Title	Formal Topology における Bishop のコンパクト性
Author(s)	河井，達治
Citation	
Issue Date	2015-03
Type	Thesis or Dissertation
Text version	ETD
URL	http://hdl.handle.net/10119/12746
Rights	
Description	Supervisor:石原　哉，情報科学研究科，博士

BISHOP COMPACTNESS IN FORMAL TOPOLOGY

Tatsuji Kawai

Japan Advanced Institute of Science and Technology

Doctoral Dissertation

BISHOP COMPACTNESS IN FORMAL TOPOLOGY

Tatsuji Kawai

Supervisor: Hajime Ishihara

School of Information Science
Japan Advanced Institute of Science and Technology

March, 2015

Abstract

Since the publication of *Foundation of Constructive Analysis*, Bishop and coworkers have developed a large body of analysis constructively. However, the gap between the notion of compactness for topological spaces and for Bishop metric spaces has been a major obstacle to finding the right notion of general topology which naturally extends that of Bishop metric space.

Independently of Bishop, Sambin initiated a study of constructive general topology using a point-free approach. His notion, *formal topology*, has been quite successful in constructivising many results of classical general topology, and has established itself as the most promising approach to general topology in constructive mathematics.

However, the precise connection between Bishop metric space and formal topology has not been established, and this prevents us from applying the wealth of results obtained in formal topology to Bishop metric spaces. This thesis tries to improve this unsatisfactory situation by establishing a precise connection between the notions of compactness and local compactness for Bishop metric spaces and the corresponding notions for formal topologies.

As the first main result of this thesis, we obtained a point-free characterisation of compact metric spaces in terms of formal topology. We identified the full subcategory of formal topologies which is essentially equivalent to that of compact metric spaces. We show that the notion of compact overt enumerably completely regular formal topology characterises that of compact metric space up to isomorphism.

Our second main result generalises the above mentioned characterisation to the class of Bishop locally compact metric spaces. We show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space up to isomorphism. As an application of these characterisations, we prove a point-free version of the well-known fact that any Bishop locally compact metric space has a one-point compactification. The point-free result immediately yields the corresponding result for Bishop locally compact metric spaces.

Keywords: Constructive mathematics; Formal topologies; Point-free characterisations; Compact metric spaces; Locally compact metric spaces

Acknowledgements

First and foremost, I would like to thank my supervisor Hajime Ishihara for his excellent guidance and Giovanni Sambin for acting as a source of inspiration for my research during the last three years.

Next, I would like to thank Francesco Ciraulo, Milly Maietti, and Samuele Maschio for stimulating discussions, valuable advice, and warm hospitality during my visit to Padua.

I acknowledge financial support from Japan Advanced Institute of Science and Technology (JAIST), the Department of Mathematics of University of Padua, and Japan Society for the Promotion of Science (JSPS). In particular, I would like to thank JSPS for their generous funding as a Research Fellow during the last two years.

Many people gave me stimulating suggestions and comments on the topic of the thesis and research in general. I would especially like to thank Andrej Bauer, Nao Hirokawa, Basil Karádais, Takako Nemoto, Kazuhiro Ogata, Erik Palmgren, Giuseppe Rosolini, Peter Schuster, Helmut Schwichtenberg, and Bas Spitters.

I also thank my colleagues in JAIST, Yoshiyuki Ito, Takayuki Kihara, and Kazuto Yoshimura for creating a comfortable research environment.

Lastly, I thank my father, my mother, my sister and my brother for their constant encouragement and support. Without them, this thesis would not have been completed.

Contents

Chapter 1

Introduction

1.1 Bishop constructive mathematics

1.1.1 Foundation of Constructive Analysis

By the publication of his book *Foundation of Constructive Analysis*, Bishop gave a renewed impetus to constructive mathematics by reconstructing a large part of analysis constructively [8]. Unlike the previous approaches by Brouwer's Intuitionism and Markov's Constructive Recursive Mathematics, Bishop adopted a set of principles which is compatible with classical mathematics, but also with the other two schools of constructive mathematics.

The principles of Brouwer's Intuitionism which Bishop rejected are *Continuity principle* and the *Fan theorem*. The former states that all functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ are continuous, which is clearly inconsistent with classical mathematics[1]. The *Fan theorem* is equivalent to the statement that the Cantor space $2^{\mathbb{N}}$ is compact[2]. The Fan theorem is classically valid, but it is incompatible with one of the principles of Constructive Recursive Mathematics, namely Church's Thesis (CT). Church's Thesis states that every total function $f : \mathbb{N} \to \mathbb{N}$ is recursive. Since CT is incompatible with the Fan theorem, it is incompatible with classical mathematics. Hence, Bishop also rejected CT. By showing that a significant body of analysis can be constructivised without using these principles, Bishop established another style of constructive mathematics, called *Bishop constructive mathematics*, which is compatible with classical mathematics as well as the other two schools of constructive mathematics.

Apart from the use of intuitionistic logic and the rejection of the above mentioned principles, one notable feature of Bishop constructive mathematics is predicativity: it is not permissible to define a set A in terms of a collection of which A is to be an element[3].

[1] In Brouwer's Intuitionism, $\mathbb{N}^{\mathbb{N}}$ is the collection of all *choice sequences* (See [31]).

[2] In fact, the Fan theorem is a consequence of Brouwer's principle called *the monotone bar induction*, but it is widely recognised that as far as mathematical applications are concerned, the Fan theorem suffices.

[3] For example, defining the transitive closure of a relation $R \subseteq X \times X$ by the intersection of all the transitive relations on X which include R is not permissible.

In particular, the notion of power set, which makes such definitions possible, is rejected.

In this thesis, we study general topology in the style of constructive mathematics initiated by Bishop.

1.1.2 Limitations

Although Bishop and his coworkers developed a large body of analysis constructively, they did not developed general topology beyond the theory of metric spaces.

The notion of point-wise continuous function was considered to be useless by Bishop since we cannot prove that every point-wise continuous function from $[0, 1]$ to $\mathbb{R}$ is uniformly continuous without recourse to the Fan theorem. For this reason, Bishop defined a function on a locally compact metric space to be continuous if it is uniformly continuous on each compact subset. As far as the theory of metric spaces is concerned, this was a very successful step.

The classical notion of topological space was rejected by Bishop since the notion of uniformly continuous function cannot be formulated for topological spaces. Moreover, without the Fan theorem, there would be no nontrivial example of a compact topological space constructively. In fact, the classical example of a compact space, the unit interval $[0, 1]$, cannot be compact in Bishop constructive mathematics since the statement that $[0, 1]$ is compact is incompatible with Church's Thesis. These obstacles prevented Bishop from extending his theory of metric spaces to a more general notion of space.

1.2 Formal topology

Independently of Bishop, Sambin, together with Martin-Löf, proposed the notion of *formal topology* [50] with the aim of developing general topology in the constructive type theory of Martin-Löf [43].

1.2.1 Locale theory

The precursor of formal topology is locale theory [34]. Locale theory is based on the observation that many topological properties of a space can be characterised in terms of the lattice of its open subsets without mentioning points. The main idea of locale theory is to take the structure of frame, of which the lattice of open subsets of a topological space is one example, as the central object of study. For this reason, locale theory is called a point-free topology.

A frame is a poset $(A, \leq)$ which has a top $\top$, binary meets $\wedge$, and the join $\bigvee$ for each subset of A, and moreover a binary meet distributes over an arbitrary join, i.e. $a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$. A frame morphism $f : A \to B$ is a function which preserves the structure of frame. The category of locales is the *opposite* of the category of frames. In view of the fact that in point-set topology, the inverse image function $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ of a continuous function $f : (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$ determines a frame morphism from

$\mathcal{O}(Y)$ to $\mathcal{O}(X)$, the category of locales can be seen as a point-free counterpart of that of topological spaces.

A large body of literature on locale theory suggests that the point-free topology is not only possible, but often more fruitful than the point-set counterpart [34, 36]. More importantly, it has been noticed that by formulating classical topological notions in the language of locale, many classical theorems admit intuitionistic proofs [35].

1.2.2 Formal topology

The notion of locale, however, is problematic from a constructive point of view, since there is no non-trivial complete lattice which forms a set in constructive mathematics [24]. Instead, Sambin developed formal topology based on the presentation of locale as a formal space [26]. A formal topology is a triple $(S, \lhd, \leq)$ where $(S, \leq)$ is a preordered set and $\lhd$ is a relation between S and the subsets of S satisfying the conditions in Definition 2.1.1. Intuitively, the set S can be thought of as a set of basic open subsets of an imaginary topological space, and $a \lhd U$ can be read as 'the basic open a is covered by the union of basic opens in U'. The preorder $\leq$ can be thought of as a covering relation between basic opens. A motivating example of a formal topology is the one determined by a concrete space. A *concrete space* is a triple $(X, \Vdash, S)$ where X and S are sets and $\Vdash$ is a relation between X and S such that

$$X = \mathsf{ext}\, S,$$
$$\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}(a \downarrow b),$$

where

$$\mathsf{ext}\, a \stackrel{\text{def}}{=} \{x \in X \mid x \Vdash a\},$$
$$\mathsf{ext}\, U \stackrel{\text{def}}{=} \bigcup_{a \in U} \mathsf{ext}\, a,$$
$$a \downarrow b \stackrel{\text{def}}{=} \{c \in S \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}$$

for all $a, b \in S$ and $U \subseteq S$. A concrete space is nothing but a set X equipped with a family of basic open subsets $(\mathsf{ext}\, a)_{a \in S}$. Then, every concrete space $(X, \Vdash, S)$ determines a formal topology $\mathcal{S}_X = (S, \lhd_X, \leq_X)$ by

$$a \leq_X b \stackrel{\text{def}}{\iff} \mathsf{ext}\, a \subseteq \mathsf{ext}\, b,$$
$$a \lhd_X U \stackrel{\text{def}}{\iff} \mathsf{ext}\, a \subseteq \mathsf{ext}\, U.$$

Hence, in $\mathcal{S}_X$ the intuitive readings of S, $\leq$ and $\lhd$ agree with the actual definitions. However, the point of formal topology is in forgetting the points, and taking the structure $(S, \lhd, \leq)$ as a primitive. By doing so, formal topology has achieved significant success in constructivising many important results of the classical point-set topology [51]. One notable example is the Tychonoff theorem for compact formal topologies [16, 59], which in the classical point-set topology is equivalent to the Axiom of Choice. Moreover, important examples of spaces, e.g. real numbers $\mathbb{R}$, the unit interval $[0,1]$, and the Cantor space have desirable properties when they are formulated in formal topology. For example, the formal notion of $\mathbb{R}$ and $[0,1]$, called the formal reals $\mathcal{R}$ and the formal unit interval $\mathcal{I}[0,1]$ respectively, are locally compact and compact as formal topologies respectively. If the classical point-set notion of topology were adopted, these results cannot be obtained without recourse to the Fan theorem.

Hence, formal topology is considered to be a viable substitute for the classical notion of topological space. From the point of view of Bishop's theory of metric spaces, however, it still remains to be seen whether formal topology can be regarded as an extension of Bishop metric space.

1.3 Connection with Bishop metric space

1.3.1 Classical adjunction

In locale theory, the notion of locale and that of topological space are related by the adjunction between the category of locales and that of topological spaces. Given a locale A, a point of A is a locale map $\mathbf{1} \to A$ from the terminal object $\mathbf{1}$, the power set lattice of a one-point set $\mathbf{1} = \mathsf{Pow}(\{*\})$. The collection of points $Pt(A)$ of A can be equipped with a suitable topology, and this assignment $A \mapsto Pt(A)$ extends to a functor from the category **Loc** of locales to that of topological spaces **Top**. On the opposite direction, the operation of taking the lattice of open subsets of a topological space extends to a functor from **Top** to **Loc**. The two functors form an adjunction between **Top** and **Loc**, and it restricts to an equivalence between the category of sober topological spaces and that of spatial locales. Hence, on these subcategories the theory of locales and that of topological spaces are essentially equivalent. Classically, the class of spatial locales is very large, and many spatial locales are related to familiar point-set spaces. For example, the localic reals[4] correspond to the reals $\mathbb{R}$ and the class of compact regular locales corresponds to that of compact Hausdorff spaces [34]. In locale theory, this adjunction is a standard framework in which to compare the point-free approach with the point-set counterpart.

The above adjunction can be formulated in the context of formal topology by suitably modifying the definition of topological space [4]. Constructively, however, the adjunction is of little use in relating formal topology with the point-set topology since there seems to be no non-trivial spatial formal topology except for those formal topologies which are determined by concrete spaces. For example, the statement that the formal reals $\mathcal{R}$ is spatial is equivalent to the compactness of $[0,1]$, which is inconsistent with Church's Thesis (See Section 2.5.3). The lack of non-trivial examples of spatial formal topologies implies that there is little hope of establishing precise connections between point-free notions and point-set notions through this adjunction.

[4]The locale version of the formal reals $\mathcal{R}$.

1.3.2 From Bishop metric space to formal topology

A pioneering work by Palmgren [48] initiated a series of researches relating Bishop's notion of metric space with that of formal topology [49, 53, 19]. Based on the notion of localic completion of a metric space by Vickers [58], Palmgren extended it to a full and faithful functor (i.e. an embedding) from the category of locally compact metric spaces into that of locally compact regular formal topologies. The embedding has an important property that a metric space is totally bounded iff its localic completion is compact as a formal topology [58]. Moreover, important examples of metric spaces such as $\mathbb{R}$ and $[0,1]$ are related to its point-free counterparts via the embedding. For example, the localic completion of $\mathbb{R}$ is the formal reals $\mathcal{R}$.

From Palmgren's embedding, we can draw the following conclusions.

- The notion of morphism between formal topologies is compatible with that of continuous function as defined by Bishop.
- The notions of compactness and local compactness for formal topologies are compatible with the corresponding notions for Bishop metric spaces.

Hence, if we look at the relation between Bishop metric space and formal topology through this embedding, we see that formal topology resolves major issues which have prevented generalisation of Bishop metric space. For this reason, we regard Palmgren's embedding of locally compact metric spaces into formal topologies as a fundamental construction which connects Bishop metric space and formal topology.

1.4 Aim of the thesis

From the point of view of Bishop's theory of metric spaces, one of the motivations for developing general topology is to make the abstract method of general topology available to the theory of metric spaces. In order for the theory of formal topologies to serve this purpose, however, we need to know precisely which class of formal topologies corresponds to Bishop metric spaces. The main aim of this thesis is to establish part of this correspondence.

In this thesis, we focus on two important classes of metric spaces: compact metric spaces and Bishop locally compact metric spaces[5]. Our aim is to establish a precise correspondence between the properties of formal topologies and the compactness for metric spaces, and similarly for the local compactness. More specifically, we try to identify two subcategories of the category of formal topologies which are essentially equivalent to the categories of compact metric spaces and Bishop locally compact metric spaces respectively. To this end, we aim to characterise the image of the categories of compact metric spaces and Bishop locally compact metric spaces under Palmgren's embedding in terms of formal topology. Since the embedding is full and faithful, if such characterisations are obtained, then we have the desired full subcategories. In this thesis, we call such characterisations

[5] See Definition 5.0.9 for the definition of Bishop locally compact metric space.

point-free characterisations of compact metric spaces and Bishop locally compact metric spaces.

1.5 Note on foundations

The publication of Bishop's book stimulated development of several formal systems suitable for formalisation of Bishop constructive mathematics. Two systems which are widely in use today are Martin-Löf's constructive type theory [43] (henceforth, simply the type theory) and Aczel's Constructive Zermelo-Fraenkel Set theory (CZF) [1]. The type theory makes the intuitionistic reading of the logical connectives explicit, and hence, it is regarded as the most fundamental framework for constructive mathematics. On the other hand, constructive justification of CZF has been given by a series interpretations into the type theory [1, 2, 3].

In this thesis, we adopt CZF as the foundational framework mainly because it allows us to use the familiar set theoretical language. The set theory CZF is a first-order theory similar to the classical set theory ZF, but formulated in intuitionistic logic. CZF is a predicative set theory in that it does not have the Powerset axiom and restricts the separation scheme to restricted formulae.

In addition to the standard axioms of CZF, we also require several extra axioms of CZF. First, we need two choice principles: the Countable Choice and the Dependent Choice. The latter is stronger than the former, but we prefer to distinguish the use of the two axioms in order to isolate the axioms which are needed to prove a particular result. Next, we need a weaker version of the Regular Extension Axiom, called wREA. The axiom wREA allows us to define a set by a generalised inductive definition. In this thesis, wREA is used to define the notion of inductively generated formal topology in Section 2.2.

Throughout this thesis, we work informally in CZF using familiar set constructions which are known to be possible in CZF. One exception is Section 3.2.3, where we explicitly indicate the use of the principle of Fullness. The standard axioms of CZF and the above mentioned extra axioms are listed in Appendix A. A more detailed treatment of these axioms can be found in [6].

1.6 Overview

Chapter 2 provides background on formal topology which will be needed in later chapters. First, we recall the adjunction between the category of formal topologies and that of set-based locales. Next, we introduce the notion of inductively generated formal topology and the method of proof by induction. Then, we establish well-known facts about subtopologies, regularities, compactness, and local compactness. Finally, we describe the adjunction between the category of constructive topological spaces and that of formal topologies, and recall the well-known fact that spatiality of certain formal topologies are equivalent to some versions of bar inductions.

In Chapter 3, we consider extensions of Palmgren's embedding to the setting of uniform spaces. Except for Section 3.1.7, this chapter is a digression from the main line of this thesis. The aim of this chapter is to see how much of the results on localic completions of metric spaces can be extended to a wider class of point-set spaces where the notion of uniform continuity is still meaningful. We consider two extensions: one to the class of uniform spaces defined by sets of pseudometrics and the other to the class of uniform spaces defined by covering uniformities. In Section 3.1, we consider an extension of the embedding to the class of uniform spaces defined by sets of pseudometrics. We define the notion of localic completion of a uniform space which naturally extends Vickers's notion of localic completion of a metric space [58]. We show that localic completions of uniform spaces retain most of the well-known properties of localic completions of metric spaces [48]. In particular, we extend the construction of a localic completion to a full and faithful functor from the category of locally compact uniform spaces to that of locally compact regular formal topologies. We also show that the functor preserves countable products of inhabited compact uniform spaces, the result which is crucial in Chapter 4. In Section 3.2, we consider an extension of Palmgren's embedding to the class of uniform spaces defined by covering uniformities. We define the notion of covering completion of a uniform space analogous to that of localic completion of a uniform space defined in Section 3.1. We show that a uniform space is totally bounded iff its covering completion is compact. Then, we extend the construction of a covering completion to a full and faithful functor from the category of compact uniform spaces to that of compact 2-regular formal topologies. In Section 3.3, we compare the notion of localic completion with that of covering completion. We show that the two notions are equivalent on the class of uniform spaces defined by sets of pseudometrics, and that the two functors associated with these notions are naturally isomorphic on the category of compact uniform spaces defined by sets of pseudometrics.

In Chapter 4, we give a point-free characterisation of compact metric spaces. In Section 4.1, we show that the class of compact overt subtopologies of the localic completions of locally compact metric spaces characterises the image of the class of compact metric spaces under Palmgren's embedding. To obtain this result, we also extend the notion of located subtopology of compact regular formal topologies by Spitters [53] to the class of locally compact formal topologies. In Section 4.2, we characterise the class of enumerably completely regular formal topologies as the subtopologies of the countable product of formal unit intervals. Combining these results, in Section 4.3 we show that the notion of compact overt enumerably completely regular formal topology characterises that of compact metric space. In Section 4.4, we give an application of the point-free characterisation. We show that any inhabited compact enumerably completely regular formal topology is a surjective image of the formal Cantor space, a point-free analogue of the famous result due to Brouwer [55].

In Chapter 5, we give a point-free characterisation of Bishop locally compact metric spaces. In Section 5.1, we introduce the notion of the open complement of a located subtopology for the class of locally compact formal topologies. Then, we show that every inhabited open complement of a located subtopology of the localic completion of a compact metric space is in the image of Bishop locally compact metric spaces under

Palmgren's embedding. In Section 5.2, we define the notion of one-point compactification of an overt enumerably locally compact regular formal topology, and show that every overt enumerably locally compact regular formal topology has a one-point compactification. In Section 5.3, we show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space.

In Chapter 6, we summarise our results and give possible directions for further research.

In Appendix, we give further background. Appendix A lists the axioms of CZF. Appendix B contains a proof of the Tychonoff theorem for formal topologies due to Vickers [59]. Appendix C describes an embedding of the category **OLCM** into that of formal topologies due to Palmgren [49] which we exploit in Chapter 5. Appendix D gives some background on Bishop metric spaces.

Chapter 2

Formal Topologies

In this Chapter, we introduce the notion of formal topology and review some basic facts about formal topologies which are relevant for later chapters.

In Section 2.1, we give the definition of formal topology and relate that notion with the impredicative theory of locale by establishing an equivalence between the category of formal topologies and that of set-based locales. Next, we introduce the notion of overtness for formal topologies which distinguishes intuitionistic point-free topologies from the classical counterpart.

The notion of inductively generated formal topology and the method of proof by induction associated with it play important roles in later chapters. In Section 2.2, we establish basic facts about inductively generated formal topologies and review some of the categorical constructions that can be performed on the class of inductively generated formal topologies.

In Section 2.3, we consider the notion of subtopology. The standard notions of open and closed subtopologies are introduced together with another notion of closed subtopology, called overt weakly closed subtopology. We pay particular attention to the connection between the notions of 'closed' and 'overt weakly closed', which we often exploit in Chapter 4 and Chapter 5. Then in Section 2.4, we review important topological properties of formal topologies: regularities, compactness and local compactness, and we establish well-known facts about these properties. We use the formal reals as a running example to illustrate these notions.

Finally in Section 2.5, we describe a constructive version of the classical adjunction between the category of topological spaces and that of locales. Then, we introduce the notion of spatiality of a formal topology and recall the well-known fact that spatiality of certain formal topologies are equivalent to some versions of bar inductions.

Preliminaries

We define notations which will be used in this thesis. We use class notation throughout this thesis (See [6] for details). Informally, a class is a collection of sets, or a properties that can be specified by the language of CZF. Hence, a class A is identified with a formula

$\varphi(x)$ with a free variable x. In this case, we write $A = \{x \mid \varphi(x)\}$ and $x \in A$ for $\varphi(x)$.

Let S be a set. Then, $\mathsf{Pow}(S)$ denotes the class of subsets of S, i.e. $\mathsf{Pow} = \{x \mid x \subseteq S\}$. Constructively, $\mathsf{Pow}(S)$ is not a set unless $S = \emptyset$. $\mathsf{Fin}(S)$ denotes the *set* of finitely enumerable subsets of S, where a set A is finitely enumerable if there exists a surjection $f\colon \{0, \dots, n-1\} \to A$ for some $n \in \mathbb{N}$. $\mathsf{Fin}(S)$ is the least set such that

1. $\emptyset \in \mathsf{Fin}(S)$,

2. $A \in \mathsf{Fin}(S)$ & $a \in S \implies A \cup \{a\} \in \mathsf{Fin}(S)$.

$\mathsf{Fin}^+(S)$ denotes the set of inhabited finitely enumerable subsets of S, where a set X is *inhabited* if there exists an element $x \in X$.

For any set S, S^* denotes the set of finite lists of elements of S. The elements of S^* are denoted by $\langle a_0, \dots, a_{n-1}\rangle$, where $a_0, \dots, a_{n-1} \in S$. In particular, $\langle\rangle$ denotes the null list. The length $|l|$ of $l \in S^*$ is defined by $|\langle\rangle| = 0$ and $|l * \langle a\rangle| = |l| + 1$. For each $k < |l|$, $l(k)$ denotes the k-th element of l. The concatenation of two finite lists $a, b \in S^*$ is denoted by $a * b$. The prefix relation $a \preccurlyeq b$ on S^* is given by $a \preccurlyeq b \overset{\text{def}}{\iff} (\exists c \in S^*)\, a * c = b$. Given any sequence $\alpha : \mathbb{N} \to S$ of elements of S and $n \in \mathbb{N}$, $\overline{\alpha}n$ denotes the initial segment of α of length n; it is defined by $\overline{\alpha}0 = \langle\rangle$ and $\overline{\alpha}(n+1) = \overline{\alpha}n * \langle\alpha(n)\rangle$.

For subsets $U, V \subseteq S$ of a set S, we define

$$\neg U \overset{\text{def}}{=} \{a \in S \mid \neg(a \in U)\},$$
$$U \between V \overset{\text{def}}{\iff} (\exists a \in S)\, a \in U \cap V.$$

For $a \in S$ and $U, V \subseteq S$, we sometimes use the following notations.

$$V(a) \overset{\text{def}}{\iff} a \in V,$$
$$V(U) \overset{\text{def}}{\iff} V \between U.$$

Given a relation $r \subseteq X \times S$ between sets X and S and their subsets $D \subseteq X$ and $U \subseteq S$, the direct image and the inverse image under r, respectively, are defined by

$$rD \overset{\text{def}}{=} \{a \in S \mid (\exists x \in D)\, x \mathrel{r} a\},$$
$$r^- U \overset{\text{def}}{=} \{x \in X \mid (\exists a \in U)\, x \mathrel{r} a\}.$$

Furthermore, we introduce the following notations.

$$r^* U \overset{\text{def}}{=} \{x \in X \mid r\{x\} \subseteq U\},$$
$$r^{-*} D \overset{\text{def}}{=} \left\{a \in S \mid r^-\{a\} \subseteq D\right\}.$$

We sometimes write rx for $r\{x\}$ and similarly for the other operations determined by a relation.

2.1 Formal topologies

Formal topology aims to develop general topology in constructive foundations. The structure which arises from the formal side of a concrete space motivates the following definition.

Definition 2.1.1. A *formal topology* $\mathcal{S}$ is a triple $\mathcal{S} = (S, \lhd, \leq)$ where $(S, \leq)$ is a preordered set and $\lhd$ is a relation between elements of S and subsets of S such that

$$\mathcal{A}U \stackrel{\text{def}}{=} \{a \in S \mid a \lhd U\}$$

is a set for each $U \subseteq S$ and satisfies

(Ref) $U \lhd U$,

(Tra) $a \lhd U \ \& \ U \lhd V \implies a \lhd V$,

(Loc) $a \lhd U \ \& \ a \lhd V \implies a \lhd U \downarrow V$,

(Ext) $a \leq b \implies a \lhd b$

for all $a, b \in S$ and $U, V \subseteq S$. Here, we define

$$U \lhd V \stackrel{\text{def}}{\iff} (\forall a \in U)\, a \lhd V,$$
$$U \downarrow V \stackrel{\text{def}}{=} \{c \in S \mid (\exists a \in U)\, (\exists b \in V)\, c \leq a \ \& \ c \leq b\}.$$

We write $a \downarrow U$ for $\{a\} \downarrow U$ and $U \lhd a$ for $U \lhd \{a\}$. The set S is called the *base* of $\mathcal{S}$, and the relation $\lhd$ is called a *cover* on $(S, \leq)$ (or the cover of $\mathcal{S}$).

Remark 2.1.2. Some authors use the term *formal cover* for the structure defined above and reserve the term 'formal topology' for the notion of overt formal topology to be introduced in Definition 2.1.12 [42, 41]. The above definition corresponds to the notion of $\leq$-*formal cover* defined in [15], and it is shown to be equivalent to the other definitions of formal topology (or formal cover)[1]. In this thesis, we follow the terminology used in [48, 49, 53] to which this thesis is most relevant.

Notation 2.1.3. We shall use letters $\mathcal{S}, \mathcal{S}', \dots$ to denote formal topologies. Given a formal topology $\mathcal{S}$, we use letters S, $\lhd$, and $\leq$ to denote the base, the cover, and the preorder of $\mathcal{S}$. To avoid confusion, we often append subscripts (or superscripts) to the base, the cover and the preorder, e.g. S', $\lhd_{\mathcal{S}}$ and $\leq_X$.

Given a formal topology $\mathcal{S}$, a subset $U \subseteq S$ is said to be *saturated* if $U = \mathcal{A}U$. The collection $\mathit{Sat}(\mathcal{S})$ of saturated subsets of S can be identified with the class $\mathsf{Pow}(S)$ together with the equality

$$U =_{\mathcal{S}} V \stackrel{\text{def}}{\iff} \mathcal{A}U = \mathcal{A}V$$

[1] See [15] for various definitions of formal topology and equivalence between them.

for all $U, V \subseteq S$. The class $\mathit{Sat}(\mathcal{S})$ forms a *frame*, a partially ordered class $(A, \leq)$ with arbitrary set-indexed joins $\bigvee_{i \in I}$ and finite meets $\wedge$ (including the top element $\top$) which distribute over set-indexed joins, i.e.

