

The logics listed on the right sides of the equivalences except Σ^1_1 are called fixed-point logics, which have been introduced to strengthen the expressive power of first-order logic by adding the operations that represent recursive procedures. This theorem provides the logical characterizations of complexity classes, therefore we are in a position to obtain logical analogies of major problems in complexity theory. For example, the P $\neq$ NP-problem now amounts to the question whether two logics FO(IFP) and Σ^1_1 have the same expressive power in finite structures or not.

From the point of view indicated in the theorem mentioned above, one will be convinced that it is of great use to inquire the expressive powers of the logics in finite structures, for it resolves itself to check up the relations among complexity classes. However, as already stated above, many important methods to serve this purpose in classical model theory become useless when restricting oneself to finite structures. Still, the gametheoretic method of Ehrenfeucht survives, which in fact is almost the only technique available in finite model theory. The Ehrenfeucht method provides a simple characterization of the definability in first-order logic in terms of a game. Therefore it has been one of the major issue in this area to investigate the applications of the Ehrenfeucht method and to generalize the method for some extensions of first-order logic. For example, the straightforward generalization for MSO (monadic second-order logic), that is, second-order logic in which only unary relation variables are allowed, has been known already (see [5]).

Our goal in the present paper is to propose some generalizations of the Ehrenfeucht method and to give some applications. This paper is divided into several chapters, and the contents of each chapter are constructed in the following way.

In Chapter 1 we introduce some basic notions related to first-order logic, that are needed in the following chapters.

In Chapter 2 we give further informations concerned with the theorem described above. There the precise definitions of second-order logic and fixed-point logics are given, and then what is known as the theory of descriptive complexity is surveyed.

In Chapter 3 we present the Ehrenfeucht method for first-order logic and give some examples of mathematical properties that are known to be inexpressible in first-order logic. For instance, it has been shown in [5] that the class of finite structures of even cardinality is not expressible in first-order logic.

Chapter 4 forms the core of this paper, where we attempt to generalize the method of Ehrenfeucht, mainly for second-order logic. After we mention the method for MSO we give a new application of it, where we show that, even in MSO, the class of finite structures of even cardinality is not expressible. Further, we show that the method for MSO can be extended to the method for general second-order logic, removing the limitation on the arity of relation variables. And the extension enables us to introduce a new method for Σ^1_1, which provides us a gametheoretic characterization of Σ^1_1-axiomatizability. From the viewpoint of the theorem mentioned above, it corresponds precisely to the computability in NPTIME. Therefore it contains a potential for throwing a new light on the problems concerned with NPTIME in complexity theory.

The contents of Chapter 2 and Chapter 3 are based on [5].

Chapter 1

Preliminaries

The purpose of this section is to fix notations and terminology for the basic notions related to first-order logic. For more details, see [4][15].

1.1 Structures

A *vocabulary* is a finite nonempty set that consists of *relation symbols* P, Q, R, $\dots$ and constant symbols c, d, $\dots$. Every relation symbol is equipped with a natural number, called its *arity*. Vocabularies are denoted by τ, σ, $\dots$. A vocabulary is *relational* if it does not contain constants.

A *structure* $\mathcal{A}$ of a vocabulary τ (for short: a τ-*structure*) consists of the following things:

(a) A nonempty set A, called the *universe* of $\mathcal{A}$.

(b) For each n-ary relation symbol R of τ, an n-ary relation $R^{\mathcal{A}}$ in A.

(c) For each constant symbol c of τ, an element $c^{\mathcal{A}}$ of A.

Two τ-structures $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, written $\mathcal{A} \cong \mathcal{B}$, if there is an *isomorphism* from $\mathcal{A}$ to $\mathcal{B}$, that is, a bijection $\pi : A \to B$ preserving relations and constants:

– for n-ary $R \in \tau$ and $a_1, \dots, a_n \in A$,

$$R^A a_1 \dots a_n \quad \text{iff} \quad R^B \pi(a_1) \dots \pi(a_n)$$

– for $c \in \tau$, $\pi(c^A) = c^B$.

We give some examples of structures.

Graphs

Let $\tau = \{E\}$ with a binary relation symbol E. A *graph* is a τ-structure $\mathcal{G} = (G, E^G)$ satisfying

(a) for all $a \in G$, not $E^G aa$

(b) for all $a, b \in G$, if $E^G ab$ then $E^G ba$.

If only (a) is required, we call it a *digraph* (or, *a directed graph*).

Let $\mathcal{G}$ be a digraph. If $n \geq 1$ and

$$E^G a_0 a_1, E^G a_1 a_2, \ldots E^G a_{n-1} a_n$$

then a sequence $a_0, \ldots, a_n$ is a *path* from a_0 to a_n of *length* n. If $a_0 = a_n$ then a path $a_0, \ldots, a_n$ is a *cycle*. $\mathcal{G}$ is *acyclic* if there is no cycle. A path $a_0, \ldots, a_n$ is *Hamiltonian* if $G = \{a_0, \ldots, a_n\}$ and $a_i \neq a_j$ for $i \neq j$. If $E^G a_n a_0$ in addition, we call it a *Hamiltonian circuit*.

Let $\mathcal{G}$ be a graph. Write $a \sim b$ if $a = b$ or if there is a path from a to b. Clearly, $\sim$ is an equivalence relation. The equivalence class of a is called the *(connected) component* of a. $\mathcal{G}$ is *connected* if $a \sim b$ for all $a, b \in G$, that is, if there is only one connected component.

We denote by $d(a, b)$ the length of a shortest path from a to b; more precisely,

$$d(a, b) = \infty \text{ iff } a \not\sim b; \qquad d(a, b) = 0 \text{ iff } a = b;$$

and otherwise,

$$\begin{aligned} d(a, b) \;\; = \;\; & \min\{n \geq 1 \mid \text{there are } a_0, \ldots, a_n \in G \text{ such that} \\ & \qquad a = a_0, b = a_n, \text{ and } E^G a_i a_{i+1} \text{ for } i < n\} \end{aligned}$$

Obviously,

$$d(a, c) \leq d(a, b) + d(b, c),$$

where we use the natural conventions for ∞.

Orderings

Let $\tau = \{<\}$ with a binary relation symbol $<$. A τ-structure $\mathcal{A} = (A, <^A)$ is called an *ordering* if for all a, b, $c \in A$:

(a) not $a <^A a$

(b) $a <^A b$ or $b <^A a$ or $a = b$

(c) if $a <^A b$ and $b <^A c$ then $a <^A c$

Sometimes we consider finite orderings also as $\{<, S, \min, \max\}$-structures. Here S is a binary relation symbol representing the successor relation, and min and max are constants for the first and the last element of the ordering, respectively (note that the successor relation is always possible in a *finite* ordering, the first and the last elements are as well). Thus, a finite $\{<, S, \min, \max\}$-structure $\mathcal{A}$ is an ordering if, in addition to (a), (b), (c), for all $a, b \in A$:

(d) $S^A ab$ iff ($a <^A b$ and for all c, if $a <^A c$ then $b <^A c$ or $b = c$)

(e) $\min^A <^A a$ or $\min^A = a$

(f) $a <^A \max^A$ or $a = \max^A$

Suppose that τ_0 is a vocabulary with $\{<\} \subseteq \tau_0 \subseteq \{<, S, \min, \max\}$ and let σ be an arbitrary vocabulary with $\tau_0 \subseteq \sigma$. A finite σ-structure $\mathcal{A}$ is said to be *ordered* if the reduct $\mathcal{A}|\tau_0$ (i.e. the τ_0-structure obtained from $\mathcal{A}$ by omitting the interpretations of the symbols in $\sigma \backslash \tau_0$) is an ordering.

1.2 First-Order Logic

Fix a vocabulary τ. A *first-order language* has symbols as the following.

- the *variables* $v_1, v_2, v_3, \ldots$
- the symbols $\neg$, $\vee$, and $\exists$
- the *equality symbol* $=$
-), (
- the symbols in τ

A *term* of vocabulary τ is a variable or a constant in τ. Henceforth, we will often use letters x, y, z, ... for variables and t, t_1, t_2, ... for terms.

The *formulas* of first-order logic of vocabulary τ are defined by the following inductive definition:

(a) If t_0 and t_1 are terms, then $t_0 = t_1$ is a formula.

(b) If R in τ is n-ary and $t_1, \ldots, t_n$ are terms, then $Rt_1 \ldots t_n$ is a formula.

(c) If φ is a formula, then $\neg\varphi$ is a formula.

(d) If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.

(e) If φ is a formula and x a variable, then $\exists x \varphi$ is a formula.

We denote by $\mathrm{FO}[\tau]$ the set of formulas of first-order logic of vocabulary τ. Formulas obtained by (a) or (b) are called *atomic* formulas. We use some defined symbols; $(\varphi \wedge \psi)$ is an abbreviation of $\neg(\neg\varphi \vee \neg\psi)$; $(\varphi \rightarrow \psi)$ is an abbreviation of $(\neg\varphi \vee \psi)$; $(\varphi \leftrightarrow \psi)$ is an abbreviation of $((\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi))$; and $\forall x\varphi$ is an abbreviation of $\neg\exists x\neg\varphi$.
We will often omit parentheses in formulas if they are not essential like the outermost parenthesis in the disjunction $(\varphi \vee \psi)$.

The set $\mathrm{free}(\varphi)$ of *free variables* of a formula φ is defined by:

- If φ is atomic then the set $\mathrm{free}(\varphi)$ of free variables of φ is the set of variables occurring in φ
- $\mathrm{free}(\neg\varphi) := \mathrm{free}(\varphi)$
- $\mathrm{free}(\varphi \vee \psi) := \mathrm{free}(\varphi) \cup \mathrm{free}(\psi)$
- $\mathrm{free}(\exists x\varphi) := \mathrm{free}(\varphi) \setminus \{x\}$

We use the notation $\varphi(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n$ are distinct and $\mathrm{free}(\varphi)$ $\subseteq \{x_1, \ldots, x_n\}$ without implying that all of $x_1, \ldots, x_n$ are actually free in φ. Often we abbreviate an n-tuple $x_1, \ldots, x_n$ of variables by $\bar{x}$, for example, writing $\varphi(\bar{x})$ for $\varphi(x_1, \ldots, x_n)$.

Let $\mathcal{A}$ be a τ-structure. An *assignment* in $\mathcal{A}$ is a function α with domain of the set of variables and with values in $\mathcal{A}$, $\alpha : \{v_n \mid n \geq 1\} \rightarrow A$. Extend α to a function defined for all terms by setting $\alpha(c) := c^{\mathcal{A}}$ for c in τ. We denote by $\alpha\frac{a}{x}$ the assignment that agree with α except $\alpha\frac{a}{x}(x) = a$.
We define the relation

$$\mathcal{A} \models \varphi[\alpha]$$

("φ is *true* in $\mathcal{A}$ under α") as follows:

$$\begin{array}{lll}
\mathcal{A} \models t_1 = t_2[\alpha] & \text{iff} & \alpha(t_1) = \alpha(t_2) \\
\mathcal{A} \models Rt_1 \ldots t_n[\alpha] & \text{iff} & R^{\mathcal{A}}\alpha(t_1) \ldots \alpha(t_n) \\
\mathcal{A} \models \neg\varphi[\alpha] & \text{iff} & \text{not } \mathcal{A} \models \varphi[\alpha] \\
\mathcal{A} \models (\varphi \vee \psi)[\alpha] & \text{iff} & \mathcal{A} \models \varphi[\alpha] \text{ or } \mathcal{A} \models \psi[\alpha] \\
\mathcal{A} \models \exists x\varphi[\alpha] & \text{iff} & \text{there is an } a \in A \text{ such that } \mathcal{A} \models \varphi[\alpha\frac{a}{x}]
\end{array}$$

Note that the truth or falsity of $\mathcal{A} \models \varphi[\alpha]$ depends only on the values of α for those variables which are free in φ. That is, if $\alpha_1(x) = \alpha_2(x)$ for all $x \in \mathrm{free}(\varphi)$, then $\mathcal{A} \models \varphi[\alpha_1]$ iff $\mathcal{A} \models \varphi[\alpha_2]$. Thus, if $\varphi = \varphi(x_1, \ldots, x_n)$ and $a_1 = \alpha(x_1), \ldots, a_n = \alpha(x_n)$, then we may write $\mathcal{A} \models \varphi[a_1, \ldots, a_n]$ for $\mathcal{A} \models \varphi[\alpha]$. In particular, if φ is a sentence, then the truth or falsity of $\mathcal{A} \models \varphi[\alpha]$ is completely independent of α. Thus, we may write $\mathcal{A} \models \varphi$ ($\mathcal{A}$ is a model of φ). For a set Φ of formulas, $\mathcal{A} \models \Phi[\alpha]$ means that $\mathcal{A} \models \varphi[\alpha]$ for all $\varphi \in \Phi$. Φ is *satisfiable* if there is a structure A and an assignment α in $\mathcal{A}$ such that $\mathcal{A} \models \Phi[\alpha]$.

A formula ψ is a *consequence* of Φ, written $\Phi \models \psi$, if $\mathcal{A} \models \psi[\alpha]$ whenever $\mathcal{A} \models \Phi[\alpha]$. The formula ψ is *logically valid*, written $\models \psi$, if $\emptyset \models \psi$, that is, if ψ is true in all structures under all assignments. Formulas φ and ψ are *logically equivalent* if $\models \varphi \leftrightarrow \psi$. When only taking into consideration finite structures, we use the notations $\Phi \models_{\text{fin}} \psi$ and $\models_{\text{fin}} \psi$.
At some places it will be convenient to assume that first-order logic contains two zero-ary relation symbols T, F. In every structure, T and F are interpreted as TRUE and FALSE, respectively. Hence, the atomic formula T is logically equivalent to $\exists x(x = x)$ and F to $\neg\exists x(x = x)$. If $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ we sometimes write $\bigwedge \Phi$ for $\varphi_1 \wedge \ldots \wedge \varphi_n$ and $\bigvee \Phi$ for $\varphi_1 \vee \ldots \vee \varphi_n$. In case $\Phi = \emptyset$ we set $\bigwedge \Phi =$T and $\bigvee \Phi =$F.

The *quantifier rank* $\mathrm{qr}(\varphi)$ of a formula φ is the maximum number of nested quantifiers occurring in it:

$$\begin{array}{rcllrcl} \mathrm{qr}(\varphi) & := & 0 \quad \text{if } \varphi \text{ is atomic;} & & \mathrm{qr}(\neg\varphi) & := & \mathrm{qr}(\varphi); \\ \mathrm{qr}(\varphi \vee \psi) & := & \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}; & & \mathrm{qr}(\exists x\varphi) & := & \mathrm{qr}(\varphi) + 1 \end{array}$$

It can be shown that every first-order formula is logically equivalent to a formula in prenex normal form, that is, a formula of the form $Q_1 x_1 \ldots Q_s x_s \psi$, where $Q_1, \ldots, Q_s \in \{\forall, \exists\}$ and ψ is quantifierfree. Such a formula is called Σ_n, if the string of quantifiers consists of n consecutive blocks, where in each block all quantifiers are of the same type (i.e. all universal or all existential), adjacent blocks contain quantifiers of different type, and the first block is existential. Π_n-formulas are defined in the same way, but now we require that the first block consists of universal quantifiers. A Δ_n-formula is a formula logically equivalent to both a Σ_n-formula and a Π_n-formula.

Given a formula $\varphi(x, \bar{z})$ and $n \geq 1$, $\exists^{\geq n} x \varphi(x, \bar{z})$ is an abbreviation for the formula

$$\exists x_1 \ldots \exists x_n (\bigwedge_{1 \leq i \leq n} \varphi(x_i, \bar{z}) \wedge \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j)$$

expressing that there are at least n elements x with $\varphi(x, \bar{z})$. Similarly, $\exists^{\leq n} x \varphi(x, \bar{z})$ is defined by $\neg\exists^{\geq n+1} x \varphi(x, \bar{z})$, and $\exists^{=n} x \varphi(x, \bar{z})$ is defined by $\exists^{\geq n} x \varphi(x, \bar{z}) \wedge \exists^{\leq n} x \varphi(x, \bar{z})$. Moreover, we set

$$\varphi_{\geq n} := \exists^{\geq n} x (x = x); \quad \varphi_{\leq n} := \exists^{\leq n} x (x = x); \quad \varphi_{= n} := \exists^{= n} x (x = x)$$

Clearly,

$$\mathcal{A} \models \varphi_{\geq n} \quad \text{iff} \quad \|A\| \geq n$$

and similarly for $\varphi_{\leq n}$ and $\varphi_{=n}$.