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i.$$

The class $\mathit{Sat}(\mathcal{S})$ is ordered by $U \leq_{\mathit{Sat}(\mathcal{S})} V \overset{\text{def}}{\iff} U \lhd V$. The top of $\mathit{Sat}(\mathcal{S})$ is given by S, the meet is given by $\mathcal{A}\,U \wedge \mathcal{A}\,V \overset{\text{def}}{=} \mathcal{A}(U \downarrow V)$ and the join is given by $\bigvee_{i \in I} \mathcal{A}\,U_i \overset{\text{def}}{=} \mathcal{A} \bigcup_{i \in I} U_i$ for any $U, V \subseteq S$ and for any set-indexed family $(U_i)_{i \in I}$ of subsets of S. Moreover, the frame $\mathit{Sat}(\mathcal{S})$ is *set-based* in the sense as follows: a *set-based* frame is a frame $(A, \top, \wedge, \bigvee)$ together with a set-indexed family $(x_a)_{a \in S}$ of elements of A such that for each $x \in A$

1. $S_x = \{a \in S \mid x_a \leq x\}$ is a set,
2. $x = \bigvee_{a \in S_x} x_a$.

We call such a family the *base* of the frame A. In the case of $\mathit{Sat}(\mathcal{S})$, the base is given by the family $(\mathcal{A}\,\{a\})_{a \in S}$, since we have $\mathcal{A}\,U = \mathcal{A} \bigcup_{a \in U} \mathcal{A}\,\{a\}$ for all $U \subseteq S$.

Conversely, any set-based frame $(A, \top, \wedge, \bigvee)$ with a base $(x_a)_{a \in S}$ determines a formal topology $\mathcal{S}_A = (S, \lhd_A, \leq_A)$ by

$$\begin{aligned} a \leq_A b &\overset{\text{def}}{\iff} x_a \leq x_b, \\ a \lhd_A U &\overset{\text{def}}{\iff} x_a \leq \bigvee_{b \in U} x_b \end{aligned} \tag{2.1}$$

for all $a, b \in S$ and $U \subseteq S$, where the order $\leq$ is that of A.

Thus, a formal topology can be thought of as a presentation of its associated set-based frame. This can be made more precise once we define the notion of morphism between formal topologies.

Definition 2.1.4. Let $\mathcal{S}$ and $\mathcal{S}'$ be formal topologies. A relation $r \subseteq S \times S'$ is called a *formal topology map* from $\mathcal{S}$ to $\mathcal{S}'$ if

(FTM1) $S \lhd r^- S'$,

(FTM2) $r^- a \downarrow r^- b \lhd r^-(a \downarrow' b)$,

(FTM3) $a \lhd' U \implies r^- a \lhd r^- U$

for all $a, b \in S'$ and $U \subseteq S'$. Note that under (FTM3), the condition (FTM2) is equivalent to

$$\mathcal{A}\,r^- \{a\} \cap \mathcal{A}\,r^- \{b\} \subseteq \mathcal{A}\,r^-(\mathcal{A}'\,\{a\} \cap \mathcal{A}'\,\{b\}).$$

Hence, a formal topology map does not depend on the preorders of its domain and codomain.

The collection $\mathrm{Hom}(\mathcal{S}, \mathcal{S}')$ of formal topology maps from $\mathcal{S}$ to $\mathcal{S}'$ is ordered by

$$r \leq s \overset{\text{def}}{\iff} (\forall a \in S')\, r^{-}a \lhd s^{-}a.$$

Two formal topology maps $r, s : \mathcal{S} \to \mathcal{S}'$ are defined to be *equal*, denoted by $r = s$, if $r \leq s$ and $s \leq r$. Note that Definition 2.1.4 is well-defined with respect to this equality.

Remark 2.1.5. Given a formal topology map $r : \mathcal{S} \to \mathcal{S}'$, the relation $r_{\mathcal{A}} \subseteq S \times S'$ given by

$$a\, r_{\mathcal{A}}\, b \overset{\text{def}}{\iff} a \lhd r^{-}b$$

is the largest formal topology map which is equivalent to r. In some literature [4, 28, 48], a formal topology map is defined in terms of the largest representative of the equivalence class of formal topology maps. We have adopted Definition 2.1.4 since it simplifies some arguments.

If $r : \mathcal{S} \to \mathcal{S}'$ and $s : \mathcal{S}' \to \mathcal{S}''$ are formal topology maps, the composition of r and s, denoted by $s \circ r$, is given by the composition of the underlying relations. It is easy to check that $s \circ r$ is a formal topology map from $\mathcal{S}$ to $\mathcal{S}''$. Note that compositions respect the equality on formal topology maps. This follows from (FTM3).

For each formal topology $\mathcal{S}$, let $id_{\mathcal{S}}$ denote the formal topology map from $\mathcal{S}$ to $\mathcal{S}$ whose underlying relation is the identity relation id_S on the base S of $\mathcal{S}$. A formal topology map $r : \mathcal{S} \to \mathcal{S}'$ is an *isomorphism* if there exists a formal topology map $s : \mathcal{S}' \to \mathcal{S}$ such that

$$r \circ s = id_{\mathcal{S}'}, \qquad s \circ r = id_{\mathcal{S}}.$$

Here, the equalities are interpreted as the equality on formal topology maps. As usual, s is called the inverse of r.

The formal topologies and formal topology maps form a category **FTop** with the identities and compositions as defined above.

2.1.1 Equivalence of formal topologies and set-based locales

In this section, we recall the well-known fact that the category of formal topologies and the opposite of the category of set-based frames, namely the category of set-based locales, are equivalent. The equivalence establishes a precise connection between the theory of formal topologies with that of locales [34]. For the record, the category **Frm** of set-based frames consists of set-based frames and operations between these frames which preserve set-indexed joins and finite meets. The opposite of **Frm** is called the category of set-based locales, and is denoted by $\mathbf{Frm}^{Op}$. An object of $\mathbf{Frm}^{Op}$ is called a set-based locale although it is just a set-based frame.

From **FTop** to $\mathbf{Frm}^{Op}$

Given a formal topology map $r : \mathcal{S} \to \mathcal{S}'$, the operation $\mathcal{A}\, r^-(-) : Sat(\mathcal{S}') \to Sat(\mathcal{S})$ given by $U \mapsto \mathcal{A}\, r^- U$ defines a frame morphism from $Sat(\mathcal{S}')$ to $Sat(\mathcal{S})$. Note that $\mathcal{A}\, r^-(-)$ respects the equality on $Sat(\mathcal{S}')$. Indeed, if $U =_{\mathcal{S}'} V$, then we have

$$\mathcal{A}\, r^- U = \mathcal{A}\, r^- \mathcal{A}' U = \mathcal{A}\, r^- \mathcal{A}' V = \mathcal{A}\, r^- V$$

by (FTM3). Then, the conditions (FTM1), (FTM2), and (FTM3) are equivalent to saying that $\mathcal{A}\, r^-(-)$ preserves the top, the binary meets, and the set-indexed joins of $Sat(\mathcal{S}')$ respectively. Note also that equal formal topology maps give rise to the same frame morphism. The assignments $\mathcal{S} \mapsto Sat(\mathcal{S})$ and $r \mapsto \mathcal{A}\, r^-(-)$ determines a functor from **FTop** to $\mathbf{Frm}^{Op}$, which we denote by $F : \mathbf{FTop} \to \mathbf{Frm}^{Op}$.

From $\mathbf{Frm}^{Op}$ to **FTop**

Given a frame map $f : L \to L'$ between set-based frames $L = (L, \top, \wedge, \bigvee, (x_a)_{a \in S})$ and $L' = (L', \top', \wedge', \bigvee', (y_b)_{b \in S'})$ with bases $(x_a)_{a \in S}$ and $(y_b)_{b \in S'}$ respectively, the relation $r_f \subseteq S' \times S$ given by

$$b\; r_f\; a \overset{\text{def}}{\iff} y_b \leq' f(x_a)$$

determines a formal topology map from $\mathcal{S}_{L'}$ to $\mathcal{S}_L$, where $\mathcal{S}_{L'}$ and $\mathcal{S}_L$ are defined by (2.1). Indeed, since

$$b \lhd_{L'} r_f^- U \iff y_b \leq' {\bigvee_{a \in U}}' \; {\bigvee_{\substack{c \in S', \\ y_c \leq' f(x_a)}}}' y_c \iff y_b \leq' {\bigvee_{a \in U}}' f(x_a), \tag{2.2}$$

and

$$\begin{aligned}
{\bigvee_{b \in S'}}' y_b = \top' \leq' f(\top) = f\left(\bigvee_{a \in S} x_a\right) = {\bigvee_{a \in S}}' f(x_a), \\
f(x_a) \wedge' f(x_b) = f(x_a \wedge x_b) = f\left(\bigvee_{c \in a \downarrow b} x_c\right) = {\bigvee_{c \in a \downarrow b}}' f(x_c), \\
x_a \leq \bigvee_{b \in U} x_b \implies f(x_a) \leq' f\left(\bigvee_{b \in U} x_b\right) = {\bigvee_{b \in U}}' f(x_b),
\end{aligned}$$

the relation r_f is a formal topology map from $\mathcal{S}_{L'}$ to $\mathcal{S}_L$. Using (2.2), it is easy to see that the assignment $f \mapsto r_f$ preserves compositions of frame morphisms. This, together with the assignment $L \mapsto \mathcal{S}_L$ of a formal topology to each set-based frame, determines a functor from $\mathbf{Frm}^{Op}$ to **FTop**, which we denote by $G : \mathbf{Frm}^{Op} \to \mathbf{FTop}$.

Equivalence of FTop and Frm^{Op}

Given a formal topology $\mathcal{S}$, the formal topology $GF(\mathcal{S}) = (S, \lhd_{F(\mathcal{S})}, \leq_{F(\mathcal{S})})$ is given by

$$\begin{aligned} a \leq_{F(\mathcal{S})} b &\overset{\text{def}}{\iff} a \lhd b, \\ a \lhd_{F(\mathcal{S})} U &\overset{\text{def}}{\iff} \mathcal{A}\{a\} \subseteq \mathcal{A} \bigcup_{b \in U} \mathcal{A}\{b\} \iff a \lhd U. \end{aligned}$$

Since the covers of $\mathcal{S}$ and $GF(\mathcal{S})$ are equal, the identity relation id_S on S is an isomorphism $id_S : \mathcal{S} \to GF(\mathcal{S})$. Moreover, for any formal topology map $r : \mathcal{S} \to \mathcal{S}'$, the formal topology map $GF(r) : GF(\mathcal{S}) \to GF(\mathcal{S}')$ is given by

$$a\, GF(r)\, b \overset{\text{def}}{\iff} \mathcal{A}\{a\} \subseteq \mathcal{A}\, r^{-} \mathcal{A}'\{b\} \iff a \lhd r^{-} b. \tag{2.3}$$

Since the covers of $\mathcal{S}$ and $GF(\mathcal{S})$ are the same and similarly for $\mathcal{S}'$ and $GF(\mathcal{S}')$, the underlying relation r of $r : \mathcal{S} \to \mathcal{S}'$ is a formal topology map from $GF(\mathcal{S})$ to $GF(\mathcal{S}')$, which we denote by $\bar{r}$. Then, (2.3) says that $GF(r)$ is equal to $\bar{r}$. It easily follows that the family of morphisms $id_{\mathcal{S}} : \mathcal{S} \to GF(\mathcal{S})$ for each $\mathcal{S}$ is a natural isomorphism between the identity functor $Id_{\mathbf{FTop}}$ on **FTop** and $G \circ F$.

Conversely, for each set-based frame $L = (L, \top, \wedge, \bigvee)$ with a base $(x_a)_{a \in S}$, define an operation $\varepsilon_L : FG(L) \to L$ by

$$\varepsilon_L(U) \overset{\text{def}}{=} \bigvee_{a \in U} x_a$$

for all $U \subseteq S$. Note that ε_L respects the equality on $Sat(G(L))$ since $U =_{G(L)} V \iff \bigvee_{a \in U} x_a = \bigvee_{b \in V} x_b$ for any $U, V \subseteq S$. We have

$$\begin{aligned} \varepsilon_L(S) &= \bigvee_{a \in S} x_a = \top, \\ \varepsilon_L(U \wedge V) &= \varepsilon_L(U \downarrow V) = \bigvee_{a \in U \downarrow V} x_a = \bigvee_{\substack{a \in U, \\ b \in V}} x_a \wedge x_b \\ &= \left(\bigvee_{a \in U} x_a \right) \wedge \left(\bigvee_{b \in V} x_b \right) = \varepsilon_L(U) \wedge \varepsilon_L(V), \\ \varepsilon_L \left(\bigvee_{i \in I} U_i \right) &= \varepsilon_L \left(\bigcup_{i \in I} U_i \right) = \bigvee_{i \in I} \bigvee_{a \in U_i} x_a = \bigvee_{i \in I} \varepsilon_L(U_i). \end{aligned}$$

Thus, ε_L is a morphism of frames. The inverse θ_L of ε_L is given by

$$\theta_L(x) \overset{\text{def}}{=} \{a \in S \mid x_a \leq x\}.$$

Hence, ε_L is an isomorphism between $FG(L)$ and L.

Given any morphism $f : L \to L'$ between set-based frames, where L has a base $(x_a)_{a \in S}$, we have by (2.2)

$$(\varepsilon_{L'} \circ FG(f))(U) = \bigvee_{a\in U}{}' f(x_a) = f\left(\bigvee_{a\in U} x_a\right) = (f \circ \varepsilon_L)(U)$$

for all $U \subseteq S$, where we identified the morphism $FG(f) : FG(L') \to FG(L)$ of set-based locales with the underlying frame morphism from $FG(L)$ to $FG(L')$. Hence, ε_L represents a component of a natural isomorphism from $Id_{\mathbf{Frm}^{Op}}$ to $F \circ G$.

Therefore, the functors F and G together with the natural isomorphisms id and ε establish an equivalence of categories **FTop** and $\mathbf{Frm}^{Op}$.

Remark 2.1.6. For any formal topology map $r : \mathcal{S} \to \mathcal{S}'$, since $Sat(\mathcal{S}')$ is set-based, the frame morphism $\mathcal{A}\, r^-(-) : Sat(\mathcal{S}') \to Sat(\mathcal{S})$ has a right adjoint $f : Sat(\mathcal{S}) \to Sat(\mathcal{S}')$. By definition, we must have

$$f(U) = \bigvee{}' \left\{a \in S' \mid \mathcal{A}\, r^- \{a\} \lhd U\right\} = \mathcal{A}'\, r^{-*} U = r^{-*} U.$$

Hence, the right adjoint of $\mathcal{A}\, r^-(-)$ is given by the operation $r^{-*}(-)$.

2.1.2 Overt formal topologies

The notion of unary positivity predicate was originally included in the definition of formal topology [50], but it was later dropped from the definition. However, a formal topology with a unary positivity predicate, called an overt formal topology, is of particular interest to constructive mathematics, since classically every formal topology is overt, but constructively this is not the case (See Example 2.3.10). Hence, we expect that an overt formal topology may carry constructively meaningful properties. Indeed, in Chapter 4, we will see that overtness is related to the notion of locatedness, which is one of the most important properties of metric spaces in constructive mathematics [8, 54][2].

First, we introduce the notion of splitting subset of formal topologies [51] (also called lower powerpoint in [61]), which can be thought of as a point-free analogue of the notion of closed subset of topological spaces (See Theorem 4.1.3).

Definition 2.1.7. Let $\mathcal{S}$ be a formal topology. A *splitting subset* of $\mathcal{S}$ is a subset $V \subseteq S$ which *splits* the cover $\lhd$ in the following sense:

$$a \in V \;\&\; a \lhd U \implies V \between U$$

for all $a \in S$ and $U \subseteq S$. Given a formal topology $\mathcal{S}$, the class of splitting subsets of $\mathcal{S}$ is denoted by $Red(\mathcal{S})$.

[2]In this thesis, we call a unary positivity predicate just a positivity. For many years, the notion of binary positivity predicate has been proposed [30], and a comprehensive monograph presenting a new notion of formal topology with a binary positivity predicate is in preparation [52]. However, since we do not use that notion of binary positivity predicate in this thesis, the term 'positivity' always means a unary positivity predicate.

A positivity predicate of a formal topology is a splitting subset satisfying an additional condition.

Definition 2.1.8. Let $\mathcal{S}$ be a formal topology. A *positivity predicate* (or just a positivity) of $\mathcal{S}$ is a splitting subset Pos of $\mathcal{S}$ which satisfies

(Pos) $a \lhd \{x \in S \mid x = a \;\&\; \mathsf{Pos}(a)\}$

for all $a \in S$, where we write $\mathsf{Pos}(a)$ for $a \in \mathsf{Pos}$. Note that we have

$$\{x \in S \mid x = a \;\&\; \mathsf{Pos}(a)\} = \{a\} \cap \mathsf{Pos}\,.$$

An intuitive reading of $\mathsf{Pos}(a)$ is that 'the basic open subset a is inhabited'.

The condition (Pos) is first given in the following form [50].

Lemma 2.1.9. *Let $\mathcal{S}$ be a formal topology, and let Pos be a subset of S. Then, Pos satisfies (Pos) iff*

$$(\mathsf{Pos}(a) \to a \lhd U) \implies a \lhd U \tag{2.4}$$

for all $a \in S$ and $U \subseteq S$.

Proof. Suppose that Pos satisfies (Pos). Let $a \in S$ and $U \subseteq S$, and suppose that $\mathsf{Pos}(a)$ implies $a \lhd U$. By (Pos), we have $a \lhd \{a\} \cap \mathsf{Pos}$. Let $b \in \{a\} \cap \mathsf{Pos}$. Then, $b = a$ and $\mathsf{Pos}(a)$. Thus, $a \lhd U$, and hence $a \lhd \{a\} \cap \mathsf{Pos} \lhd U$ by transitivity.

Conversely, suppose that Pos satisfies (2.4). Let $a \in S$. By letting $U = \{a\} \cap \mathsf{Pos}$ in (2.4), since $\mathsf{Pos}(a)$ implies $a \lhd \{a\} \cap \mathsf{Pos}$, we have $a \lhd \{a\} \cap \mathsf{Pos}$, i.e. Pos satisfies (Pos). □

The following is an immediate consequence of (Pos).

Proposition 2.1.10. *If $\mathcal{S}$ is an overt formal topology, then its positivity is the largest splitting subset of $\mathcal{S}$.*

Corollary 2.1.11. *Let $\mathcal{S}$ be a formal topology, and let $\mathsf{Pos}, \mathsf{Pos}' \subseteq S$ be positivities of $\mathcal{S}$. Then, $\mathsf{Pos} = \mathsf{Pos}'$.*

Hence, for an overt formal topology $\mathcal{S}$, we can refer to its positivity as *the* positivity of $\mathcal{S}$.

Definition 2.1.12. A formal topology is *overt* if it is equipped with a (necessarily unique) positivity predicate. A formal topology is *inhabited* if it is overt and its positivity is inhabited.

There is a close connection between splitting subsets and overt formal topologies. See Section 2.3.2.

Remark 2.1.13. Classically, every formal topology $\mathcal{S}$ is overt, and its positivity can be defined by $\mathsf{Pos} = \{a \in S \mid \neg(a \lhd \emptyset)\}$. Constructively, not all formal topologies are overt. See Example 2.3.10. Other weak counterexamples of non-overt formal topologies are given in [17].

2.1.3 Points of formal topologies

Since formal topology is a point-free topology, it does not include the notion of point as primitive. Instead, a point of a formal topology is defined as a certain filter on its base.

Definition 2.1.14. Given a formal topology $\mathcal{S}$, a subset $\alpha \subseteq S$ is a *formal point* of $\mathcal{S}$ if

(P1) $S \between \alpha$,

(P2) $a, b \in \alpha \implies \alpha \between (a \downarrow b)$,

(P3) $a \in \alpha \;\&\; a \lhd U \implies \alpha \between U$

for all $a, b \in S$ and $U \subseteq S$. The class of formal points of a formal topology $\mathcal{S}$ is denoted by $\mathcal{P}t(\mathcal{S})$.

Remark 2.1.15. Constructively, we cannot assume that $\mathcal{P}t(\mathcal{S})$ is a set (See [23, Corollary 7.1]).

By (P3), a formal point is a splitting subset.

Proposition 2.1.16. *Let $\mathcal{S}$ be an overt formal topology with a positivity* Pos. *Then, for any formal point $\alpha \in \mathcal{P}t(\mathcal{S})$ we have $\alpha \subseteq$* Pos.

Proof. By Proposition 2.1.10. □

A formal point of a formal topology $\mathcal{S}$ can equivalently be defined as a global point of $\mathcal{S}$, i.e. a formal topology map from the terminal object $\mathbf{1}$ in **FTop**, where $\mathbf{1} = (\{*\}, \lhd, =)$ is the discrete topology on a singleton $\{*\}$ given by

$$a \lhd U \overset{\text{def}}{\iff} a \in U$$

for any $a \in \{*\}$ and $U \subseteq \{*\}$. There exists a bijective correspondence between the formal points of $\mathcal{S}$ and the formal topology maps from $\mathbf{1}$ to $\mathcal{S}$. A formal topology map $r \colon \mathbf{1} \to \mathcal{S}$ corresponds to a formal point $\alpha_r \overset{\text{def}}{=} \{a \in S \mid * \; r \; a\}$. Conversely, a formal point α of $\mathcal{S}$ corresponds to a formal topology map $r_\alpha \colon \mathbf{1} \to \mathcal{S}$ defined by

$$* \; r_\alpha \; a \overset{\text{def}}{\iff} a \in \alpha. \tag{2.5}$$

Hence, for any formal point $\alpha \in \mathcal{P}t(\mathcal{S})$ and a formal topology map $r : \mathcal{S} \to \mathcal{S}'$, the composition $r \circ r_\alpha : \mathbf{1} \to \mathcal{S}'$ determines a formal point of $\mathcal{S}'$, which we denote by $\mathcal{P}t(r)(\alpha)$. By unfolding the definition, we have

$$\mathcal{P}t(r)(\alpha) = r\alpha = \{b \in S' \mid (\exists a \in \alpha)\, a \; r \; b\}. \tag{2.6}$$

2.2 Inductively generated formal topologies

The notion of inductively generated formal topology allows us to define a formal topology by a set of axioms, and this gives us many examples of formal topologies. More importantly, it allows us to prove properties of a formal topology by induction, the method which is heavily used in later chapters. Also, inductively generated formal topologies in general have more desirable properties than those formal topologies which arise from concrete spaces (See Section 2.5.3). Lastly, the inductively generated formal topologies form the only class of formal topologies which so far admits various constructions of limits[3].

2.2.1 Inductive generation of covers

Definition 2.2.1. An *axiom-set* is a preordered set $S = (S, \leq)$ together with a pair (I, C), where $(I(a))_{a \in S}$ is a family of sets indexed by S, and C is a family $(C(a,i))_{a \in S, i \in I(a)}$ of subsets of S indexed by $\sum_{a \in S} I(a)$. For each $a \in S$ and $i \in I(a)$, the pair $(a, C(a,i))$ is called an *axiom* of (I, C).

In the following, we often use the phrase '(I, C) is an axiom-set on $(S, \leq)$' to mean that the pair $((S, \leq), (I, C))$ is an axiom-set.

Theorem 2.2.2 (Coquand et al. [18, Theorem 3.3])**.** *Let (I, C) be an axiom-set on $(S, \leq)$. Then, there exists a cover $\lhd_{I,C}$ on $(S, \leq)$ inductively generated by the following rules:*

$$\frac{a \in U}{a \lhd_{I,C} U} \text{ (reflexivity)}; \qquad \frac{a \leq b \quad b \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-left)};$$
$$\frac{a \leq b \quad i \in I(b) \quad a \downarrow C(b,i) \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-infinity)}.$$

The relation $\lhd_{I,C}$ is the least cover on $(S, \leq)$ which satisfies $a \lhd_{I,C} C(a,i)$ for each $a \in S$ and $i \in I(a)$.

Proof. See Coquand et al. [18, Theorem 3.3]. □

Definition 2.2.3. Let (I, C) be an axiom-set on $(S, \leq)$. Then the cover $\lhd_{I,C}$ on $(S, \leq)$ given in Theorem 2.2.2 is called *the cover inductively generated by* (I, C). A formal topology $\mathcal{S} = (S, \lhd, \leq)$ is *inductively generated* if there exists an axiom-set (I, C) on $(S, \leq)$ such that $\lhd = \lhd_{I,C}$.

Remark 2.2.4. The statement that the relation $\lhd_{I,C}$ is inductively generated by (reflexivity), ($\leq$-left), and ($\leq$-infinity) is equivalent to saying that for each subset $U \subseteq S$, the set

$$\mathcal{A}U \stackrel{\text{def}}{=} \{a \in S \mid a \lhd_{I,C} U\}$$

is the least subset of S such that

[3] In suitable extensions of CZF and Martin-Löf's type theory, the category of inductively generated formal topologies can be shown to be cocomplete [5, 46].

1. $U \subseteq \mathcal{A}U$,
2. $a \leq b \;\&\; b \in \mathcal{A}U \implies a \in \mathcal{A}U$,
3. $a \leq b \;\&\; a \downarrow C(b,i) \subseteq \mathcal{A}U \implies a \in \mathcal{A}U$

for all $a, b \in S$ and $i \in I(b)$.

Remark 2.2.5. Theorem 2.2.2 was obtained in the setting of Martin-Löf's type theory. To define the notion of inductively generated formal topology in CZF, we need the axiom wREA which allows us to define a set by an inductive definition. In CZF, given an axiom-set (I, C) on $(S, \leq)$, the formal topology $(S, \lhd_{I,C}, \leq)$ inductively generated by (I, C) can be defined by

$$a \lhd_{I,C} U \stackrel{\text{def}}{\iff} a \in I(\Phi_U)$$

for all $a \in S$ and $U \subseteq S$, where $I(\Phi_U)$ is the set inductively defined by the inductive definition Φ_U given by

$$\begin{aligned} \Phi_U \stackrel{\text{def}}{=} \; & \{(\emptyset, a) \mid a \in U\} \\ & \cup \{(\{b\}, a) \mid a \leq b\} \\ & \cup \{(a \downarrow C(b,i), a) \mid a \leq b \;\&\; i \in I(b)\}. \end{aligned}$$

See Appendix A for the axiom wREA and the notion of inductive definition.

Two axiom-sets on the same preorder may generate the same cover. In that case, they are said to be equivalent.

Definition 2.2.6. Let (I, C) and (J, D) be axiom-sets on $(S, \leq)$. We say that (I, C) and (J, D) are *equivalent* if

- $(\forall i \in I(a))\, a \lhd_{J,D} C(a,i)$,
- $(\forall j \in J(a))\, a \lhd_{I,C} D(a,j)$

for all $a \in S$.

Localised axiom-sets are particularly convenient to work with.

Definition 2.2.7. Let (I, C) be an axiom-set on $(S, \leq)$. Then, we say that (I, C) is *localised* if

$$(\forall a, b \in S)\, a \leq b \implies (\forall i \in I(b))\, (\exists j \in I(a))\, C(a,j) \subseteq a \downarrow C(b,i).$$

Proposition 2.2.8. *Let (I, C) be a localised axiom-set on $(S, \leq)$, and let $\mathcal{S} = (S, \lhd, \leq)$ be the formal topology generated by (I, C). Then, for each $U \subseteq S$, the set*

$$\mathcal{A}U \stackrel{\text{def}}{=} \{a \in S \mid a \lhd U\}$$

is the least subset of S such that

1. $U \subseteq \mathcal{A}U$,

2. $a \leq b \;\&\; b \in \mathcal{A}U \implies a \in \mathcal{A}U$,

3. $a \;\&\; C(a,i) \subseteq \mathcal{A}U \implies a \in \mathcal{A}U$

for all $a, b \in S$ *and* $i \in I(a)$.

Proof. By Remark 2.2.4, the set $\mathcal{A}U$ satisfies $1-3$. To see that $\mathcal{A}U$ is the least such subset, it suffices to show that any subset $V \subseteq S$ satisfying $1-3$ also satisfies ($\leq$-infinity). Let $V \subseteq S$ satisfy $1-3$ above. Let $a \leq b$ and $i \in I(b)$, and suppose that $a \downarrow C(b,i) \subseteq V$. Since (I,C) is localised, there exists $j \in I(a)$ such that $C(a,j) \subseteq a \downarrow C(b,i)$. Thus $C(a,j) \subseteq V$, and hence $a \in V$ by 3. □

Thus, we have the following induction principle: let (I,C) be a localised axiom-set on $(S,\leq)$. Then, for any subset $U \subseteq S$ and a predicate Φ on S, if

(ID1) $\dfrac{a \in U}{\Phi(a)}$,

(ID2) $\dfrac{a \leq b \quad \Phi(b)}{\Phi(a)}$,

(ID3) $\dfrac{i \in I(a) \quad (\forall c \in C(a,i))\, \Phi(c)}{\Phi(a)}$

for all $a, b \in S$, then $a \lhd_{I,C} U \implies \Phi(a)$ for all $a \in S$. An application of the above principle is called a *proof by induction on the cover* $\lhd_{I,C}$.