1.3 Model Classes

Fix a vocabulary τ. For a sentence φ of $\mathrm{FO}(\tau)$ we use notations $\mathrm{arbMod}(\varphi)$ and $\mathrm{Mod}(\varphi)$, which denote the class of arbitrary (finite and infinite) models of φ and the class of finite models of φ, respectively. If π is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, $\varphi(x_1, \ldots, x_n) \in \mathrm{FO}[\tau]$, and $a_1, \ldots, a_n \in A$, then

$$\mathcal{A} \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \mathcal{B} \models \varphi[\pi(a_1), \ldots, \pi(a_n)]$$

In particular, if φ is a sentence, then

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{B} \models \varphi$$

Hence, $\mathrm{arbMod}(\varphi)$ and $\mathrm{Mod}(\varphi)$ are closed under isomorphism, that is,

$$\mathcal{A} \in \mathrm{Mod}(\varphi) \text{ and } \mathcal{A} \cong \mathcal{B} \text{ imply } \mathcal{B} \in \mathrm{Mod}(\varphi) \tag{1.1}$$

and similarly for $\mathrm{arbMod}(\varphi)$. Later we will discuss various logics extended from first-order logic. In all of these logics only structural properties, that is, properties invariant under isomorphisms, will be expressible. In fact, (1.1) says that only classes of structures closed under isomorphisms can be axiomatizable. We therefore agree upon the following convention: All classes K of structures considered will tacitly be assumed to be *closed under isomorphism*, that is

$$\mathcal{A} \in K \text{ and } \mathcal{A} \cong \mathcal{B} \text{ imply } \mathcal{B} \in K$$

In the following, axiomatizability by a single sentence in finite structures will be a major issue.

Definition 1.3.1 Let K be a class of τ-structures and $\mathcal{L}$ be a logic. K is *axiomatizable* in $\mathcal{L}$, if there is a sentence φ of $\mathcal{L}$ of vocabulary τ such that $K = \mathrm{Mod}(\varphi)$. □

Chapter 2

Some Extensions of First-Order Logic

2.1 Second-Order Logic

Second-order logic, SO, is an extension of first-order logic which allows to quantify over relations. In addition to the symbols of first-order logic, its vocabulary contains, for each $n \geq 1$, countably many *relation variables* $V_1^n, V_2^n, \ldots$. We use letters $X, Y, \ldots$ for relation variables.
We define the set of second-order formulas of a vocabulary τ to be the set generated by the rules for first-order formulas extended by:

- If X is n-ary and $t_1, \ldots, t_n$ are terms then $Xt_1 \ldots t_n$ is a formula.
- If φ is a formula and X is a relation variable then $\exists X\varphi$ is a formula.

To define the free occurrence of a variable or of a relation variable in a second-order formula, we add the following rule:

- $\text{free}(\exists X\varphi) := \text{free}(\varphi) \setminus \{X\}$

Before we define the notion of satisfaction, we need to extend the assignment function α to a function defined also for relation variables, but now $\alpha(X)$, a value of a relation variable X, is a relation over A of arity corresponding to X. The satisfaction relation $\models$ is extended by the following rule:

$$\mathcal{A} \models \exists X\varphi[\alpha] \quad \text{iff} \quad \text{there is a relation } R \text{ over } A \text{ of arity corresponding to } X \text{ such that } \mathcal{A} \models \varphi[\alpha\tfrac{R}{X}]$$

Then, given $\varphi = \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ with free (individual and relation) variables among $x_1, \ldots, x_n, Y_1, \ldots, Y_k$, a τ-structure $\mathcal{A}$, elements $a_1, \ldots, a_n \in A$, and relations $R_1, \ldots, R_k$ over A of arities corresponding to $Y_1, \ldots, Y_k$, respectively,

$$\mathcal{A} \models \varphi[a_1, \ldots, a_n, R_1, \ldots, R_k]$$

means that $a_1, \ldots, a_n$ together with $R_1, \ldots, R_k$ satisfy φ in $\mathcal{A}$.

When we allow only *unary* relation variables ("set variables"), we get the fragment MSO of second-order logic known as *monadic second-order logic*.

Using equivalences such as

$$\models \neg\exists X\varphi \leftrightarrow \forall X\neg\varphi, \qquad \models (\varphi \vee \forall Y\psi) \leftrightarrow \forall Y(\varphi \vee \psi) \quad \text{if } Y \text{ is not free in } \varphi$$

we can show that each (M)SO-formula is logically equivalent to an (M)SO-formula in prenex normal form, that is, to a formula of the form

$$Q_1\alpha_1, \ldots, Q_s\alpha_s\psi$$

where $Q_1, \ldots, Q_s \in \{\forall, \exists\}$, and where $\alpha_1, \ldots, \alpha_s$ are first-order or second-order variables and ψ is quantifierfree. Moreover, since

$$\begin{aligned} &\models \exists x Q_1\alpha_1, \ldots, Q_s\alpha_s\psi \quad \leftrightarrow \quad \exists X Q_1\alpha_1, \ldots, Q_s\alpha_s\psi(\exists^{=1}xXx \wedge \forall x(Xx \rightarrow \psi)) \\ &\models \forall x Q_1\alpha_1, \ldots, Q_s\alpha_s\psi \quad \leftrightarrow \quad \forall X Q_1\alpha_1, \ldots, Q_s\alpha_s\psi(\exists^{=1}xXx \rightarrow \forall x(Xx \rightarrow \psi)) \end{aligned}$$

every (M)SO-formula is logically equivalent to one in prenex normal form in which each second-order quantifier precedes all first-order quantifiers. Such a formula is called (M)Σ^1_n, if the string of second-order quantifiers consists of n consecutive blocks, where in each block all quantifiers are of the same type (i.e. all universal or all existential), adjacent blocks contain quantifiers of different type, and the first block is existential. (M)Π^1_n-formulas are defined in the same way, but now we require that the first block consists of universal quantifiers.
Clearly, the negation of a Σ^1_n-formula is logically equivalent to a Π^1_n-formula, and the negation of a Π^1_n-formula is logically equivalent to a Σ^1_n-formula. Δ^1_n denotes the set of formulas that are logically equivalent to both a Σ^1_n-formula and a Π^1_n-formula.

2.2 Logics with Fixed-Point Operators

In this section we introduce fixed-point logics, such as inflationary fixed-point logic, partial fixed-point logic, deterministic transitive closure logic and transitive closure logic. These logics are obtained from first-order logic by adding operations well-suited to describe iterative and recursive procedures.

FO(IFP) and FO(PFP)

Let M be a finite nonempty set. $\text{Pow}(M)$ denotes the power set of M. A function $F : \text{Pow}(M) \rightarrow \text{Pow}(M)$, defined by $F_0 = \emptyset$ and $F_{n+1} = F(F_n)$, induces a sequence $\emptyset,\ F(\emptyset),\ F(F(\emptyset)), \ldots$ of subsets of M. Suppose there is an $n_0 \geq 0$ such that $F_{n_0+1} = F_{n_0}$, that is, $F(F_{n_0}) = F_{n_0}$, then $F_m = F_{n_0}$ for all $m \geq n_0$. Then F_{n_0} is denoted by F_∞, and we say that the *fixed-point* F_∞ of F *exists*. In case the fixed-point F_∞ does not exist, we agree to set $F_\infty := \emptyset$.
F is said to be *inflationary* if for all $X \subseteq M$, then $X \subseteq F(X)$.

Lemma 2.2.1 (a) If F_∞ exists then $F_\infty = F_{2^{\|M\|}-1}$.

(b) If F is inflationary then F_∞ exists and $F_\infty = F_{\|M\|}$.

(c) If F is arbitrary and $F' : \text{Pow}(M) \to \text{Pow}(M)$ is given by $F'(X) := X \cup F(X)$ then F' is inflationary.

Proof. (a) Suppose F_∞ exists. As $\text{Pow}(M)$ has $2^{\|M\|}$ elements, there are $m < 2^{\|M\|}$ and $l \geq 1$ such that $F_m = F_{m+l}$. If $l = 1$ then $F_m = F_{m+1}$, and hence, $F_m = F_\infty = F_{2^{\|M\|}-1}$. If $F_m \neq F_{m+1}$ then for all $k \geq 0$ we have $F_{m+k\cdot l} \neq F_{m+k\cdot l+1}$, so F_∞ does not exist.

(b) By assumption, $F_0 \subseteq F_1 \subseteq \ldots \subseteq M$. Since M has $\|M\|$ elements, this sequence must get constant before $F_{\|M\|}$.

(c) Trivial. □

Let $\varphi(x_1, \ldots, x_k, X)$ be a formula in the vocabulary τ, where the relation variable X has arity k, and let $\mathcal{A}$ be a τ-structure. Then φ and $\mathcal{A}$ give rise to an operation $F^\varphi : \text{Pow}(A^k) \to \text{Pow}(A^k)$ defined by

$$F^\varphi(R) \ := \ \{(a_1, \ldots, a_k) \mid \mathcal{A} \models \varphi(a_1, \ldots, a_k, R)\}.$$

Note that for the function $(F^\varphi)'$ obtained from F^φ as in part (c) of the preceding lemma we have $(F^\varphi)' = F^{(X\bar{x} \vee \varphi)}$.

One obtains *Inflationary Fixed-Point Logic* FO(IFP) and *Partial Fixed-Point Logic* FO(PFP) by closing first-order logic FO under inflationary and arbitrary fixed-points of definable operations, respectively. We state the precise definitions.

For a vocabulary τ the class FO(IFP)[τ], which is the set of FO(IFP) formulas of vocabulary τ, is given by the following calculus.

- An atomic second-order formula over τ is a formula.
- If φ , ψ are formulas, then $\neg\varphi$, $\varphi \vee \psi$, $\exists x \varphi$ are formulas.
- If φ is a formula, then $[\text{IFP}_{\bar{x},X}\varphi]\bar{t}$ is a formula.
  (where the length of $\bar{x}$ and $\bar{t}$ are the same and coincide with the arity of X)

For FO(PFP) we replace the last rule by

- If φ is a formula, then $[\text{PFP}_{\bar{x},X}\varphi]\bar{t}$ is a formula.
  (where the length of $\bar{x}$ and $\bar{t}$ are the same and coincide with the arity of X)

Sentences are formulas without free first-order and second-order variables, where the free occurrence of variables is defined in the standard way, adding, for example, for FO(IFP) the clause

$$\text{free}([\text{IFP}_{\bar{x},X}\varphi]\bar{t}) \quad := \quad \text{free}(\bar{t}) \cup (\text{free}(\varphi) \backslash \{\bar{x}, X\})$$

The semantics is defined inductively with respect to the calculus above, the meanings of sentences $[\text{IFP}_{\bar{x},X}\varphi]\bar{t}$ and $[\text{PFP}_{\bar{x},X}\varphi]\bar{t}$ being as follows.

$$\mathcal{A} \models [\text{IFP}_{\bar{x},X}\varphi]\bar{t} \quad \text{iff} \quad \bar{t} \in F_\infty^{(X\bar{x} \vee \varphi)}$$

$$\mathcal{A} \models [\text{PFP}_{\bar{x},X}\varphi]\bar{t} \quad \text{iff} \quad \bar{t} \in F_\infty^{\varphi}$$

Example 2.2.2 Let $\mathcal{G} = (G, E^G)$ be a graph and

$$\varphi(x, y, X) \quad := \quad (Exy \vee \exists z(Xxz \wedge Ezy))$$

with xy corresponding to $\bar{x}$ above. Then for $n \geq 1$,

$$F_n^\varphi = \{(a, b) \mid \text{there is a path of length } \leq n \text{ from } a \text{ to } b\}$$

and hence,

$$F_\infty^\varphi = \{(a, b) \mid \text{there is a path from } a \text{ to } b\}$$

Therefore, the formula

$$\psi_0(x, y) \quad := \quad [\mathrm{IFP}_{xy,X} Exy \vee \exists z(Xxz \wedge Ezy))]xy$$

of FO(IFP) expresses that x, y are connected by a path. Hence, the class of connected graphs is axiomatizable in FO(IFP) by $\forall x \forall y(\neg x = y \rightarrow \psi_0(x, y))$ (and the graph axioms). However, the class of connected graphs is not axiomatizable in first-order logic (even in $\mathrm{M}\Sigma_1^1$), as we will show in the next chapter (see Proposition 3.4.5). □

FO(TC) and FO(DTC)

Let R be a binary relation on a set M, $R \subseteq M^2$. The *transitive closure* $\mathrm{TC}(R)$ of R is defined by

$$\begin{aligned} \mathrm{TC}(R) \quad = \quad \{(a,b) \in M^2 \mid & \text{ there exist } n > 0 \text{ and } e_0, \ldots, e_n \in M \text{ such that} \\ & a = e_0, b = e_n, \text{ and for all } i < n, (e_i, e_{i+1}) \in R\} \end{aligned}$$

And the *deterministic transitive closure* $\mathrm{DTC}(R)$ is defined by

$$\begin{aligned} \mathrm{DTC}(R) \quad = \quad \{(a,b) \in M^2 \mid & \text{ there exist } n > 0 \text{ and } e_0, \ldots, e_n \in M \text{ such that} \\ & a = e_0, b = e_n, \text{ and for all } i < n, e_{i+1} \text{ is the unique } e \text{ for} \\ & \text{which } (e_i, e) \in R\} \end{aligned}$$

Transitive Closure Logic FO(TC) and *Deterministic Transitive Closure Logic* FO(DTC) are obtained by closing FO under the transitive closure and the deterministic transitive closure of definable relations, respectively.
For a vocabulary τ the class of formulas of $\mathrm{FO(TC)}[\tau]$, which is the set of FO(TC) formulas of vocabulary τ, is given by the following calculus.

- An atomic first-order formula over τ is a formula.
- If φ , ψ are formulas, then $\neg\varphi$, $\varphi \vee \psi$, $\exists x\varphi$ are formulas.
- If φ is a formula, then $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi]\bar{s}\bar{t}$ is a formula. (where the variables in $\bar{x}\bar{y}$ are pairwise distinct and where the tuples $\bar{x}, \bar{y}, \bar{s}$ and $\bar{t}$ are all of the same length.)

For FO(DTC) we replace the last rule by

- If φ is a formula, then $[\mathrm{DTC}_{\bar{x},\bar{y}}\varphi]\bar{s}\bar{t}$ is a formula.
  (where the variables in $\bar{x}\bar{y}$ are pairwise distinct and where the tuples $\bar{x}, \bar{y}, \bar{s}$ and $\bar{t}$ are all of the same length.)

We define

$$\mathrm{free}([\mathrm{TC}_{\bar{x},\bar{y}}\varphi]\bar{s}\bar{t}) \quad := \quad \mathrm{free}(\bar{s}) \cup \mathrm{free}(\bar{t}) \cup (\mathrm{free}(\varphi)\backslash\{\bar{x},\bar{y}\})$$

and similarly for FO(DTC).

The semantics is defined inductively with respect to the calculus above, the meanings of $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi]\bar{s}\bar{t}$ and $[\mathrm{DTC}_{\bar{x},\bar{y}}\varphi]\bar{s}\bar{t}$ being as follows.

$$\mathcal{A} \models [\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x},\bar{y})]\bar{s}\bar{t} \quad \text{iff} \quad (\bar{s},\bar{t}) \in \mathrm{TC}(\{(\bar{a},\bar{b}) \mid \mathcal{A} \models \varphi(\bar{a},\bar{b})\})$$

$$\mathcal{A} \models [\mathrm{DTC}_{\bar{x},\bar{y}}\varphi(\bar{x},\bar{y})]\bar{s}\bar{t} \quad \text{iff} \quad (\bar{s},\bar{t}) \in \mathrm{DTC}(\{(\bar{a},\bar{b}) \mid \mathcal{A} \models \varphi(\bar{a},\bar{b})\})$$

Example 2.2.3 (a) A graph is connected if it is a model of

$$\forall x \forall y(\neg x = y \rightarrow [\mathrm{TC}_{x,y}Exy]xy)$$

(b) For $\tau = \{<, S, \min, \max\}$ the sentence

$$\neg[\mathrm{DTC}_{x,y}\exists z(Sxz \wedge Szy)]\min\ \max$$

of FO(DTC) together with the ordering axioms axiomatizes the class of orderings of even cardinality. On the contrary, the evenness property is not definable in first-order logic, as we will show in the next chapter (see Example 3.3.5). □

To compare the expressive power of logics we introduce the following relations.

Definition 2.2.4 Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics.

(a) $\mathcal{L}_1 \leq \mathcal{L}_2$ ($\mathcal{L}_1$ *is at most as expressive as* $\mathcal{L}_2$) if for every τ and every sentense $\varphi \in \mathcal{L}_1[\tau]$ there is a sentense $\psi \in \mathcal{L}_2[\tau]$ such that $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$.

(b) $\mathcal{L}_1 \equiv \mathcal{L}_2$ ($\mathcal{L}_1$ *and* $\mathcal{L}_2$ *have the same expressive power*) if $\mathcal{L}_1 \leq \mathcal{L}_2$ and $\mathcal{L}_2 \leq \mathcal{L}_1$.

(c) $\mathcal{L}_1 < \mathcal{L}_2$ if $\mathcal{L}_1 \leq \mathcal{L}_2$ and not $\mathcal{L}_2 \leq \mathcal{L}_1$. □

Clearly, FO $<$ SO where FO and SO stand for first-order logic and second-order logic, respectively.

Proposition 2.2.5 (a) FO(IFP) $\leq$ FO(PFP)

(b) FO(DTC) $\leq$ FO(TC)

(c) FO(TC) $\leq$ FO(IFP)

Proof. Note that

(a) $\models_{\mathrm{fin}} [\mathrm{IFP}_{\bar{x},X}\varphi]\bar{t} \leftrightarrow [\mathrm{PFP}_{\bar{x},X}(X\bar{x} \vee \varphi)]\bar{t}$

(b) $\models_{\mathrm{fin}} [\mathrm{DTC}_{\bar{x},\bar{y}}\varphi(\bar{x},\bar{y})]\bar{s}\bar{t} \leftrightarrow [\mathrm{TC}_{\bar{x},\bar{y}}(\varphi(\bar{x},\bar{y}) \wedge \forall\bar{z}(\varphi(\bar{x},\bar{z}) \rightarrow \bar{z} = \bar{y}))]\bar{s}\bar{t}$

(c) $\models_{\mathrm{fin}} [\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x},\bar{y})]\bar{s}\bar{t} \leftrightarrow [\mathrm{IFP}_{\bar{x}\bar{y},X}(\varphi(\bar{x},\bar{y}) \vee \exists\bar{v}(X\bar{x}\bar{v} \wedge \varphi(\bar{v},\bar{y})))]\bar{s}\bar{t}$ □

2.3 Descriptive Complexity Theory

Descriptive complexity theory analyzes the complexity of all queries definable in a given logic, the central question being the following: Given a complexity class $\mathcal{C}$, is there a logic $\mathcal{L}$ such that the queries definable in $\mathcal{L}$ are precisely the queries in $\mathcal{C}$? In this section, we briefly glance over the descriptive characterizations of complexity classes using fixed-point logics just introduced in the preceding section.

First, we should recall basic definitions and results from computation theory, to fix our computation model for structures as inputs, and to introduce the corresponding complexity class.

In the following, we fix a finite alphabet Σ. A *Turing machine* M is a finite device that performs operations on a tape which is bounded to the left and unbounded to the right and divided into *cells*. The machine operates stepwise, each step leading from one situation to a new one. In any situation every cell of the tape either contains a single symbol from Σ or is blank. In the latter case we say that it contains the symbol "blank". There is one exception: the leftmost or "virtual" cell always contains an endmark, the "virtual" letter α (which is not in Σ). M has a reading/writing head which, in any situation, scans a single cell of the tape and, in any step of a computation, erases or replaces the scanned symbol by another one and moves one cell to the left or to the right or remains at its place.

In every situation, M is in one of the states of a finite set $\mathrm{State}(M)$, the *set of states* of M. $\mathrm{State}(M)$ contains a special state s_0, the *initial* state, and special states s_+, the *accepting* state, and s_-, the *rejecting* state. We assume that s_0, s_+, and s_- are pairwise distinct. The action or behaviour of M in a situation depends on the current state of M and on the symbol currently being scanned by the head. It is given by $\mathrm{Instr}(M)$, the *set of instructions* of M. Each instruction has the form

$$sa \rightarrow s'bh \tag{2.1}$$

where

- $s, s' \in \mathrm{State}(M),\ s \neq s_+,\ s \neq s_-$
- $a, b \in \Sigma \cup \{\alpha, \mathrm{blank}\}$ and $(a = \alpha \text{ iff } b = \alpha)$
- $h \in \{-1, 0, 1\}$, and if $a = \alpha$ then $h \neq -1$

The instruction (2.1) means: If you are in state s and your head scans a cell with symbol a, replace a by b, move your head one cell to the left ($h = -1$), or to the right ($h = 1$), or don't move ($h = 0$); finally, change to state s'.

A machine M is "deterministic", if for all $s \in \mathrm{State}(M)$ and $a \in \Sigma \cup \{\alpha, \mathrm{blank}\}$ there is at most one instruction of the form (2.1) in $\mathrm{Instr}(M)$.

As usual, Σ^* denotes the set of words over Σ and Σ^+ the set of nonempty words over Σ. Let $u \in \Sigma^*$, $u = a_1 \ldots a_r$ with $a_i \in \Sigma$. M is *started* with u if M begins a computation in state s_0 in the situation.

The computation proceeds stepwise, each step corresponding to the execution of one instruction of M. The machine stops when it is in a state s scanning a symbol $a \in \Sigma \cup \{\alpha, \text{blank}\}$ such that there is no instruction of the form (2.1) in $\text{Instr}(M)$. If $s = s_+$ we speak of an *accepting run*, if $s = s_-$ of a *rejecting run*. *M accepts u* if there is at least one accepting run of M started with u, and *M rejects u* if all runs are finite and rejecting.

Subsets of Σ^+ are called *languages*. A language $L \subseteq \Sigma^+$ is *accepted* by M if for all $u \in \Sigma^+$,

$$M \text{ accepts } u \qquad \text{iff} \qquad u \in L$$

L is *decided* by M if, in addition

$$M \text{ rejects } u \qquad \text{iff} \qquad u \notin L$$

Clearly, if M decides L then M accepts L. L is said to be *decidable* if it is decided by some deterministic Turing machine, and *acceptable* or *enumerable* if it is accepted by some nondeterministic Turing machine.

For a function $f : N \to N$ we say that M is *f time-bounded*, if for all $u \in \Sigma^+$ accepted by M there is an accepting run of M started with u which has length at most $f(|u|)$ ($|u|$ denotes the length of the word u). And M is *f space-bounded*, if for all $u \in \Sigma^+$ accepted by M there is an accepting run which uses at most $f(|u|)$ cells before stopping. $N[x]$ denotes the set of polynomials with coefficients from N. A language $L \subseteq \Sigma^+$ is in PTIME ("polynomial time") or in PSPACE ("polynomial space"), if it is accepted by a deterministic machine that is p time-bounded or p space-bounded, respectively, for some polynomial $p \in N[x]$. The classes NPTIME ("nondeterministic polynomial time") and NPSPACE ("nondeterministic polynomial space") are defined similarly, now allowing nondeterministic machines.

Immediately from the definitions one gets

$$\text{PTIME} \subseteq \text{NPTIME} \quad \text{and} \quad \text{PTIME} \subseteq \text{PSPACE} \subseteq \text{NPSPACE}$$

and one can show that

$$\text{NPTIME} \subseteq \text{PSPACE} \quad \text{and} \quad \text{PSPACE} = \text{NPSPACE}$$

Hence,

$$\text{PTIME} \subseteq \text{NPTIME} \subseteq \text{PSPACE} \; (= \text{NPSPACE})$$

We have adopted the Turing machine model which belongs to the most popular one in theoretical computer science. Our choice is motivated by the fact that Turing machine computations allow for simple descriptions and for natural definitions of complexity classes. However, as Turing machines deal only with strings, any data must be coded by strings. Hence, when we regard finite structures as inputs to machines, we require that structures are *ordered* so as to make it easy to represent them by sequence of strings. Recall the definition:

Definition 2.3.1 Let $\{<\} \subseteq \tau_0 \subseteq \{<, S, \min, \max\}$ and $\tau_0 \subseteq \tau$. A τ-structure $\mathcal{A}$ is *ordered* if the reduct $\mathcal{A}|\tau_0$ is an ordering.

Suppose $\tau = \tau_0 \cup \tau_1$, say $\tau_1 = \{R_1, \ldots, R_k, c_1, \ldots, c_l\}$, and let $\mathcal{A}$ be an ordered τ-structure. We say that a Turing machine M is *started with* $\mathcal{A}$, if the input tapes contain the information on $\mathcal{A}$. The informations of $\mathcal{A}$ must include its size $\|A\|$, relations R_i^A for $R_i \in \tau_1$, and constants c_i^A for $c_i \in \tau_1$. Of course, there is no canonical way of representing structures by strings. Still, one can agree that any way of representation will do as long as it is a "natural" way, for it makes no difference within the complexity classes under consideration. We will not mention a specific method of representation here. For details, see [5].

Let K be a class of ordered τ-structures. M *accepts* K if M accepts exactly those ordered τ-structures that lie in K. For classes of structures the definitions of PTIME ("polynomial time"), NPTIME ("nondeterministic polynomial time"), PSPACE ("polynomial space") are introduced in the obvious way. For example,

- K is in PTIME iff there is a deterministic machine M and a polynomial $p \in N[x]$ such that M accepts K and M is p time-bounded.

And we define

- K is in NLOGSPACE, "nondeterministic logarithmic space" (LOGSPACE, "deterministic logarithm space") iff there is a (deterministic) machine M and $d \geq 1$ such that M accepts K and is $d \cdot \log$ space-bounded ($\log n$ stands for the least natural number $\geq \log_2 n$).

Now we are in a position to look over the main results obtained so far in the field. The following theorem provides the bridge between logic and complexity theory

Theorem 2.3.2 Let K be a class of ordered τ-structures.

$K \in$ LOGSPACE	iff	K is axiomatizable in FO(DTC)
$K \in$ NLOGSPACE	iff	K is axiomatizable in FO(TC)
$K \in$ PTIME	iff	K is axiomatizable in FO(IFP)
$K \in$ NPTIME	iff	K is axiomatizable in Σ_1^1
$K \in$ PSPACE	iff	K is axiomatizable in FO(PFP)

(Σ_1^1 denotes the fragment of second-order logic consisting of the sentences of the form $\exists X_1 \cdots \exists X_n \psi$, where ψ is first-order) □

These characterizations of complexity classes are due to Immerman [13] (LOGSPACE, NLOGSPACE), Immerman [12] and Vardi [16] (PTIME), Fagin [7] (NPTIME), Abiteboul and Vianu [1] (PSPACE). Theorem 2.3.2 allows us to convert problems, methods, and results in complexity theory into logic and vice versa. We give some examples.

The following consequences are immediate from Theorem 2.3.2.

Corollary 2.3.3 (a) PTIME = PSPACE
iff FO(IFP) $\equiv$ FO(PFP) on ordered structures

(b) PTIME = NPTIME
iff FO(IFP) $\equiv \Sigma_1^1$ on ordered structures □

Moreover,

Corollary 2.3.4 The following are equivalent:

(i) PTIME = NPTIME

(ii) FO(IFP) $\equiv$ SO on ordered structures

Proof. If (ii) holds then $\Sigma_1^1 \leq$ FO(IFP) on ordered structures, thus NPTIME $\leq$ PTIME. Conversely, if NPTIME = PTIME then, on ordered structures, $\Sigma_1^1 \equiv$ FO(IFP). As Σ_1^1 is closed under existential quantifications and FO(IFP) under negations and disjunctions, an easy induction yields SO $\equiv$ FO(IFP): For a logic $\mathcal{L}$ we write $\varphi \mathrel{\tilde{\in}} \mathcal{L}$ to express that φ is equivalent to an $\mathcal{L}$-sentence. Assume $\varphi \in$ SO, $\varphi = \neg\psi$ and suppose $\psi \mathrel{\tilde{\in}}$ FO(IFP), then we have $\neg\psi \mathrel{\tilde{\in}}$ FO(IFP). In the cases $\varphi = \psi \vee \chi$ and $\varphi = \exists x \psi$ we argue similarly. Suppose $\varphi = \exists X \psi$ and $\psi \mathrel{\tilde{\in}}$ FO(IFP). By the assumption FO(IFP) $\equiv \Sigma_1^1$, we have $\psi \mathrel{\tilde{\in}} \Sigma_1^1$ and hence $\exists X \psi \mathrel{\tilde{\in}} \Sigma_1^1 \equiv$ FO(IFP). □

Whereas the preceding corollaries contain the translation of *problems* from complexity theory to logics we now turn to the translation of a *result*. In complexity theory one shows

$$\text{LOGSPACE} \subseteq \text{NLOGSPACE} \subseteq \text{PTIME} \subseteq \text{NPTIME} \subseteq \text{PSPACE}$$

and

$$\text{LOGSPACE} \neq \text{PSPACE}$$

Hence, by Theorem 2.3.2,

Corollary 2.3.5 On ordered structures,

(a) FO(DTC) $\leq$ FO(TC) $\leq$ FO(IFP) $\leq \Sigma_1^1 \leq$ FO(PFP)

(b) FO(DTC) $\not\equiv$ FO(PFP) □

Note that most of the $\leq$-relations in (a) are immediate from the Proposition 2.2.5.

Chapter 3

Ehrenfeucht-Fraïssé Game

The Ehrenfeucht-Fraïssé method is one of the few tools of model theory that survive when we restrict our attention to finite structures. In this chapter we present the method in its gametheoretic and its algebraic form due to Ehrenfeucht and Fraïssé, respectively. Although we describe the method in a context of finite model theory, one will easily verifies that the method also applies to the case when we take infinite structures into consideration.

We fix a vocabulary τ through this chapter, so that we will not refer to the vocabulary unless stated otherwise.

The style of the descriptions in this chapter is due to [5].

3.1 Elementary Classes

In this section we observe some easy remarks concerning the expressive power of first-order logic in the finite.

Proposition 3.1.1 Every finite structure can be characterized in first-order logic up to isomorphism, that is, for every finite structure $\mathcal{A}$ there is a sentence $\varphi_{\mathcal{A}}$ of first-order logic such that for all $\mathcal{B}$ we have

$$\mathcal{B} \models \varphi_{\mathcal{A}} \quad \text{iff} \quad \mathcal{A} \cong \mathcal{B}$$

Proof. Suppose $A = \{a_1, \ldots, a_n\}$. Set $\bar{a} = a_1, \ldots, a_n$. Let

$$\begin{aligned} \Theta_n \ := \ & \{\psi \mid \psi \text{ has the form } Rx_1 \ldots x_k, x = y, \text{ or } c = x \\ & \text{and variables among } v_1, \ldots, v_n\} \end{aligned}$$

and

$$\begin{aligned} \varphi_{\mathcal{A}} \ := \ & \exists v_1 \ldots \exists v_n (\bigwedge \{\psi \mid \psi \in \Theta_n,\ \mathcal{A} \models \psi[\bar{a}]\} \wedge \\ & \bigwedge \{\neg\psi \mid \psi \in \Theta_n,\ \mathcal{A} \models \neg\psi[\bar{a}]\} \wedge \\ & \forall v_{n+1} (v_{n+1} = v_1 \vee \ldots \vee v_{n+1} = v_n)) \end{aligned}$$

□

Corollary 3.1.2 Let K be a class of finite structures. Then there is a set Φ of first-order sentences such that

$$K = \text{Mod}(\Phi)$$

that is, K is the class of finite models of Φ.

Proof. Let K be a class of finite structures. For each n, there are only finite number of pairwise nonisomorphic structures of cardinality n. Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ be a maximal subset K of pairwise nonisomorphic structures of cardinality n. Set

$$\psi_n \ := \ (\varphi_{=n} \rightarrow (\varphi_{\mathcal{A}_1} \vee \ldots \vee \varphi_{\mathcal{A}_K}))$$

Then $K = \text{Mod}(\{\psi_n \mid n \geq 1\})$. □

In the following, we want to know whether a class K of finite structures is axiomatizable by a *single* first-order sentence, that is, whether K is elementary in the sense of the following definition.

Definition 3.1.3 Let K be a class of finite structures. K is called *axiomatizable in first-order logic* or *elementary*, if there is a sentence φ of first-order logic such that $K = \text{Mod}(\varphi)$. □

For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in N$ we write $\mathcal{A} \equiv_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *m-equivalent*, if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$, that is, for any sentence φ of quantifier rank $\leq m$

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{B} \models \varphi$$

The following theorem contains a necessary condition for a class K to be elementary.

Proposition 3.1.4 Let K be a class of finite structures. Suppose that for every m there are finite structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} \in K, \mathcal{B} \notin K, \text{ and } \mathcal{A} \equiv_m \mathcal{B}$$

Then K is not axiomatizable in first-order logic.

Proof. Let φ be any first-order sentence. Set $m := \text{qr}(\varphi)$. By our assumption there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$, and $\mathcal{A} \equiv_m \mathcal{B}$; hence, $K \neq \text{Mod}(\varphi)$. □

3.2 Ehrenfeucht's Theorem

In this section we present a gametheoretic characterization of the relation $\equiv_m$. Along with Proposition 3.1.4, it can be used for checking up the axiomatizability of a class of structures in first-order logic.

The notion of partial isomorphism plays a central role in this characterization.

Definition 3.2.1 Assume $\mathcal{A}$ and $\mathcal{B}$ are structures. Let p be a map with $\text{do}(p) \subseteq A$ and $\text{rg}(p) \subseteq B$, where $\text{do}(p)$ and $\text{rg}(p)$ denote the domain and the range of p, respectively. Then p is said to be a *partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if

- p is injective
- for every $c \in \tau : c^A \in \text{do}(p)$ and $p(c^A) = c^B$
- for every n-ary $R \in \tau$ and all $a_1, \ldots, a_n \in \text{do}(p)$,

$$R^A a_1 \ldots a_n \quad \text{iff} \quad R^B p(a_1) \ldots p(a_n)$$

We write Part($\mathcal{A}$,$\mathcal{B}$) for the set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. □

In the following we identify a map p with its graph $\{(a, p(a)) \mid a \in \text{do}(p)\}$. Then $p \subseteq q$ means that q is an extension of p.