Remark 2.2.9. Every axiom-set $((S,\leq),(I,C))$ can be localised, i.e. there exists a localised axiom-set (J,D) on $(S,\leq)$ which is equivalent to (I,C); in fact, there is a canonical choice (I',C') of a localised axiom-set, called the *localisation of* (I,C), which is given by

$$\begin{aligned} I'(a) &\overset{\text{def}}{=} \{(b,i) \mid b \in S \;\&\; a \leq b \;\&\; i \in I(b)\}\,, \\ C'(a,(b,i)) &\overset{\text{def}}{=} a \downarrow C(b,i). \end{aligned} \tag{2.7}$$

The following operation on axiom-sets allows us to force new axioms to a given inductively generated formal topology.

Definition 2.2.10. Let (I,C) and (J,D) be axiom-sets on $(S,\leq)$. The *sum* of (I,C) and (J,D), denoted by $(I,C)+(J,D)$, is the axiom-set (K,E) on $(S,\leq)$ given by

$$\begin{aligned} K(a) &\overset{\text{def}}{=} I(a) + J(a), \\ E(a,(0,i)) &\overset{\text{def}}{=} C(a,i), \\ E(a,(1,j)) &\overset{\text{def}}{=} D(a,j). \end{aligned}$$

It is straightforward to check that the sum respects the equivalence of axiom-sets, and that the sum of localised axiom-sets is again localised. Note that the sum $(I, C) + (J, D)$ generates a formal topology which is a *subtopology* of both the formal topology generated by (I, C) and the one generated by (J, D) (See Section 2.3).

Notation 2.2.11. In the rest of this thesis, an axiom-set on a preorder $(S, \leq)$ is given by a set $\Phi \subseteq S \times \mathsf{Pow}(S)$ which corresponds to the set of axioms of the axiom-set (I, C) given by

$$I(a) \stackrel{\text{def}}{=} \{(b, U) \in \Phi \mid b = a\},$$
$$C(a, (b, U)) \stackrel{\text{def}}{=} U.$$

Moreover, an axiom $(a, U) \in \Phi$ is often presented in the form $a \lhd U$ using the same symbol $\lhd$ denoting the cover generated by the axiom-set. See Section 2.2.4 for the use of such informal notations.

2.2.2 Morphisms

A formal topology map with an inductively generated codomain $\mathcal{S}$ can be characterised completely by the axiom-set generating $\mathcal{S}$.

Proposition 2.2.12. *Let $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ be a formal topology inductively generated by an axiom-set (I, C), and let $\mathcal{S}'$ be a formal topology. Then, a relation $r \subseteq S' \times S$ is a formal topology map $r \colon \mathcal{S}' \to \mathcal{S}$ iff*

(FTMi1) $S' \lhd' r^- S$,

(FTMi2) $r^- a \downarrow' r^- b \lhd' r^- (a \downarrow b)$,

(FTMi3) $r^- a \lhd' r^- C(a, i)$,

(FTMi4) $a \leq b \implies r^- a \lhd' r^- b$

for all $a, b \in S$ and $i \in I(a)$.

Proof. The only if part is trivial.

The converse is proved by induction on $\lhd_{I,C}$. Suppose that r satisfies (FTMi1) – (FTMi4). It suffices to show that r satisfies (FTM3), i.e.

$$a \lhd_{I,C} U \implies r^- a \lhd' r^- U$$

for all $a \in S$ and $U \subseteq S$. Given $U \subseteq S$, define a predicate Φ on S by

$$\Phi(a) \stackrel{\text{def}}{\iff} r^- a \lhd' r^- U.$$

We show that $a \lhd_{I,C} U \implies \Phi(a)$ by induction on $\lhd_{I,C}$. We must check the conditions (ID1) – (ID3), using the localisation of (I, C) given by (2.7).

The condition (ID1) is trivial, and (ID2) follows from (FTMi4). For (ID3), let $a, b \in S$ such that $a \leq b$, and let $i \in I(b)$. Suppose that $\Phi(c)$ holds for all $c \in a \downarrow C(b,i)$. Then, for any $c \in C(b,i)$, we have

$$r^{-}c \downarrow' r^{-}a \lhd' r^{-}(c \downarrow a) \lhd' r^{-}U$$

by (FTMi2). Thus,

$$r^{-}C(b,i) \downarrow' r^{-}a \lhd' r^{-}U,$$

and hence

$$r^{-}a \lhd' r^{-}C(b,i) \downarrow' r^{-}a \lhd' r^{-}U$$

by (FTMi3) and (FTMi4). Therefore, $\Phi(a)$. □

By the correspondence between the formal topology maps from **1** to $\mathcal{S}$ and the formal points of $\mathcal{S}$, we have the following.

Corollary 2.2.13. *Let $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ be a formal topology inductively generated by an axiom-set (I, C). Then, a subset $\alpha \subseteq S$ is a formal point of $\mathcal{S}$ iff*

(Pi1) $S \between \alpha$,

(Pi2) $a, b \in \alpha \implies \alpha \between (a \downarrow b)$,

(Pi3) $a \in \alpha \implies \alpha \between C(a,i)$,

(Pi4) $a \leq b \ \& \ a \in \alpha \implies b \in \alpha$

for all $a, b \in S$ and $i \in I(a)$.

2.2.3 Overt formal topologies

In this section, we show how one can inductively generate an overt formal topology.

First, we give a characterisation of splitting subsets of an inductively generated formal topology.

Definition 2.2.14. Let (I, C) be an axiom-set on $(S, \leq)$. Given a subset $V \subseteq S$, we say that V *splits* (I, C) if

(Spl1) $a \in V \ \& \ a \leq b \implies b \in V$,

(Spl2) $a \leq b \ \& \ a \in V \ \& \ i \in I(b) \implies V \between (a \downarrow C(b,i))$.

Proposition 2.2.15 (Ciraulo and Sambin [14, Lemma 5.4])**.** *Let $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ be a formal topology inductively generated by an axiom-set (I, C). Then, a subset $V \subseteq S$ is a splitting subset of $\mathcal{S}$ iff V splits (I, C).*

Proof. If V is a splitting subset of $\mathcal{S}$, then V satisfies (Spl1) and (Spl2) by ($\leq$-left) and ($\leq$-infinity).

Conversely, suppose that V splits (I, C). We show that

$$a \lhd_{I,C} U \implies (a \in V \to V \between U)$$

by induction on $\lhd_{I,C}$. Given $U \subseteq S$, let Φ be the predicate on S given by

$$\Phi(a) \stackrel{\text{def}}{\iff} a \in V \to V \between U.$$

We check the conditions (ID1) – (ID3), using the localisation of (I, C).

The condition (ID1) is trivial. For (ID2), let $a, b \in S$, and suppose that $a \leq b$ and $\Phi(b)$. Let $a \in V$. By (Spl1), we have $b \in V$, and thus $V \between U$. Hence $\Phi(a)$. For (ID3), let $a, b \in S$ such that $a \leq b$. Let $i \in I(b)$, and suppose that $\Phi(c)$ for all $c \in a \downarrow C(b, i)$. Let $a \in V$. By (Spl2), we have $V \between (a \downarrow C(b, i))$, so there exists $c \in a \downarrow C(b, i)$ such that $c \in V$. Since $\Phi(c)$, we have $V \between U$. Hence $\Phi(a)$. □

Corollary 2.2.16. *Let (I, C) and (J, D) be equivalent axiom-sets on $(S, \leq)$. Then, a subset $V \subseteq S$ splits (I, C) iff V splits (J, D).*

For a localised axiom-set, we have a simpler characterisation of its splitting subsets.

Proposition 2.2.17. *Let (I, C) be a localised axiom-set on $(S, \leq)$. Then, a subset $V \subseteq S$ splits (I, C) iff it satisfies (Spl1) and*

(Spl2') $a \in V \;\&\; i \in I(a) \implies V \between C(a, i)$.

Proof. ($\Rightarrow$): By reflexivity of $\leq$.

($\Leftarrow$): Suppose that V satisfies (Spl1) and (Spl2'). Let $a, b \in S$, and suppose that $a \leq b$ and $a \in V$. Let $i \in I(b)$. Since (I, C) is localised, there exists $j \in I(a)$ such that $C(a, j) \subseteq a \downarrow C(b, i)$. By (Spl2'), we have $V \between C(a, j)$, and hence $V \between (a \downarrow C(b, j))$. Thus V satisfies (Spl2). □

Lastly, note that given two axiom-sets (I, C) and (J, D) on $(S, \leq)$, a subset $V \subseteq S$ splits $(I, C) + (J, D)$ iff V splits (I, C) and (J, D).

Any subset which splits an axiom-set determines an overt formal topology [18].

Proposition 2.2.18. *Let (I, C) be an axiom-set on $(S, \leq)$ and let $\mathsf{Pos} \subseteq S$ be a subset which splits (I, C). Let (I', C') be an axiom-set obtained from (I, C) by adding one axiom*

$$a \lhd_{I',C'} \{a\} \cap \mathsf{Pos} \tag{2.8}$$

for each $a \in S$. That is, (I', C') is the sum of (I, C) and the axiom-set (J, D) given by

$$J(a) = \{*\},$$
$$D(a, *) = \{a\} \cap \mathsf{Pos}.$$

Then, the formal topology $\mathcal{S} = (S, \lhd_{I',C'}, \leq)$ generated by (I', C') is overt and has the positivity Pos.

Moreover, the relation $\lhd_{I',C'}$ is the least among the covers $\lhd'$ on $(S, \leq)$ such that $a \lhd' C(a, i)$ for each $a \in S$ and $i \in I(a)$, and for which the triple $(S, \lhd', \leq)$ is an overt formal topology with the positivity Pos.

Proof. For the first statement, since Pos clearly splits (J, D), Pos is a splitting subset of $\mathcal{S}$. Since Pos satisfies (2.8) for each $a \in S$, Pos is the positivity of $\mathcal{S}$. Since the relation $\lhd_{I',C'}$ is the least cover on $(S, \leq)$ such that

1. $a \lhd_{I',C'} \{a\} \cap \mathsf{Pos}$ for all $a \in S$,
2. $a \lhd_{I',C'} C(a, i)$ for all $a \in S$ and $i \in I(a)$,

the second statement is obvious. □

2.2.4 Examples

We give examples of inductively generated formal topologies. Note that every axiom-set in the following examples is localised. Moreover, except for Example 2.2.19, every formal topology in the examples is overt, and in every case the positivity is the whole base.

Example 2.2.19 (Tree [26]). Let A be a set. A *tree* over A is a subset $T \subseteq A^*$ which is closed under predecessor:

$$a \preccurlyeq b \ \&\ b \in T \implies a \in T \tag{2.9}$$

for all $a, b \in A^*$[4].

Define an order $\leq$ on T by

$$a \leq b \stackrel{\text{def}}{\iff} b \preccurlyeq a$$

for all $a, b \in T$. The tree topology $\mathcal{T} = (T, \lhd_{\mathcal{T}}, \leq)$ is inductively generated by the axiom-set on $(T, \leq)$ consisting an axiom

$$a \lhd_{\mathcal{T}} \{a * \langle x \rangle \in T \mid x \in A\}$$

for each $a \in T$. A formal point of $\mathcal{T}$ may be thought of as a path in T.

Example 2.2.20 (The formal Cantor space [18]). A special case of Example 2.2.19 where $A = \{0, 1\}$ and $T = A^*$ is called the *formal Cantor space*. Explicitly, let $C = \{0, 1\}^*$ be ordered by (2.9). The formal Cantor space $\mathbf{C} = (C, \lhd_{\mathbf{C}}, \leq)$ is inductively generated by the axiom-set on $(C, \leq)$ consisting an axiom

$$a \lhd_{\mathbf{C}} \{a * \langle 0 \rangle, a * \langle 1 \rangle\}$$

for each $a \in C$.

Example 2.2.21 (The formal Baire space [18]). Another special case of Example 2.2.19 where $A = \mathbb{N}$ and $T = A^*$ is called the *formal Baire space*. Explicitly, let $B = \mathbb{N}^*$ be ordered by (2.9). The formal Baire space $\mathbf{B} = (B, \lhd_{\mathbf{B}}, \leq)$ is inductively generated by the axiom-set on $(B, \leq)$ consisting of an axiom

$$a \lhd_{\mathbf{B}} \{a * \langle n \rangle \mid n \in \mathbb{N}\} \tag{2.10}$$

for each $a \in B$.

[4] By this definition, a tree may be empty.

The formal points of the formal Cantor space and the formal Baire space are homeomorphic to the point-set Cantor space and the Baire space respectively (See Section 2.5.3).

Example 2.2.22 (The formal reals [44, 18]). Let $\mathbb{Q}$ be the set of rationals, and let $S_{\mathcal{R}}$ be the set of open intervals with rational end points, i.e.

$$S_{\mathcal{R}} = \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}.$$

Define a preorder $\leq_{\mathcal{R}}$ and a strict order $<_{\mathcal{R}}$ on $S_{\mathcal{R}}$ by

$$\begin{aligned}(p,q) \leq_{\mathcal{R}} (r,s) &\overset{\text{def}}{\iff} r \leq p \;\&\; q \leq s,\\ (p,q) <_{\mathcal{R}} (r,s) &\overset{\text{def}}{\iff} r < p \;\&\; q < s\end{aligned}$$

for all $(p,q),(r,s) \in S_{\mathcal{R}}$. The *formal reals* $\mathcal{R} = (S_{\mathcal{R}}, \lhd_{\mathcal{R}}, \leq_{\mathcal{R}})$ is inductively generated by the axiom-set on $(S_{\mathcal{R}}, \leq_{\mathcal{R}})$ consisting of axioms

(R1) $(p,q) \lhd_{\mathcal{R}} \{(r,s) \in S_{\mathcal{R}} \mid (r,s) <_{\mathcal{R}} (p,q)\}$,

(R2) $(p,q) \lhd_{\mathcal{R}} \{(p,s),(r,q)\}$ for each $p < r < s < q$.

Explicitly, $\lhd_{\mathcal{R}}$ is generated by the axiom-set (I, C) on $S_{\mathcal{R}}$ defined by

$$\begin{aligned}I\left((p,q)\right) &\overset{\text{def}}{=} \{*\} + \{(r,s) \in S_{\mathcal{R}} \mid p < r < s < q\},\\ C((p,q),*) &\overset{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid (r,s) <_{\mathcal{R}} (p,q)\},\\ C((p,q),(r,s)) &\overset{\text{def}}{=} \{(p,s),(r,q)\}.\end{aligned}$$

The formal points $\mathcal{P}t(\mathcal{R})$ is isomorphic to the Dedekind cuts [26]. A *Dedekind cut* is a pair $(L,U) \in \mathsf{Pow}(\mathbb{Q}) \times \mathsf{Pow}(\mathbb{Q})$ of subsets of the rationals such that

$$\begin{array}{ll} L \between \mathbb{Q} \;\&\; U \between \mathbb{Q} & \text{(boundedness)}\\ (\forall p \in L)\,(\forall q \in U)\, p < q & \text{(disjointness)}\\ (\forall p \in L)\,(\exists p' \in L)\, p < p' \;\&\; (\forall q \in U)\,(\exists q' \in U)\, q' < q & \text{(openness)}\\ (\forall p, q \in \mathbb{Q})\, p < q \implies p \in L \vee q \in U & \text{(locatedness)}\end{array}$$

Note that from disjointness, openness, and locatedness, we have

$$\begin{aligned}(\forall p, p' \in \mathbb{Q})\, p \in L \;\&\; p' \leq p &\implies p' \in L,\\ (\forall q, q' \in \mathbb{Q})\, q \in U \;\&\; q \leq q' &\implies q' \in U.\end{aligned}$$

The order $\leq$ on the class of Dedekind cuts is defined by

$$(L,U) \leq (S,T) \overset{\text{def}}{\iff} T \subseteq U,$$

or equivalently one can define $(L,U) \leq (S,T) \iff L \subseteq S$. The addition is defined by

$$(L,U) + (S,T) \stackrel{\text{def}}{=} (\{p+s \mid p \in L \;\&\; s \in S\}, \{q+t \mid q \in U \;\&\; t \in T\}).$$

The rationals $\mathbb{Q}$ are embedded into the Dedekind cuts by

$$q \mapsto q_* \stackrel{\text{def}}{=} (\{r \in \mathbb{Q} \mid r < q\}, \{r \in \mathbb{Q} \mid q < r\}).$$

Then, one defines a strict order $<$ by

$$(L,U) < (S,T) \stackrel{\text{def}}{\iff} \left(\exists \varepsilon \in \mathbb{Q}^{>0}\right) (L,U) + \varepsilon_* \leq (S,T).$$

One can show that

$$\begin{aligned} r_* \leq (L,U) &\iff (\forall p \in U)\, r < p, \\ (L,U) \leq r_* &\iff (\forall q \in L)\, q < r, \\ r_* < (L,U) &\iff r \in L, \\ (L,U) < r_* &\iff r \in U \end{aligned}$$

for any Dedekind cut (L,U) and $r \in \mathbb{Q}$. See [54, Chapter 5] for more details about the Dedekind cuts.

Now, given any Dedekind cut $(L,U) \in \mathsf{Pow}(\mathbb{Q}) \times \mathsf{Pow}(\mathbb{Q})$, the subset

$$\alpha_{(L,U)} \stackrel{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid p \in L \;\&\; q \in U\}$$

is a formal point of $\mathcal{R}$. Conversely, given $\alpha \in \mathcal{P}t(\mathcal{R})$, the pair (L_α, U_α) given by

$$\begin{aligned} L_\alpha &\stackrel{\text{def}}{=} \{p \in \mathbb{Q} \mid (\exists q \in \mathbb{Q})\, (p,q) \in \alpha\}, \\ U_\alpha &\stackrel{\text{def}}{=} \{q \in \mathbb{Q} \mid (\exists p \in \mathbb{Q})\, (p,q) \in \alpha\} \end{aligned}$$

is a Dedekind cut. Then, we have

$$\begin{aligned} p_* < (L_\alpha, U_\alpha) < q_* &\iff (p,q) \in \alpha, \\ p_* \leq (L_\alpha, U_\alpha) \leq q_* &\iff (\forall (r,s) \in \alpha)\, r < q \;\&\; p < s \end{aligned}$$

for all $(p,q) \in S_{\mathcal{R}}$ and $\alpha \in \mathcal{P}t(\mathcal{R})$.

2.2.5 Products and equalisers

One of the most important facts about inductively generated formal topologies is that the category of inductively generated formal topologies is complete, i.e. it has equalisers and products of arbitrary set-indexed families of inductively generated formal topologies.

Products

Following Vickers [59], we define a product of a set-indexed family of inductively generated formal topologies as follows. Let $(\mathcal{S}_i)_{i\in I}$ be a set-indexed family of inductively generated formal topologies, each of the form $\mathcal{S}_i = (S_i, \lhd_i, \leq_i)$, and let (K_i, C_i) be the axiom-set which generates $\mathcal{S}_i$. Define a preorder $(S_\Pi, \leq_\Pi)$ by

$$S_\Pi \stackrel{\text{def}}{=} \mathsf{Fin}\left(\sum_{i\in I} S_i\right),$$

$$A \leq_\Pi B \stackrel{\text{def}}{\iff} (\forall (i,b) \in B)\,(\exists (j,a) \in A)\, i = j \;\&\; a \leq_i b$$

for all $A, B \in S_\Pi$. The axiom-set on $(S_\Pi, \leq_\Pi)$ is given by

(S1) $S_\Pi \lhd_\Pi \{\{(i,a)\} \in S_\Pi \mid a \in S_i\}$ for each $i \in I$,

(S2) $\{(i,a),(i,b)\} \lhd_\Pi \{\{(i,c)\} \in S_\Pi \mid c \leq_i a \;\&\; c \leq_i b\}$ for each $i \in I$ and $a, b \in S_i$,

(S3) $\{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in C_i(a,k)\}$ for each $i \in I$, $a \in S_i$, and $k \in K_i(a)$.

Let $\prod_{i\in I} \mathcal{S}_i = (S_\Pi, \lhd_\Pi, \leq_\Pi)$ be the formal topology inductively generated by the above axiom-set. For each $i \in I$, the projection $p_i \colon \prod_{i\in I} \mathcal{S}_i \to \mathcal{S}_i$ is defined by

$$A \; p_i \; a \stackrel{\text{def}}{\iff} A = \{(i,a)\}$$

for all $A \in S_\Pi$ and $a \in S_i$. By the definition of $\prod_{i\in I} \mathcal{S}_i$, the relation p_i is a formal topology map. Then, it is straightforward to show that the family of projections $\left(p_i \colon \prod_{i\in I} \mathcal{S}_i \to \mathcal{S}_i\right)_{i\in I}$ is a product of $(\mathcal{S}_i)_{i\in I}$.

Given any family $(r_i \colon \mathcal{S} \to \mathcal{S}_i)_{i\in I}$ of formal topology maps, we have a unique formal topology map $r : \mathcal{S} \to \prod_{i\in I} \mathcal{S}_i$ such that $r_i = p_i \circ r$ for each $i \in I$. The map r is defined by

$$a \; r \; A \stackrel{\text{def}}{\iff} (\forall (i,b) \in A)\, a \lhd r_i^- \{b\}$$

for all $a \in S$ and $A \in S_\Pi$. More explicitly, we have

$$a \, r \, \{(i_0, a_0), \dots, (i_{n-1}, a_{n-1})\} \iff a \lhd r_{i_0}^- \{a_0\} \downarrow \cdots \downarrow r_{i_{n-1}}^- \{a_{n-1}\}.$$

The following localised form of the above axiom-set will be useful when proving properties of a product by induction.

Lemma 2.2.23. *The following axiom-set on $(S_\Pi, \leq_\Pi)$ is equivalent to the axioms (S1) – (S3).*

(S1') $A \lhd_\Pi \{A \cup \{(i,a)\} \in S_\Pi \mid a \in S_i\}$ *for each* $i \in I$,

(S2') $A \cup \{(i,a),(i,b)\} \lhd_\Pi \{A \cup \{(i,c)\} \in S_\Pi \mid c \leq_i a' \;\&\; c \leq_i b'\}$ *for each* $i \in I$, $a \leq_i a'$ *and* $b \leq_i b'$,

(S3') $A \cup \{(i,a)\} \lhd_\Pi \{A \cup \{(i,b)\} \in S_\Pi \mid b \in C_i(a',k)\}$ *for each* $i \in I$, $a \leq_i a'$ *and* $k \in K_i(a')$

for each $A \in S_\Pi$. *Moreover, the axiom-set (S1') – (S3') is localised.*

Proof. These facts follow from the definition of the order $\leq_\Pi$. □

Lemma 2.2.24. *For each* $i \in I$, *we have*

$$a \lhd_i U \implies \{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in U\}$$

for all $a \in S_i$ *and* $U \subseteq S_i$.

Proof. By induction on $\lhd_i$. Given $U \subseteq S_i$, define a predicate Φ on S_i by

$$\Phi(a) \overset{\text{def}}{\iff} \{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in U\}.$$

We must check the conditions (ID1) – (ID3). The condition (ID1) is trivial. (ID2) follows from the definition of $\leq_\Pi$. For (ID3), let $a, b \in S_i$ such that $a \leq_i b$ and let $k \in K_i(b)$. Suppose that $\Phi(c)$ for all $c \in a \downarrow C_i(b,k)$. By (S3), we have

$$\{(i,b)\} \lhd_\Pi \{\{(i,d)\} \in S_\Pi \mid d \in C_i(b,k)\}.$$

Then, we have

$$\begin{aligned}
\{(i,a)\} &\lhd_\Pi \{(i,a),(i,b)\} \\
&\lhd_\Pi \{\{(i,a),(i,d)\} \in S_\Pi \mid d \in C_i(b,k)\} \\
&\lhd_\Pi \{\{(i,c)\} \in S_\Pi \mid c \in C_i(b,k) \downarrow a\} \\
&\lhd_\Pi \{\{(i,e)\} \in S_\Pi \mid e \in U\}
\end{aligned}$$

by (S2) and by the fact that $a \leq_i b$. Hence $\Phi(a)$. □

Corollary 2.2.25. *Let* $\{i_0, \dots, i_{n-1}\} \in \mathsf{Fin}(I)$, *and for each* $k < n$, *let* $a_k \in S_{i_k}$ *and* $U_k \subseteq S_{i_k}$ *such that* $a_k \lhd_{i_k} U_k$. *Then,*

$$\{(i_0,a_0),\dots,(i_{n-1},a_{n-1})\} \lhd_\Pi \{\{(i_0,b_0),\dots,(i_{n-1},b_{n-1})\} \in S_\Pi \mid (\forall k < n)\, b_k \in U_k\}.$$

A binary product of a pair of inductively generated formal topologies admits a simple construction. Given two inductively generated formal topologies $\mathcal{S} = (S, \lhd_S, \leq_S)$ and $\mathcal{T} = (T, \lhd_T, \leq_T)$ generated by axiom-sets (I, C) and (J, D) respectively, their product $\mathcal{S} \times \mathcal{T}$ is an inductively generated formal topology with the preorder $(S \times T, \leq)$ defined by

$$(a,b) \leq (a',b') \overset{\text{def}}{\iff} a \leq_S a' \;\&\; b \leq_T b'$$

and the axiom-set (K, E) on $(S \times T, \leq)$ defined by

$$\begin{aligned}
K((a,b)) &\overset{\text{def}}{=} I(a) + J(b), \\
E((a,b),(0,i)) &\overset{\text{def}}{=} C(a,i) \times \{b\}, \\
E((a,b),(1,j)) &\overset{\text{def}}{=} \{a\} \times D(b,j).
\end{aligned}$$

The projection $p_S : \mathcal{S} \times \mathcal{T} \to \mathcal{S}$ is given by

$$(a, b)\ p_S\ a' \overset{\text{def}}{\iff} (a, b) \lhd_{K,E} \{a'\} \times T$$

for all $a, a' \in S$ and $b \in T$, and the other projection is similarly defined.

Given two formal topology maps $r : \mathcal{S}' \to \mathcal{S}$ and $s : \mathcal{S}' \to \mathcal{T}$, the canonical map $\langle r, s\rangle : \mathcal{S}' \to \mathcal{S} \times \mathcal{T}$ is given by

$$c\ \langle r, s\rangle\ (a, b) \overset{\text{def}}{\iff} c \lhd' r^- a \ \&\ c \lhd' s^- b$$

for all $c \in S'$, $a \in S$, and $b \in T$.

The following is analogous to Corollary 2.2.25, and can be proved by induction.

$$a \lhd_S U \ \&\ b \lhd_T V \implies (a, b) \lhd_{K,E} U \times V$$

for all $a \in S$, $b \in T$, $U \subseteq S$, and $V \subseteq T$.

Equalisers

Equalisers of inductively generated formal topologies were shown to exist by Palmgren [47]. Let $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ be an inductively generated formal topology generated by an axiom-set (I, C), and let $r, s : \mathcal{S} \to \mathcal{T}$ be formal topology maps. Let $\lhd_E$ be the cover on $(S, \leq)$ generated by (I, C) together with the following additional axioms:

$$\begin{aligned} a &\lhd_E r^- b && (a\ s\ b \ \&\ b \in T), \\ a &\lhd_E s^- b && (a\ r\ b \ \&\ b \in T). \end{aligned}$$

Put $\mathcal{E} = (S, \lhd_E, \leq)$. Then, the formal topology $\mathcal{E}$ together with the canonical inclusion $i_{\mathcal{E}} : \mathcal{E} \to \mathcal{S}$, where $i_{\mathcal{E}} = id_S$, is an equaliser of r and s. Note that the additional axioms force $r^- b =_{\mathcal{E}} s^- b$ for all $b \in T$.

Pullbacks

From the constructions of a product and an equaliser, we obtain a construction of a pullback. Specifically, given inductively generated formal topologies $\mathcal{S}_1$ and $\mathcal{S}_2$ generated by axiom-sets (I_1, C_1) and (I_2, C_2) respectively, a pullback $\mathcal{S}_1 \times_{\mathcal{T}} \mathcal{S}_2$ of formal topology maps $r : \mathcal{S}_1 \to \mathcal{T}$, $s : \mathcal{S}_2 \to \mathcal{T}$ is generated by the axiom-set of the product $\mathcal{S}_1 \times \mathcal{S}_2$ together with the following additional axioms:

$$\begin{aligned} (a, b) &\lhd S_1 \times s^- c && (a\ r\ c \ \&\ c \in T), \\ (a, b) &\lhd r^- c \times S_2 && (b\ s\ c \ \&\ c \in T). \end{aligned}$$

Then, the restrictions of the projections $p_i : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_i \quad (i = 1, 2)$ to $\mathcal{S}_1 \times_{\mathcal{T}} \mathcal{S}_2$ form a pullback square.

2.3 Subtopologies

The notion of subtopology for formal topologies is a point-free analogue of that of subspace for topological spaces. In this section, we introduce the notion of open subtopology and two notions of closed subtopology. The connection between the two notions of closed subtopology is of central importance in Chapter 4 and Chapter 5.

Definition 2.3.1. A *subtopology* of a formal topology $\mathcal{S} = (S, \lhd, \leq)$ is a formal topology $\mathcal{S}' = (S, \lhd', \leq)$ where $(S, \leq)$ is the same preorder as that of $\mathcal{S}$ and $\lhd'$ is a cover on $(S, \leq)$ such that

$$a \lhd U \implies a \lhd' U$$

for all $a \in S$ and $U \subseteq S$. If $\mathcal{S}'$ is a subtopology of $\mathcal{S}$, we write $\mathcal{S}' \sqsubseteq \mathcal{S}$. Given two subtopologies $\mathcal{S}'$ and $\mathcal{S}''$ of $\mathcal{S}$, we say that $\mathcal{S}'$ is *smaller* than $\mathcal{S}''$ (or $\mathcal{S}''$ is *larger* than $\mathcal{S}'$) if $\mathcal{S}' \sqsubseteq \mathcal{S}''$. Note that if $\mathcal{S}' \sqsubseteq \mathcal{S}''$, then we have $\mathcal{P}t(\mathcal{S}') \subseteq \mathcal{P}t(\mathcal{S}'')$. This follows from (P3).

Given a formal topology map $r \colon \mathcal{S} \to \mathcal{S}'$, the relation $\lhd_r \subseteq S' \times \mathsf{Pow}(S')$ given by

$$a \lhd_r U \overset{\text{def}}{\iff} r^- a \lhd r^- U$$

defines a cover on $(S', \leq')$. The formal topology $\mathcal{S}_r = (S', \lhd_r, \leq')$ is called the *image* of $\mathcal{S}$ under r. Note that $\mathcal{S}_r \sqsubseteq \mathcal{S}'$ by (FTM3).

A formal topology map $r \colon \mathcal{S} \to \mathcal{S}'$ is an *embedding* if the frame morphism $\mathcal{A} r^-(-) : \mathit{Sat}(\mathcal{S}') \to \mathit{Sat}(\mathcal{S})$ is surjective. This is equivalent to saying that the right adjoint $r^{-*}(-) : \mathit{Sat}(\mathcal{S}) \to \mathit{Sat}(\mathcal{S}')$ of the frame morphism $\mathcal{A} r^-(-) : \mathit{Sat}(\mathcal{S}') \to \mathit{Sat}(\mathcal{S})$ is the right inverse of $\mathcal{A} r^-(-)$, i.e.

$$\mathcal{A} r^- r^{-*} \mathcal{A} U = \mathcal{A} U$$

for all $U \subseteq S$. Clearly, the condition is equivalent to $a \lhd r^- r^{-*} \mathcal{A}\{a\}$ for all $a \in S$.

Dually, a formal topology map $r \colon \mathcal{S} \to \mathcal{S}'$ is a *surjection* if the frame morphism $\mathcal{A} r^-(-) : \mathit{Sat}(\mathcal{S}') \to \mathit{Sat}(\mathcal{S})$ is injective. This is equivalent to saying that the right adjoint $r^{-*}(-)$ is the left inverse of $\mathcal{A} r^-(-)$, i.e.

$$r^{-*} \mathcal{A} r^- V = \mathcal{A}' V$$

for all $V \subseteq S'$. The condition is equivalent to $r^{-*} \mathcal{A} r^- V \lhd' V$ for all $V \subseteq S'$, i.e. the image of $\mathcal{S}$ under r is equal to $\mathcal{S}'$.

Proposition 2.3.2. *A formal topology map $r : \mathcal{S} \to \mathcal{S}'$ is an isomorphism iff r is an embedding and a surjection.*

Proof. Obviously, r is an embedding and a surjection iff $\mathcal{A} r^-(-) : \mathit{Sat}(\mathcal{S}') \to \mathit{Sat}(\mathcal{S})$ is an isomorphism of frames. Then, the conclusion follows from the equivalence between the categories of formal topologies and that of set-based locales. □

Corollary 2.3.3. *If $r \colon \mathcal{S} \to \mathcal{S}'$ is an embedding, then r restricts to an isomorphism between $\mathcal{S}$ and the image $\mathcal{S}_r$.*

Overtness is preserved by formal topology maps.

Lemma 2.3.4. *Let $\mathcal{S}$ be an overt formal topology with a positivity* Pos*, and let* $r\colon \mathcal{S} \to \mathcal{S}'$ *be a formal topology map. Then, the image* $\mathcal{S}_r$ *of* $\mathcal{S}$ *under* r *is overt with the positivity*

$$r\,\mathsf{Pos} \stackrel{\text{def}}{=} \{a \in S' \mid (\exists b \in \mathsf{Pos})\, b\; r\; a\}\,.$$

Proof. Write $\mathcal{S}_r = (S', \lhd_r, \leq')$. Let $b \in S'$ and $V \subseteq S'$, and suppose that $b \lhd_r U$. Let $b \in r\,\mathsf{Pos}$. Then, $r^- b \between \mathsf{Pos}$. Since $r^- b \lhd r^- V$, we have $r^- V \between \mathsf{Pos}$, i.e. $V \between r\,\mathsf{Pos}$, and thus $r\,\mathsf{Pos}$ splits $\lhd_r$. We use (2.4) to show that $r\,\mathsf{Pos}$ satisfies (Pos). Let $b \in S'$ and $V \subseteq S'$, and suppose that $b \in r\,\mathsf{Pos} \implies b \lhd_r V$. Let $a \in r^- b$, and suppose that $a \in \mathsf{Pos}$. Then, $b \in r\,\mathsf{Pos}$, and so $b \lhd_r V$, i.e. $r^- b \lhd r^- V$. Thus, $a \lhd r^- V$, and hence $r^- b \lhd r^- V$. Therefore $b \lhd_r V$, as required. □

2.3.1 Open subtopologies

An open subtopology of a formal topology corresponds to a saturated subset.

Definition 2.3.5. Let $\mathcal{S}$ be a formal topology, and let $V \subseteq S$. The *open subtopology* of $\mathcal{S}$ determined by V is a subtopology $\mathcal{S}_V$ of $\mathcal{S}$ whose cover $\lhd_V$ is given by

$$a \lhd_V U \stackrel{\text{def}}{\iff} a \downarrow V \lhd U$$

for all $a \in S$ and $U \subseteq S$. We have $V \lhd W$ iff $\mathcal{S}_V \sqsubseteq \mathcal{S}_W$. Thus, the class of open subtopologies is order isomorphic to the frame $\mathit{Sat}(\mathcal{S})$ of saturated subsets of $\mathcal{S}$. The open subtopology determined by $V \subseteq S$ is denoted by $\mathcal{S}_V$.

Lemma 2.3.6. *Let $\mathcal{S}$ be a formal topology. Let $\mathcal{S}_V$ be the open subtopology of $\mathcal{S}$ determined by $V \subseteq S$. Then*

1. *$\mathcal{S}_V$ is the largest subtopology $\mathcal{S}'$ of $\mathcal{S}$ such that $S \lhd' V$.*

2. *$\mathcal{P}t(\mathcal{S}_V) = \{\alpha \in \mathcal{P}t(\mathcal{S}) \mid \alpha \between V\}$.*

3. *If $\mathcal{S}$ is overt with a positivity* Pos*, then $\mathcal{S}_V$ is overt with the positivity* Pos_V *given by*

$$\mathsf{Pos}_V \stackrel{\text{def}}{=} \{a \in S \mid \mathsf{Pos} \between (a \downarrow V)\}\,. \tag{2.11}$$

4. *If $\mathcal{S}$ is an inductively generated formal topology, then $\mathcal{S}_V$ is generated by the axiom-set of $\mathcal{S}$ together with the following extra axiom:*

$$S \lhd_V V. \tag{2.12}$$

Proof. 1. Since $S \downarrow V \lhd V$, we have $S \lhd_V V$. Let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$ such that $S \lhd' V$. Let $a \in S$ and $U \subseteq S$, and suppose that $a \lhd_V U$. Then, $a \downarrow V \lhd U$. Thus, $a \downarrow V \lhd' U$. Hence, $a \lhd' a \downarrow S \lhd' a \downarrow V \lhd' U$. Therefore, $\mathcal{S}' \sqsubseteq \mathcal{S}_V$.

2. Let $\alpha \in \mathcal{P}t(\mathcal{S}_V)$. Since $S \lhd_V V$, we have $\alpha \between V$ by (P1) and (P3). Conversely, let $\alpha \in \mathcal{P}t(\mathcal{S})$ such that $\alpha \between V$. It suffices to show that α satisfies (P3). Let $a \in S$ and $U \subseteq S$, and suppose that $a \lhd_V U$ and $a \in \alpha$. Then, $\alpha \between (a \downarrow V)$ by (P2), and thus, $\alpha \between U$ by (P3).

3. Suppose that $\mathcal{S}$ is overt with a positivity Pos, and let $\mathsf{Pos}_V \subseteq S$ as defined by (2.11). Suppose that $a \lhd_V U$ and $a \in \mathsf{Pos}_V$, i.e. $a \downarrow V \lhd U$ and $\mathsf{Pos} \between (a \downarrow V)$. Then, $a \downarrow V \lhd U \downarrow V$. Since Pos splits the cover $\lhd$, we have $\mathsf{Pos} \between (U \downarrow V)$, i.e. $\mathsf{Pos}_V \between U$. Hence, Pos_V splits the cover $\lhd_V$. Moreover, for any $a \in S$, we have $a \downarrow V \lhd (a \downarrow V) \cap \mathsf{Pos}$ by (Pos) of Pos. Thus, $a \lhd_V (a \downarrow V) \cap \mathsf{Pos} \lhd_V \{a\} \cap \mathsf{Pos}_V$. Hence, Pos_V satisfies (Pos).

4. Suppose that $\mathcal{S}$ is inductively generated by an axiom-set (I, C). Let (I', C') be the axiom-set given by

$$\begin{aligned} I'(a) &\overset{\text{def}}{=} I(a) + \{*\}, \\ C'(a,i) &\overset{\text{def}}{=} C(a,i) \qquad\qquad \text{if } i \in I(a), \\ C'(a,*) &\overset{\text{def}}{=} V. \end{aligned}$$

Let $\mathcal{S}'$ be the formal topology generated by (I', C'). We must show that $\mathcal{S}' = \mathcal{S}_V$. Since $\mathcal{S}_V$ satisfies all the axioms of (I', C'), we have $\mathcal{S}_V \sqsubseteq \mathcal{S}'$. Conversely, since $\mathcal{S}'$ is a subtopology of $\mathcal{S}$ and $S \lhd' V$, we have $\mathcal{S}' \sqsubseteq \mathcal{S}_V$ by 1. □

Example 2.3.7 (cf. Example 2.2.22). Let $\mathcal{R}$ be the formal reals, and let $\mathcal{R}_{(0,1)}$ be the open subtopology of $\mathcal{R}$ determined by $\{(0,1)\}$. By Lemma 2.3.6.3, $\mathcal{R}_{(0,1)}$ is overt, and its positivity $\mathsf{Pos}_{(0,1)}$ is given by

$$\begin{aligned} \mathsf{Pos}_{(0,1)} &= \{(p,q) \in S_{\mathcal{R}} \mid (\exists (r,s) \in S_{\mathcal{R}})\, (r,s) \in (p,q) \downarrow (0,1)\} \\ &= \{(p,q) \in S_{\mathcal{R}} \mid p < 1 \;\&\; 0 < q\}. \end{aligned}$$

Moreover, by Lemma 2.3.6.2, we have

$$\mathcal{P}t(\mathcal{R}_{(0,1)}) = \{\alpha \in \mathcal{P}t(\mathcal{R}) \mid (0,1) \in \alpha\}.$$

Hence, by the correspondence between $\mathcal{P}t(\mathcal{R})$ and the Dedekind cuts, $\mathcal{P}t(\mathcal{R}_{(0,1)})$ corresponds to the open interval $(0,1)$.

2.3.2 Closed and overt weakly closed subtopologies

In the classical point-set topology, the notion of closed subset can be defined in two ways: as the complement of an open subset, or as the set of adherent points of some subset Y, i.e. those points x for which every neighbourhood meets Y. Constructively, the two notions are not equivalent. The point-free analogue of the former notion is that of closed subtopology, while the latter corresponds to the notion of overt weakly closed subtopology ([11, 61, 28]).

Closed subtopologies

Definition 2.3.8. Let $\mathcal{S}$ be a formal topology, and let $V \subseteq S$. The *closed subtopology* of $\mathcal{S}$ determined by V is a subtopology $\mathcal{S}^{S-V}$ of $\mathcal{S}$ whose cover $\lhd^{S-V}$ is given by

$$a \lhd^{\mathcal{S}-V} U \overset{\text{def}}{\iff} a \lhd V \cup U$$

for all $a \in S$ and $U \subseteq S$. We have $V \lhd W$ iff $\mathcal{S}^{\mathcal{S}-W} \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$, so the class of closed subtopologies is order isomorphic to $Sat(\mathcal{S})^{Op}$. The closed subtopology determined by $V \subseteq S$ is denoted by $\mathcal{S}^{\mathcal{S}-V}$.

Lemma 2.3.9. *Let $\mathcal{S}$ be a formal topology. Let $\mathcal{S}^{\mathcal{S}-V}$ be the closed subtopology of $\mathcal{S}$ determined by $V \subseteq S$. Then*

1. *$\mathcal{S}^{\mathcal{S}-V}$ is the largest subtopology $\mathcal{S}'$ of $\mathcal{S}$ such that $V \lhd' \emptyset$.*
2. *$\mathcal{P}t(\mathcal{S}^{\mathcal{S}-V}) = \{\alpha \in \mathcal{P}t(\mathcal{S}) \mid \neg\,(\alpha \between V)\}$.*
3. *If $\mathcal{S}$ is an inductively generated formal topology, then $\mathcal{S}^{\mathcal{S}-V}$ is generated by the axiom-set of $\mathcal{S}$ together with the following extra axiom:*

$$V \lhd^{\mathcal{S}-V} \emptyset.$$

Proof. 1. Since $V \lhd V \cup \emptyset$, we have $V \lhd^{\mathcal{S}-V} \emptyset$. Let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$ such that $V \lhd' \emptyset$. Let $a \in S$ and $U \subseteq S$, and suppose that $a \lhd^{\mathcal{S}-V} U$. Then, $a \lhd V \cup U$. Thus, $a \lhd' V \cup U$. Hence, $a \lhd' V \cup U \lhd' \emptyset \cup U = U$. Therefore, $\mathcal{S}' \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$.

2. Let $\alpha \in \mathcal{P}t(\mathcal{S}^{\mathcal{S}-V})$. Since $V \lhd^{\mathcal{S}-V} \emptyset$, we have $\neg\,(\alpha \between V)$ by (P3). Conversely, let $\alpha \in \mathcal{P}t(\mathcal{S})$ such that $\neg\,(\alpha \between V)$. It suffices to show that α satisfies (P3). Let $a \in S$ and $U \subseteq S$, and suppose that $a \lhd^{\mathcal{S}-V} U$ and $a \in \alpha$. Then, $a \lhd V \cup U$, and thus $\alpha \between (V \cup U)$ by (P3). Hence, either $\alpha \between V$ or $\alpha \between U$. In the former case, we have a contradiction. Hence, we conclude $\alpha \between U$, as required.

3. The proof is similar to that of Lemma 2.3.6.4, and hence omitted. □

Unlike open subtopologies, not all closed subtopologies are overt constructively. The following counterexample is essentially due to Ciraulo and Sambin [14, Proposition 4.3].

Example 2.3.10. We consider a closed subtopology of $\mathbf{1} = (\{*\}, \in, =)$, the discrete topology on a singleton $\{*\}$ (See Section 2.1.3).

Let φ be any restricted formula. Let $\mathcal{S} = (\{*\}, \lhd, =)$ be the closed subtopology of $\mathbf{1}$ determined by the subset $\{* \mid \varphi\}$. Thus, the cover of $\mathcal{S}$ is given by

$$* \lhd U \overset{\text{def}}{\iff} * \in \{* \mid \varphi\} \cup U$$

for all $U \subseteq \{*\}$. We show that if $\mathcal{S}$ were overt, then $\neg\neg\varphi \to \varphi$ would follow. Suppose that $\mathcal{S}$ is overt with a positivity Pos. By Lemma 2.1.9, Pos satisfies

$$(* \in \mathsf{Pos} \to * \lhd \emptyset) \to * \lhd \emptyset. \tag{2.13}$$

We have $* \lhd \emptyset \iff * \in \{* \mid \varphi\} \iff \varphi$. Moreover, $* \in \mathsf{Pos} \iff \neg\varphi$. To see this, suppose that $* \in \mathsf{Pos}$. Then, we have $\mathsf{Pos} = \{*\}$. Since Pos is splitting, we have $* \lhd U \to * \in U$ for all $U \subseteq \{*\}$, i.e. $\{* \mid \varphi\} \subseteq U$ for all $U \subseteq \{*\}$. In particular, we have $\{* \mid \varphi\} \subseteq \emptyset$, which is equivalent to $\neg\varphi$. Conversely, suppose that $\neg\varphi$. Since $* \lhd \{*\} \cap \mathsf{Pos}$ and $\{* \mid \varphi\} = \emptyset$, we have $* \in \{*\} \cap \mathsf{Pos}$, and thus $* \in \mathsf{Pos}$. Hence, (2.13) is equivalent to $(\neg\varphi \to \varphi) \to \varphi$. Since $\neg\varphi \to \varphi$ is equivalent to $\neg\neg\varphi$, we have $\neg\neg\varphi \to \varphi$. Therefore, constructively $\mathcal{S}$ cannot be overt.

Definition 2.3.11. Let $\mathcal{S}$ be a formal topology, and let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$. Then, the *closure* of $\mathcal{S}'$ in $\mathcal{S}$ is the closed subtopology $\mathcal{S}^{\mathcal{S}-Z}$ determined by the set

$$Z \overset{\text{def}}{=} \{a \in S \mid a \lhd' \emptyset\}\,. \tag{2.14}$$

The closure of a formal topology has an expected property.

Lemma 2.3.12. *Let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$. Then, the closure of $\mathcal{S}'$ in $\mathcal{S}$ is the smallest closed subtopology of $\mathcal{S}$ which is larger than $\mathcal{S}'$.*

Proof. Let $Z \subseteq S$ as defined by (2.14), so that $\mathcal{S}^{\mathcal{S}-Z}$ is the closure of $\mathcal{S}'$. Let $V \subseteq S$ and suppose that $\mathcal{S}' \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$. Then, $V \lhd' \emptyset$, and so $V \subseteq Z$. Hence, $\mathcal{S}^{\mathcal{S}-Z} \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$. □

The closure of an overt formal topology is determined by the complement of its positivity.

Proposition 2.3.13. *Let $\mathcal{S}'$ be an overt subtopology of $\mathcal{S}$ with a positivity Pos. Then, the closure of $\mathcal{S}'$ in $\mathcal{S}$ is the closed subtopology $\mathcal{S}^{\mathcal{S}-\neg Pos}$.*

Proof. Let $Z = \{a \in S \mid a \lhd' \emptyset\}$. It suffices to show that $\neg\,\mathsf{Pos} = Z$. Since Pos is the positivity of $\mathcal{S}'$, we have $\neg\,\mathsf{Pos} \lhd' \emptyset$, and thus $\neg\,\mathsf{Pos} \subseteq Z$. Conversely, if $a \lhd' \emptyset$ and $a \in \mathsf{Pos}$, then we have $\mathsf{Pos} \between \emptyset$, a contradiction. Hence $Z \subseteq \neg\,\mathsf{Pos}$. □

The class of overt closed subtopologies is of fundamental importance in this thesis. The following says that an overt closed subtopology is overt weakly closed (See Definition 2.3.18).

Corollary 2.3.14. *Let $\mathcal{S}'$ be an overt closed subtopology of $\mathcal{S}$ with a positivity Pos. Then, $\mathcal{S}'$ is the largest subtopology of $\mathcal{S}$ with the positivity Pos.*

Proof. Let $\mathcal{S}''$ be an overt subtopology of $\mathcal{S}$ with the positivity Pos. By Proposition 2.3.13, we have $\mathcal{S}'' \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg\,\mathsf{Pos}}$. But since $\mathcal{S}'$ is closed, we have $\mathcal{S}' = \mathcal{S}^{\mathcal{S}-\neg\,\mathsf{Pos}}$, and hence $\mathcal{S}'' \sqsubseteq \mathcal{S}'$. □

Example 2.3.15 (cf. Example 2.2.22). Let $\mathcal{R}$ be the formal reals. *The formal unit interval* $\mathcal{I}[0,1]$ is the closed subtopology of $\mathcal{R}$ determined by the set $(-\infty, 0) \cup (1, \infty)$ where

$$(-\infty, 0) \overset{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid q \leq 0\}\,,$$
$$(1, \infty) \overset{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid p \geq 1\}\,.$$

That is, $\mathcal{I}[0,1]$ is determined by the set

$$\{(p,q) \in S_{\mathcal{R}} \mid 1 \leq p \vee q \leq 0\}\,.$$

By Lemma 2.3.6.3 and Proposition 2.3.13, $\mathcal{I}[0,1]$ is the closure of $\mathcal{R}_{(0,1)}$ (See Example 2.3.7). The formal points of $\mathcal{I}[0,1]$ will be studied in Example 2.3.24.

Given a formal topology $\mathcal{S}$, the subtopology $\mathcal{S}_0$ with the cover $\lhd_0 = S \times \mathsf{Pow}(S)$ is the smallest subtopology of $\mathcal{S}$. If $\mathcal{S}'$ and $\mathcal{S}''$ are subtopologies of $\mathcal{S}$, the join $\mathcal{S}' \vee \mathcal{S}''$ is given by the cover $\lhd_\vee$ defined by

$$a \lhd_\vee U \overset{\text{def}}{\iff} a \lhd' U \ \& \ a \lhd'' U.$$

The meet of an arbitrary pair of subtopologies is not known to exist predicatively. However, the meet $\mathcal{S}_V \wedge \mathcal{S}^{\mathcal{S}-W}$ of an open subtopology $\mathcal{S}_V$ and a closed subtopology $\mathcal{S}^{\mathcal{S}-W}$ exists [61]. Its cover $\lhd_\wedge$ is given by

$$a \lhd_\wedge U \overset{\text{def}}{\iff} a \downarrow V \lhd W \cup U.$$

A closed subtopology represents the complement of an open subtopology in the following sense.

Proposition 2.3.16 (Vickers [61, Proposition 12]). *Let $\mathcal{S}$ be a formal topology, and let $V \subseteq S$. Then, the open subtopology $\mathcal{S}_V$ and the closed subtopology $\mathcal{S}^{\mathcal{S}-V}$ are boolean complement of each other in the lattice of subtopologies of $\mathcal{S}$, i.e. $\mathcal{S}_V \wedge \mathcal{S}^{\mathcal{S}-V} = \mathcal{S}_0$ and $\mathcal{S}_V \vee \mathcal{S}^{\mathcal{S}-V} = \mathcal{S}$.*

Proof. Let $a \in S$ and $U \subseteq S$, and suppose that $a \lhd_\vee U$, i.e. $a \lhd_V U$ and $a \lhd^{\mathcal{S}-V} U$. Then, $a \downarrow V \lhd U$ and $a \lhd V \cup U$. Thus, $a \lhd (a \downarrow V) \cup U$, and hence $a \lhd U$. Therefore $\mathcal{S}_V \vee \mathcal{S}^{\mathcal{S}-V} = \mathcal{S}$. For the meet, we have $S \lhd_\wedge \emptyset$. Hence, $\mathcal{S}_V \wedge \mathcal{S}^{\mathcal{S}-V} = \mathcal{S}_0$. □

The open subtopologies and the closed subtopologies are pullback stable, i.e. they are preserved by pullbacks.

Lemma 2.3.17. *Let $r \colon \mathcal{S}' \to \mathcal{S}$ be a formal topology map, and let $V \subseteq S$. Then,*

1. *r factors through the inclusion $\mathcal{S}_V \to \mathcal{S}$ iff $S' \lhd' r^- V$.*
2. *r factors through the inclusion $\mathcal{S}^{\mathcal{S}-V} \to \mathcal{S}$ iff $r^- V \lhd' \emptyset$.*

Proof. Let $\mathcal{S}_r$ be the image of $\mathcal{S}'$ under r. Then

$$\begin{aligned} r \text{ factors through } \mathcal{S}_V &\iff \mathcal{S}_r \sqsubseteq \mathcal{S}_V \\ &\iff S \lhd_r V \\ &\iff r^- S \lhd' r^- V \\ &\iff S' \lhd' r^- V, \end{aligned}$$

and

$$\begin{aligned} r \text{ factors through } \mathcal{S}^{\mathcal{S}-V} &\iff \mathcal{S}_r \sqsubseteq \mathcal{S}^{\mathcal{S}-V} \\ &\iff V \lhd_r \emptyset \\ &\iff r^- V \lhd' r^- \emptyset \\ &\iff r^- V \lhd' \emptyset. \end{aligned}$$ □

It follows from Lemma 2.3.17 and the characterisations of open and closed subtopologies that $\mathcal{S}'_{r^-V}$ and $\mathcal{S}'^{\mathcal{S}'-r^-V}$ are the pullbacks (i.e. the inverse images) of $\mathcal{S}_V$ and $\mathcal{S}^{\mathcal{S}-V}$ along r respectively.

Overt weakly closed subtopologies

Definition 2.3.18. Let $\mathcal{S}'$ be an overt subtopology of a formal topology $\mathcal{S}$, and let Pos be the positivity of $\mathcal{S}'$. Then, $\mathcal{S}'$ is said to be *overt weakly closed* if $\mathcal{S}'$ is the largest overt subtopology of $\mathcal{S}$ with the positivity Pos.

Remark 2.3.19. The notion of overt weakly closed subtopology given in Definition 2.3.18 is stronger than the one given in [28, Definition 3.5.13], although impredicatively, they are equivalent. However, our main examples of overt weakly closed subtopologies, namely overt closed subtopologies, satisfy the condition of Definition 2.3.18. Moreover, if a formal topology $\mathcal{S}$ in Definition 2.3.18 is inductively generated, the two notions coincide.

Proposition 2.3.20. *Any overt closed subtopology is overt weakly closed.*

Proof. Immediate from Corollary 2.3.14. □

For an inductively generated formal topology $\mathcal{S}$, its overt weakly closed subtopology can be identified with a splitting subset of $\mathcal{S}$.

Theorem 2.3.21. *Let $\mathcal{S}$ be an inductively generated formal topology. Then, there exists an order isomorphism between the splitting subsets of $\mathcal{S}$ and the overt weakly closed subtopologies of $\mathcal{S}$.*

Proof. Write $\mathcal{S} = (S, \lhd, \leq)$, and suppose that $\mathcal{S}$ is generated by an axiom-set (I, C) on $(S, \leq)$. Given any splitting subset $V \subseteq S$, let $\mathcal{S}^V$ be the formal topology generated by the axiom-set (I, C) together with the extra axiom

$$a \lhd^V \{a\} \cap V \tag{2.15}$$

for each $a \in S$. Then, $\mathcal{S}^V$ is the largest overt subtopology of $\mathcal{S}$ with the positivity V by Proposition 2.2.18. Hence, $\mathcal{S}^V$ is an overt weakly closed subtopology of $\mathcal{S}$. Conversely, for any overt weakly closed subtopology of $\mathcal{S}$, its positivity is a splitting subset of $\mathcal{S}$.

The assignment $V \mapsto \mathcal{S}^V$ is clearly bijective. Moreover, for any splitting subsets V and W of $\mathcal{S}$, we have $V \subseteq W \implies \mathcal{S}^V \sqsubseteq \mathcal{S}^W$ by (2.15). Conversely, if $\mathcal{S}^V \sqsubseteq \mathcal{S}^W$, then $V \subseteq W$ since V is a splitting subset of $\mathcal{S}^W$. Therefore, the assignment is an order isomorphism. □

Notation 2.3.22. If $\mathcal{S}$ is an inductively generated formal topology and V is a splitting subset of $\mathcal{S}$, the overt weakly closed subtopology determined by V is denoted by $\mathcal{S}^V$.