Remark 3.2.2 (a) The empty map, $p = \emptyset$, is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ just in case the vocabulary contains no constants.

(b) If $p \neq \emptyset$ is a map with $\text{do}(p) \subseteq A$ and $\text{rg}(p) \subseteq B$, then p is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff $\text{do}(p)$ contains c^A for all constants $c \in \tau$ and $p : \text{do}(p)^{\mathcal{A}} \cong \text{rg}(p)^{\mathcal{B}}$ (where $\text{do}(p)^{\mathcal{A}}$ and $\text{rg}(p)^{\mathcal{B}}$ denote the substructures of $\mathcal{A}$ and $\mathcal{B}$ with universes $\text{do}(p)$ and $\text{rg}(p)$, respectively).

(c) For $\bar{a} = a_1 \ldots a_n \in A$ and $\bar{b} = b_1 \ldots b_n \in B$ the following statements are equivalent:

(i) The clauses

$$p(a_i) = b_i \text{ for } i = 1, \ldots, s$$

and

$$p(c^A) = c^B \text{ for } c \text{ in } \tau$$

fix a map, which is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (henceforth denoted by $\bar{a} \mapsto \bar{b}$, a notation that suppresses the constants).

(ii) For all quantifierfree $\varphi(v_1, \ldots, v_s)$: $\mathcal{A} \models \varphi[\bar{a}]$ iff $\mathcal{B} \models \varphi[\bar{b}]$.

(iii) For all atomic $\varphi(v_1, \ldots, v_s)$: $\mathcal{A} \models \varphi[\bar{a}]$ iff $\mathcal{B} \models \varphi[\bar{b}]$.

Proof. Clearly, (ii) implies (iii), and (ii) follows from (iii), since every quantifierfree formula is a boolean combination of atomic formulas: in fact, we can prove it by induction on the length of φ. If φ is atomic, then the conclusion is immediate. If $\varphi := \neg\psi$, then

$$\begin{aligned} \mathcal{A} \models \varphi[\bar{a}] \quad &\text{iff} \quad \mathcal{A} \models \neg\psi[\bar{a}] \\ &\text{iff} \quad \mathcal{B} \models \neg\psi[\bar{b}] \quad \text{(ind.hyp.)} \\ &\text{iff} \quad \mathcal{B} \models \varphi[\bar{b}] \end{aligned}$$

If $\varphi := \psi \vee \chi$ then

$$\begin{aligned}
\mathcal{A} \models \varphi[\bar{a}] \quad & \text{iff} \quad \mathcal{A} \models (\psi \vee \chi)[\bar{a}] \\
& \text{iff} \quad \mathcal{A} \models \psi[\bar{a}] \text{ or } \mathcal{A} \models \chi[\bar{a}] \\
& \text{iff} \quad \mathcal{B} \models \psi[\bar{b}] \text{ or } \mathcal{B} \models \chi[\bar{b}] \quad \text{(ind.hyp.)} \\
& \text{iff} \quad \mathcal{B} \models (\psi \vee \chi)[\bar{b}] \\
& \text{iff} \quad \mathcal{B} \models \varphi[\bar{b}]
\end{aligned}$$

Next, we show the equivalence of (i) and (iii). Note that for an arbitrary structure $\mathcal{D}$ and $\bar{d}$ in $\mathcal{D}$,

$$\begin{aligned}
d_i = d_j \quad & \text{iff} \quad \mathcal{D} \models (v_i = v_j)[\bar{d}] \\
c^D = d_j \quad & \text{iff} \quad \mathcal{D} \models (c = v_j)[\bar{d}] \\
R^D c^D d_i d_j \quad & \text{iff} \quad \mathcal{D} \models R c v_i v_j[\bar{d}] \qquad (c, R \in \tau,\ R \text{ ternary})
\end{aligned}$$

Using equivalences above, it is easy to prove that (i) implies (iii); for $\varphi := R c v_i v_j$,

$$\begin{aligned}
\mathcal{A} \models \varphi[\bar{a}] \quad & \text{iff} \quad \mathcal{A} \models R c v_i v_j[\bar{a}] \\
& \text{iff} \quad R^A c^A a_i a_j \\
& \text{iff} \quad R^B c^B b_i b_j \quad \text{(by (i))} \\
& \text{iff} \quad \mathcal{B} \models R c v_i v_j[\bar{b}] \\
& \text{iff} \quad \mathcal{B} \models \varphi[\bar{b}]
\end{aligned}$$

(iii) $\Rightarrow$ (i) : It suffices to show that the conditions in 3.2.1 are satisfied. Let φ be $v_i = v_j$, then

$$\begin{aligned}
a_i = a_j \quad & \text{iff} \quad \mathcal{A} \models (v_i = v_j)[\bar{a}] \\
& \text{iff} \quad \mathcal{B} \models (v_i = v_j)[\bar{b}] \quad \text{(by (iii))} \\
& \text{iff} \quad b_i = b_j
\end{aligned}$$

Such equivalences imply that p is a map and injective. It is easy to show that the third condition in 3.2.1 is also satisfied; for example, ternary $R \in \tau$ and $c^A, a_i, a_j \in \text{do}(p)$,

$$\begin{aligned}
R^A c^A a_i a_j \quad & \text{iff} \quad \mathcal{A} \models R c v_i v_j[\bar{a}] \\
& \text{iff} \quad \mathcal{B} \models R c v_i v_j[\bar{b}] \quad \text{(by (iii))} \\
& \text{iff} \quad R^B c^B b_i b_j
\end{aligned}$$

□

Definition 3.2.3 Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \in N$. The *Ehrenfeucht game* $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ is played by two players called the *spoiler* and the *duplicator*.

▷ Each player has to make m moves in the course of a play.

- In his i-th move, the spoiler first selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and an element in this structure.
- If the spoiler chooses e_i in $\mathcal{A}$ then the duplicator must choose an element f_i in $\mathcal{B}$. If the spoiler chooses f_i in $\mathcal{B}$ then the duplicator must choose an element e_i in $\mathcal{A}$.

▷ At the end, elements $e_1, \ldots, e_m$ in $\mathcal{A}$ and $f_1, \ldots, f_m$ in $\mathcal{B}$ have been chosen.

- The duplicator *wins* if $\bar{a}\bar{e} \mapsto \bar{b}\bar{f}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (in case $m = 0$ we just require that $\bar{a} \mapsto \bar{b}$ is a partial isomorphism).
- Otherwise, the spoiler wins.

▷ We say that a player, the spoiler or the duplicator, has a winning strategy in $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, or shortly that he *wins* $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, if it is possible for him to win whatever choices are made by the other player. □

If $s = 0$ (hence $\bar{a}$ and $\bar{b}$ are empty), the game is denoted by $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$.

Lemma 3.2.4 (a) If $\mathcal{A} \cong \mathcal{B}$ then the duplicator wins $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$.

(b) If the duplicator wins $\mathrm{G}_{m+1}(\mathcal{A}, \mathcal{B})$ and $\|A\| \leq m$ then $\mathcal{A} \cong \mathcal{B}$.

(c) If the duplicator has a winning strategy in $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$ and in $\mathrm{G}_m(\mathcal{B}, \mathcal{C})$, then the duplicator wins $\mathrm{G}_m(\mathcal{A}, \mathcal{C})$.

Proof. (a) Suppose $\pi : \mathcal{A} \cong \mathcal{B}$. A winning strategy for the duplicator consists in the following way; if the spoiler chooses $a \in A$ then the duplicator chooses $\pi(a)$, and if the spoiler chooses $b \in B$ then the duplicator answers with $\pi^{-1}(b)$.

(b) Suppose that the duplicator has a winning strategy in $\mathrm{G}_{m+1}(\mathcal{A}, \mathcal{B})$, and assume that $A = \{a_1, \ldots, a_m\}$. Let us consider a play in which the spoiler, in his first m moves, chooses $a_1, \ldots, a_m$, and let $b_1, \ldots, b_m$ be the responses of the duplicator according to his winning strategy. Then $p : \bar{a} \mapsto \bar{b}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ with $\mathrm{do}(p) = A$. Thus, it suffices to prove that p is surjective. Otherwise, we have $\mathrm{rg}(p) \neq B$. Then the spoiler, in the last move of the play, chooses some element $b \in B \backslash \mathrm{rg}(p)$. As there is no answer for the duplicator leading to win, we get a contradiction.

(c) Assume that Γ and Δ are winning strategy for the duplicator in the games $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$ and $\mathrm{G}_m(\mathcal{B}, \mathcal{C})$, respectively. In order to win the game $\mathrm{G}_m(\mathcal{A}, \mathcal{C})$, the duplicator does the following. Suppose that the spoiler starts by choosing an element $a \in A$. To this move, the duplicator applies Γ, as though it were a first move in the game $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$. The answer b produced by Γ is given as an input to Δ as though it were a first move in $\mathrm{G}_m(\mathcal{B}, \mathcal{C})$. Finally, the answer c given by Δ is returned by the duplicator as his real answer in the game $\mathrm{G}_m(\mathcal{A}, \mathcal{C})$. A similar procedure is carried out when the spoiler chooses an element in $\mathcal{C}$. Eventually, the relations built by Γ and Δ must be partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$, and from $\mathcal{B}$ to $\mathcal{C}$, respectively. Therefore, their composition will be a partial isomorphism as well. □

The following lemma collects some facts about the Ehrenfeucht game. Their proofs are immediate from the definition.

Lemma 3.2.5 Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \in N$.

(a) The duplicator wins $\mathrm{G}_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ iff $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.

(b) For $m > 0$, the duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ iff (for all $a \in A$ there is $b \in B$ such that the duplicator wins $\mathrm{G}_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$, and for all $b \in B$ there is $a \in A$ such that the duplicator wins $\mathrm{G}_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$).

(c) If the duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ and if $m' < m$, then the duplicator wins $\mathrm{G}_{m'}(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$. □

Let $\mathcal{A}$ be given. For $\bar{a} = a_1 \dots a_n \in A$ and $m \geq 0$ we introduce a formula $\varphi_{\bar{a}}^m(v_1, \dots, v_s)$ that describes the gametheoretic properties of $\bar{a}$ in any game $\mathrm{G}_m(\mathcal{A}, \bar{a}, \dots)$.

Definition 3.2.6 Let $\bar{v}$ be $v_1, \dots, v_s$.

$$\varphi_{\bar{a}}^0(\bar{v}) \;:=\; \bigwedge\{\varphi(\bar{v}) \mid \varphi \text{ atomic or negated atomic, } \mathcal{A} \models \varphi[\bar{a}]\}$$

and for $m > 0$,

$$\varphi_{\bar{a}}^m(\bar{v}) \;:=\; \bigwedge_{a \in A} \exists v_{s+1} \varphi_{\bar{a}a}^{m-1}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi_{\bar{a}a}^{m-1}(\bar{v}, v_{s+1})$$

□

$\varphi_{\bar{a}}^m$ is called the *m-isomorphism type* of $\bar{a}$ in $\mathcal{A}$. If the structure $\mathcal{A}$ is not clear from the context, we use the notation $\varphi_{\mathcal{A},\bar{a}}^m$ for $\varphi_{\bar{a}}^m$. We also allow $s = 0$, the case of the empty sequence $\emptyset$ of elements in A, and write $\varphi_{\mathcal{A}}^m$ for the sentence $\varphi_{\mathcal{A},\emptyset}^m$.

Actually, the conjunctions and disjunctions in the above definition are finite, as we see in the following lemma.

Lemma 3.2.7 For $s, m \geq 0$, the set $\{\varphi_{\mathcal{A},\bar{a}}^m \mid \mathcal{A}$ a structure and $\bar{a} \in A^s\}$ is finite.

Proof. The proof is by induction on m. Since we have assumed that a vocabulary contains only finite symbols, $\{\varphi(\bar{v}) \mid \varphi$ atomic or negated atomic$\}$ is finite. For $m > 0$, assume that $\{\varphi_{\mathcal{A},\bar{a}a}^{m-1} \mid \mathcal{A}$ a structure and $\bar{a}a \in A^{s+1}\}$ is finite. As the number of their combinations is finite, the set of formulas $\{\varphi_{\mathcal{A},\bar{a}}^m\}$, a member of which is composed by the formulas $\varphi_{\mathcal{A},\bar{a}a}^{m-1}$, is finite. □

Lemma 3.2.8 (a) $\mathrm{qr}(\varphi_{\bar{a}}^m) = m$

(b) $\mathcal{A} \models \varphi_{\bar{a}}^m[\bar{a}]$

(c) For any $\mathcal{B}$ and $\bar{b}$ in B,

$$\mathcal{B} \models \varphi_{\bar{a}}^0[\bar{b}] \quad \text{iff} \quad \bar{a} \mapsto \bar{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$$

Proof. (a) trivial.

(b) The proof is by induction on m. For $m = 0$, the conclusion is immediate from the definition. For $m > 0$, suppose that for all $a \in A$, $\mathcal{A} \models \varphi^{m-1}_{\bar{a}a}[\bar{a}a]$. Then for all $a \in A$, we have $\mathcal{A} \models \exists v_{s+1}\varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{a}]$. Hence, we proved the first half $\mathcal{A} \models \bigwedge_{a\in A} \exists v_{s+1}\varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{a}]$. For the latter, we get from the induction hypothesis that, for all $a' \in A$, $\mathcal{A} \models \bigvee_{a\in A} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{a}a']$. That is, $\mathcal{A} \models \forall v_{s+1} \bigvee_{a\in A} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{a}]$.

(c) By 3.2.2(c), it suffices to show that, for any $\mathcal{B}$ and $\bar{b}$ in B,

$$\mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}] \quad \text{iff} \quad (\text{for all atomic } \varphi(v_1, \ldots, v_s):\ \mathcal{A} \models \varphi[\bar{a}] \Leftrightarrow \mathcal{B} \models \varphi[\bar{b}])$$

If the condition on the right side holds, then we get from the definition that $\varphi^0_{\bar{a}} = \varphi^0_{\bar{b}}$. Since $\mathcal{B} \models \varphi^0_{\bar{b}}[\bar{b}]$, we have $\mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}]$. Now suppose that $\mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}]$. Assume ψ is atomic. If $\mathcal{A} \models \psi[\bar{a}]$, then ψ is a member of the conjunction in $\varphi^0_{\bar{a}}$. So the assumption $\mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}]$ yields $\mathcal{B} \models \psi[\bar{b}]$. If $\mathcal{A} \models \neg\psi[\bar{a}]$, then $\mathcal{B} \models \neg\psi[\bar{b}]$ follows in a similar way. □

Theorem 3.2.9 (Ehrenfeucht's Theorem) [6] Given $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^s$ and $\bar{b} \in B^s$, and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$

(ii) $\mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}]$

(iii) $\bar{a}$ and $\bar{b}$ satisfy the same formulas of quantifier rank $\leq m$, that is, if $\varphi(x_1, \ldots, x_s)$ is of quantifier rank $\leq m$, then

$$\mathcal{A} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathcal{B} \models \varphi[\bar{b}] \tag{3.1}$$

Proof. (iii) implies (ii) since $\mathrm{qr}(\varphi^m_{\bar{a}}) = m$ and $\mathcal{A} \models \varphi^m_{\bar{a}}[\bar{a}]$. We prove the equivalence of (i) and (ii) by induction on m. For $m = 0$

$$\begin{aligned} \text{the duplicator wins } \mathrm{G}_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \ &\text{iff} \ \ \bar{a} \mapsto \bar{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \quad (\text{by 3.2.5(a)}) \\ &\text{iff} \ \ \mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}] \quad (\text{by 3.2.8(c)}) \end{aligned}$$

For $m > 0$,

$$\begin{aligned} & \text{the duplicator wins } \mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \\ \text{iff} \ \ & \text{for all } a \in A \text{ there is } b \in B \text{ such that the duplicator wins} \\ & \mathrm{G}_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b), \text{ and for all } b \in B \text{ there is } a \in A \text{ such that} \\ & \text{the duplicator wins } \mathrm{G}_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b) \quad (\text{by 3.2.5(b)}) \\ \text{iff} \ \ & \text{for all } a \in A \text{ there is } b \in B \text{ with } \mathcal{B} \models \varphi^{m-1}_{\bar{a}a}[\bar{b}b], \text{ and} \\ & \text{for all } b \in B \text{ there is } a \in A \text{ with } \mathcal{B} \models \varphi^{m-1}_{\bar{a}a}[\bar{b}b] \quad (\text{ind. hyp.}) \\ \text{iff} \ \ & \mathcal{B} \models \textstyle\bigwedge_{a\in A}\exists v_{s+1}\varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1}\bigvee_{a\in A}\varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{b}] \\ \text{iff} \ \ & \mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}] \end{aligned}$$

(i) $\Rightarrow$ (iii) : The proof proceeds by induction on m. For $m = 0$

$$\begin{aligned} & \text{the duplicator wins } \mathrm{G}_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \\ \text{iff} \quad & \bar{a} \mapsto \bar{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \quad \text{(by 3.2.5(a))} \\ \text{iff} \quad & \text{for all quantifier free } \varphi(v_1, \dots, v_s) : \ \mathcal{A} \models \varphi[\bar{a}] \ \Leftrightarrow \ \mathcal{B} \models \varphi[\bar{b}] \quad \text{(by 3.2.2(c))} \end{aligned}$$

For $m > 0$, assume that the duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$. Suppose that $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$ and $\mathrm{qr}(\varphi) \leq m$. Assume, for instance, $\mathcal{A} \models \varphi[\bar{a}]$. Then there is $a \in A$ such that $\mathcal{A} \models \psi[\bar{a}, a]$. As the duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, there is $b \in B$ such that the duplicator wins $\mathrm{G}_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$. Since $\mathrm{qr}(\psi) \leq m - 1$, the induction hypothesis yields $\mathcal{B} \models \psi[\bar{b}, b]$, hence $\mathcal{B} \models \varphi[\bar{b}]$. The opposite direction can be shown similarly. Clearly, the set of formulas $\varphi(x_1, \dots, x_s)$ satisfying (3.1) is closed under $\neg$ and $\vee$. □

Corollary 3.2.10 For structures $\mathcal{A}$, $\mathcal{B}$ and $m \geq 0$ the following are equivalent:

(i) The duplicator wins $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$

(ii) $\mathcal{B} \models \varphi^m_{\mathcal{A}}$

(iii) $\mathcal{A} \equiv_m \mathcal{B}$ □

The corollary together with the following result contain the desired characterization of classes axiomatizable in first-order logic.