By Proposition 2.2.18 and Corollary 2.2.13, the formal points of $\mathcal{S}^V$ can be characterised by

$$\mathcal{P}t(\mathcal{S}^V) = \{\alpha \in \mathcal{P}t(\mathcal{S}) \mid \alpha \subseteq V\}\,.$$

The following is a corollary of Proposition 2.3.20.

Corollary 2.3.23. *Let $\mathcal{S}$ be an inductively generated formal topology, and let $\mathcal{S}' \sqsubseteq \mathcal{S}$ be an overt closed subtopology with a positivity Pos. Then $\mathcal{S}' = \mathcal{S}^{\mathsf{Pos}} = \mathcal{S}^{\mathcal{S} - \neg \mathsf{Pos}}$.*

Example 2.3.24 (cf. Example 2.3.15). The formal unit interval $\mathcal{I}[0,1]$ is an overt closed subtopology of the formal reals $\mathcal{R}$ with the positivity $\mathsf{Pos}_{\mathcal{I}[0,1]}$ given by

$$\mathsf{Pos}_{\mathcal{I}[0,1]} \stackrel{\text{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid p < 1 \ \&\ 0 < q\}\,. \tag{2.16}$$

Hence, $\mathcal{I}[0,1]$ can equivalently be defined as the overt weakly closed subtopology of $\mathcal{R}$ determined by $\mathsf{Pos}_{\mathcal{I}[0,1]}$. Thus, we have

$$\mathcal{P}t(\mathcal{I}[0,1]) = \{\alpha \in \mathcal{P}t(\mathcal{R}) \mid (\forall (p,q) \in \alpha)\, p < 1 \ \&\ 0 < q\}\,.$$

Hence, by the correspondence between $\mathcal{P}t(\mathcal{R})$ and the Dedekind cuts, $\mathcal{P}t(\mathcal{I}[0,1])$ corresponds to the unit interval $[0,1]$ (See Example 2.2.22).

2.4 Regularities and compactness

The notions of regularity and compactness for formal topologies are point-free analogues of the corresponding notions in the point-set topology. The definitions of these notions for formal topologies are straightforward adaptations of those in locale theory. In addition to these standard notions, we also introduce a weaker notion of regularity, called 2-regularity.

2.4.1 Regularities

In formal topology, several notions of regularity have been proposed [20, 28]. In this thesis, we consider two of them: regularity and 2-regularity. The former notion is adapted from locale theory, while the latter was proposed to accommodate formal topologies which arise from covering uniformities [28].

Definition 2.4.1. Let $\mathcal{S}$ be a formal topology. For each $a \in S$ and $U \subseteq S$, define

$$a^* \stackrel{\text{def}}{=} \{b \in S \mid b \downarrow a \lhd \emptyset\}\,,$$
$$U^* \stackrel{\text{def}}{=} \{b \in S \mid b \downarrow U \lhd \emptyset\}\,.$$

Note that $U^* = \bigcap_{a \in U} a^*$ and $U \lhd V \implies V^* \subseteq U^*$.

Given $a, b \in S$, we say that a is *well-covered* by b, denoted by $a \lll b$, if $S \lhd a^* \cup \{b\}$. We extend the relation $\lll$ to the subsets of S by

$$U \lll V \stackrel{\text{def}}{\iff} S \lhd U^* \cup V$$

for all $U, V \subseteq S$.

The first item in the following list clarifies what it means to be 'well-covered'.

Lemma 2.4.2. *Let $\mathcal{S}$ be a formal topology, and let $U, U', V, V' \subseteq S$.*

1. *$U \lll V$ iff the closure of $\mathcal{S}_U$ is smaller than $\mathcal{S}_V$.*

2. $U \lll V \implies U \lhd V$.

3. $U' \lhd U \lll V \lhd V' \implies U' \lll V'$.

4. If $r \colon \mathcal{S} \to \mathcal{S}'$ is a formal topology map, then

$$U \lll' V \implies r^- U \lll r^- V$$

for all $U, V \subseteq S'$.

Proof. 1. We have $U \lll V \iff S \lhd U^* \cup V \iff \mathcal{S}^{\mathcal{S}-U^*} \sqsubseteq \mathcal{S}_V$. Since $\mathcal{S}^{\mathcal{S}-U^*}$ is the closure of $\mathcal{S}_U$, the conclusion follows.

2 and 3 follow from 1.

4. Let $U, V \subseteq S'$, and suppose that $U \lll' V$. Then $S' \lhd' U^* \cup V$, so $S \lhd r^- S' \lhd r^- U^* \cup r^- V$. Let $a \in r^- U^*$. Then, there exists $b \in U^*$ such that $a \; r \; b$. Thus, $a \downarrow r^- U \lhd r^- b \downarrow r^- U \lhd r^-(b \downarrow U) \lhd r^- \emptyset \lhd \emptyset$. Hence $r^- U^* \subseteq (r^- U)^*$, so $S \lhd (r^- U)^* \cup r^- V$. Therefore $r^- U \lll r^- V$. □

The following definition is an inessential modification of the standard notion of regularity [51].

Definition 2.4.3. A formal topology $\mathcal{S}$ is *regular* if there exists a function $wc \colon S \to \mathsf{Pow}(S)$ such that for each $a \in S$

1. $(\forall b \in wc(a))\, b \lll a$,
2. $a \lhd wc(a)$.

Remark 2.4.4. Since the relation $\lll$ on S is a set, $\mathcal{S}$ is regular iff

$$a \lhd \{b \in S \mid b \lll a\} \tag{2.17}$$

for all $a \in S$. In most literature, the condition (2.17) is adopted as the definition of regularity [51, 20, 48, 53].

The following notion of 2-regular formal topology is due to Fox [28].

Definition 2.4.5 (Fox [28, Definition 3.6.4])**.** A formal topology $\mathcal{S}$ is *2-regular* if it is overt and $a \lhd \mathsf{wc}_2(a)$ holds for each $a \in S$. Here, $\mathsf{wc}_2(a)$ is a subset of S defined by

$$\mathsf{wc}_2(a) \stackrel{\text{def}}{=} \{b \in S \mid S \lhd \{c \in S \mid \mathsf{Pos}(c \downarrow b) \to c \lhd a\}\},$$

where Pos is the positivity of $\mathcal{S}$.

Intuitively, $b \in \mathsf{wc}_2(a)$ iff every basic open $c \in S$ which intersects b is covered by a. In particular, $b \in \mathsf{wc}_2(a)$ implies $b \lhd a$ by (Pos).

2-regularity is weaker than regularity in the following sense.

Proposition 2.4.6. *Every overt regular formal topology is 2-regular.*

Proof. Let $\mathcal{S} = (S, \lhd, \leq)$ be an overt regular formal topology with a positivity Pos. It suffices to show that

$$a \lll b \implies a \in \mathsf{wc}_2(b)$$

for all $a, b \in S$. So suppose that $a \lll b$. Let

$$C \stackrel{\text{def}}{=} \{c \in S \mid \mathsf{Pos}(c \downarrow a) \to c \lhd b\}.$$

It suffices to show that $a^* \cup \{b\} \subseteq C$. Clearly, $b \in C$. Let $c \in a^*$, and suppose that $\mathsf{Pos}(c \downarrow a)$. Since $c \downarrow a \lhd \emptyset$, we have $\mathsf{Pos} \between \emptyset$, a contradiction. Hence $c \lhd b$. Therefore $a^* \cup \{b\} \subseteq C$, as required. □

Constructively, we cannot show that every 2-regular formal topology is regular [28].

Example 2.4.7. Let $\mathcal{S} = (S, \lhd_d, =)$ be the discrete formal topology on the set S where the cover $\lhd_d$ is given by $a \lhd_d U \stackrel{\text{def}}{\iff} a \in U$. It is easy to see that $\mathcal{S}$ is 2-regular, however, if $\mathcal{S}$ were regular, then we would have $a = b$ or $\neg(a = b)$ for all $a, b \in S$. Hence, for any set S its equality is decidable, which is constructively impossible.

The following plays an important role in later chapters.

Proposition 2.4.8. *Let $r, s : \mathcal{T} \to \mathcal{S}$ be formal topology maps between overt formal topologies, where $\mathcal{S}$ is 2-regular. Then, we have*

$$r \leq s \implies s \leq r,$$

i.e. every formal topology map between overt formal topologies with 2-regular codomain is maximal.

Proof. Let $\mathsf{Pos}_{\mathcal{T}}$ and $\mathsf{Pos}_{\mathcal{S}}$ be positivities of $\mathcal{T}$ and $\mathcal{S}$ respectively. Suppose that $r \leq s$. Let $a \in S$ and $b \in T$, and suppose that $b \; s \; a$. Since $\mathcal{S}$ is 2-regular, we have $b \lhd_{\mathcal{T}} s^- \mathsf{wc}_2(a)$. Let $b' \in s^- \mathsf{wc}_2(a)$. Then, there exists $a' \in \mathsf{wc}_2(a)$ such that $b' \; s \; a'$. Since r is a formal topology map, we have $T \lhd_{\mathcal{T}} r^- \{c \in S \mid \mathsf{Pos}_{\mathcal{S}}(c \downarrow a') \to c \lhd_{\mathcal{S}} a\}$, so

$$b' \lhd_{\mathcal{T}} \left(r^- \{c \in S' \mid \mathsf{Pos}_{\mathcal{S}}(c \downarrow a') \to c \lhd_{\mathcal{S}} a\} \downarrow s^- a'\right) \cap \mathsf{Pos}_{\mathcal{T}}.$$

Let $b'' \in$ RHS. Then, there exists $c \in S$ such that $\mathsf{Pos}_{\mathcal{S}}(c \downarrow a') \to c \lhd_{\mathcal{S}} a$ and $b'' \lhd_{\mathcal{T}} r^- c$. Thus, we have

$$b'' \lhd_{\mathcal{T}} r^- c \downarrow s^- a' \lhd_{\mathcal{T}} s^- c \downarrow s^- a' \lhd_{\mathcal{T}} s^-(c \downarrow a').$$

Since $\mathsf{Pos}_{\mathcal{T}}(b'')$, we have $\mathsf{Pos}_{\mathcal{S}}(c \downarrow a')$, and hence $c \lhd_{\mathcal{S}} a$, so that $r^- c \lhd_{\mathcal{T}} r^- a$. Thus, $b'' \lhd_{\mathcal{T}} r^- a$. Therefore $b \lhd_{\mathcal{T}} r^- a$, from which we conclude $s \leq r$. □

If $\mathcal{S}$ is regular, a similar fact holds without requiring that $\mathcal{T}$ and $\mathcal{S}$ be overt [47].

Proposition 2.4.9. *Let $r, s : \mathcal{T} \to \mathcal{S}$ be formal topology maps to a regular formal topology $\mathcal{S}$. Then, $r \leq s \implies s \leq r$.*

Proof. Suppose that $r \leq s$. Let $a \in S$ and $b \in T$, and suppose that $b\ s\ a$. Since $\mathcal{S}$ is regular, we have $b \lhd_{\mathcal{T}} s^{-} wc(a)$. Let $b' \in s^{-} wc(a)$. Then, there exists $a' \in wc(a)$ such that $b'\ s\ a'$. Since r is a formal topology map, we have

$$\begin{aligned} T \lhd_{\mathcal{T}} r^{-}\left(\{a'\}^{*} \cup \{a\}\right) &= r^{-}\{a'\}^{*} \cup r^{-}\{a\} \\ &\lhd_{\mathcal{T}} s^{-}\{a\}'^{*} \cup r^{-}\{a\} \lhd_{\mathcal{T}} \left(s^{-}\{a'\}\right)^{*} \cup r^{-}\{a\}, \end{aligned}$$

where we used the assumption $r \leq s$. Hence

$$\begin{aligned} b' &\lhd_{\mathcal{T}} \left(\left(s^{-}\{a'\}\right)^{*} \cup r^{-}\{a\}\right) \downarrow s^{-}\{a'\} \\ &\lhd_{\mathcal{T}} \left((s^{-}\{a'\})^{*} \downarrow s^{-}\{a'\}\right) \cup r^{-}\{a\} \lhd_{\mathcal{T}} r^{-}\{a\}. \end{aligned}$$

Therefore $b \lhd_{\mathcal{T}} r^{-}a$, from which we conclude $s \leq r$. □

By the correspondence between the formal topology maps $\mathbf{1} \to \mathcal{S}$ and $\mathcal{P}t(\mathcal{S})$, we have the following.

Corollary 2.4.10. *Every formal point of 2-regular (or regular) formal topology is maximal, i.e. if $\mathcal{S}$ is a 2-regular (resp. regular) formal topology, then for any $\alpha, \beta \in \mathcal{P}t(\mathcal{S})$, we have*

$$\alpha \subseteq \beta \implies \beta \subseteq \alpha.$$

Some of the closure properties of regular formal topologies which are well-known in locale theory are also valid in formal topology [34].

Proposition 2.4.11.

1. *A subtopology of a regular formal topology is regular.*

2. *A product of a family of inductively generated regular formal topologies is regular.*

Proof. 1. If $\mathcal{S}$ is regular and $\mathcal{S}'$ is a subtopology of $\mathcal{S}$, we have $a \lll b$ in $\mathcal{S}$ implies $a \lll b$ in $\mathcal{S}'$, from which the conclusion follows.

2. Let $(\mathcal{S}_i)_{i\in I}$ be a family of regular inductively generated formal topologies, and let $(wc_i)_{i\in I}$ be a family such that for each $i \in I$, $wc_i : S_i \to \mathsf{Pow}(S_i)$ is a function which makes $\mathcal{S}_i$ regular. By Corollary 2.2.25, given any $A = \{(i_0, a_0), \ldots, (i_{n-1}, a_{n-1})\} \in S_\Pi$, we have

$$\{(i_0, a_0), \ldots, (i_{n-1}, a_{n-1})\} \lhd_\Pi \{\{(i_0, b_0), \ldots, (i_{n-1}, b_{n-1})\} \mid (\forall k < n)\, b_k \in wc_{i_k}(a_k)\}.$$

Thus, it suffices to show that for any $B = \{(i_0, b_0), \ldots, (i_{n-1}, b_{n-1})\}$ such that $b_k \in wc_{i_k}(a_k)$ for all $k < n$, we have $B \lll A$ in $\mathcal{S}_\Pi$. Let $B = \{(i_0, b_0), \ldots, (i_{n-1}, b_{n-1})\} \in S_\Pi$ such that $b_k \in wc_{i_k}(a_k)$ for all $k < n$. Then, for each $k < n$, since $S_{i_k} \lhd_{i_k} b_k^{*} \cup \{a_k\}$, we have

$$S_\Pi \lhd_\Pi \{\{(i_k, c)\} \in S_\Pi \mid c \in b_k^{*} \cup \{a_k\}\}.$$

Thus, we have

$$S_\Pi \lhd_\Pi \{\{(i_0, c_0), \ldots, (i_{n-1}, c_{n-1})\} \in S_\Pi \mid (\forall k < n)\, c_k \in b_k^{*} \cup \{a_k\}\}.$$

Let $C = \{(i_0, c_0), \ldots, (i_{n-1}, c_{n-1})\} \in \text{RHS}$. Then, either $c_k = a_k$ for all $k < n$, or $c_k \in b_k^*$ for some $k < n$. In the former case, we have $C = A$. In the latter case, let $k < n$ such that $c_k \in b_k^*$. Then,

$$\begin{aligned} C \downarrow B \lhd_\Pi & \, C \cup B \\ \lhd_\Pi & \, \{(i_k, c_k), (i_k, b_k)\} \\ \lhd_\Pi & \, \{\{(i_k, d)\} \in S_\Pi \mid d \in c_k \downarrow b_k\} \\ \lhd_\Pi & \, \{\{(i_k, d)\} \in S_\Pi \mid d \in \emptyset\} \lhd_\Pi \emptyset. \end{aligned}$$

Hence, $C \in B^*$, and thus $S_\Pi \lhd_\Pi B^* \cup \{A\}$. Therefore $B \lll A$. □

2.4.2 Compactness and local compactness

Compactness and local compactness for formal topologies are defined by the covering compactness. Compactness for formal topologies is defined by expressing the usual notion of covering compactness in terms of formal topology. However, there is a predicativity problem in defining the notion of local compactness, and we need to include a base of the way-below relation in the definition.

Contrary to the case of point-set topology, compactness and local compactness for formal topologies are compatible with the corresponding notions for metric spaces [48] (See also Theorem 3.1.39).

Definition 2.4.12. A formal topology $\mathcal{S}$ is *compact* if

$$S \lhd U \implies (\exists U_0 \in \mathsf{Fin}(U))\, S \lhd U_0$$

for all $U \subseteq S$.

Some of the closure properties of compact locales carry over to formal topology ([34, Chapter III, Proposition 1.2],[23]).

Proposition 2.4.13.

1. *A closed subtopology of a compact formal topology is compact.*
2. *A compact subtopology of a regular formal topology is closed.*
3. *The image of a compact formal topology under a formal topology map is compact.*

Proof. 1. Let $\mathcal{S}$ be a compact formal topology, and let $\mathcal{S}^{S-V}$ be the closed subtopology of $\mathcal{S}$ determined by $V \subseteq S$. Let $U \subseteq S$, and suppose that $S \lhd^{S-V} U$. Then, $S \lhd V \cup U$. Since $\mathcal{S}$ is compact, there exists $U_0 \in \mathsf{Fin}(U)$ such that $S \lhd V \cup U_0$, i.e. $S \lhd^{S-V} U_0$. Therefore $\mathcal{S}^{S-V}$ is compact.

2. Let $\mathcal{S}'$ be a compact subtopology of a regular formal topology $\mathcal{S}$. We show that $\mathcal{S}'$ coincides with its closure, i.e. $\mathcal{S}' = \mathcal{S}^{S-Z}$ where $Z = \{a \in S \mid a \lhd' \emptyset\}$. We first show that $S \lhd' U$ implies $S \lhd Z \cup U$. Suppose that $S \lhd' U$. Since $\mathcal{S}$ is regular, we have

$S \lhd' \{a \in S \mid (\exists b \in U)\, a \lll b\}$. Put $V = \{a \in S \mid (\exists b \in U)\, a \lll b\}$. Since $\mathcal{S}'$ is compact, there exists $V_0 \in \mathsf{Fin}(V)$ such $S \lhd' V_0$. Since $S \lhd a^* \cup U$ for each $a \in V_0$, we have $S \lhd V_0^* \cup U \subseteq S^* \cup U = Z \cup U$. Now, suppose that $a \lhd' U$, and let $b \lll a$ in $\mathcal{S}$. Then $S \lhd' b^* \cup \{a\} \lhd' b^* \cup U$, so $S \lhd Z \cup b^* \cup U$. Thus $b \lhd (Z \cup b^* \cup U) \downarrow b \lhd (b^* \downarrow b) \cup ((Z \cup U) \downarrow b) \lhd Z \cup U$. Since $\mathcal{S}$ is regular, we have $a \lhd \{b \in S \mid b \lll a\} \lhd Z \cup U$. Hence $a \lhd^{\mathcal{S}-Z} U$. Therefore, $\lhd' \subseteq \lhd^{\mathcal{S}-Z}$, so we have $\lhd' = \lhd^{\mathcal{S}-Z}$.

3. Let $\mathcal{S}$ be a compact formal topology, and let $r : \mathcal{S} \to \mathcal{S}'$ be a formal topology map. Let $\mathcal{S}_r = (S', \lhd_r, \leq')$ be the image of $\mathcal{S}$ under r. Let $V \subseteq S'$, and suppose that $S' \lhd_r V$. Since r is a formal topology map, we have $S \lhd r^- V$. Since $\mathcal{S}$ is compact, there exists $U \in \mathsf{Fin}(r^- V)$ such that $S \lhd U$. Thus, there exists $V_0 \in \mathsf{Fin}(V)$ such that $U \subseteq r^- V_0$, and hence $r^- S' \lhd r^- V_0$, i.e. $S' \lhd_r V_0$. Therefore, $\mathcal{S}_r$ is compact. □

The most celebrated result about compact formal topologies is the Tychonoff theorem. In the classical point-set topology, the Tychonoff theorem is equivalent to the Axiom of Choice [37]. However, Johnstone [33] showed that in locale theory the Tychonoff theorem can be proved without the Axiom of Choice[5]. Moreover, for inductively generated formal topologies, a fully constructive proof is possible.

Theorem 2.4.14. *Let $(\mathcal{S}_i)_{i \in I}$ be a set-indexed family of inductively generated formal topologies such that $\mathcal{S}_i$ is compact for each $i \in I$. Then, the product $\prod_{i \in I} \mathcal{S}_i$ is compact.*

Proof. For interested readers, we give the proof due to Vickers [59] in Appendix B. □

The following predicative notion of local compactness is due to Curi [20, 21].

Definition 2.4.15. Let $\mathcal{S}$ be a formal topology, and let $a, b \in S$. We say that a is *way-below* b, denoted by $a \ll b$, if

$$b \lhd U \implies (\exists U_0 \in \mathsf{Fin}(U))\, a \lhd U_0$$

for all $U \subseteq S$. We extend the relation $\ll$ to the subsets of S as in the case of well-covered relation $\lll$. A formal topology $\mathcal{S}$ is *locally compact* if there exists a function $wb \colon S \to \mathsf{Pow}(S)$ such that for each $a \in S$

1. $(\forall b \in wb(a))\, b \ll a$,
2. $a \lhd wb(a)$.

[5] There are many proofs of the Tychonoff theorem in point-free topologies. The first point-free proof without the Axiom of Choice is given by Johnstone [33, 34], but he used classical logic. The first intuitionistic (but impredicative) proof seems to be due to Vermeulen [56]. In formal topology, the Tychonoff theorem only makes sense for inductively generated formal topologies, since construction of a product of arbitrary formal topologies is yet unknown. For inductively generated formal topologies, the Tychonoff theorem was first proved by Negri and Valentini [45] on condition that the equality of the indexed-set of a family is decidable. Later, fully constructive proofs without requiring the decidability were given by Coquand [16] and Vickers [59].

Since we have $U \lhd V \ll W \implies U \ll W$, the way-below relation can be characterised by a restricted formula:

$$a \ll b \iff (\exists U \in \mathsf{Fin}(S))\, a \lhd U \;\&\; U \subseteq wb(b).$$

Note that without the function wb, a quantification over $\mathsf{Pow}(S)$ is needed to define $\ll$. This is the main reason for including wb in the definition of local compactness for formal topologies. In this thesis, we call such a function wb a *base* of the way-below relation. It is shown by Curi [21] that the property of having a base of the way-below relation is invariant under isomorphisms.

Remark 2.4.16. Any locally compact formal topology is inductively generated by an axiom-set. Given a locally compact formal topology $\mathcal{S} = (S, \lhd, \leq)$ with a base $wb : S \to \mathsf{Pow}(S)$ of the way-below relation, we can define an axiom-set (I, C) on $(S, \leq)$ which generates $\mathcal{S}$ by

$$\begin{aligned} I(a) &\stackrel{\text{def}}{=} \{U \in \mathsf{Fin}(S) \mid a \lhd U\} + \{*\}\,, \\ C(a, U) &\stackrel{\text{def}}{=} U, \\ C(a, *) &\stackrel{\text{def}}{=} wb(a). \end{aligned}$$

Indeed, if $\mathcal{S}_{I,C}$ is the formal topology generated by the axiom-set (I, C), then we have $\mathcal{S} \sqsubseteq \mathcal{S}_{I,C}$ since $\mathcal{S}$ satisfies all the axioms of $\mathcal{S}_{I,C}$. Conversely, suppose that $a \lhd U$. Then for each $b \in wb(a)$, there exists $U_0 \in \mathsf{Fin}(U)$ such that $b \lhd U_0$, so that $b \lhd_{I,C} U_0$. Hence $a \lhd_{I,C} wb(a) \lhd_{I,C} U$. Therefore $\mathcal{S}_{I,C} \sqsubseteq \mathcal{S}$.

In a locally compact formal topology, the way-below relation interpolates [22].

Lemma 2.4.17. *Let $\mathcal{S}$ be a locally compact formal topology. For any $U, V \subseteq S$ such that $U \ll V$, there exists $W \in \mathsf{Fin}(S)$ such that $U \ll W \ll V$.*

Proof. Let $wb : S \to \mathsf{Pow}(S)$ be a base of the way-below relation. Let $U, V \subseteq S$, and suppose that $U \ll V$. Since

$$V \lhd \bigcup \{wb(a) \mid (\exists b \in V)\, a \in wb(b)\}\,,$$

there exist $\{a_0, \dots, a_{n-1}\}$, $\{b_0, \dots, b_{n-1}\} \in \mathsf{Fin}(S)$ such that $U \lhd \{a_0, \dots, a_{n-1}\}$, $a_i \ll b_i$ for each $i < n$, and $\{b_0, \dots, b_{n-1}\} \ll V$. Hence $U \ll \{b_0, \dots, b_{n-1}\} \ll V$. ☐

Definition 2.4.18. Let $\mathcal{S}$ be a formal topology. A subset $U \subseteq S$ is *bounded* if $U \ll S$. A subtopology $\mathcal{S}'$ of $\mathcal{S}$ is *bounded* if there exists a bounded subset $U \subseteq S$ such that $\mathcal{S}' \sqsubseteq \mathcal{S}_U$, where $\mathcal{S}_U$ is the open subtopology of $\mathcal{S}$ determined by U.

Proposition 2.4.19. *Let $\mathcal{S}$ be a locally compact regular formal topology. Then, a subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ is compact iff $\mathcal{S}'$ is closed and bounded.*

Proof. Let $wb : S \to \mathsf{Pow}(S)$ be a base of the way-below relation of $\mathcal{S}$. Suppose that $\mathcal{S}'$ is compact. Since $\mathcal{S}$ is regular, $\mathcal{S}'$ is closed by Proposition 2.4.13.2, and since $S \lhd' \{a \in S \mid (\exists b \in S)\, a \in wb(b)\}$, there exists $U \in \mathsf{Fin}(S)$ such that $S \lhd' U$ and $U \ll S$. Then, $\mathcal{S}' \sqsubseteq \mathcal{S}_U$, so $\mathcal{S}'$ is bounded.

Conversely, suppose that $\mathcal{S}'$ is closed and bounded. Since $\mathcal{S}'$ is closed, there exists $V \subseteq S$ such that $\mathcal{S}' = \mathcal{S}^{\mathcal{S}-V}$. Let $U \subseteq S$ be a bounded subset of $\mathcal{S}$ such that $\mathcal{S}' \sqsubseteq \mathcal{S}_U$. Let $W \subseteq S$, and suppose that $S \lhd' W$, i.e. $S \lhd V \cup W$. Since $U \ll S$, there exists $W_0 \in \mathsf{Fin}(W)$ such that $U \lhd' W_0$, and since $\mathcal{S}' \sqsubseteq \mathcal{S}_U$, we have $S \lhd' W_0$. Therefore $\mathcal{S}'$ is compact. □

We note some connections between the well-covered relation and the way-below relation.

Lemma 2.4.20. *Let $\mathcal{S}$ be a formal topology. For any $U, V \subseteq S$, if U is bounded, then $U \lll V \implies U \ll V$.*

Proof. Let $U, V \subseteq S$, and suppose that $U \ll S$ and $U \lll V$. Let $W \subseteq S$ such that $V \lhd W$. Then, we have $S \lhd U^* \cup V \lhd U^* \cup W$. Since U is bounded, there exists $W_0 \in \mathsf{Fin}(W)$ such that $U \lhd U^* \cup W_0$. Then, $U \lhd (U^* \cup W_0) \downarrow U \lhd (U^* \downarrow U) \cup (W_0 \downarrow U) \lhd W_0$. Therefore $U \ll V$. □

Note that a formal topology $\mathcal{S}$ is compact iff $S \ll S$.

Corollary 2.4.21. *Let $\mathcal{S}$ be a compact formal topology. Then, for any $U, V \subseteq S$, we have $U \lll V \implies U \ll V$.*

Lemma 2.4.22. *Let $\mathcal{S}$ be a regular formal topology. Then, for any $U, V \subseteq S$, we have $U \ll V \implies U \lll V$.*

Proof. Let $U, V \subseteq S$, and suppose that $U \ll V$. Since $\mathcal{S}$ is regular, we have

$$V \lhd \{a \in S \mid (\exists b \in V)\, a \lll b\}.$$

Thus, there exists $W = \{a_0, \ldots, a_{n-1}\}$ such that $U \lhd W$ and $a_i \lll V$ for each $i < n$. Then, $W \lll V$, and so $U \lll V$. □

As a corollary, we have a well-known result [20].

Proposition 2.4.23. *A compact regular formal topology is locally compact, and the relations $\ll$ and $\lll$ coincide.*

2.4.3 Examples

We give examples of regular formal topologies, compact formal topologies, and locally compact formal topologies.

Example 2.4.24 (cf. Example 2.2.20). The formal Cantor space $\mathbf{C}$ is compact and regular. In fact, $\mathbf{C}$ is an example of a *finitary* formal topology. A formal topology $\mathcal{S}$ is *finitary* if

$$a \lhd U \implies (\exists U_0 \in \mathsf{Fin}(U))\, a \lhd U_0$$

for all $a \in S$ and $U \subseteq S$. The fact that $\mathbf{C}$ is finitary can easily be proved by induction on $\lhd_{\mathbf{C}}$. Then, in particular we have

$$\langle\rangle \lhd_{\mathbf{C}} U \implies (\exists U_0 \in \mathsf{Fin}(U))\, \langle\rangle \lhd_{\mathbf{C}} U_0.$$

Since $\langle\rangle$ is the greatest element of the underlying order structure $(C, \leq)$ of $\mathbf{C}$, we see that $\mathbf{C}$ is compact.

To see that $\mathbf{C}$ is regular, we use the fact that

$$a \lhd_{\mathbf{C}} a[n]$$

for all $n \in \mathbb{N}$, where $a[n] \overset{\text{def}}{=} \{b \in C \mid |b| = |a| + n \;\&\; b \leq a\}$ (See Lemma 2.5.12). Then, it is straightforward to show that

$$C \lhd_{\mathbf{C}} a^* \cup \{a\}$$

for all $a \in C$, using the decidability of the equality on C. Hence, $\mathbf{C}$ is regular with the function $wc_{\mathbf{C}} : C \to \mathsf{Pow}(C)$ given by

$$wc_{\mathbf{C}}(a) \overset{\text{def}}{=} \{a\}\,.$$

Example 2.4.25 (cf. Example 2.2.22). The formal reals $\mathcal{R}$ is regular and locally compact.

To see that $\mathcal{R}$ is regular, we first show that the axiom (R2) is equivalent to the following axiom:

(R2') $(p, q) \lhd_{\mathcal{R}'} \left\{(r, s) \in S_{\mathcal{R}} \mid s - r = 2^{-k}\right\}$ for each $k \in \mathbb{N}$.

Let $\lhd_{\mathcal{R}'}$ be the cover generated by (R2'). To see that (R2) and (R2') are equivalent, we first assume (R2). Then, we have

$$(p, q) \lhd_{\mathcal{R}} \left\{(r, s) \in S_{\mathcal{R}} \mid s - r = (2/3)^{-n}(q - p) \;\&\; (r, s) \leq_{\mathcal{R}} (p, q)\right\}$$

for each $(p, q) \in S_{\mathcal{R}}$ and $n \in \mathbb{N}$. Then, (R2') follows by taking n large enough.

Conversely, assume (R2'). Given $p, q, r, s \in \mathbb{Q}$ such that $p < r < s < q$, take $k \in \mathbb{N}$ such that $2^{-k} < s - r$. Then,

$$(p, q) \lhd_{\mathcal{R}'} \left\{(p', q') \in S_{\mathcal{R}} \mid q' - p' = 2^{-k}\right\} \downarrow (p, q) \lhd_{\mathcal{R}'} \{(p, s), (r, q)\}\,.$$

Therefore, (R2) and (R2') are equivalent as axiom-sets.

Using (R2'), it is straightforward to show that

$$(r, s) <_{\mathcal{R}} (p, q) \implies (r, s) \lll (p, q)$$

for all $(p,q),(r,s) \in S_{\mathcal{R}}$. Hence by (R1), $\mathcal{R}$ is regular with the function $wc_{\mathcal{R}} : S_{\mathcal{R}} \to \mathsf{Pow}(S_{\mathcal{R}})$ given by

$$wc_{\mathcal{R}}((p,q)) \stackrel{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid (r,s) <_{\mathcal{R}} (p,q)\}. \tag{2.18}$$

Next, we show that $\mathcal{R}$ is locally compact. By (R1), it suffices to show that

$$(p,q) \lhd_{\mathcal{R}} U \implies (\forall (r,s) <_{\mathcal{R}} (p,q))\,(\exists U_0 \in \mathsf{Fin}(U))\,(r,s) \lhd_{\mathcal{R}} U_0$$

for all $(p,q) \in S_{\mathcal{R}}$ and $U \subseteq S_{\mathcal{R}}$. This is proved by induction on $\lhd_{\mathcal{R}}$. Given $U \subseteq S_{\mathcal{R}}$, let Φ be the predicate on $S_{\mathcal{R}}$ given by

$$\Phi(a) \stackrel{\text{def}}{\iff} (\forall b <_{\mathcal{R}} a)\,(\exists U_0 \in \mathsf{Fin}(U))\, b \lhd_{\mathcal{R}} U_0.$$

Then, we show that $a \lhd_{\mathcal{R}} U \implies \Phi(a)$ by induction, checking the conditions (ID1) – (ID3). All conditions are easy to check. For example, to see that (ID3) holds for the axiom (R2), let $p,q,r,s \in \mathbb{Q}$ such that $p < r < s < q$, and suppose that $\Phi((p,s))$ and $\Phi((r,q))$. Let $(u,v) <_{\mathcal{R}} (p,q)$. Then, either $(u,v) <_{\mathcal{R}} (p,s)$, $(u,v) <_{\mathcal{R}} (r,q)$, or $(r,s) \leq_{\mathcal{R}} (u,v)$. In the first two cases, the conclusion is immediate. For the third case, take $(u',v') \in S_{\mathcal{R}}$ such that $(u',v') <_{\mathcal{R}} (r,s)$. Then, we have $(u,v') <_{\mathcal{R}} (p,s)$ and $(u',v) <_{\mathcal{R}} (r,q)$, so there exist $U_0, U_1 \in \mathsf{Fin}(U)$ such that $(u,v') \lhd_{\mathcal{R}} U_0$ and $(u',v) \lhd_{\mathcal{R}} U_1$. Then, $(u,v) \lhd_{\mathcal{R}} U_0 \cup U_1$ by (R2), and hence, $\Phi((p,q))$.

Therefore, $(r,s) <_{\mathcal{R}} (p,q) \implies (r,s) \ll (p,q)$, and thus $\mathcal{R}$ is locally compact with the function $wc_{\mathcal{R}} : S_{\mathcal{R}} \to \mathsf{Pow}(S_{\mathcal{R}})$ given by (2.18).

Example 2.4.26 (cf. Example 2.3.15 and Example 2.3.24). The formal unit interval $\mathcal{I}[0,1]$ is compact. In formal topology, this fact was first proved by Cederquist and Negri [13]. We give a different proof based on [26, Lemma 4.8].

Since $\mathcal{I}[0,1]$ is a closed subtopology of the formal reals $\mathcal{R}$, it suffices to show that $\mathcal{I}[0,1]$ is bounded (See Proposition 2.4.19). To see this, take any $(p,q) \in S_{\mathcal{R}}$ such that $p < 0$ & $1 < q$. Recall that $\mathcal{I}[0,1]$ is overt with the positivity $\mathsf{Pos}_{\mathcal{I}[0,1]}$ given by (2.16). Then, using (R2), it is straightforward to show that

$$S_{\mathcal{R}} \lhd_{\mathcal{I}[0,1]} (p,q),$$

i.e. $\mathcal{I}[0,1] \sqsubseteq \mathcal{R}_{\{(p,q)\}}$. Now, let $U \subseteq S_{\mathcal{R}}$ such that $S_{\mathcal{R}} \lhd_{\mathcal{R}} U$. Take any $(p',q') \in S_{\mathcal{R}}$ such that $(p,q) <_{\mathcal{R}} (p',q')$. Since $(p,q) <_{\mathcal{R}} (p',q') \implies (p,q) \ll (p',q')$, there exists $U_0 \in \mathsf{Fin}(U)$ such that $(p,q) \lhd_{\mathcal{R}} U_0$. Thus, $\{(p,q)\} \ll S_{\mathcal{R}}$, and hence $\mathcal{I}[0,1]$ is bounded. Therefore $\mathcal{I}[0,1]$ is compact. Note that $\mathcal{I}[0,1]$ is also regular by Proposition 2.4.11.1, and the function $wc_{\mathcal{R}}$ associated with $\mathcal{R}$ makes $\mathcal{I}[0,1]$ regular.

2.5 Spatiality of formal topologies

Classically, the operation which assigns to each topological space the lattice of its open subsets and the operation which takes the points of each locale form an adjunction between the category of topological spaces and that of locales [34]. The adjunction restricts to an equivalence between the category of sober topological spaces and that of spatial locales. Sobriety of a space allows us to identify the space with its corresponding locale. Conversely, spatiality of a locale allows us to identify the locale with the lattice of open subsets of some topological space. Classically, the class of spatial locales contains such important locales as locally compact ones. Moreover, important examples of spaces and locales correspond to each other via this equivalence. For example, the lattice of open subsets of the reals $\mathbb{R}$ is isomorphic to the localic reals, and conversely, the set of points of the localic reals is homeomorphic to $\mathbb{R}$.

In this section, we review the constructive reformulation of the above adjunction due to Aczel [4] and the associated notion of spatiality. Contrary to the classical case, the adjunction is of little use constructively; important examples of formal topologies fail to be spatial. This is demonstrated by the well-known equivalence between spatiality of certain formal topologies and some versions of bar inductions.

2.5.1 Constructive topological spaces

In view of the fact that the class of formal points of a formal topology does not necessarily form a set, the notion of concrete space is too restrictive (cf. Section 1.2.2). The desire to set up an adjunction between the category of concrete spaces and that of formal topologies leads to the following notion of topological space [4].

Definition 2.5.1. A *constructive topological space* (or ct-space) is a triple $X = (X, \Vdash, S)$, where X is a class, S is a set, and $\Vdash$ is a relation between X and S such that

1. $\mathsf{ext}\, a \cap \mathsf{ext}\, b \subseteq \mathsf{ext}(a \downarrow b)$,
2. $X = \mathsf{ext}\, S$,

where

$$\begin{aligned} \mathsf{ext}\, a &\stackrel{\text{def}}{=} \{x \in X \mid x \Vdash a\}\,, \\ a \downarrow b &\stackrel{\text{def}}{=} \{c \in S \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}\,, \\ \mathsf{ext}\, U &\stackrel{\text{def}}{=} \bigcup_{a \in U} \mathsf{ext}\, a \end{aligned}$$

for all $a, b \in S$ and $U \subseteq S$. Moreover, we require that the classes

$$\begin{aligned} \Diamond x &\stackrel{\text{def}}{=} \{a \in S \mid x \Vdash a\}\,, \\ &\{y \in X \mid \Diamond y = \Diamond x\} \end{aligned}$$

are sets for each $x \in X$. In the following, we simply write $X, X', \dots$ for ct-spaces $(X, \Vdash, S), (X', \Vdash', S'), \dots$.

A function $f : X \to X'$ between the underlying classes of ct-spaces is *continuous* if

$$(\forall x \in X)\, (\forall b \in \Diamond' f(x))\, (\exists a \in \Diamond x)\, \mathsf{ext}\, a \subseteq f^{-1}\left[\mathsf{ext}'\, b\right].$$

A ct-space X is *standard* if the class

$$\mathcal{A}_X\, U \stackrel{\text{def}}{=} \{a \in S \mid \mathsf{ext}\, a \subseteq \mathsf{ext}\, U\}$$

is a set for each $U \subseteq S$. A continuous function $f : X \to X'$ between ct-spaces is *standard* if the class

$$\left\{a \in S \mid \mathsf{ext}\, a \subseteq f^{-1}\left[\mathsf{ext}'\, b\right]\right\}$$

is a set of each $b \in S'$.

It is straightforward to show that the standard ct-spaces and standard continuous functions between them form a category, which we denote by **Top**.

2.5.2 The adjunction between Top and FTop

We recall the constructive version of the classical adjunction between the category of topological spaces and that of locales (See [4] for detailed proofs).

From Top to FTop

Given a standard ct-space $X = (X, \Vdash, S)$, let $\Omega(X)$ be the triple $(S, \lhd_X, \leq_X)$ consisting of the preorder $\leq_X$ on S and the relation $\lhd_X \subseteq S \times \mathsf{Pow}(S)$ defined by

$$a \leq_X b \stackrel{\text{def}}{\iff} \mathsf{ext}\, a \subseteq \mathsf{ext}\, b,$$
$$a \lhd_X U \stackrel{\text{def}}{\iff} \mathsf{ext}\, a \subseteq \mathsf{ext}\, U.$$

It is straightforward to show that $\Omega(X)$ is a formal topology.

Moreover, for any standard continuous function $f : X \to X'$ between standard ct-spaces, the relation $r_f \subseteq S \times S'$ given by

$$a\ r_f\ b \stackrel{\text{def}}{\iff} \mathsf{ext}\, a \subseteq f^{-1}\left[\mathsf{ext}'\, b\right]$$

is a formal topology map from $\Omega(X)$ to $\Omega(X')$.

It is straightforward to show that the assignments $X \mapsto \Omega(X)$ and $f \mapsto r_f$ define a functor from **Top** to **FTop**, which we denote by $\Omega : \mathbf{Top} \to \mathbf{FTop}$.

From FTop to Top

Given any formal topology $\mathcal{S} = (S, \lhd, \leq)$, the triple $(\mathcal{P}t(\mathcal{S}), \Vdash_{\mathcal{S}}, S)$ defined by

$$\alpha \Vdash_{\mathcal{S}} a \stackrel{\text{def}}{\iff} a \in \alpha$$

is a ct-space. This follows from the properties (P1) – (P3) of formal points. Moreover, for any formal topology map $r : \mathcal{S} \to \mathcal{S}'$, the function $\mathcal{P}t(r) : \mathcal{P}t(\mathcal{S}) \to \mathcal{P}t(\mathcal{S}')$ given by

$$\mathcal{P}t(r)(\alpha) \stackrel{\text{def}}{=} r\alpha = \{b \in S' \mid (\exists a \in \alpha)\, a\ r\ b\}$$

is a continuous function from $\mathcal{P}t(\mathcal{S}) \to \mathcal{P}t(\mathcal{S}')$. Note that $\mathcal{P}t(r)$ maps a formal point of $\mathcal{S}$ to a formal point of $\mathcal{S}'$ (cf. (2.6)).

Note that the ct-space $(\mathcal{P}t(\mathcal{S}), \Vdash_{\mathcal{S}}, S)$ may not be standard. This is a motivation for introducing the following somewhat ad hoc notion [4, Section 5].

Definition 2.5.2. A formal topology $\mathcal{S}$ is *standard* if the ct-space $\mathcal{P}t(\mathcal{S})$ is standard.

Let $\mathbf{FTop}_{\mathrm{STD}}$ be the full subcategory of $\mathbf{FTop}$ consisting of standard formal topologies. Then, the assignments $(S, \lhd, \leq) \mapsto (\mathcal{P}t(\mathcal{S}), \Vdash_{\mathcal{S}}, S)$ and $r \mapsto \mathcal{P}t(r)$ restrict to a functor from $\mathbf{FTop}_{\mathrm{STD}}$ to $\mathbf{Top}$, which we denote by $\mathcal{P}t : \mathbf{FTop}_{\mathrm{STD}} \to \mathbf{Top}$. Note that a formal topology map between standard formal topologies gives rise to a standard continuous function via $\mathcal{P}t$.

Lemma 2.5.3. *If $X = (X, \Vdash, S)$ is a standard ct-space, then $\Omega(X)$ is standard, i.e. $\mathcal{P}t(\Omega(X))$ is a standard ct-space.*

Proof. Write $\mathcal{P}t(\Omega(X)) = (X', \Vdash', S)$. Then, we can show that $\mathsf{ext}\, a \subseteq \mathsf{ext}\, U \iff \mathsf{ext}'\, a \subseteq \mathsf{ext}'\, U$ for all $a \in S$ and $U \subseteq S$, using the fact that $\Diamond x \in \mathcal{P}t(\Omega(X))$ for all $x \in X$ in case X is standard. Hence, if X is standard, then $\mathcal{P}t(\Omega(X))$ is standard. □

Hence, the functor $\Omega : \mathbf{Top} \to \mathbf{FTop}$ restricts to a functor $\Omega : \mathbf{Top} \to \mathbf{FTop}_{\mathrm{STD}}$.

Theorem 2.5.4 (Aczel [4, Theorem 21])**.** *The functor $\Omega : \mathbf{Top} \to \mathbf{FTop}_{\mathrm{STD}}$ is left adjoin to $\mathcal{P}t : \mathbf{FTop}_{\mathrm{STD}} \to \mathbf{Top}$.*

The unit $\eta_X : X \to \mathcal{P}t(\Omega(X))$ of the adjunction is given by

$$\eta_X(x) \stackrel{\text{def}}{=} \Diamond x$$

for each standard ct-space X and $x \in X$. It satisfies the following universal property. Let X be a standard ct-space. Then, for any standard formal topology $\mathcal{S}$ and standard continuous function $f : X \to \mathcal{P}t(\mathcal{S})$, there exists a unique formal topology map $\hat{f} : \Omega(X) \to \mathcal{S}$ given by

$$a \; \hat{f} \; b \stackrel{\text{def}}{\iff} \mathsf{ext}_X\, a \subseteq f^{-1}\left[\mathsf{ext}_{\mathcal{P}t(\mathcal{S})}\, b\right]$$

such that $\mathcal{P}t(\hat{f}) \circ \eta_X = f$.

Then, the counit $\varepsilon_{\mathcal{S}} : \Omega(\mathcal{P}t(\mathcal{S})) \to \mathcal{S}$ at a standard formal topology $\mathcal{S}$ can be computed by putting $X = \mathcal{P}t(\mathcal{S})$ and $f = id_{\mathcal{P}t(\mathcal{S})}$. Explicitly, $\varepsilon_{\mathcal{S}}$ is given by

$$a \; \varepsilon_{\mathcal{S}} \; b \stackrel{\text{def}}{\iff} a \lhd_{\mathcal{P}t(\mathcal{S})} b.$$

Note that $\varepsilon_{\mathcal{S}} : \Omega(\mathcal{P}t(\mathcal{S})) \to \mathcal{S}$ is an embedding, and the image of $\Omega(\mathcal{P}t(\mathcal{S}))$ under $\varepsilon_{\mathcal{S}}$ is a formal topology $(S, \lhd_{\mathcal{P}t(\mathcal{S})}, \leq)$.

2.5.3 Spatiality

A formal topology is spatial if its cover is determined by some standard ct-space through the adjunction $\Omega \vdash \mathcal{P}t$. More precise definition is the following.

Definition 2.5.5. A standard formal topology $\mathcal{S}$ is *spatial* if the counit $\varepsilon_{\mathcal{S}}$ of the adjunction $\Omega \vdash \mathcal{P}t$ is an isomorphism, i.e.

$$a \lhd_{\mathcal{P}t(\mathcal{S})} U \implies a \lhd U$$

for all $a \in S$ and $U \subseteq S$.

By the definition of ct-space $\mathcal{P}t(\mathcal{S})$, $\mathcal{S}$ is spatial iff for all $a \in S$ and $U \subseteq S$

$$[(\forall \alpha \in \mathcal{P}t(\mathcal{S}))\, a \in \alpha \to \alpha \between U] \implies a \lhd U.$$

Proposition 2.5.6. *Let $r : \mathcal{S} \to \mathcal{S}'$ be a surjective formal topology map between standard formal topologies. Then, if $\mathcal{S}$ is spatial, then so is $\mathcal{S}'$.*

Proof. Suppose that $\mathcal{S}$ is spatial. Let $b \in S'$ and $U \subseteq S'$ such that $b \lhd_{\mathcal{P}t(\mathcal{S}')} U$. Since r is surjective and $\mathcal{S}$ is spatial, it suffices to show that $r^{-}b \lhd_{\mathcal{P}t(\mathcal{S})} r^{-}U$. Let $\alpha \in \mathcal{P}t(\mathcal{S})$ such that $\alpha \in \mathsf{ext}_{\mathcal{P}t(\mathcal{S})}\, r^{-}b$. Then, $\mathcal{P}t(r)(\alpha) \in \mathsf{ext}_{\mathcal{P}t(\mathcal{S}')}\, b$, and thus $\mathcal{P}t(r)(\alpha) \in \mathsf{ext}_{\mathcal{P}t(\mathcal{S}')}\, U$, i.e. $\alpha \in \mathsf{ext}_{\mathcal{P}t(\mathcal{S})}\, r^{-}U$. Hence, $r^{-}b \lhd_{\mathcal{P}t(\mathcal{S})} r^{-}U$, as required. □

We give some examples of formal topologies which cannot be spatial constructively. For each example, the statement that the formal topology in question is spatial is equivalent to some version of bar induction. These results are well-known in local theory [26]. In formal topology, spatiality of the formal Cantor space and the formal Baire space were studied by Gambino and Schuster [29], and that of the formal reals was studied by Diener and Hedin [25].

Formal Baire space

We show that the spatiality of the formal Baire space $\mathbf{B}$ is equivalent to the monotone bar induction (See Example 2.2.21).

The following is intuitively clear.

Lemma 2.5.7. *For any $u \in B$, the open subtopology $\mathbf{B}_u$ of $\mathbf{B}$ determined by $\{u\}$ is isomorphic to $\mathbf{B}$.*

Proof. Let $u \in B$. Define a relation $r \subseteq B \times B$ by

$$a \; r \; b \overset{\text{def}}{\iff} a = u * b.$$

We show that r is a formal topology map $r : \mathbf{B}_u \to \mathbf{B}$.

(FTMi1): By the definition of the cover of $\mathbf{B}_u$, the condition is equivalent to $u \lhd_{\mathbf{B}} r^{-}B$, which is obvious.

(FTMi2): Let $a, b, c \in B$ and suppose that $a \in r^- b \downarrow r^- c$. Then, $a \leq u * b$ and $a \leq u * c$. We can assume without loss of generality that $b \preccurlyeq c$. Then, $a \lhd_{\mathbf{B}_u} u * c \in r^-(b \downarrow c)$.

(FTMi3): Let $b \in B$. We must show that $r^- b \lhd_{\mathbf{B}_u} r^- \{b * \langle n \rangle \mid n \in \mathbb{N}\}$. Let $a \in r^- b$. Then, $a = u * b$, so that $a \lhd_{\mathbf{B}_u} \{a * \langle n \rangle \mid n \in \mathbb{N}\} \subseteq r^- \{b * \langle n \rangle \mid n \in \mathbb{N}\}$.

(FTMi4): Let $b, c \in B$ such that $b \leq c$, and suppose that $a \in r^- b$. Then, $a = u * b$, so that there exists $a' \geq a$ such that $a' = u * c$. Hence, $a \lhd_{\mathbf{B}_u} a' \in r^- c$.

Therefore r is a formal topology map. The map r is an embedding. Indeed, let $a \in B$, and $b \in a \downarrow u$. Then, there exists $c \in B$ such that $b = u * c$, and so $b \; r \; c$. Since $r^- c = \{b\}$, we have $c \in r^{-*} \mathcal{A}_{\mathbf{B}_u} \{a\}$. Hence $a \lhd_{\mathbf{B}_u} r^- r^{-*} \mathcal{A}_{\mathbf{B}_u} \{a\}$. Therefore, r is an embedding.

It remains to be shown that r is surjective, i.e.

$$r^- b \lhd_{\mathbf{B}_u} r^- U \implies b \lhd_{\mathbf{B}} U$$

for all $b \in B$ and $U \subseteq B$, which is equivalent to

$$a \lhd_{\mathbf{B}_u} r^- U \implies [(\forall b \in B)\, a = u * b \to b \lhd_{\mathbf{B}} U]$$

for all $a \in B$ and $U \subseteq B$. Define a predicate Φ on B by

$$\Phi(a) \overset{\text{def}}{\iff} (\forall a' \leq a)\, [(\forall b \in B)\, a' = u * b \to b \lhd_{\mathbf{B}} U] \,.$$

We show that $a \lhd_{\mathbf{B}_u} r^- U \implies \Phi(a)$ by induction on $\lhd_{\mathbf{B}_u}$. This suffices to show that r is surjective.

(ID1): Suppose that $a \in r^- U$. Let $a' \leq a$ and let $b \in B$ such that $a' = u * b$. Since $a \in r^- U$, there exists $b' \in U$ such that $a = u * b'$. Then, $b \leq b'$, and hence, $b \lhd_{\mathbf{B}} U$. Therefore $\Phi(a)$

(ID2): This is trivial.

(ID3): We have to check the axiom (2.10) and the extra axiom (2.12) for $\mathbf{B}_u$ in the localised form.

(2.10): Suppose that $\Phi(a * \langle n \rangle)$ for all $n \in \mathbb{N}$. Let $a' \leq a$ and let $b \in B$ such that $a' = u * b$. Then, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $a' * \langle n \rangle \leq a * \langle m \rangle$. Thus, $b * \langle n \rangle \lhd_{\mathbf{B}} U$ for all $n \in \mathbb{N}$, and hence $b \lhd_{\mathbf{B}} U$. Therefore, $\Phi(a)$

(2.12): Let $a \leq c$, and suppose that $\Phi(c')$ for all $c' \in a \downarrow u$. Let $a' \leq a$ and $b \in B$ such that $a' = u * b$. Then, $\Phi(a')$, and so $b \lhd_{\mathbf{B}} U$. Hence, $\Phi(a)$.

This completes the inductive proof. □

Definition 2.5.8. *The monotone bar induction* is the statement: for any $U \subseteq \mathbb{N}^*$, if

1. $a \in U \;\&\; b \leq a \implies b \in U$,
2. $[(\forall n \in \mathbb{N})\, a * \langle n \rangle \in U] \implies a \in U$,
3. $\left(\forall \alpha \in \mathbb{N}^{\mathbb{N}}\right) (\exists n \in \mathbb{N})\, \overline{\alpha} n \in U$,

then $\langle\rangle \in U$.

Before proving Theorem 2.5.9, we make explicit the relation between the ct-space of the formal points of $\mathbf{B}$ and the usual point-set notion of Baire space. The Baire space can be defined as a ct-space $\mathbb{N}^{\mathbb{N}} = (\mathbb{N}^{\mathbb{N}}, \Vdash, B)$ given by

$$\alpha \Vdash a \overset{\text{def}}{\iff} \overline{\alpha}|a| = a,$$

where B is the base of $\mathbf{B}$, i.e. $\mathbb{N}^*$. Let $\mathcal{P}t(\mathbf{B}) = (\mathcal{P}t(\mathbf{B}), \Vdash_{\mathbf{B}}, B)$ be the ct-space of $\mathbf{B}$ determined by the functor $\mathcal{P}t : \mathbf{FTop}_{\mathrm{STD}} \to \mathbf{Top}$. By Corollary 2.2.13, a point $\alpha \in \mathcal{P}t(\mathbf{B})$ is an inhabited subset $\alpha \subseteq B$ such that

1. $a, b \in \alpha \implies a \preccurlyeq b \vee b \preccurlyeq a$,
2. $a \preccurlyeq b \;\&\; b \in \alpha \implies a \in \alpha$,
3. $a \in \alpha \implies (\exists n \in \mathbb{N})\, a * \langle n \rangle \in \alpha$.

These conditions imply that $(\forall n \in \mathbb{N})\,(\exists! a \in \alpha)\,|a| = n$. Thus, there exists a function $f : \mathbb{N} \to \alpha$ such that $(\forall n \in \mathbb{N})\, f(n) \in \alpha \;\&\; |f(n)| = n$. Then, f determines a sequence $\beta \in \mathbb{N}^{\mathbb{N}}$ by

$$\beta(n) \overset{\text{def}}{=} f(n+1)(n).$$

Conversely, given any sequence $\beta \in \mathbb{N}^{\mathbb{N}}$, the subset $\alpha \subseteq B$ given by

$$\alpha \overset{\text{def}}{=} \left\{ \overline{\beta}n \in B \mid n \in \mathbb{N} \right\}$$

is a formal point of $\mathbf{B}$. The two mappings can easily be shown to be continuous and inverse of each other. Hence, they are homeomorphisms between $\mathcal{P}t(\mathbf{B})$ and $\mathbb{N}^{\mathbb{N}}$.

Theorem 2.5.9. *The following are equivalent:*

1. *The monotone bar induction;*
2. $B \lhd_{\mathcal{P}t(\mathbf{B})} U \implies B \lhd_{\mathbf{B}} U$ *for any* $U \subseteq B$*;*
3. $\mathbf{B}$ *is spatial.*

Proof. ($1 \Leftrightarrow 2$): The conditions 1 and 2 in Definition 2.5.8 are equivalent to saying that U is a saturated subset of $\mathbf{B}$. The condition $B \lhd_{\mathcal{P}t(\mathbf{B})} U$ is equivalent to the condition 3 in Definition 2.5.8 according to the homeomorphism described above, and $B \lhd_{\mathcal{P}t(\mathbf{B})} U$ is equivalent to $B \lhd_{\mathcal{P}t(\mathbf{B})} \mathcal{A}_{\mathbf{B}} U$. Hence, 2 is equivalent to the statement of the monotone bar induction.