Proposition 3.2.11 For a class K of finite structures the following are equivalent:

(i) K is not axiomatizable in first-order logic.

(ii) For each m there are finite structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} \in K, \ \mathcal{B} \notin K \text{ and } \mathcal{A} \equiv_m \mathcal{B}$$

Proof. (ii) $\Rightarrow$ (i) was proven in 3.1.4. For the converse, suppose that (ii) does not hold, that is, for some m and all finite $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \in K \text{ and } \mathcal{A} \equiv_m \mathcal{B} \text{ imply } \mathcal{B} \in K$$

Then $K = \mathrm{Mod}(\bigvee\{\varphi^m_{\mathcal{A}} \mid \mathcal{A} \in K\})$, and thus K is axiomatizable. □

3.3 Fraïssé's Theorem

To apply Ehrenfeucht's characterization to concrete examples, it is more convenient to use an algebraic version due to Fraïssé.

Given structures $\mathcal{A}$, $\mathcal{B}$ and $m \geq 0$, let $W_m(\mathcal{A}, \mathcal{B}) :=$

$$\{\bar{a} \mapsto \bar{b} \mid s \geq 0, \bar{a} \in A^s, \bar{b} \in B^s, \text{ the duplicator wins } \mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})\}$$

be the set of winning positions for the duplicator. The sequence of the $W_m(\mathcal{A}, \mathcal{B})$ has the back and forth properties as introduced in the following definition.

Definition 3.3.1 Structures $\mathcal{A}$ and $\mathcal{B}$ are said to be *m-isomorphic*, written $\mathcal{A} \cong_m \mathcal{B}$, if there is a sequence $(I_j)_{j\leq m}$ with the following properties:

(a) Every I_j is a nonempty set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

(b) (*Forth property*) For every $j < m$, $p \in I_{j+1}$, and $a \in A$, there is $q \in I_j$ such that $q \supseteq p$ and $a \in \mathrm{do}(q)$.

(c) (*Back property*) For every $j < m$, $p \in I_{j+1}$, and $b \in B$, there is $q \in I_j$ such that $q \supseteq p$ and $b \in \mathrm{rg}(q)$.

If $(I_j)_{j\leq m}$ has the properties (a), (b) and (c), we write $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *m-isomorphic via* $(I_j)_{j\leq m}$. □

Using the results of the proceeding section we obtain:

Theorem 3.3.2 For structures $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$.

(ii) $\bar{a} \mapsto \bar{b} \in W_m(\mathcal{A}, \mathcal{B})$ and $(W_j(\mathcal{A}, \mathcal{B}))_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$.

(iii) There is $(I_j)_{j\leq m}$ with $\bar{a} \mapsto \bar{b} \in I_m$ such that $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$.

(iv) $\mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}]$.

(v) $\bar{a}$ satisfies in $\mathcal{A}$ the same formulas of quantifier rank $\leq m$ as $\bar{b}$ in $\mathcal{B}$.

Proof. By 3.2.5, (i) implies (ii), and obviously, (ii) implies (iii). Therefore it suffices to show the implication (iii) $\Rightarrow$ (i), for the remaining equivalences are clear from 3.2.9. For (iii) $\Rightarrow$ (i) suppose that $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ and $\bar{a} \mapsto \bar{b} \in I_m$. We describe a winning strategy in $\mathrm{G}_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ for the duplicator: let $p_{i-1} \in I_{m-i+1}$, hence $p_{i-1} : \bar{a}e_1 \dots e_{i-1} \mapsto \bar{b}f_1 \dots f_{i-1}$, and if the spoiler's choice is $e_i \in A$ (or $f_i \in B$), then in the i-th move the duplicator chooses the element f_i (or e_i, respectively) such that $p_i \supseteq p_{i-1}$ and $p_i : \bar{a}e_1 \dots e_i \mapsto \bar{b}f_1 \dots f_i$; it is always possible for him to choose such an element because of the back and forth properties of $(I_j)_{j\leq m}$. Looking at $i := m$ we see that the duplicator wins. □

For $s = 0$ the preceding theorem yields the following extension of 3.2.10.

Corollary 3.3.3 For structures $\mathcal{A}$, $\mathcal{B}$ and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$.

(ii) $(W_j(\mathcal{A}, \mathcal{B}))_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$.

(iii) $\mathcal{A} \cong_m \mathcal{B}$.

(iv) $\mathcal{B} \models \varphi^m_{\mathcal{A}}$.

(v) $\mathcal{A} \equiv_m \mathcal{B}$. □

The equivalence of (iii) and (v) is known as *Fraïssé's Theorem* ([8]). The proof of the preceding theorem shows that Ehrenfeucht's Theorem and Fraïssé's Theorem are different formulations of the same fact. Therefore one often calls it the *Ehrenfeucht-Fraïssé game* or the *Ehrenfeucht-Fraïssé method*.

Example 3.3.4 Let τ be the empty vocabulary and $\mathcal{A}$ and $\mathcal{B}$ be τ-structures (nonempty sets). Suppose $\|A\| \geq m$ and $\|B\| \geq m$. Then $\mathcal{A} \cong_m \mathcal{B}$. In fact, $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ with $I_j := \{p \in \text{Part}(\mathcal{A}, \mathcal{B}) \mid \|\text{do}(p)\| \leq m - j\}$.

As a consequence, the class EVEN(τ) of finite τ-structures of even cardinality is not axiomatizable in first-order logic. In fact, for each $m \geq 0$, let $\mathcal{A}_m$ be a structure of cardinality m. Then, $\mathcal{A}_m \in \text{EVEN}(\tau)$ iff $\mathcal{A}_{m+1} \notin \text{EVEN}(\tau)$, but $\mathcal{A}_m \cong_m \mathcal{A}_{m+1}$. Now apply 3.2.11.

We can show that EVEN(τ) is not axiomatizable also for an arbitrary τ. Now we construct a structure $\mathcal{A}$ in which, for all relation symbol $R \in \tau$, the relation R^A is empty, and all the constants are fixed to an element in A. Considering only such kind of structures, it is verified similarly that for $\|A\| \geq m$ and $\|B\| \geq m$, then $\mathcal{A} \cong_m \mathcal{B}$. □

Example 3.3.5 Let $\tau = \{<, \min, \max\}$ be the vocabulary for finite orderings and $m \geq 1$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite orderings, $\|A\| > 2^m$ and $\|B\| > 2^m$. Then $\mathcal{A} \cong_m \mathcal{B}$, as we show in the following. Hence, the class of finite orderings of even cardinality is not axiomatizable in first-order logic.
Proof. Given any ordering $\mathcal{C}$, we define its distance function d by

$$d(a, a') \quad := \quad \|\{b \in C \mid (a < b \leq a') \text{ or } (a' < b \leq a)\}\|$$

And for $j \geq 0$, we introduce the "truncated" j-distance function d_j on $C \times C$ by

$$d_j(a, a') \quad := \quad \begin{cases} d(a, a') & \text{if } d(a, a') < 2^j \\ \infty & \text{otherwise} \end{cases}$$

Now, suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite orderings with $\|A\|, \|B\| > 2^m$. For $j \leq m$ set

$$I_j \quad := \quad \{p \in \text{Part}(\mathcal{A}, \mathcal{B}) \mid d_j(a, a') = d_j(p(a), p(a')) \text{ for } a, a' \in \text{do}(p)\}$$

Then $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$: By assumption on the cardinalities of $\mathcal{A}$ and $\mathcal{B}$ we have $\{(\min^A, \min^B), (\max^A, \max^B)\} \in I_j$ for every $j \leq m$. To give a proof of the forth property of $(I_j)_{j\leq m}$ (the back property can be proven analogously), suppose $j < m$, $p \in I_{j+1}$, and $a \in A$. We distinguish two cases, depending on whether or not the following condition

$$\text{there is an } a' \in \text{do}(p) \text{ such that } d(a, a') < 2^j \tag{3.2}$$

is satisfied. If (3.2) holds then there is exactly one $b \in B$ for which $p \cup \{(a, b)\}$ is a partial isomorphism preserving d_j-distances. Now assume that (3.2) does not hold and let $\text{do}(p) = \{a_1, \ldots, a_r\}$ with $\min^A = a_1 < \ldots < a_r = \max^A$. We restrict ourselves to the case $a_i < a < a_{i+1}$ for some i. Then, $d_j(a_i, a) = \infty$ and $d_j(a, a_{i+1}) = \infty$; hence, $d_{j+1}(a_i, a_{i+1}) = \infty$ and therefore, $d_{j+1}(p(a_i), p(a_{i+1})) = \infty$. Thus there is b such that $p(a_i) < b < p(a_{i+1})$ with $d_j(p(a_i), b) = \infty$ and $d_j(b, p(a_{i+1})) = \infty$. One verifies that $q := p \cup \{(a, b)\}$ is a partial isomorphism in I_j. □

3.4 Hanf's Theorem

We assume that all vocabularies in this section are relational. For nonempty subset M of a structure $\mathcal{A}$, $\mathcal{M}$ denotes the substructure of $\mathcal{A}$ with universe M.

Given a structure $\mathcal{A}$, we define the binary relation E^A on A by

$$E^A := \{(a,b) \mid a \neq b, \text{ and there are } R \text{ in } \tau \text{ and } \bar{c} \in A \text{ such that } R^A\bar{c} \text{ where } a \text{ and } b \text{ are components of the tuple } \bar{c}\}$$

The $\mathcal{G}(\mathcal{A}) := (A, E^A)$ is called the *graph of* $\mathcal{A}$. Obviously, if $\mathcal{A}$ itself is a graph then $\mathcal{G}(\mathcal{A}) := \mathcal{A}$. For a in A and $r \in N$, $S(r,a)$ (or $S^{\mathcal{A}}(r,a)$) denotes the *r-sphere* of a,

$$S(r,a) \quad := \quad \{b \in A \mid d(a,b) \leq r\}$$

$\mathcal{S}(r,a)$ (or $\mathcal{S}^{\mathcal{A}}(r,a)$) stands for the substructure of $\mathcal{A}$ with universe $S(r,a)$. Note that for $b, c \in S(r,a)$ we have $d(b,c) \leq 2r$. For $\bar{a} = a_1 \ldots a_s$ we set $S(r,\bar{a}) := S(r,a_1) \cup \ldots \cup S(r,a_s)$.

We define the *r-sphere type* of a point a in $\mathcal{A}$ to be the isomorphism type of $(\mathcal{S}(r,a), a)$, that is, points a in $\mathcal{A}$ and b in $\mathcal{B}$ have the same r-sphere type iff $(\mathcal{S}^{\mathcal{A}}(r,a), a) \cong (\mathcal{S}^{\mathcal{B}}(r,b), b)$.

Theorem 3.4.1 (Hanf's Theorem) [10] Let $\mathcal{A}$ and $\mathcal{B}$ be structures and let $m \in N$. Suppose that for some $e \in N$, every 3^m-sphere in $\mathcal{A}$ and $\mathcal{B}$ contains less than e elements, and that for each $n \leq 3^m$ and n-sphere type ι, (i) or (ii) holds

(i) $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements of n-sphere type ι.

(ii) both $\mathcal{A}$ and $\mathcal{B}$ have more than $m \cdot e$ elements of n-sphere type ι.

Then $A \equiv_m B$.

Proof. We show that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$, where I_j is the set

$$\{\bar{a} \mapsto \bar{b} \in \text{Part}(\mathcal{A},\mathcal{B}) \mid (\mathcal{S}(3^j,\bar{a}),\bar{a}) \cong (\mathcal{S}(3^j,\bar{b}),\bar{b}) \text{ and length}(\bar{a}) \leq m - j\},$$

and for length$(\bar{a}) = 0$, we set $(\mathcal{S}(3^j,\bar{a}),\bar{a}) = \emptyset$ and agree that $\emptyset \cong \emptyset$. First, we have $\emptyset \mapsto \emptyset \in I_m$. Concerning the back and forth properties it is enough, by symmetry, to prove the forth property. Thus suppose that $0 \leq j < m$, $a \in A$ and $\bar{a} \mapsto \bar{b} \in I_{j+1}$, say,

$$\pi : (\mathcal{S}(3^{j+1},\bar{a}),\bar{a}) \cong (\mathcal{S}(3^{j+1},\bar{b}),\bar{b}) \tag{3.3}$$

Case 1 : $a \in S(2 \cdot 3^j, \bar{a})$
Then $S(3^j, \bar{a}a) \subseteq S(3^{j+1}, \bar{a})$. Setting $b := \pi(a)$, we have $\pi : (\mathcal{S}(3^j,\bar{a}a),\bar{a}a) \cong (\mathcal{S}(3^j,\bar{b}b),\bar{b}b)$, hence $\bar{a}a \mapsto \bar{b}b \in I_j$.
Case 2 : $a \notin S(2 \cdot 3^j, \bar{a})$ (and hence, $S(3^j,a) \cap S(3^j,\bar{a}) = \emptyset$)
Let ι be the 3^j-sphere type of a. By 3.3, $\mathcal{S}(2 \cdot 3^j, \bar{a})$ and $\mathcal{S}(2 \cdot 3^j, \bar{b})$ contain the same number of 3^j-sphere type ι which, in our assumption on the cardinality of spheres, is $\leq$ length$(\bar{a}) \cdot e \leq m \cdot e$. Therefore, there must be an element $b \notin S(2 \cdot 3^j, \bar{b})$ with 3^j-sphere

type ι. Choose $\pi' : (\mathcal{S}(3^j, a), a) \cong (\mathcal{S}(3^j, b), b)$. Then the corresponding restriction of $\pi \cup \pi'$ is an isomorphism of $(\mathcal{S}(3^j, \bar{a}a), \bar{a}a)$ onto $(\mathcal{S}(3^j, \bar{b}b), \bar{b}b)$. □

We give an application of the theorem. Note that a graph $\mathcal{G}$ is connected if $\mathcal{G}$ is a model of the following second-order sentence:

$$\forall P((\exists x Px \land \forall x \forall y((Px \land Exy) \rightarrow Py)) \rightarrow \forall z Pz)$$

We are going to show that the class of connected graphs is not axiomatizable by a $\mathrm{M}\Sigma_1^1$-sentence, that is, by a second-order sentence of the form $\exists P_1 \ldots \exists P_r \psi$, where $P_1, \ldots, P_r$ are unary and ψ is first-order.

Let $\mathcal{D}_l = (D_l, E_l)$ be a digraph consisting of a cycle of length $l+1$, that is,

$$D_l := \{0, \ldots, l\}, \quad E_l := \{(i, i+1) \mid i < l\} \cup \{(l, 0)\}$$

Lemma 3.4.2 Suppose $\tau = \{E, P_1, \ldots, P_r\}$ where $P_1, \ldots, P_r$ are unary, and let $m \geq 0$. Then there is an $l_0 \in N$ such that for any $l \geq l_0$ and any τ-structure of the form $(\mathcal{D}_l, P_1, \ldots, P_r)$ there are $a, b \in D_l$ with disjoint and isomorphic 3^m-spheres.