($2 \Leftrightarrow 3$): $\mathbf{B}$ is spatial iff for any $U \subseteq B$

$$a \lhd_{\mathcal{P}t(\mathbf{B})} U \implies a \lhd_{\mathbf{B}} U$$

for all $a \in B$. In particular this holds for $a = \langle\rangle$. Since $B \lhd_B \langle\rangle$, we have

$$B \lhd_{\mathcal{P}t(\mathbf{B})} U \implies B \lhd_{\mathbf{B}} U.$$

Hence, 3 implies 2.

Conversely, assume that 2 holds, and let $a \in B$. Let $r : \mathbf{B} \to \mathbf{B}_a$ be the isomorphism which exists by Lemma 2.5.7. Let $U \subseteq B$, and suppose that $a \lhd_{\mathcal{P}t(\mathbf{B})} U$, i.e. $\langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}_a)} U$ by the characterisation of $\mathcal{P}t(\mathbf{B}_a)$. Since we have $r = \Omega(\mathcal{P}t(r))$ as morphisms from $\Omega(\mathcal{P}t(\mathbf{B}))$ to $\Omega(\mathcal{P}t(\mathbf{B}_a))$, we have $\langle\rangle \lhd_{\mathcal{P}t(\mathbf{B})} r^{-}U$. Thus, $\langle\rangle \lhd_{\mathbf{B}} r^{-}U$. Since $r^{-}\langle\rangle \lhd_{\mathbf{B}} \langle\rangle$, we have $\langle\rangle \lhd_{\mathbf{B}_a} U$ because r is surjective. Thus, $a \lhd_{\mathbf{B}} U$, and hence 3 holds. □

Formal Cantor space

We show that the spatiality of the formal Cantor space is equivalent to the Fan theorem.

Definition 2.5.10. Let $\mathbf{B}$ be the formal Baire space. A *fan* in $\mathbf{B}$ is an inhabited decidable subset $T \subseteq B$ such that

1. $l \leq l' \mathbin{\&} l \in T \implies l' \in T$,
2. $l \in T \implies (\exists n \in \mathbb{N})\, l * \langle n \rangle \in T$,
3. $(\forall l \in T)\, (\exists m \in \mathbb{N})\, (\forall n \in \mathbb{N})\, l * \langle n \rangle \in T \implies n \leq m$

for all $l, l' \in B$. In terms of formal topology, the conditions 1 and 2 are equivalent to saying that T is a splitting subset of $\mathbf{B}$ (See Proposition 2.2.15)[6]. Thus, a fan in $\mathbf{B}$ is an inhabited decidable splitting subset of $\mathbf{B}$ which satisfies the condition 3, which says that T is finitely branching as a tree. In particular, the base $C = \{0,1\}^*$ of the formal Cantor space is a fan in $\mathbf{B}$.

Definition 2.5.11. The *Fan theorem* is a statement: for any fan T in $\mathbf{B}$ and for any $U \subseteq B$,

$$\langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}^T)} U \implies (\exists U_0 \in \mathsf{Fin}(U))\, \langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}^T)} U_0,$$

where $\mathbf{B}^T = (B, \lhd^T, \leq)$ is the overt weakly closed subtopology of $\mathbf{B}$ determined by T.

Note that a formal point of the overt weakly closed subtopology of $\mathbf{B}$ determined by a fan corresponds to a path in the fan. The proof is similar to that of the homeomorphism $\mathcal{P}t(\mathbf{B}) \cong \mathbb{N}^{\mathbb{N}}$. Thus, the above statement is equivalent to the usual statement of the Fan theorem [54, Chapter 4, Section 7].

Let T be a splitting subset of $\mathbf{B}$. For each $a \in B$ and $n \in \mathbb{N}$, define

$$a[n] \overset{\text{def}}{=} \{b \in T \mid |b| = |a| + n \mathbin{\&} b \leq a\}.$$

Lemma 2.5.12. *Let T be a splitting subset of $\mathbf{B}$, and let $\mathbf{B}^T$ be the overt weakly closed subtopology of $\mathbf{B}$ determined by T. Then*

$$a \lhd^T a[n]$$

for all $a \in B$ and $n \in \mathbb{N}$.

[6] An inhabited decidable splitting subset of $\mathbf{B}$ is usually called a *spread* [54, Chapter 4].

Proof. By induction on $\mathbb{N}$. The base case (i.e. $n = 0$) follows from the axiom of $\mathbf{B}^T$. Let $k \in \mathbb{N}$, and suppose that $a \lhd^T a[k]$. For each $b \in a[k]$, we have

$$b \lhd^T \{b * \langle n \rangle \mid n \in \mathbb{N}\} \cap T \lhd^T a[k+1].$$

Hence, $a \lhd^T a[k+1]$. □

In the following lemma, note that if T is a fan, then each $a \in T$ can be extended to a path in T in a canonical way by taking the least successor at each node.

Lemma 2.5.13. *Let T be a fan in $\mathbf{B}$. Then, for any $a \in B$ and $U \in \mathsf{Fin}(B)$, we have*

$$a \lhd_{\mathcal{P}t(\mathbf{B}^T)} U \iff a \lhd^T U.$$

Note that U is finitely enumerable.

Proof. Suppose that $a \lhd_{\mathcal{P}t(\mathbf{B}^T)} U$. Let $n = \max\{|b| \mid b \in U\}$, and suppose that $a \in T$. We have two cases. If $|a| \geq n$, then we extend a to a path $\alpha \in \mathbb{N}^{\mathbb{N}}$ in T, which we identify with a formal point of $\mathbf{B}^T$. By the assumption, there exists $b \in U$ such that $b \in \alpha$. Since $|b| \leq n$, we have $a \leq b$. Hence, $a \lhd^T U$. If $|a| < n$, then $a \lhd^T a[n - |a|]$. For each $b \in a[n - |a|]$, by the similar argument as in the first case, we have $b \lhd^T U$. Hence $a \lhd^T \{a\} \cap T \lhd^T U$.

The converse is trivial since $\Omega(\mathcal{P}t(\mathbf{B}^T))$ is a subtopology of $\mathbf{B}^T$ (See the remark at the end of Section 2.5.2). □

Lemma 2.5.14. *Let T be a fan in $\mathbf{B}$. Then, for any $a \in B$ and $U \subseteq B$, we have*

$$a \lhd^T U \implies (\exists U_0 \in \mathsf{Fin}(U))\, a \lhd^T U_0.$$

Proof. By induction on $\lhd^T$. The only non-trivial case is the condition (ID3) for the axiom of the formal Baire space, where we use the fact that $a \lhd^T a[1]$ and $a[1]$ is finitely enumerable for each $a \in B$ because T is decidable. □

The proof of the following theorem requires some results from Chapter 4.

Theorem 2.5.15. *The following are equivalent:*

1. *The Fan theorem;*
2. $\langle\rangle \lhd_{\mathcal{P}t(\mathbf{C})} U \implies (\exists U_0 \in \mathsf{Fin}(U))\, \langle\rangle \lhd_{\mathcal{P}t(\mathbf{C})} U_0$*;*
3. $C \lhd_{\mathcal{P}t(\mathbf{C})} U \implies C \lhd_{\mathbf{C}} U$*;*
4. $\mathbf{C}$ *is spatial.*

where $\mathbf{C}$ is the formal Cantor space.

Proof. ($1 \Rightarrow 2$): 2 is an instance of the Fan theorem applied to the formal Cantor space.

($2 \Leftrightarrow 3$): By Lemma 2.5.13 and Lemma 2.5.14.

($3 \Rightarrow 4$): Similar to the case ($2 \Leftrightarrow 3$) in the proof of Theorem 2.5.9.

($4 \Rightarrow 1$): Suppose that $\mathbf{C}$ is spatial. Let T be a fan in $\mathbf{B}$. By Lemma 4.4.6 and Lemma 4.4.7, the formal topology $\mathbf{B}^T$ is a surjective image of $\mathbf{C}$. Thus, $\mathbf{B}^T$ is spatial by Lemma 2.5.6. In particular, we have

$$\langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}^T)} U \implies \langle\rangle \lhd^T U$$

which is equivalent to

$$\langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}^T)} U \implies (\exists U_0 \in \mathsf{Fin}(U))\, \langle\rangle \lhd_{\mathcal{P}t(\mathbf{B}^T)} U_0$$

by Lemma 2.5.13 and Lemma 2.5.14. □

The second statement of Theorem 2.5.15 expresses the compactness of the point-set Cantor space.

Formal Reals

We show that the spatiality of the formal reals $\mathcal{R}$ is equivalent to the compactness of the unit interval $[0,1]$ of the reals $\mathbb{R}$ (cf. Example 2.2.22).

Theorem 2.5.16. *The following are equivalent:*

1. $[0,1]$ *is topologically compact (Heine-Borel);*
2. *The formal unit interval* $\mathcal{I}[0,1]$ *is spatial;*
3. *The formal reals* $\mathcal{R}$ *is spatial.*

Proof. We use some notations introduced in Example 2.2.22 and Example 2.3.15.

($1 \Rightarrow 2$): Suppose that $[0,1]$ is compact. Let $(p,q) \in S_{\mathcal{R}}$ and $U \subseteq S_{\mathcal{R}}$. Suppose that $(p,q) \lhd_{\mathcal{P}t(\mathcal{I}[0,1])} U$, i.e. $(p,q) \cap [0,1] \subseteq \bigcup_{(r,s)\in U}(r,s)$ as intervals. We must show that $(p,q) \lhd_{\mathcal{I}[0,1]} U$. Choose $(u,v) \in S_{\mathcal{R}}$ such that $u < 0$ and $1 < v$ and $(p,q) \cap (u,v) \subseteq \bigcup_{(r,s)\in U}(r,s)$. Suppose that $(p,q) \in \mathsf{Pos}_{\mathcal{I}[0,1]}$, where $\mathsf{Pos}_{\mathcal{I}[0,1]}$ is the positivity of $\mathcal{I}[0,1]$. Then, $(p,q) \between [0,1]$. Let $(p',q') = (\max\{p,u\}, \min\{q,v\})$. Then, $(p',q') \subseteq [p',q'] \subseteq \bigcup_{(r,s)\in U}(r,s)$. Since $[p',q']$ is compact, there exists $U_0 \in \mathsf{Fin}(U)$ such that $[p',q'] \subseteq \bigcup_{(r,s)\in U_0}(r,s)$. Since U_0 is finite, we have $(p',q') \lhd_{\mathcal{I}[0,1]} U_0$ by (R2). Since $(p,q) \lhd_{\mathcal{I}[0,1]} (p',q')$ by (R2), we have $(p,q) \lhd_{\mathcal{I}[0,1]} U_0 \subseteq U$, as required.

($2 \Rightarrow 3$): Suppose that $\mathcal{I}[0,1]$ is spatial. Let $(p,q) \in S_{\mathcal{R}}$ and $U \subseteq S_{\mathcal{R}}$. Suppose that $(p,q) \lhd_{\mathcal{P}t(\mathcal{R})} U$, i.e. $(p,q) \subseteq \bigcup_{(r,s)\in U}(r,s)$ as intervals. Then $[p,q] \subseteq \bigcup_{(r,s)\in U}(r,s)$, i.e. $S_{\mathcal{R}} \lhd_{\mathcal{P}t(\mathcal{I}[p,q])} U$. Here, $\mathcal{I}[p,q]$ is the closed subtopology of $\mathcal{R}$ determined by the subset

$$\{(r,s) \in S_{\mathcal{R}} \mid s \leq p \vee q \leq r\},$$

which is easily shown to be isomorphic to $\mathcal{I}[0,1]$. Hence, $\mathcal{I}[p,q]$ is spatial, so that $S_{\mathcal{R}} \lhd_{\mathcal{I}[p,q]} U$. Since $\mathcal{I}[p,q]$ is the closure of the open subtopology $\mathcal{R}_{\{(p,q)\}}$ of $\mathcal{R}$ determined by $\{(p,q)\}$, we have $S_{\mathcal{R}} \lhd_{\{(p,q)\}} U$, i.e. $(p,q) \lhd_{\mathcal{R}} U$. Therefore, $\mathcal{R}$ is spatial.

($3 \Rightarrow 1$): Suppose that $\mathcal{R}$ is spatial. Let $U \subseteq S_{\mathcal{R}}$, and suppose that $[0,1] \subseteq \bigcup_{(r,s)\in U}(r,s)$. Then, there exist $(p,q),(p',q') \in S_{\mathcal{R}}$ such that $(0,1) <_{\mathcal{R}} (p',q') <_{\mathcal{R}} (p,q) \subseteq \bigcup_{(r,s)\in U}(r,s)$. Since $\mathcal{R}$ is spatial, we have $(p,q) \lhd_{\mathcal{R}} U$. Since $(p',q') <_{\mathcal{R}} (p,q)$ implies $(p',q') \ll (p,q)$, there exists $U_0 \in \mathsf{Fin}(U)$ such that $(p',q') \lhd_{\mathcal{R}} U_0$. Hence, $[0,1] \subseteq (p',q') \subseteq \bigcup_{(r,s)\in U_0}(r,s)$. Therefore $[0,1]$ is compact. □

The compactness of $[0,1]$ in a weaker form

$$[0,1] \subseteq \bigcup_{(p,q)\in U}(p,q) \implies (\exists U_0 \in \mathsf{Fin}(U))\, [0,1] \subseteq \bigcup_{(p,q)\in U_0}(p,q)$$

where $U \subseteq S_{\mathcal{R}}$ is enumerable is equivalent to the Fan theorem for decidable bars [39].

Since the Fan theorem for decidable bars is false under Church's Thesis [54], the formal Baire space, the formal Cantor space and the formal reals cannot be spatial constructively.

Chapter 3

Functorial Embeddings of Uniform Spaces

In this chapter, we introduce the main tool which connects the notion of compactness for Bishop metric spaces and compactness for formal topologies. The essential tool was introduced by Palmgren [48], who used the notion of localic completion of a metric space due to Vickers [58] to construct an embedding from the category of locally compact metric spaces into that of formal topologies. The embedding has important properties that it sends a locally compact metric space to a locally compact formal topology and that a metric space is totally bounded iff its localic completion is compact. Hence, the embedding shows that the notion of morphism between locally compact metric spaces and that of formal topology map are compatible, and moreover that the notions of compactness and local compactness for metric spaces and the corresponding notions for formal topologies are compatible. The main goal of this thesis is to characterise the image of this embedding in terms of formal topology, which is the topic of Chapter 4 and Chapter 5.

In this chapter, rather than just describing Palmgren's results, we study how much of his results can be extended to a wider class of point-set spaces where the notion of uniform continuity is still meaningful. In this thesis, we consider two extensions: one to the class of uniform spaces defined by sets of pseudometrics and the other to the class of uniform spaces defined by covering uniformities. The two notions of uniform space are classically equivalent; constructively, however, the latter notion is more general than the former.

The notion of uniform space defined by a set of pseudometrics is a natural generalisation of that of metric space, and this notion of uniform space seems to be favoured by Bishop [8, Chapter 4, Problems 17]. We introduce the notion of localic completion of a uniform space which extends the corresponding notion for metric spaces. Then, we present straightforward extensions of most of the results by Palmgren [48] to the setting of uniform spaces. In particular, we show that the construction of a localic completion extends to a full and faithful functor from the category of locally compact uniform spaces into that of locally compact regular formal topologies, and that a uniform space is totally bounded iff its localic completion is compact. We also show that the localic completion preserves countable products of inhabited compact uniform spaces, extending the case of binary products known for metric spaces. This result is crucial in obtaining the point-free characterisations of compact metric spaces and Bishop locally compact metric spaces in Chapter 4 and Chapter 5 respectively.

The notion of uniform space defined by a covering uniformity is more general than that of uniform space defined by a set of pseudometrics. For the covering uniformities, we define the notion of covering completion, which is analogous to that of localic completion. Generality of covering uniformities, however, makes it hard to establish some of the results obtained for metric spaces. In particular, we have to explicitly work in CZF to show that the covering completion of a uniform space defined by a covering uniformity is constructively well-defined. Nevertheless, we are able to extend the construction of a covering completion to a full and faithful functor from the category of compact uniform spaces into that of compact 2-regular formal topologies. Moreover, we show that a uniform space is totally bounded iff its covering completion is compact.

Lastly, we show that the notion of covering completion is equivalent to that of localic completion for the class of covering uniformities determined by sets of pseudometrics. The equivalence extends to a natural isomorphism between the construction of a covering completion and that of a localic completion when we regard them as functors from the category of compact uniform spaces defined by sets of pseudometrics to that of formal topologies.

3.1 Uniform spaces by sets of pseudometrics

In this section, we extend the notion of localic completion of a metric space [58] and the embedding of the category of locally compact metric spaces into that of formal topologies [48] to the setting of uniform spaces defined by sets of pseudometrics.

Remark 3.1.1. In this section, we define a pseudometric on a set X to be a function $d : X \times X \to \mathbb{R}^{\geq 0}$ which takes values in the non-negative Dedekind reals $\mathbb{R}^{\geq 0}$. The non-negative Dedekind reals consist of subsets $U \subseteq \mathbb{Q}^{>0}$ of positive rationals such that

(D1) $(\exists q \in \mathbb{Q}^{>0})\, q \in U$,

(D2) $q \in U \iff (\exists p < q)\, p \in U$,

(D3) $(\forall p, q \in \mathbb{Q}^{>0})\, p < q \implies p \notin U \vee q \in U$.

The non-negative rationals $\mathbb{Q}^{\geq 0}$ are embedded into $\mathbb{R}^{\geq 0}$ by

$$q \mapsto q_* \overset{\text{def}}{=} \left\{p \in \mathbb{Q}^{>0} \mid q < p\right\}.$$

The order and the addition on $\mathbb{R}^{\geq 0}$ are defined by

$$\begin{aligned} U + V &\overset{\text{def}}{=} \{q + p \mid q \in U \ \&\ p \in V\}, \\ U \leq V &\overset{\text{def}}{=} V \subseteq U, \\ U < V &\overset{\text{def}}{=} \left(\exists q \in \mathbb{Q}^{>0}\right) U + q_* \leq V. \end{aligned}$$

Note that we have

$$q \in U \iff U < q_*$$

for any $U \in \mathbb{R}^{\geq 0}$ and $q \in \mathbb{Q}^{>0}$. Using the Countable Choice, one can show that $\mathbb{R}^{\geq 0}$ is order isomorphic to the non-negative Cauchy reals.

3.1.1 Uniform spaces

Among the several notions of uniform space, the notion of uniform space defined by a set of pseudometrics is the most natural generalisation of that of metric space.

Definition 3.1.2. A *uniform space* is a pair $X = (X, M)$, where X is a set and M is an inhabited set of pseudometrics on X satisfying

$$x = y \iff (\forall d \in M)\, d(x, y) = 0 \tag{3.1}$$

for all $x, y \in X$. The set M is called a *uniformity* on X (or the uniformity of the uniform space (X, M)).

Each $A \in \mathsf{Fin}^+(M)$ determines a pseudometric ρ_A on X given by

$$\rho_A(x, y) \stackrel{\text{def}}{=} \max \left\{ d(x, y) \in \mathbb{R}^{\geq 0} \mid d \in A \right\}$$

for all $x, y \in X$. We have

1. $A \subseteq B \implies \rho_A(x, y) \leq \rho_B(x, y)$,
2. $\rho_A(x, y) < \varepsilon \iff (\forall d \in A)\, d(x, y) < \varepsilon$

for all $\varepsilon \in \mathbb{Q}^{>0}$, $A, B \in \mathsf{Fin}^+(M)$ and $x, y \in X$. If $A \in \mathsf{Fin}^+(M)$ is a singleton $A = \{d\}$, then we simply write d for $\rho_{\{d\}}$.

Notation 3.1.3. We sometimes write X for a uniform space (X, M) using the same symbol for the underlying set of the space. Moreover, we often say that 'X is a uniform space' leaving the underlying uniformity implicit.

Definition 3.1.4. Let $X = (X, M)$ and $Y = (Y, N)$ be uniform spaces. A function $f : X \to Y$ is *uniformly continuous* if for each $d \in N$ and $\varepsilon \in \mathbb{Q}^{>0}$, there exist $A \in \mathsf{Fin}^+(M)$ and $\delta \in \mathbb{Q}^{>0}$ such that

$$\rho_A(x, x') < \delta \implies d(f(x), f(x')) < \varepsilon \tag{3.2}$$

for all $x, x' \in X$.

For example, for any uniform space (X, M), the identity function id_X on X is a uniformly continuous function from (X, M) to (X, M).

Notation 3.1.5. We write $f : (X, M) \to (Y, N)$, or simply $f : X \to Y$, to mean that f is a uniformly continuous function from (X, M) to (Y, N). If X and Y denote uniform spaces, the context will always make clear whether $f : X \to Y$ means that f is just a function between the underlying sets or that f is a uniformly continuous function from X and Y.

Definition 3.1.6. A uniformly continuous function $f : X \to Y$ is a *uniform isomorphism* if there exists a uniformly continuous $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Definition 3.1.4 is equivalent to a more general form given below.

Lemma 3.1.7. *Let $X = (X, M)$ and $Y = (Y, N)$ be uniform spaces. A function $f : X \to Y$ is uniformly continuous iff for each $B \in \mathsf{Fin}^+(N)$ and $\varepsilon \in \mathbb{Q}^{>0}$, there exist $A \in \mathsf{Fin}^+(M)$ and $\delta \in \mathbb{Q}^{>0}$ such that*

$$\rho_A(x, x') < \delta \implies \rho_B(f(x), f(x')) < \varepsilon \tag{3.3}$$

for all $x, x' \in X$.

Proof. ($\Leftarrow$): Trivial.

($\Rightarrow$): Let $B \in \mathsf{Fin}^+(N)$ and $\varepsilon \in \mathbb{Q}^{>0}$. Write $B = \{d_0, \dots, d_n\}$. For each $i \leq n$, there exists $A_i \in \mathsf{Fin}^+(M)$ and $\delta_i \in \mathbb{Q}^{>0}$ such that (3.2) holds. Let $A = \bigcup_{i \leq n} A_i$ and $\delta = \min \{\delta_i \mid i \leq n\}$. Let $x, x' \in X$, and suppose that $\rho_A(x, x') < \delta$. Then, $\rho_{A_i}(x, x') < \delta_i$ for all $i \leq n$, and thus $d_i(f(x), f(x')) < \varepsilon$ for all $i \leq n$, i.e. $\rho_B(f(x), f(x')) < \varepsilon$. Therefore, (3.3) holds. □

Definition 3.1.8. A *subspace* of a uniform space $X = (X, M)$ is a subset Y of X together with the uniformity $M|_Y$ given by the restrictions:

$$M|_Y \stackrel{\text{def}}{=} \{d|_{Y \times Y} \mid d \in M\},$$

where $d|_{Y \times Y}$ denotes the restriction of a pseudometric $d : X \times X \to \mathbb{R}^{\geq 0}$ to $Y \times Y$. We identify each subset Y of a uniform space (X, M) with the corresponding subspace $(Y, M|_Y)$. A uniformly continuous function $f : X \to Y$ is an *embedding* if the image $f[X]$ as a subspace of Y is uniformly isomorphic to X via the (co-)restriction of f to $f[X]$.

The uniform spaces and uniformly continuous functions between them form a category **USpa**. The category of metric spaces is embedded into **USpa** by $(X, d) \mapsto (X, \{d\})$.

3.1.2 Localic completions of uniform spaces

A representation of a complete metric space by a formal topology, called a localic completion, is the object part of the embedding of the category of locally compact metric spaces into that of formal topologies by Palmgren [48]. This representation was first obtained by Vickers in locale theory [58]. In this section, we extend the notion of localic completion of a metric space to the setting of uniform spaces.

Given a uniform space $X = (X, M)$, define

$$\mathcal{E}_X \stackrel{\text{def}}{=} \mathsf{Fin}^+(M) \times \mathbb{Q}^{>0},$$
$$U_X \stackrel{\text{def}}{=} \mathcal{E}_X \times X.$$

We write $\mathsf{b}_A(x, \varepsilon)$ for the element $((A, \varepsilon), x) \in U_X$. If $A \in \mathsf{Fin}^+(M)$ is a singleton $A = \{d\}$, we write $\mathsf{b}_d(x, \varepsilon)$ for $\mathsf{b}_{\{d\}}(x, \varepsilon)$.

Define an order $\leq_X$ and a transitive relation $<_X$ on U_X by

$$\mathsf{b}_A(x, \delta) \leq_X \mathsf{b}_B(y, \varepsilon) \stackrel{\text{def}}{\iff} B \subseteq A \ \&\ \rho_B(x, y) + \delta \leq \varepsilon,$$
$$\mathsf{b}_A(x, \delta) <_X \mathsf{b}_B(y, \varepsilon) \stackrel{\text{def}}{\iff} B \subseteq A \ \&\ \rho_B(x, y) + \delta < \varepsilon.$$

We extend the relations $\leq_X$ and $<_X$ to the subsets of U_X by

$$U \leq_X V \stackrel{\text{def}}{\iff} (\forall a \in U)\,(\exists b \in V)\, a \leq_X b$$

for all $U, V \subseteq U_X$, and similarly for $<_X$.

The *localic completion* of a uniform space X is a formal topology

$$\mathcal{U}(X) = (U_X, \lhd_X, \leq_X)$$

inductively generated by the axiom-set on $(U_X, \leq_X)$ consisting of the following axioms:

(U1) $a \lhd_X \{b \in U_X \mid b <_X a\}$;

(U2) $a \lhd_X \mathcal{C}^\varepsilon_A$ for each $(A, \varepsilon) \in \mathcal{E}_X$

for each $a \in S$, where we define $\mathcal{C}^\varepsilon_A \stackrel{\text{def}}{=} \{\mathsf{b}_A(x, \varepsilon) \in U_X \mid x \in X\}$.

Lemma 3.1.9. *The axioms (U1) and (U2) are equivalent to the following axiom-set:*

(U1') $a \lhd_X \{b \in U_X \mid b <_X a\}$;

(U2') $a \lhd_X \mathcal{C}^\varepsilon_A \downarrow a$ for each $(A, \varepsilon) \in \mathcal{E}_X$.

Proof. Immediate from (Loc). □

Note that the above axiom-set is localised.

For each $\mathsf{b}_A(x, \varepsilon) \in U_X$, we use notations $\mathsf{b}_A(x, \varepsilon)_*$ or $B_A(x, \varepsilon)$ to denote the *open ball* corresponding to $\mathsf{b}_A(x, \varepsilon)$, i.e.

$$\mathsf{b}_A(x, \varepsilon)_* \stackrel{\text{def}}{=} B_A(x, \varepsilon) \stackrel{\text{def}}{=} \{y \in X \mid \rho_A(x, y) < \varepsilon\}.$$

If A is a singleton $A = \{d\}$, then we write $B_d(x, \varepsilon)$ for $B_{\{d\}}(x, \varepsilon)$. We extend the notation $(-)_*$ to the subsets of U_X by

$$V_* \stackrel{\text{def}}{=} \bigcup_{a \in V} a_*. \tag{3.4}$$

Dually, each $x \in X$ is associated with the set $\Diamond x$ of *open neighbourhoods* of x, namely

$$\Diamond x \stackrel{\text{def}}{=} \{a \in U_X \mid x \in a_*\}.$$

We also extend the notation $\Diamond(-)$ to the subsets of X by

$$\Diamond Y \stackrel{\text{def}}{=} \bigcup_{y \in Y} \Diamond y. \tag{3.5}$$

Lemma 3.1.10. *Let $X = (X, M)$ be a uniform space. Then*

1. $a' \leq_X a <_X b \leq_X b' \implies a' <_X b'$,
2. $a <_X b \implies (\exists c \in U_X)\, a <_X c <_X b$,
3. $a \leq_X b \implies a_* \subseteq b_*$

for all $a, a', b, b' \in U_X$.

Proof. 1 is obvious. 2 follows from the density of the order on $\mathbb{Q}$. 3 follows from the fact that

$$A \subseteq B \implies \rho_A(x, y) \leq \rho_B(x, y)$$

for all $x, y \in X$ and $A, B \in \mathsf{Fin}^+(M)$. □

Remark 3.1.11. The converse of Lemma 3.1.10.3 may not hold. For example, consider the unit interval $([0,1], d)$, where d denotes the usual metric on $[0,1]$. We have $B_d(1,3) \subseteq B_d(1,2)$, but $\mathsf{b}_d(1,3) \leq_{[0,1]} \mathsf{b}_d(1,2)$ is false. This difference, however, is not essential in that one can define a formal topology which is isomorphic to $\mathcal{U}(X)$ by using the inclusion ordering of open balls (See Section 3.3).