Proof. For the structures under consideration any 3^m-sphere contains exactly $2 \cdot 3^m + 1$ elements (note that $P_1, \ldots, P_r$ are unary and therefore do not influence the distances induced by the digraphs). Let i be the number of possible isomorphism types of 3^m-spheres. Set $l_0 = (i+1)(2 \cdot 3^m + 1)$. Then in a structure of cardinality $\geq l_0$ there must be two points with disjoint 3^m-spheres of the same isomorphism type. □

Lemma 3.4.3 Suppose $(\mathcal{D}_l, P_1, \ldots, P_r)$ is a τ-structure (τ is as in the preceding lemma) containing elements a and b with disjoint and isomorphic 3^m-spheres. a^- and b^- denote the elements of D_l with $E_l a^- a$ and $E_l b^- b$, respectively. Let $(D_l, E'_l, P_1, \ldots, P_r)$ be the structure obtained by splitting the cycle $(\mathcal{D}_l, P_1, \ldots, P_r)$ into two cycles by removing the edges $(a^-, a), (b^-, b)$ and adding edges $(b^-, a), (a^-, b)$ instead, that is,

$$E'_l \quad := \quad (E_l \setminus \{(a^-, a), (b^-, b)\}) \cup \{(b^-, a), (a^-, b)\}$$

Then $(\mathcal{D}_l, P_1, \ldots, P_r) \cong_m (D_l, E'_l, P_1, \ldots, P_r)$.

Proof. Immediate by Hanf's Theorem, since for each $n \leq 3^m$, both structures have the same number of n-spheres of any given isomorphism type. □

Since a partial isomorphism between digraphs is a partial isomorphism between the associated graphs, we get the following lemma from the two preceding lemmas:

Lemma 3.4.4 For $\tau = \{E, P_1, \ldots, P_r\}$ and $m \geq 0$, choose l_0 according to 3.4.2. Let $l \geq l_0$ and $(\mathcal{G}_l, P_1, \ldots, P_r)$ be a τ-structure, where $\mathcal{G}_l$ is the graph $\mathcal{G}(\mathcal{D}_l)$, that is, a cycle of length $l+1$. Let $\mathcal{G}'_l$ be the graph $\mathcal{G}((D_l, E'_l))$, where (D_l, E'_l) is defined as in the preceding lemma. Then

$$(\mathcal{G}_l, P_1, \ldots, P_r) \equiv_m (\mathcal{G}'_l, P_1, \ldots, P_r)$$

□

We are now in a position to show:

Proposition 3.4.5 The class of finite and connected graphs cannot be axiomatized by a formula of the form

$$\exists P_1 \ldots \exists P_r \psi \tag{3.4}$$

where $P_1, \ldots, P_r$ are unary relation symbols and ψ is a first-order sentence over the vocabulary $\{E, P_1, \ldots, P_r\}$.

Proof. Suppose that for the sentence (3.4) and any finite graph $\mathcal{G}$, we have:

$$\mathcal{G} \text{ is connected} \quad \text{iff} \quad \text{for some } P_1, \ldots, P_r \subseteq G: \quad (\mathcal{G}, P_1, \ldots, P_r) \models \psi$$

For $m := \mathrm{qr}(\psi)$ choose l_0 as in 3.4.2. Since $\mathcal{G}_{l_0}$ is connected, there are $P_1, \ldots, P_r$ such that $(\mathcal{G}_{l_0}, P_1, \ldots, P_r) \models \psi$. Then, $(\mathcal{G}'_{l_0}, P_1, \ldots, P_r) \models \psi$ by 3.4.4, but $\mathcal{G}'_{l_0}$ is not connected, a contradiction. □

On the other side we have

Proposition 3.4.6 The class of finite and connected graphs can be axiomatized by a formula of the form $\exists R\psi$, where R is binary and ψ is a first-order sentence over the vocabulary $\{E, R\}$.

Proof. Let ψ be a sentence expressing that R is an irreflexible and transitive relation with a minimal element, and that Exy holds for any immediate R-successor y of x; that is, ψ is the conjunction of

$$\forall x \neg Rxx \wedge \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\exists x \forall y(x = y \vee Rxy)$$
$$\forall x \forall y((Rxy \wedge \forall z \neg(Rxz \wedge Rzy)) \rightarrow Exy)$$

Let $\mathcal{G}$ be a graph. First, suppose $\mathcal{G}$ is a model of $\exists R\psi$, say $(\mathcal{G}, R^G) \models \psi$, then for any element of $\mathcal{G}$ there is a path connecting it with the minimal element; for any a in $\mathcal{G}$, we can show $e \sim a$ where e is the minimal element of R^G. Assume $e \not\sim a$, that is, there is not a path from e to a, then we have $\neg Eea$. According to the third condition above, a is not an immediate R-successor of e, therefore, there is an element a_1 with Rea_1 and Ra_1a. By our assumption $e \not\sim a$, either $e \not\sim a_1$ or $a_1 \not\sim a$ holds. Suppose $a_1 \not\sim a$, then we have again that there is an element a_2 with Ra_1a_2 and Ra_2a, by a similar argument above. Applying the argument above repeatedly, we get an sequence $a_1, a_2, \ldots$ of elements between e and a. This contradicts the fact that $\mathcal{G}$ has only finite number of elements. Hence, $\mathcal{G}$ is connected. Conversely, suppose $\mathcal{G}$ is connected. Choose an arbitrary $a \in G$. For $n \in N$ set $L_n := \{b \mid d(a, b) = n\}$ and take R as the transitive closure of $\{(b, c) \mid E^G bc \text{ and for some } n, b \in L_n \text{ and } c \in L_{n+1}\}$. □

Chapter 4

More on Games

In the preceding chapter we presented the Ehrenfeucht's gametheoretic method, which provides a simple characterization of classes axiomatizable in first-order logic. Now we investigate some generalizations of the method for languages other than first-order, mainly for second-order logic. First we restrict ourselves to the monadic second-order logic, that is, second-order logic in which only unary relation variables are allowed, and then try to remove the restriction of arity. These extensions are presented in parallel with the case of first-order logic.

4.1 Game for Monadic Second-Order Logic

For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in N$ we write $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same monadic second-order sentences of quantifier rank $\leq m$ (the quantifier rank is the maximum number of nested first-order and second-order quantifiers).

As in first-order logic, $\equiv_m^{\mathrm{MSO}}$ can be characterized by an Ehrenfeucht-Fraïssé game, MSO-$\mathrm{G}_m(\mathcal{A}, \mathcal{B})$, as shown below.

Definition 4.1.1 Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^r$, $\bar{b} \in B^r$, $\overline{P}(= P_1 \dots P_s)$ a sequence of subsets of A, $\overline{Q}(= Q_1 \dots Q_s)$ a sequence of subsets of B, and $m \in N$. The *MSO-Ehrenfeucht game* MSO-$\mathrm{G}_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$ is played by the spoiler and the duplicator as follows.

- ▷ Each player has to make m moves in the course of a play.
  - In his i-th move, the spoiler first decide whether to make a *point move* or *set move*.
    - ∗ The point moves are the same as the moves in the first-order case.
    - ∗ In a set move the spoiler chooses a subset $P' \subseteq A$ or $Q' \subseteq B$, and then the duplicator responds by a subset $Q' \subseteq B$ or $P' \subseteq A$, respectively.
- ▷ At the end, elements $e_1, \dots, e_k$ and subsets $P'_1, \dots, P'_l$ in A, and the corresponding elements $f_1, \dots, f_k$ and subsets $Q'_1, \dots, Q'_l$ in B (with $m = k + l$) are chosen.

– The duplicator wins if $\bar{a}\bar{e} \mapsto \bar{b}\bar{f} \in \mathrm{Part}((\mathcal{A}, \overline{P}, P'_1, \ldots, P'_l), (\mathcal{B}, \overline{Q}, Q'_1, \ldots, Q'_l))$.

– Otherwise, the spoiler wins. □

Our goal is to show the following.

Theorem 4.1.2 $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$ iff the duplicator wins MSO-G$_m(\mathcal{A}, \mathcal{B})$.

In order to prove it, we need some variants of the consequences acquired in the case of first-order.

The proof of the following lemma is straightforward.

Lemma 4.1.3 Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^r$, $\bar{b} \in B^r$, $\overline{P}(= P_1 \ldots P_s)$ a sequence of subsets of A, $\overline{Q}(= Q_1 \ldots Q_s)$ a sequence of subsets of B, and $m \in N$.

(a) The duplicator wins MSO-G$_0((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$ iff $\bar{a} \mapsto \bar{b} \in \mathrm{Part}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{Q}))$.

(b) For $m > 0$, the duplicator wins MSO-G$_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$ iff
for all $a \in A$ there is $b \in B$ such that the duplicator wins
MSO-G$_{m-1}((\mathcal{A}, \overline{P}), \bar{a}a, (\mathcal{B}, \overline{Q}), \bar{b}b)$,
and for all $b \in B$ there is $a \in A$ such that the duplicator wins
MSO-G$_{m-1}((\mathcal{A}, \overline{P}), \bar{a}a, (\mathcal{B}, \overline{Q}), \bar{b}b)$,
and for all $P \subseteq A$ there is $Q \subseteq B$ such that the duplicator wins
MSO-G$_{m-1}((\mathcal{A}, \overline{P}P), \bar{a}, (\mathcal{B}, \overline{Q}Q), \bar{b})$,
and for all $Q \subseteq B$ there is $P \subseteq A$ such that the duplicator wins
MSO-G$_{m-1}((\mathcal{A}, \overline{P}P), \bar{a}, (\mathcal{B}, \overline{Q}Q), \bar{b})$.
□

We define the formulas $\psi^j_{\bar{a},\overline{P}}$ similar to the j-isomorphism type $\varphi^j_{\bar{a}}$ (cf. 3.2.6), but now taking into account also the second-order set quantifiers.

Definition 4.1.4

$$\psi^0_{\bar{a},\overline{P}} := \bigwedge\{\varphi(v_1, \ldots, v_r, V_1, \ldots, V_s) \mid \varphi \text{ atomic or negated atomic}, \mathcal{A} \models \varphi[\bar{a}, \overline{P}]\}$$

$$\psi^{j+1}_{\bar{a},\overline{P}} := \bigwedge_{a \in A} \exists v_{r+1} \psi^j_{\bar{a}a,\overline{P}} \wedge \forall v_{r+1} \bigvee_{a \in A} \psi^j_{\bar{a}a,\overline{P}} \wedge \bigwedge_{P \subseteq A} \exists V_{s+1} \psi^j_{\bar{a},\overline{P}P} \wedge \forall V_{s+1} \bigvee_{P \subseteq A} \psi^j_{\bar{a},\overline{P}P}$$

□

The following lemma is proven similarly to the case of j-isomorphism types $\varphi^j_{\mathcal{A},\bar{a}}$ (cf. 3.2.7), therefore the conjunctions and the disjunctions in the definition above are finite.

Lemma 4.1.5 For $r, s, m \geq 0$, the set $\{\psi^j_{\mathcal{A},\bar{a},\overline{P}} \mid \mathcal{A}$ a structure, $\bar{a} \in A^r$ and $\overline{P}(= P_1 \ldots P_s)$ a sequence of subsets of $A\}$ is finite. □

Lemma 4.1.6 (a) $\mathrm{qr}(\psi^m_{\bar{a},\overline{P}}) = m$

(b) $\mathcal{A} \models \psi^m_{\bar{a},\overline{P}}[\bar{a}, \overline{P}]$

(c) For any $\mathcal{B}$, $\bar{b}$ in B and $\overline{Q}$ subsets of B,

$$\mathcal{B} \models \psi^0_{\bar{a},\overline{P}}[\bar{b}, \overline{Q}] \quad \text{iff} \quad \bar{a} \mapsto \bar{b} \in \mathrm{Part}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{Q}))$$

Proof. (a) trivial.

(b) The fact that $\mathcal{A}$ satisfies the first half of the formula $\psi^m_{\bar{a},\overline{P}}[\bar{a}, \overline{P}]$ can be proven by the same way as $\varphi^m_{\bar{a}}[\bar{a}]$ (cf. 3.2.8(b)). Therefore it suffices to show the latter half by induction on m, again in a similar way. For $m = 0$, the conclusion is immediate from the definition. For $m > 0$, by the induction hypothesis we assume that for all $P \subseteq A$, $\mathcal{A} \models \psi^{m-1}_{\bar{a},\overline{P}P}[\bar{a}, \overline{P}P]$. Then for all $P \subseteq A$, we have $\mathcal{A} \models \exists V_{s+1} \psi^{m-1}_{\bar{a},\overline{P}P}[\bar{a}, \overline{P}]$. Hence, we get $\mathcal{A} \models \bigwedge_{P \subseteq A} \exists V_{s+1} \psi^{m-1}_{\bar{a},\overline{P}P}[\bar{a}, \overline{P}]$, and we have from the induction hypothesis that, for all $P' \subseteq A$, $\mathcal{A} \models \bigvee_{P \subseteq A} \psi^{m-1}_{\bar{a},\overline{P}P}[\bar{a}, \overline{P}P']$, that is, $\mathcal{A} \models \forall V_{s+1} \bigvee_{P \subseteq A} \psi^{m-1}_{\bar{a},\overline{P}P}[\bar{a}, \overline{P}]$.

(c) By 3.2.8(c), it suffices to show that, for any $\mathcal{B}$, $\bar{b}$ in B and $\overline{Q}$ subsets of B,

$$\mathcal{B} \models \psi^0_{\bar{a},\overline{P}}[\bar{b}, \overline{Q}] \quad \text{iff} \quad (\mathcal{B}, \overline{Q}) \models \varphi^0_{(\mathcal{A},\overline{P}),\bar{a}}[\bar{b}]$$

This is clear from the definition, for $\psi^0_{\bar{a},\overline{P}}$ is equivalent to $\varphi^0_{(\mathcal{A},\overline{P}),\bar{a}}$ if the variables $\overline{V}$ occurring in $\psi^0_{\bar{a},\overline{P}}$ are interpreted as $\overline{Q}$. □

Now the following theorem, an analogy to Ehrenfeucht's theorem, can be obtained.

Theorem 4.1.7 Given $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^r$, $\bar{b} \in B^r$, $\overline{P}(= P_1 \dots P_s)$ a sequence of subsets of A, $\overline{Q}(= Q_1 \dots Q_s)$ a sequence of subsets of B, and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $\text{MSO-G}_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$

(ii) $\mathcal{B} \models \psi^m_{\bar{a},\overline{P}}[\bar{b}, \overline{Q}]$

(iii) $\bar{a}$, $\overline{P}$ satisfies the same formulas of MSO of quantifier rank $\leq m$ in $\mathcal{A}$ as $\bar{b}$, $\overline{Q}$ in $\mathcal{B}$, that is, if $\varphi(v_1, \dots, v_r, V_1, \dots, V_s)$ is of quantifier rank $\leq m$, then

$$\mathcal{A} \models \varphi[\bar{a}, \overline{P}] \quad \text{iff} \quad \mathcal{B} \models \varphi[\bar{b}, \overline{Q}] \tag{4.1}$$

Proof. The proof is completely parallel to that of 3.2.9. (iii) implies (ii) since $\mathrm{qr}(\psi^m_{\bar{a},\overline{P}}) = m$ and $\mathcal{A} \models \psi^m_{\bar{a},\overline{P}}[\bar{a}, \overline{P}]$. We prove the equivalence of (i) and (ii) by induction on m. For $m = 0$

$$\begin{aligned} &\text{the duplicator wins } \text{MSO-G}_0((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b}) \\ \text{iff}\quad & \bar{a} \mapsto \bar{b} \in \mathrm{Part}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{Q})) \quad \text{(by 4.1.3(a))} \\ \text{iff}\quad & \mathcal{B} \models \psi^0_{\bar{a},\overline{P}}[\bar{b}, \overline{Q}] \quad \text{(by 4.1.6(c))} \end{aligned}$$

For $m > 0$,

the duplicator wins MSO-$\mathrm{G}_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$

iff for all $a \in A$ there is $b \in B$ such that the duplicator wins MSO-$\mathrm{G}_{m-1}((\mathcal{A}, \overline{P}), \bar{a}a, (\mathcal{B}, \overline{Q}), \bar{b}b)$, for all $b \in B$ there is $a \in A$ such that the duplicator wins MSO-$\mathrm{G}_{m-1}((\mathcal{A}, \overline{P}), \bar{a}a, (\mathcal{B}, \overline{Q}), \bar{b}b)$, for all $P \subseteq A$ there is $Q \subseteq B$ such that the duplicator wins MSO-$\mathrm{G}_{m-1}((\mathcal{A}, \overline{P}P), \bar{a}, (\mathcal{B}, \overline{Q}Q), \bar{b})$, and for all $Q \subseteq B$ there is $P \subseteq A$ such that the duplicator wins MSO-$\mathrm{G}_{m-1}((\mathcal{A}, \overline{P}P), \bar{a}, (\mathcal{B}, \overline{Q}Q), \bar{b})$ (by 4.1.3(b))

iff for all $a \in A$ there is $b \in B$ with $\mathcal{B} \models \psi^{m-1}_{\bar{a}a, \overline{P}}[\bar{b}b, \overline{Q}]$,
for all $b \in B$ there is $a \in A$ with $\mathcal{B} \models \psi^{m-1}_{\bar{a}a, \overline{P}}[\bar{b}b, \overline{Q}]$,
for all $P \subseteq A$ there is $Q \subseteq B$ with $\mathcal{B} \models \psi^{m-1}_{\bar{a}, \overline{P}P}[\bar{b}, \overline{Q}Q]$, and
for all $Q \subseteq B$ there is $P \subseteq A$ with $\mathcal{B} \models \psi^{m-1}_{\bar{a}, \overline{P}P}[\bar{b}, \overline{Q}Q]$ (ind. hyp.)

iff $\mathcal{B} \models \bigwedge_{a \in A} \exists v_{r+1} \psi^{m-1}_{\bar{a}a, \overline{P}} \wedge \forall v_{r+1} \bigvee_{a \in A} \psi^{m-1}_{\bar{a}a, \overline{P}} \wedge \bigwedge_{P \subseteq A} \exists V_{s+1} \psi^{m-1}_{\bar{a}, \overline{P}P} \wedge \forall V_{s+1} \bigvee_{P \subseteq A} \psi^{m-1}_{\bar{a}, \overline{P}P}[\bar{b}, \overline{Q}]$

iff $\mathcal{B} \models \psi^{m}_{\bar{a}, \overline{P}}[\bar{b}, \overline{Q}]$

(i) $\Rightarrow$ (iii) : The proof proceeds by induction on m. For $m = 0$

the duplicator wins MSO-$\mathrm{G}_0((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$

iff $\bar{a} \mapsto \bar{b} \in \mathrm{Part}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{Q}))$ (by 4.1.3(a))

iff for all quantifier free $\varphi(\bar{v})$: $(\mathcal{A}, \overline{P}) \models \varphi[\bar{a}] \;\Leftrightarrow\; (\mathcal{B}, \overline{Q}) \models \varphi[\bar{b}]$ (by 3.2.2(c))

iff for all quantifier free $\varphi(\bar{v}, \overline{V})$: $\mathcal{A} \models \varphi[\bar{a}, \overline{P}] \;\Leftrightarrow\; \mathcal{B} \models \varphi[\bar{b}, \overline{Q}]$

For $m > 0$, assume that the duplicator wins MSO-$\mathrm{G}_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{P}), \bar{b})$. The case $\varphi(\bar{v}, \overline{V}) = \exists x \psi(\bar{v}, x, \overline{V})$ where $\mathrm{qr}(\varphi) \leq m$ was handled in the corresponding proof in 3.2.9. Suppose $\varphi(\bar{v}, \overline{V}) = \exists X \psi(\bar{v}, \overline{V}, X)$ and $\mathrm{qr}(\varphi) \leq m$. Assume, for instance, $\mathcal{A} \models \varphi[\bar{a}, \overline{P}]$. Then there is $P \subseteq A$ such that $\mathcal{A} \models \psi[\bar{a}, \overline{P}P]$. As the duplicator wins MSO-$\mathrm{G}_m((\mathcal{A}, \overline{P}), \bar{a}, (\mathcal{B}, \overline{Q}), \bar{b})$, there is $Q \subseteq B$ such that the duplicator wins MSO-G_{m-1} $((\mathcal{A}, \overline{P}P), \bar{a}, (\mathcal{B}, \overline{Q}Q), \bar{b})$. Since $\mathrm{qr}(\psi) \leq m - 1$, the induction hypothesis yields $\mathcal{B} \models \psi[\bar{b}, \overline{Q}Q]$, hence $\mathcal{B} \models \varphi[\bar{b}, \overline{Q}]$. The opposite direction can be shown similarly. Clearly, the set of formulas $\varphi(\bar{v}, \overline{V})$ satisfying (4.1) is closed under $\neg$ and $\vee$. □

Corollary 4.1.8 For structures $\mathcal{A}$, $\mathcal{B}$ and $m \geq 0$ the following are equivalent:

(i) The duplicator wins MSO-$\mathrm{G}_m(\mathcal{A}, \mathcal{B})$

(ii) $\mathcal{B} \models \psi^m_{\mathcal{A}}$

(iii) $\mathcal{A} \equiv^{\mathrm{MSO}}_m \mathcal{B}$ □

This corollary and the following proposition provide a method of examining the axiomatizability in monadic second-order logic.

Proposition 4.1.9 For a class K of finite structures the following are equivalent:

(i) K is not axiomatizable in monadic second-order logic.

(ii) For each m there are finite structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} \in K,\ \mathcal{B} \notin K \text{ and } \mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$$

Proof. (ii) $\Rightarrow$ (i) Let φ be any monadic second-order sentence. Set $m := \mathrm{qr}(\varphi)$. By our assumption there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$, and $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$; hence, $K \neq \mathrm{Mod}(\varphi)$. For the converse, suppose that (ii) does not hold, that is, for some m and all finite $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \in K \text{ and } \mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B} \text{ imply } \mathcal{B} \in K$$

Then $K = \mathrm{Mod}(\bigvee\{\psi_{\mathcal{A}}^m \mid \mathcal{A} \in K\})$, and thus K is axiomatizable. Note that the disjunction above is finite according to 4.1.5. □

Thus, a class K of finite structures is not axiomatizable in monadic second-order logic iff the duplicator has a winning strategy for the following game.

1. The spoiler selects a number $m \in N$.
2. The duplicator selects a member $\mathcal{A} \in K$.
3. The duplicator selects a member $\mathcal{B} \notin K$.
4. The spoiler and the duplicator play $\mathrm{MSO}\text{-}\mathrm{G}_m(\mathcal{A}, \mathcal{B})$.

The Ehrenfeucht's method generalized for monadic second-order logic we mentioned above is known already (see [5][14]), but its applications are hardly found among literatures. Now we give a new application of this method.

Theorem 4.1.10 Let τ be an arbitrary vocabulary. Then the class EVEN(τ) of τ-structures of even cardinality is not axiomatizable in monadic second-order logic.

Theorem 4.1.10 strengthens the result of Example 3.3.4 which was proved in [5]. We need to state some remarks before the proof of Theorem 4.1.10. Let τ be a vocabulary in which all relation symbols are unary, and let $\mathcal{A}$ be τ-structure. For each pair of elements $a, b \in \mathcal{A}$, we write $a \sim b$ if for all unary relation symbol $P \in \tau$, Pa iff Pb. Clearly $\sim$ is an equivalence relation, and elements in a common equivalence class have the same r-sphere type for any $r \in N$ (cf. section 3.4, note that all P_i are unary and therefore the r-sphere of a contains only one element a itself). Hence we call r-sphere types simply *types*.

The following remark is a variant of Hanf's Theorem 3.4.1.

Remark 4.1.11 Let τ be a vocabulary in which all relation symbols are unary, and let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and $m \in N$. Suppose that τ contains constants $c_1, \ldots, c_r$, and for each c_i,

$$c_i^A \text{ and } c_i^B \text{ have the same type,} \tag{4.2}$$

and suppose for each type ι, the following (i) or (ii) holds

(i) The equivalence classes of type ι in $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements.

(ii) The equivalence classes of type ι in $\mathcal{A}$ and $\mathcal{B}$ have at least $m + r$ elements.

Then $A \equiv_m B$.

Proof. We show that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$, where I_j is the set

$$\{\bar{a} \mapsto \bar{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \mid \mathrm{length}(\bar{a}) \leq m - j\},$$

First, we have $\emptyset \mapsto \emptyset \in I_m$. Concerning the back and forth properties, it is enough, by symmetry, to prove the forth property. Thus suppose that $0 \leq j < m$, $\bar{a} \mapsto \bar{b} \in I_{j+1}$ and $a \in A$. It suffices to consider the case that a is not a member of $\bar{a}$. Suppose a has a type ι, then there must be an element $b \in B$ with type ι by our assumption on the cardinality of equivalence classes. Hence, $\bar{a}a \mapsto \bar{b}b \in I_j$. □

Proof of 4.1.10. First, suppose τ has no relation symbols, and contains constants $c_1, \ldots, c_r$. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and let $m \in N$. Suppose that

$$\|A\| \geq (2m + r) \cdot 2^m \quad \text{and} \quad \|B\| = \|A\| + 1 \tag{4.3}$$

Then $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$ holds, since the duplicator is able to construct a winning strategy in MSO-G$_m(\mathcal{A}, \mathcal{B})$ as shown in the following. First, we change the game MSO-G$_m(\mathcal{A}, \mathcal{B})$ a little bit to a new game in which the spoiler and the duplicator are allowed to make $2m$ moves, which consist exactly of m moves of point moves and m moves of set moves. Therefore it is easily verified that, if the duplicator construct a winning strategy in this new game of $2m$ moves, then the first m moves of the strategy forms a winning strategy in MSO-G$_m(\mathcal{A}, \mathcal{B})$.

In the new game, suppose the spoiler selects set moves in all the first half m moves. Then we show that the duplicator can make such responses as the conditions (4.2) and (i) or (ii) in 4.1.11 holds. More precisely, in his i-th move ($1 \leq i \leq m$) the duplicator can make responses in such a way that for each type ι (caused by set moves) except a certain type κ, the equivalence classes of type ι in $(\mathcal{A}, P_1, \ldots, P_i)$ and $(\mathcal{B}, Q_1, \ldots, Q_i)$ have the same number of elements, and the number of elements of the type κ in $(\mathcal{B}, Q_1, \ldots, Q_i)$ is one larger than that of the corresponding type in $(\mathcal{A}, P_1, \ldots, P_i)$ and both have at least $m \cdot 2^{m-i}$ elements. This is proven by induction on i:

(In the first move) Assume that the spoiler selects $\mathcal{A}$ and chooses a subset $P_1 \subseteq A$. Then the duplicator checks which set P_1 or $\overline{P_1}(= A \backslash P_1)$ is smaller (e.g. $\overline{P_1}$), and then he chooses a subset $Q_1 \subseteq B$ which preserves the number of elements of the smaller side (e.g. $\overline{Q_1} = \overline{P_1}$), taking care of the condition (4.2). Clearly, the larger sides (e.g. P_1 and Q_1) have elements $\geq m \cdot 2^{m-1}$, and the size of Q_1 is one larger than that of P_1. If the sizes of P_1 and $\overline{P_1}$ are the same then either will do.

In case the spoiler selects $\mathcal{B}$, then the duplicator behaves similarly. Only when the sizes of Q_1 and $\overline{Q_1}$ are the same the duplicator has to choose a subset $P_1 \subseteq A$ which preserves the size of the larger side. Hence this time the smaller sides have a different number of elements. In this case the number of elements in $\mathcal{B}$ must be even, that is, $\|B\| \geq m \cdot 2^m + 2$, therefore the sizes of the smaller sides $\geq m \cdot 2^{m-1}$.

(In the i-th move) Assume that the spoiler selects $\mathcal{A}$ and chooses a subset $P_i \subseteq A$. His choice P_i divides each equivalence class of type ι (caused by $i-1$ moves) in two parts, P_i part and $\overline{P_i}$ part. Therefore the duplicator needs to pay attention only to the equivalence classes of type κ of different size in $(\mathcal{A}, P_1, \ldots, P_{i-1})$ and $(\mathcal{B}, Q_1, \ldots, Q_{i-1})$. For, if the sizes of the equivalence classes of type ι are the same, then the duplicator is enough to divide the class of ι in $(\mathcal{B}, Q_1, \ldots, Q_{i-1})$ as the spoiler divides in $(\mathcal{A}, P_1, \ldots, P_{i-1})$. And the argument of the first move also applies to the case when the duplicator divides the class of type κ, where he have to preserve the size of the smaller part. In case the spoiler selects $\mathcal{B}$, the duplicator behave in a similar way above.

When their m moves of set moves complete, $(\mathcal{A}, P_1, \ldots, P_m) \equiv_m (\mathcal{B}, Q_1, \ldots, Q_m)$ holds, since the conditions in 4.1.11 are all satisfied. Therefore the duplicator wins the latter half m moves of point moves.

In case the spoiler selects a point move in his i-th move ($i < m$) before he completes all m moves of set moves, the duplicator responds by a point of the same type, indeed he can choose such an element because $(\mathcal{A}, P_1, \ldots, P_{i-1}) \equiv_m (\mathcal{B}, Q_1, \ldots, Q_{i-1})$ according to 4.1.11. And after the point moves the duplicator consider the points picked up in the point moves as new constants. Hence at most $m + r$ constants may appear in this game, like the case the spoiler selects m moves of point moves in the first m moves. The winning strategy for the duplicator described above is also true in this case since the numbers of elements contained in $\mathcal{A}$ and $\mathcal{B}$ are large enough by the condition (4.3).

Lastly we have to consider the case that τ has relation symbols. In this case, it is enough to choose structures $\mathcal{A}$ and $\mathcal{B}$ satisfying (4.3), where all relations are interpreted as empty relations. Then, $\mathcal{A} \in \mathrm{EVEN}(\tau)$ iff $\mathcal{B} \notin \mathrm{EVEN}(\tau)$, and $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$. Thus, the class $\mathrm{EVEN}(\tau)$ is not axiomatizable in monadic second-order logic. $\square$

The theorem we stated above leads us to a question: what kind of a set of natural numbers is axiomatizable in monadic second-order logic? The following theorem is an answer to this question.

Let I be a set of natural numbers. We say $I(\tau)$ of τ-structures of cardinality in I is *definable* in $\mathcal{L}$, if there is a sentence φ of $\mathcal{L}$ such that $\mathrm{Mod}(\varphi)$ is just the class of τ-structures of cardinality in I.

Theorem 4.1.12 Let τ be an arbitrary vocabulary. Then the class $I(\tau)$ is definable in monadic second-order logic iff I is either finite or co-finite. (A set $I \subset N$ is *co-finite* if $N - I$ is finite.)

Proof. For every natural number $n \in N$, it is possible to write down a (first-order) formula $\varphi_{=n}$ expressing that its model has exactly n elements (cf. section 1.2). It follows that a finite set $I \subset N$ can be defined by the disjunction $\bigvee_{n \in I} \varphi_{=n}$; its complement $N - I$ is defined by the negation of this formula. For the converse, it suffices to show that if both I and $N - I$ are infinite, then I is not definable in monadic second-order logic. This can be shown by applying the proof of the case EVEN in 4.1.10: this time it is enough to choose structures $\mathcal{A} \in I(\tau)$ and $\mathcal{B} \notin I(\tau)$ such that the condition (4.3) holds. Actually we can choose such structures since both I and $N - I$ contain infinitary many numbers. □

The statement of the preceding theorems depend essentially on the restriction on the relation variables to be *unary*. In fact, for any τ the class EVEN(τ) is axiomatizable in second-order logic with a single *binary* relation variable X as follows,

$$\exists X(\forall x Xxx \wedge \forall x \forall y(Xxy \rightarrow Xyx) \wedge \forall x \forall y \forall z((Xxy \wedge Xyz) \rightarrow Xxz)$$
$$\wedge \forall x \exists^{=1} y(Xxy \wedge y \neq x))$$

("there is a binary relation which is an equivalence relation having only equivalence classes with exactly two elements")

4.2 Game for Existential Second-Order Logic

In the preceding section we introduced the generalized version of Ehrenfeucht's method for monadic second-order logic, which characterized the monadic second-order axiomatizability in terms of the game that allows set moves corresponding to set quantifiers. Now it is easily verified that the restriction on the arity of relation variables is not essential, by introducing a new game that allows k-ary moves corresponding to k-ary relation variables. We state this observation precisely in the following.

We introduce the fragment k-SO of second-order logic as *k-ary second-order logic* in which relation variables are allowed only if their arities $\leq k$. For structures $\mathcal{A}$, $\mathcal{B}$ and $m \in N$ we write $\mathcal{A} \equiv_m^{\text{k-SO}} \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same k-ary second-order sentences of quantifier rank $\leq m$ (the quantifier rank is the maximum number of nested first-order and second-order quantifiers). We can show in general that $\equiv_m^{\text{k-SO}}$ is characterized by the game k-SO-G$_m(\mathcal{A}, \mathcal{B})$. The rules are the same as in the monadic second-order game, but now in every move the spoiler can decide whether to make a *point move* or a *l-ary move* for $l \leq k$. The point moves are the same as the moves in the first-order case. In a l-ary move the spoiler chooses a l-ary relation $P^l \subseteq A^l$ or $Q^l \subseteq B^l$, and then the duplicator responds by a l-ary relation $Q^l \subseteq B^l$ or $P^l \subseteq A^l$, respectively. After m moves, elements $a_1, \ldots, a_r$ and relations $P_1^l, \ldots, P_{s_l}^l$ in A for $l \leq k$, and corresponding elements $b_1, \ldots, b_r$ and relations $Q_1^l, \ldots, Q_{s_l}^l$ in B for $l \leq k$ (with $m = r + s_1 + \cdots + s_k$) have been chosen. The duplicator wins if $\bar{a} \mapsto \bar{b} \in \text{Part}((\mathcal{A}, \overline{P^1}, \ldots, \overline{P^k}), (\mathcal{B}, \overline{Q^1}, \ldots, \overline{Q^k}))$.

Theorem 4.2.1 $\mathcal{A} \equiv_m^{\text{k-SO}} \mathcal{B}$ iff the duplicator wins k-SO-G$_m(\mathcal{A}, \mathcal{B})$. □

In order to prove the theorem, we again introduce the corresponding isomorphism types similar to $\psi^j_{\bar{a},\overline{P}}$. For $\bar{a}(= a_1 \ldots a_r)$ in $\mathcal{A}$ and $\overline{P^l}(= P_1^l \ldots P_{s_l}^l)$ a sequence of l-ary relations on A^l, the formulas $\chi^j_{\bar{a},\overline{P^1},\ldots,\overline{P^k}}$ are defined as;

$$\chi^0_{\bar{a},\overline{P^1},\ldots,\overline{P^k}} :=$$
$$\bigwedge\{\varphi(\bar{v}, \overline{V^1}, \ldots, \overline{V^k}) \mid \varphi \text{ atomic or negated atomic}, \mathcal{A} \models \varphi[\bar{a}, \overline{P^1}, \ldots, \overline{P^k}]\}$$

$$
\begin{aligned}
\chi^{j+1}_{\bar{a},\overline{P^1},\ldots,\overline{P^k}} \quad := \quad & \bigwedge_{a\in A} \exists v_{r+1}\chi^{j}_{\bar{a}a,\overline{P^1},\ldots,\overline{P^k}} \wedge \forall v_{r+1} \bigvee_{a\in A} \chi^{j}_{\bar{a}a,\overline{P^1},\ldots,\overline{P^k}} \\
\wedge \quad & \bigwedge_{P\subseteq A} \exists V_{s_1+1}\chi^{j}_{\bar{a},\overline{P^1}P,\ldots,\overline{P^k}} \wedge \forall V_{s_1+1} \bigvee_{P\subseteq A} \chi^{j}_{\bar{a},\overline{P^1}P,\ldots,\overline{P^k}} \\
\wedge \quad & \cdots \\
\vdots \quad & \\
\wedge \quad & \bigwedge_{P^k\subseteq A^k} \exists V_{s_k+1}\chi^{j}_{\bar{a},\overline{P^1},\ldots,\overline{P^k}P^k} \wedge \forall V_{s_k+1} \bigvee_{P^k\subseteq A^k} \chi^{j}_{\bar{a},\overline{P^1},\ldots,\overline{P^k}P^k}
\end{aligned}
$$

Then we can show the equivalence of

(i) The duplicator wins k-SO-G$_m((\mathcal{A},\overline{P^1},\ldots,\overline{P^k}),\bar{a},(\mathcal{B},\overline{Q^1},\ldots,\overline{Q^k}),\bar{b})$

(ii) $\mathcal{B}\models\chi^{m}_{\bar{a},\overline{P^1},\ldots,\overline{P^k}}[\bar{b},\overline{Q^1},\ldots,\overline{Q^k}]$

(iii) $\bar{a}$, $\overline{P^1},\ldots,\overline{P^k}$ satisfies in $\mathcal{A}$ the same formulas of k-SO of quantifier rank $\leq m$ as $\bar{b}$, $\overline{Q^1},\ldots,\overline{Q^k}$ in $\mathcal{B}$, that is, if $\varphi(\bar{v},\overline{V^1},\ldots,\overline{V^k})$ is of quantifier rank $\leq m$, then

$$\mathcal{A}\models\varphi[\bar{a},\overline{P^1},\ldots,\overline{P^k}] \quad \text{iff} \quad \mathcal{B}\models\varphi[\bar{b},\overline{Q^1},\ldots,\overline{Q^k}]$$

We will not give a full detail of the proof, for it can be proven completely parallel to that of monadic second-order logic. Consequently, we get a way of checking up the axiomatizability in second-order logic:

Theorem 4.2.2 A class K of finite structures is not axiomatizable in second-order logic iff the duplicator has a winning strategy for the following game.

1. The spoiler selects $k\in N$.
2. The spoiler selects $m\in N$.
3. The duplicator selects $\mathcal{A}\in K$.
4. The duplicator selects $\mathcal{B}\notin K$.
5. The spoiler and the duplicator play k-SO-G$_m(\mathcal{A},\mathcal{B})$. □

The game described above gives us a hint on finding a gametheoretic characterization of a fragment Σ^1_1 of second-order logic, by modifying the second-order game slightly. In the following we discuss a game for Σ^1_1, a modification of the game above:

1. The spoiler selects $k\in N$.
2. The spoiler selects $m\in N$.
3. The duplicator selects $\mathcal{A}_0\in K$.
4. The duplicator selects $\mathcal{B}_0\notin K$.

5. The spoiler sets a k-ary relation P_0 on $\mathcal{A}_0$.

6. The duplicator sets a k-ary relation Q_0 on $\mathcal{B}_0$.

7. The spoiler and the duplicator play $\mathrm{G}_m((\mathcal{A}_0, P_0), (\mathcal{B}_0, Q_0))$.

Then we are able to get the following result, an analogue of 4.2.2.

Theorem 4.2.3 A class K of finite structures is not axiomatizable in Σ_1^1 iff the duplicator has a winning strategy for the Σ_1^1 game.

We postpone the proof of this theorem until the following result is shown.

Theorem 4.2.4 For every Σ_1^1 sentence $\exists X_1 \ldots \exists X_n \varphi$ where φ is first-order, there is an equivalent Σ_1^1 sentence $\exists X \psi$ in which only a single relation variable X occurs.

Proof. By induction on n. In case $n = 1$, the conclusion is immediate. For a sentence χ of the form $\exists X_1 \ldots \exists X_{n+1} \varphi$ where φ is first-order, χ is equivalent to a sentence $\exists X \exists Y \psi$ by the induction hypothesis. We can assume that X and Y have the same arity without a loss of generality. In fact, if the arity m of X is less than n of Y, then $\exists X \exists Y \psi(X)$ is rewritten by:

$$\exists Z \exists Y (\forall x_1 \ldots \forall x_n (Z x_1 \ldots x_n \rightarrow x_1 = x_2 \wedge \ldots \wedge x_1 = x_{n-m+1}) \wedge \psi(Z\bar{t_1 \_})).$$

Here, the arity of Z is n and $\psi(Z\bar{t_1\_})$ is obtained from $\psi(X)$ by replacing subformulas $X t_1 \ldots t_m$ by $Z t_1 \ldots t_1 t_1 t_2 \ldots t_m$. Now, $\exists X \exists Y \psi(X, Y)$ with n-ary relation variables X and Y is rewritten by:

$$\exists R \exists x \exists y (x \neq y \wedge \forall x_1 \ldots \forall x_n \forall z (R(x_1, \ldots, x_n, z) \rightarrow (z = x \vee z = y)) \wedge \psi(R\_x, R\_y)) \quad (4.4)$$

the arity of R is $n+1$, and $\psi(R\_x, R\_y)$ is obtained from $\psi(X, Y)$ by replacing subformulas $X\bar{t}$ and $Y\bar{s}$ by $R\bar{t}x$ and $R\bar{s}y$, respectively. Yet, the formulas above are equivalent only on structures with at least two elements. So, structures of cardinality 1 must be taken into consideration separately. Thus the formula must be rewritten as follows,

$$\begin{aligned} &\exists R((\exists x \forall y (x = y)) \rightarrow (\psi(T, T) \vee \psi(T, F) \vee \psi(F, T) \vee \psi(F, F)) \\ &\quad \wedge \neg(\exists x \forall y (x = y)) \rightarrow \psi_0) \end{aligned}$$

where ψ_0 consists of the first-order part of (4.4). □

Proof of 4.2.3. If K is axiomatizable in Σ_1^1, then, by the preceding result 4.2.4, it is axiomatizable by a sentence $\exists X \psi(X)$ where ψ is first-order. Therefore the spoiler has a winning strategy in the new Σ_1^1 game as follows: 1. he choose a number k as the arity of X. 2. he choose a number m as the quantifier rank of ψ. 5. he selects such a k-ary relation P_0 on $\mathcal{A}_0$ as $\mathcal{A}_0 \models \psi[P_0]$ is satisfied (note that $\mathcal{A}_0 \models \exists X \psi$). On the other hand, the duplicator has to choose $\mathcal{B}_0$ such that $\mathcal{B}_0 \not\models \exists X \psi$, and hence for any relation Q_0 over $\mathcal{B}_0$, $\mathcal{B}_0 \not\models \psi[Q_0]$. Therefore, whatever choices are made by the duplicator, $(\mathcal{A}_0, P_0) \models \psi$

but $(\mathcal{B}_0, Q_0) \not\models \psi$. This implies $(\mathcal{A}_0, P_0) \not\equiv_m (\mathcal{B}_0, Q_0)$, and hence the spoiler wins the game $\mathrm{G}_m((\mathcal{A}_0, P_0), (\mathcal{B}_0, Q_0)$ (cf. 3.2.10). Conversely, suppose the spoiler has a winning strategy in the new Σ_1^1 game. Then it is enough to show that K is axiomatizable in Σ_1^1. First, let us make the meaning of "the duplicator has a winning strategy" exact: For all $k \in N$, for all $n \in N$, there is $\mathcal{A}_0$ and $\mathcal{B}_0$ such that for all k-ary relation P_0 over $\mathcal{A}_0$, there is a k-ary relation Q_0 over $\mathcal{B}_0$ such that

$$\mathcal{A}_0 \in K, \mathcal{B}_0 \notin K \text{ and } (\mathcal{A}_0, P_0) \equiv_m (\mathcal{B}_0, Q_0).$$

Hence its negation "the spoiler has a winning strategy" is described precisely as follows: For some k and n, for all $\mathcal{B}_0$

$$\begin{aligned}&\text{"there is a } \mathcal{A}_0 \text{ such that for all } P_0 \text{ there is a } Q_0\\ &\quad\text{such that } \mathcal{A}_0 \in K \text{ and } (\mathcal{A}_0, P_0) \equiv_m (\mathcal{B}_0, Q_0)\text{"} \quad \text{implies} \quad \mathcal{B}_0 \in K \end{aligned} \tag{4.5}$$

Thus,

$$K = \mathrm{Mod}\Big(\bigvee_{\mathcal{A} \in K} \bigwedge_{P \subseteq A^k} \exists P \{\varphi^m_{(\mathcal{A},P)} \mid \mathcal{A} \in K, P \subseteq A^k\}\Big) \tag{4.6}$$

For suppose $\mathcal{A} \in K$, then for all $P \subseteq A^k$, $\mathcal{A} \models \exists P \varphi^m_{(\mathcal{A},P)}$, hence $\mathcal{A}$ satisfies the sentence on the right side of (4.6). For the converse, suppose $\mathcal{B}$ satisfies the sentence on the right side of (4.6). Then for some $\mathcal{A} \in K$, for all $P \subseteq A^k$ there is a $Q \subseteq B^k$ such that $(\mathcal{B}, Q) \models \varphi^m_{(\mathcal{A},P)}$, that is, $(\mathcal{A}, P) \equiv_m (\mathcal{B}, Q)$ (cf. 3.2.10). According to (4.5), this implies $\mathcal{B} \in K$. Note that the number of formulas $\varphi^m_{(\mathcal{A},P)}$ is finite (cf. 3.2.7), hence the conjunctions and the disjunction in (4.6) is finite. After the formula is changed into prenex normal form, we get a Σ_1^1 sentence. □

Next, we consider another game, a modified Σ_1^1 game, which looks easier for the duplicator to win. The rules of the modified game are obtained from the rules of the Σ_1^1 game by reversing the order of two of the moves:

1. The spoiler selects $k \in N$.
2. The spoiler selects $m \in N$.
3. The duplicator selects $\mathcal{A}_0 \in K$.
4. The spoiler sets a k-ary relation P_0 on $\mathcal{A}_0$.
5. The duplicator selects $\mathcal{B}_0 \notin K$.
6. The duplicator sets a k-ary relation Q_0 on $\mathcal{B}_0$.
7. The spoiler and the duplicator play $\mathrm{G}_m((\mathcal{A}_0, P_0), (\mathcal{B}_0, Q_0))$.

Thus, in the modified game, the spoiler must commit himself to setting a k-ary relation on $\mathcal{A}_0$ before knowing what $\mathcal{B}_0$ is. In spite of the fact that it seems to be harder for the spoiler to win the modified Σ_1^1 game than the Σ_1^1 game, we have the following analogue of 4.2.3.

Theorem 4.2.5 A class K of finite structures is not axiomatizable in Σ_1^1 iff the duplicator has a winning strategy for the modified Σ_1^1 game.

Proof. For showing the implication from right to left, the first half of the proof of 4.2.3 applies to this case. Therefore it suffices to show the converse. This time the meaning of "spoiler has a winning strategy" is: for some k and for some n, for all $\mathcal{A}_0$ there is a k-ary relation $P_{\mathcal{A}_0}$ over $\mathcal{A}_0$ such that for all $\mathcal{B}_0$,

$$\text{"there is a } k\text{-ary relation } Q_0 \text{ over } \mathcal{B}_0 \text{ such that } \mathcal{A}_0 \in K \text{ and } (\mathcal{A}_0, P_{\mathcal{A}_0}) \equiv_m (\mathcal{B}_0, Q_0) \text{"} \quad \text{implies} \quad \mathcal{B}_0 \in K \tag{4.7}$$

Therefore, K is axiomatizable by a Σ_1^1 sentence $\exists P \bigvee\{\varphi^m_{(\mathcal{A},P_\mathcal{A})} \mid \mathcal{A} \in K\}$; suppose $\mathcal{A} \in K$, then clearly $\mathcal{A} \models \exists P \varphi^m_{(\mathcal{A},P_\mathcal{A})}$, hence $\mathcal{A}$ satisfies the sentence above. For the converse, suppose $\mathcal{B}$ satisfies the sentence. Then for some $\mathcal{A} \in K$, there is a $Q \subseteq B^k$ such that $(\mathcal{B}, Q) \models \varphi^m_{(\mathcal{A},P_\mathcal{A})}$, that is, $(\mathcal{A}, P_\mathcal{A}) \equiv_m (\mathcal{B}, Q)$. According to (4.7), this implies $\mathcal{B} \in K$. □

4.3 Concluding Remarks

In this chapter we presented the gametheoretic characterizations concerned with second-order definability, such as monadic second-order logic, general second-order logic, and existential second-order logic.

In the first section we introduced the game of monadic second-order logic, and we gave an application that the class of finite structures of even cardinality is not expressible in monadic second-order logic, showing that the duplicator has a winning strategy in the corresponding monadic second-order game.

Further, we showed that $I(\tau)$ of τ-structures of cardinality in a set I of natural numbers is definable in monadic second-order logic if and only if I is either finite or co-finite. This result gives rise to a question: what kind of a set of natural numbers I is definable in *binary* second-order logic or *binary* Σ_1^1? As shown in the end of the section, the set EVEN of even natural numbers is definable in Σ_1^1 with a single binary relation variable. And in the same way, one can show that the set of multiples of any given number is also definable in Σ_1^1 with a single binary relation variable. Well then, what about the set PRIME of prime numbers? In other words, which of the two players, the spoiler and the duplicator, has a winning strategy in the corresponding game? Here the game is obtained from the Σ_1^1 game (introduced in the second section) setting the arity $k = 2$ and the class $K = \text{PRIME}(\tau)$.

In case we set the arity $k = 1$ (i.e. monadic) in the game above, then the *duplicator* has a winning strategy in the game by the result mentioned above, for PRIME is neither finite nor co-finite. Still, in view of descriptive complexity theory (cf. section 2.3), PRIME is definable in Σ_1^1 by a sentence of the form $\exists X \varphi$ where φ is first-order and X is of arity seven (X symbolizes the computation of Turing machine that accepts prime numbers). Therefore the *spoiler* has a winning strategy in the corresponding game with arity $k = 7$. However, we don't know which of the two wins the game with arity between one and seven. This is a problem with challenge.

In the second section we extended the game of monadic second-order logic to the game applicable to general second-order logic. And the rest of the section was devoted to describe the game of the fragment Σ_1^1 of second-order logic in its two formulations.

The reason why we were particularly concerned with the expressibility in Σ_1^1 comes from the fact that, in view of descriptive complexity theory, the expressibility in Σ_1^1 is the same as the computability in NPTIME. Therefore the characterization of Σ_1^1 may contribute to solve the problems concerned with NPTIME in complexity theory. For example, the question as to whether $\Sigma_1^1 = \Pi_1^1$ is equivalent to the famous problem of whether NPTIME = co-NPTIME. In other words, we have NPTIME $\neq$ co-NPTIME if we find a class of finite structures which is in Π_1^1 and not in Σ_1^1. Although this is an open problem, Proposition 3.4.5 shows that *monadic* Σ_1^1 is different from *monadic* Π_1^1, in fact, the class of connected graphs is not monadic Σ_1^1 although it is monadic Π_1^1.

In this way, the author has tried, in vain, to find out a good mathematical property that is not expressible in Σ_1^1, in other words, a property for which the duplicator is able to construct his winning strategy in the corresponding Σ_1^1 game. The difficulty is that the spoiler is allowed to set an *arbitrary* k-ary relation on a structure. Therefore if the spoiler chooses a random k-ary relation (even restricted to binary), it seems to be impossible for the duplicator to find out a correct response to it. To overcome the problem, we need to simplify the Σ_1^1 game by putting restrictions on the spoiler's choice of a k-ary relation, such as the arity, the figure of a relation, and so on. This problem is left as a future work.

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