Proposition 3.1.12. *For any uniform space X, its localic completion $\mathcal{U}(X)$ is overt, and the base U_X is the positivity of $\mathcal{U}(X)$.*

Proof. The condition (Pos) is trivial, so it suffices to show that U_X is a splitting subset of $\mathcal{U}(X)$. We apply Proposition 2.2.17, using (U1') and (U2').

The condition (Spl1) is trivial. Thus, it remains to check the condition (Spl2') for (U1') and (U2'). For (U1'), let $a \in U_X$, and write $a = \mathsf{b}_A(x, \varepsilon)$. Then, choosing any $\delta \in \mathbb{Q}^{>0}$ such that $\delta < \varepsilon$, we have $\mathsf{b}_A(x, \delta) <_X a$. For (U2'), let $(A, \varepsilon) \in \mathcal{E}_X$ and $a \in U_X$. Write $a = \mathsf{b}_B(x, \delta)$. Then, $\mathsf{b}_{A \cup B}(x, \min\{\varepsilon, \delta\}) \in \mathcal{C}_A^{\varepsilon} \downarrow a$. □

Lemma 3.1.13. *For any uniform space X, we have*

$$a <_X b \implies a \lll b$$

for all $a, b \in U_X$.

Proof. Let $a, b \in U_X$, and suppose that $a <_X b$. Write $a = \mathsf{b}_A(x, \varepsilon)$, and choose $\theta \in \mathbb{Q}^{>0}$ such that $\mathsf{b}_A(x, \varepsilon + 3\theta) <_X b$. Let $c = \mathsf{b}_A(z, \theta) \in \mathcal{C}_A^\theta$. Then, either $\rho_A(x, z) > \varepsilon + \theta$ or $\rho_A(x, z) < \varepsilon + 2\theta$. In the former case, for any $d \in a \downarrow c$, we have $\rho_A(x, z) < \varepsilon + \theta$, a contradiction. Thus, $a \downarrow c \lhd_X \emptyset$ and so $c \in a^*$. In the latter case, we have

$$\rho_A(z, x) + \theta \leq \varepsilon + 3\theta,$$

so that $\mathsf{b}_A(z, \theta) \leq_X \mathsf{b}_A(x, \varepsilon + 3\theta) <_X b$. Hence, $U_X \lhd_X a^* \cup \{b\}$ by (U2). Therefore, $a \lll b$. □

Hence, by (U1) we have the following.

Proposition 3.1.14 (cf. Palmgren [48, Theorem 3.7]). *The localic completion of a uniform space is regular.*

Definition 3.1.15. Let X be a uniform space, and let $Y \subseteq X$. A *closure* of Y in X, denoted by $\mathsf{cl}(Y)$, is the subset of X given by

$$\mathsf{cl}(Y) \stackrel{\text{def}}{=} \{x \in X \mid (\forall (A, \varepsilon) \in \mathcal{E}_X) (\exists y \in Y) \, \rho_A(x, y) < \varepsilon\}.$$

A subset $Y \subseteq X$ is *dense* if $\mathsf{cl}(Y) = X$, and *closed* if $\mathsf{cl}(Y) = Y$.

As the term 'completion' suggests, we have the following.

Proposition 3.1.16 (cf. Palmgren [48, Theorem 2.7]). *Let X be a uniform space, and let $Y \subseteq X$ be a dense subset of X. Then,*

$$\mathcal{U}(X) \cong \mathcal{U}(Y).$$

Proof. In the following, we identify an element of U_Y with the corresponding element of U_X. First, for each $a \in U_X$, define

$$O(a) \stackrel{\text{def}}{=} \{b \in U_Y \mid b <_X a\},$$

and for each subset $U \subseteq U_X$, define $O(U) \stackrel{\text{def}}{=} \bigcup_{a \in U} O(a)$. Define a relation $r \subseteq \mathcal{U}(Y) \times \mathcal{U}(X)$ by

$$a \; r \; b \stackrel{\text{def}}{\iff} a \lhd_Y O(b)$$

for all $a \in U_Y$ and $b \in U_X$. The relation r is easily shown to be a formal topology map from $\mathcal{U}(Y)$ to $\mathcal{U}(X)$. It is also easy to show that r is an embedding.

To show that r is surjective, we first note that

$$(\forall a, b \in U_X) \, a <_X b \implies (\exists c \in U_Y) \, a <_X c <_X b.$$

To see this, let $a, b \in U_X$ such that $a <_X b$. Write $a = \mathsf{b}_A(x, \varepsilon)$ and $b = \mathsf{b}_B(x', \delta)$. Choose $\theta \in \mathbb{Q}^{>0}$ such that

$$\rho_B(x, x') + \varepsilon + 2\theta < \delta.$$

Since Y is dense, there exists $y \in Y$ such that $\rho_A(x, y) < \theta$. Then

$$\begin{aligned}\rho_B(x', y) + \varepsilon + \theta &\leq \rho_B(x', x) + \rho_A(x, y) + \varepsilon + \theta \\ &< \rho_B(x', x) + \varepsilon + 2\theta < \delta.\end{aligned}$$

Thus, $a <_X \mathsf{b}_A(y, \varepsilon + \theta) <_X b$, as required.

Using the above fact and (U1) for $\lhd_X$, it is straightforward to show that

$$a \lhd_Y V \implies a \lhd_X V$$

for all $a \in U_Y$ and $V \subseteq U_Y$ by induction on $\lhd_Y$.

Now, let $a \in U_X$ and $V \subseteq U_X$, and suppose that $r^- a \lhd_Y r^- V$. Then, $O(a) \lhd_Y O(V)$. Thus, $a \lhd_X O(a) \lhd_X O(V) <_X V$, and hence, $a \lhd_X V$. Therefore r is surjective, so r is an isomorphism by Proposition 2.3.2. □

3.1.3 The formal points of a localic completion

We define a uniformity on the formal points $\mathcal{P}t(\mathcal{U}(X))$ of the localic completion of a uniform space X, and show that $\mathcal{P}t(\mathcal{U}(X))$ is a completion of X.

Lemma 3.1.17. *Let X be a uniform space. Then, a subset $\alpha \subseteq U_X$ is a formal point of $\mathcal{U}(X)$ iff*

(UP1) $a, b \in \alpha \implies (\exists c \in \alpha)\, c <_X a \;\&\; c <_X b,$

(UP2) $a \leq_X b \;\&\; a \in \alpha \implies b \in \alpha,$

(UP3) $(\forall (A, \varepsilon) \in \mathcal{E}_X)\, (\exists x \in X)\, \mathsf{b}_A(x, \varepsilon) \in \alpha$

for all $a, b \in U_X$.

Proof. By Lemma 2.2.13. □

Let $X = (X, M)$ be a uniform space. For each $d \in M$, define an operation $\widetilde{d} : \mathcal{P}t(\mathcal{U}(X)) \times \mathcal{P}t(\mathcal{U}(X)) \to \mathsf{Pow}(\mathbb{Q}^{>0})$ by

$$\widetilde{d}(\alpha, \beta) \overset{\text{def}}{=} \left\{ q \in \mathbb{Q}^{>0} \mid (\exists\, \mathsf{b}_d(x, \varepsilon) \in \alpha)\, (\exists\, \mathsf{b}_d(y, \delta) \in \beta)\, d(x, y) + \varepsilon + \delta < q \right\}.$$

Lemma 3.1.18. *For each $\alpha, \beta \in \mathcal{P}t(\mathcal{U}(X))$, the set $\widetilde{d}(\alpha, \beta)$ is a non-negative Dedekind real.*

Proof. The set $\widetilde{d}(\alpha, \beta)$ satisfies (D1) by (UP3). The condition (D2) is obvious. For (D3), let $p, q \in \mathbb{Q}^{>0}$, and suppose that $p < q$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $p + 5\theta < q$. By (UP3), there exist $x, y \in X$ such that $\mathsf{b}_d(x, \theta) \in \alpha$ and $\mathsf{b}_d(y, \theta) \in \beta$. Then, we have either $d(x, y) < q - 2\theta$ or $d(x, y) > q - 3\theta$. In the former case, we have $q \in \widetilde{d}(\alpha, \beta)$. In the latter case, suppose that $p \in \widetilde{d}(\alpha, \beta)$. Then, there exist $\mathsf{b}_d(x', \varepsilon') \in \alpha$ and $\mathsf{b}_d(y', \delta') \in \beta$ such that $d(x', y') + \varepsilon' + \delta' < p$. Thus,

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq \varepsilon' + \theta + d(x', y') + \theta + \delta' \\ &< p + 2\theta < q - 3\theta, \end{aligned}$$

a contradiction. Hence $p \notin \widetilde{d}(\alpha, \beta)$. □

Lemma 3.1.19. *The function $\widetilde{d} : \mathcal{P}t(\mathcal{U}(X)) \times \mathcal{P}t(\mathcal{U}(X)) \to \mathbb{R}^{\geq 0}$ is a pseudometric on $\mathcal{P}t(\mathcal{U}(X))$.*

Proof. For any $\alpha \in \mathcal{P}t(\mathcal{U}(X))$, we have $\widetilde{d}(\alpha, \alpha) = 0$ by (UP3). $\widetilde{d}$ is obviously symmetric. For the triangle inequality, let $\alpha, \beta, \gamma \in \mathcal{P}t(\mathcal{U}(X))$ and $q \in \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma)$. Then, there exist $p, r \in \mathbb{Q}^{>0}$, $\mathsf{b}_d(x, \varepsilon) \in \alpha$, $\mathsf{b}_d(y, \delta), \mathsf{b}_d(y', \delta') \in \beta$, and $\mathsf{b}_d(z, \theta) \in \gamma$ such that

$$\begin{aligned} &d(x, y) + \varepsilon + \delta < p, \\ &d(y', z) + \delta' + \theta < r, \\ &q = p + r. \end{aligned}$$

Then,

$$\begin{aligned} d(x, z) + \varepsilon + \theta &\leq d(x, y) + d(y, y') + d(y', z) + \varepsilon + \theta \\ &< d(x, y) + \delta + \delta' + d(y', z) + \varepsilon + \theta \\ &< p + r = q. \end{aligned}$$

Thus $q \in \widetilde{d}(\alpha, \gamma)$, and hence $\widetilde{d}(\alpha, \gamma) \leq \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma)$. □

For each $A \in \mathsf{Fin}^+(M)$, let $\widetilde{\rho}_A$ be the pseudometric on $\mathcal{P}t(\mathcal{U}(X))$ given by

$$\widetilde{\rho}_A(\alpha, \beta) \stackrel{\text{def}}{=} \max \left\{ \widetilde{d}(\alpha, \beta) \mid d \in A \right\}$$

for all $\alpha, \beta \in \mathcal{P}t(\mathcal{U}(X))$.

Lemma 3.1.20. *For each $x \in X$, the set $\Diamond x$ is a formal point of X.*

Proof. We must show that $\Diamond x$ satisfies (UP1) – (UP3). The conditions (UP2) and (UP3) are obvious. For (UP1), let $a, b \in \Diamond x$. Write $a = \mathsf{b}_A(y, \varepsilon)$ and $b = \mathsf{b}_B(z, \delta)$. Then, $\rho_A(x, y) < \varepsilon$ and $\rho_B(x, z) < \delta$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\rho_A(x, y) + \theta < \varepsilon$ and $\rho_B(x, z) + \theta < \delta$. Then, $\mathsf{b}_{A \cup B}(x, \theta) <_X \mathsf{b}_A(y, \varepsilon)$ and $\mathsf{b}_{A \cup B}(x, \theta) <_X \mathsf{b}_B(z, \delta)$, from which (UP1) follows. □

Lemma 3.1.21. *For any $x \in X$ and $\alpha \in \mathcal{P}t\,(\mathcal{U}(X))$, we have*

1. $\mathsf{b}_A(x, \varepsilon) \in \alpha \iff (\forall d \in A)\, \mathsf{b}_d(x, \varepsilon) \in \alpha$,

2. $\mathsf{b}_d(x,\varepsilon) \in \alpha \iff \widetilde{d}(\Diamond x, \alpha) < \varepsilon$,

3. $\mathsf{b}_A(x,\varepsilon) \in \alpha \iff \widetilde{\rho}_A(\Diamond x, \alpha) < \varepsilon$.

Proof. 1. Suppose that $\mathsf{b}_A(x,\varepsilon) \in \alpha$. Let $d \in A$. Since $\mathsf{b}_A(x,\varepsilon) \leq_X \mathsf{b}_d(x,\varepsilon)$, we have $\mathsf{b}_d(x,\varepsilon) \in \alpha$. Conversely, suppose that $\mathsf{b}_d(x,\varepsilon) \in \alpha$ for all $d \in A$. Then, there exists $\mathsf{b}_B(y,\delta) \in \alpha$ such that $\mathsf{b}_B(y,\delta) \leq_X \mathsf{b}_d(x,\varepsilon)$ for all $d \in A$. Thus, $\mathsf{b}_B(y,\delta) \leq_X \mathsf{b}_A(x,\varepsilon)$, and hence $\mathsf{b}_A(x,\varepsilon) \in \alpha$ by (UP2).

2. Suppose that $\mathsf{b}_d(x,\varepsilon) \in \alpha$. By (U1), there exists $\varepsilon' < \varepsilon$ such that $\mathsf{b}_d(x,\varepsilon') \in \alpha$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\varepsilon' + \theta < \varepsilon$. Since $\mathsf{b}_d(x,\theta) \in \Diamond x$, we have $\widetilde{d}(\Diamond x, \alpha) < \varepsilon$. Conversely, suppose that $\widetilde{d}(\Diamond x, \alpha) < \varepsilon$. Then, there exist $\mathsf{b}_d(y,\delta) \in \Diamond x$ and $\mathsf{b}_d(z,\gamma) \in \alpha$ such that $d(y,z)+\delta+\gamma < \varepsilon$. Then, $d(x,z)+\gamma \leq d(x,y)+d(y,z)+\gamma < \delta+d(y,z)+\gamma < \varepsilon$, and hence $\mathsf{b}_d(z,\gamma) <_X \mathsf{b}_d(x,\varepsilon)$. Therefore, $\mathsf{b}_d(x,\varepsilon) \in \alpha$ by (UP2).

3. Immediate from 1 and 2. □

Lemma 3.1.22. *For each $\alpha, \beta \in \mathcal{P}t(\mathcal{U}(X))$, we have*

$$\Big((\forall d \in M)\, \widetilde{d}(\alpha,\beta) = 0\Big) \iff \alpha = \beta.$$

Proof. ($\Rightarrow$): Suppose that $\widetilde{d}(\alpha,\beta) = 0$ for all $d \in M$. Let $a = \mathsf{b}_A(x,\varepsilon) \in \alpha$. By Lemma 3.1.21.3, we have $\widetilde{\rho}_A(\Diamond x, \alpha) < \varepsilon$, and thus $\widetilde{\rho}_A(\Diamond x, \beta) < \varepsilon$. Hence $a \in \beta$, and so $\alpha \subseteq \beta$. Therefore $\alpha = \beta$ by Proposition 3.1.14 and Corollary 2.4.10.

($\Leftarrow$): This follows from (UP3). □

Define $\widetilde{M} \overset{\text{def}}{=} \Big\{\widetilde{d} \mid d \in M\Big\}$. By Lemma 3.1.19 and Lemma 3.1.22, the pair $\Big(\mathcal{P}t(\mathcal{U}(X)), \widetilde{M}\Big)$ is a uniform space.

Remark 3.1.23. Since the collection $\mathcal{P}t(\mathcal{U}(X))$ does not necessarily form a set, the definition of the uniform space $\mathcal{P}t(\mathcal{U}(X))$ is problematic from a constructive point of view. However, the argument preceding Theorem 3.1.31 shows that $\mathcal{P}t(\mathcal{U}(X))$ forms a set under the assumption of the Countable Choice.

Define a function $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ by

$$i_X(x) \overset{\text{def}}{=} \Diamond x. \tag{3.6}$$

Corollary 3.1.24. *The function $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ is a uniform embedding; in fact i_X is an* isometrical *embedding in the sense that*

$$d(x,y) = \widetilde{d}(i_X(x), i_X(y))$$

for all $x, y \in X$ and $d \in M$.

Proof. Let $x, y \in X$ and $d \in M$. By Lemma 3.1.21 we have

$$d(x,y) < \varepsilon \iff \mathsf{b}_d(x,\varepsilon) \in \Diamond y \iff \widetilde{d}(\Diamond x, \Diamond y) < \varepsilon$$

for all $\varepsilon \in \mathbb{Q}^{>0}$. Hence $d(x,y) = \widetilde{d}(\Diamond x, \Diamond y)$. □

Lemma 3.1.25. *The image* $i_X[X] = \{\Diamond x \mid x \in X\}$ *is a dense subset of* $\mathcal{P}t(\mathcal{U}(X))$.

Proof. Let $(A, \varepsilon) \in \mathcal{E}_X$, and $\alpha \in \mathcal{P}t(\mathcal{U}(X))$. By (UP3), there exists $x \in X$ such that $\mathsf{b}_A(x, \varepsilon) \in \alpha$, and thus $\widetilde{\rho}_A(\Diamond x, \alpha) < \varepsilon$ by Lemma 3.1.21. Hence $i_X[X]$ is dense. □

In summary, we have shown the following.

Proposition 3.1.26. *For any uniform space* X*, the function* $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ *given by* (3.6) *is a dense isometrical embedding.*

Next, we show that the uniform space $\mathcal{P}t(\mathcal{U}(X))$ is complete by showing that $\mathcal{P}t(\mathcal{U}(X))$ is uniformly isomorphic to the usual construction of a completion of X as a closed subspace of a product of complete uniform spaces [38, 62]. The result shows that the embedding $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ given by (3.6) is a completion of X.

Notation 3.1.27. We introduce notations for localic completions of metric spaces, i.e. those uniform spaces (X, M) in which M is a singleton $\{d\}$. Given a metric space $X = (X, d)$, we write $\mathcal{M}(X)$ for $\mathcal{U}(X)$, and M_X for the base U_X. We write $\mathsf{b}^d(x, \varepsilon)$ instead of $\mathsf{b}_d(x, \varepsilon)$ for an element of M_X. When the context makes it clear, we sometimes omit the superscript and just write $\mathsf{b}(x, \varepsilon)$. If X is a metric space, the orders on M_X are given by

$$\mathsf{b}(x, \varepsilon) \leq_X \mathsf{b}(y, \delta) \iff d(x, y) + \varepsilon \leq \delta,$$
$$\mathsf{b}(x, \varepsilon) <_X \mathsf{b}(y, \delta) \iff d(x, y) + \varepsilon < \delta.$$

We identify the uniform space $\mathcal{P}t(\mathcal{M}(X)) = \left(\mathcal{P}t(\mathcal{M}(X)), \left\{\widetilde{d}\right\}\right)$ with the metric space $\left(\mathcal{P}t(\mathcal{M}(X)), \widetilde{d}\right)$.

Remark 3.1.28. Palmgren [48] showed that the metric space $\mathcal{P}t(\mathcal{M}(X))$ is complete by constructing a uniform isomorphism between $\mathcal{P}t(\mathcal{M}(X))$ and the standard completion of X given by the set of fundamental sequences on X[1]. Hence, the embedding $i_X : X \to \mathcal{P}t(\mathcal{M}(X)$ is a completion of X.

First, we recall some basic facts about complete uniform spaces.

Definition 3.1.29. A *Cauchy filter* on a uniform space X is a set $\mathcal{F}$ of subsets of X such that

(CF1) $(\forall U \in \mathcal{F})\, U \between X$,

(CF2) $(\forall U, V \in \mathcal{F})\, (\exists W \in \mathcal{F})\, W \subseteq U \cap V$,

(CF3) $(\forall\, (A, \varepsilon) \in \mathcal{E}_X)\, (\exists x \in X)\, (\exists U \in \mathcal{F})\, U \subseteq B_A(x, \varepsilon)$.

[1] The isomorphism requires the Countable Choice. Another way to see that $\mathcal{P}t(\mathcal{M}(X))$ is complete is given in Section 3.2.3.

A Cauchy filter $\mathcal{F}$ on X *converges* to a point $x \in X$ if

$$(\forall (A,\varepsilon) \in \mathcal{E}_X)\,(\exists U \in \mathcal{F})\, U \subseteq B_A(x,\varepsilon).$$

In this case, x is called a *limit* of $\mathcal{F}$. Note that a limit of a Cauchy filter, when it exists, is unique. This follows from (3.1) and (CF3).

A uniform space X is *complete* if every Cauchy filter on X converges to some point.

The product of a set-indexed family $(X_i)_{i\in I}$ of uniform spaces, each of the form $X_i = (X_i, M_i)$, consists of the cartesian product $\prod_{i\in I} X_i$ and the set

$$M_\Pi \overset{\text{def}}{=} \sum_{i\in I} M_i \tag{3.7}$$

of pseudometrics on $\prod_{i\in I} X_i$, where we identify each element $(i,d) \in M_\Pi$ with the pseudometric on $\prod_{i\in I} X_i$ given by

$$(i,d)(f,g) \overset{\text{def}}{=} d(f(i), g(i))$$

for all $f, g \in \prod_{i\in I} X_i$. The uniform space $\prod_{i\in I} X_i = (\prod_{i\in I} X_i, M_\Pi)$ together with the projections $\pi_i : \prod_{i\in I} X_i \to X_i$ for each $i \in I$ forms a product of the family $(X_i)_{i\in I}$ in **USpa**[2].

As in the case of metric spaces, we have the following results. The classical proofs go through without change.

Proposition 3.1.30.

1. *A closed subspace of a complete uniform space is complete.*

2. *A product of a set-indexed family of complete uniform spaces is complete.*

Proof. 1. Let X be a complete uniform space, and let Y be a closed subset of X. Let $\mathcal{F} \subseteq \mathsf{Pow}(Y)$ be a Cauchy filter on Y. Then, $\mathcal{F}$ is a Cauchy filter on X, so $\mathcal{F}$ converges to some $x \in X$, i.e.

$$(\forall (A,\varepsilon) \in \mathcal{E}_X)\,(\exists U \in \mathcal{F})\, U \subseteq B_A(x,\varepsilon).$$

Then, $(\forall (A,\varepsilon) \in \mathcal{E}_X)\, B_A(x,\varepsilon) \between Y$ by (CF1). Since Y is closed, we must have $x \in Y$. Hence, Y is complete.

2. Let $((X_i, M_i))_{i\in I}$ be a set-indexed family of complete uniform spaces. Let $\mathcal{F} \subseteq \mathsf{Pow}\left(\prod_{i\in I} X_i\right)$ be a Cauchy filter on $\prod_{i\in I} X_i$. We show that for each $i \in I$, the set $\mathcal{F}_i$ given by

$$\mathcal{F}_i \overset{\text{def}}{=} \{\pi_i[U] \mid U \in \mathcal{F}\},$$

where $\pi_i[U] = \left\{x_i \in X_i \mid (x_i)_{i\in I} \in U\right\}$, is a Cauchy filter on X_i. We must check the conditions (CF1) – (CF3). (CF1) and (CF2) are immediate from the corresponding

[2] When constructing a product, we always assume that the index set of the given family is inhabited; otherwise, M_Π would be the empty set.

properties of $\mathcal{F}$. For (CF3), let $(A,\varepsilon) \in \mathcal{E}_{X_i}$. By (CF3), there exist $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ and $U \in \mathcal{F}$ such that $U \subseteq B_{\{i\}\times A}((x_i)_{i\in I}, \varepsilon)$. Then, $\pi_i[U] \subseteq B_A(x_i, \varepsilon)$.

Let $(x_i)_{i\in I}$ be the element of $\prod_{i\in I} X_i$ such that x_i is the limit of $\mathcal{F}_i$ for each $i \in I$. Let $(A,\varepsilon) \in \mathcal{E}_{\prod X_i}$. Write $A = \{(i_0, d_0), \dots, (i_n, d_n)\}$. Then, for each $k \leq n$, there exists $U_k \in \mathcal{F}$ such that $\pi_{i_k}[U_k] \subseteq B_{d_k}(x_{i_k}, \varepsilon)$. By (CF2), there exists $U \in \mathcal{F}$ such that $U \subseteq \bigcap_{k\leq n} U_k$. Then, $U \subseteq B_A((x_i)_{i\in I}, \varepsilon)$. Hence, $\mathcal{F}$ converges to $(x_i)_{i\in I}$. Therefore $\prod_{i\in I} X_i$ is complete. □

In the following, we fix a uniform space $X = (X, M)$. For each $d \in M$, let $X_d = (X_d, d)$ be the metric space in which X_d is the set X equipped with the equality:

$$x = y \overset{\text{def}}{\iff} d(x,y) = 0.$$

Then, the function $j^d : X \to X_d$ given by $j^d(x) \overset{\text{def}}{=} x$ is a uniformly continuous function from (X, M) to (X_d, d). Let $i^d_X : X_d \to \mathcal{P}t(\mathcal{M}(X_d))$ be the dense embedding given by (3.6). In the following, we write $\Diamond^d x$ for $i^d_X(x)$. Then, we have a family

$$\left(i^d_X \circ j^d : X \to \mathcal{P}t(\mathcal{M}(X_d))\right)_{d\in M}$$

of uniformly continuous functions. Let $\mu_X : X \to \prod_{d\in M} \mathcal{P}t(\mathcal{M}(X_d))$ be the canonical map into the product $\prod_{d\in M} \mathcal{P}t(\mathcal{M}(X_d))$ so that $\pi_d \circ \mu_X = i^d_X \circ j^d$ for each $d \in M$. By definition, μ_X is given by

$$\mu_X(x) \overset{\text{def}}{=} \left(\Diamond^d x\right)_{d\in M}$$

for all $x \in X$. Since an element of the uniformity of $\prod_{d\in M} \mathcal{P}t(\mathcal{M}(X_d))$ is of the form $(d, \widetilde{d})$ for some $d \in M$, we have

$$d(x,y) = \widetilde{d}(\Diamond^d x, \Diamond^d y) = (d, \widetilde{d})\left((\Diamond^{d'} x)_{d'\in M}, (\Diamond^{d'} y)_{d'\in M}\right)$$

for each $d \in M$ and $x, y \in X$ by Lemma 3.1.21. Hence, μ_X is a uniform embedding. Let $\widetilde{X}$ be the closure of the image $\mu_X[X]$ in $\prod_{d\in M} \mathcal{P}t(\mathcal{M}(X_d))$. By Lemma 3.1.21, the underlying set of the uniform space $\widetilde{X}$ is given by

$$\widetilde{X} = \left\{ (\alpha_d)_{d\in M} \in \prod_{d\in M} \mathcal{P}t(\mathcal{M}(X_d)) \mid (\forall (A,\varepsilon) \in \mathcal{E}_X)\,(\exists x \in X)\,(\forall d \in A)\,\mathsf{b}^d(x,\varepsilon) \in \alpha_d \right\}.$$

Then, the uniform space $\widetilde{X}$ is complete by Proposition 3.1.30, and μ_X restricts to a dense embedding $\mu_X : X \to \widetilde{X}$. Hence, $\mu_X : X \to \widetilde{X}$ is a completion of X[3].

[3] Note that the classical proof of the following fact is already constructive (See also Proposition 3.2.29).

> A uniformly continuous function from a dense subset of a uniform space to a complete uniform space uniquely extends to the whole space.

We construct an isometry $\Phi : \widetilde{X} \to \mathcal{P}t(\mathcal{U}(X))$ which makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\ \mu_X\ } & \widetilde{X} \\ & \searrow^{i_X} & \cong \downarrow \Phi \\ & & \mathcal{P}t(\mathcal{U}(X)) \end{array} \tag{3.8}$$

For each $(\alpha_d)_{d\in M} \in \widetilde{X}$, define

$$\Phi\left((\alpha_d)_{d\in M}\right) \stackrel{\text{def}}{=} \left\{\mathsf{b}_A(x,\varepsilon) \in U_X \mid (\forall d \in A)\, \mathsf{b}^d(x,\varepsilon) \in \alpha_d\right\}.$$

Put $\alpha = \Phi\left((\alpha_d)_{d\in M}\right)$. We show that $\alpha \in \mathcal{P}t(\mathcal{U}(X))$ (See Lemma 3.1.17). The condition (UP3) follows from the definition of $\widetilde{X}$. For the condition (UP1), let $\mathsf{b}_A(x,\varepsilon), \mathsf{b}_B(y,\delta) \in \alpha$. By Lemma 3.1.21, we have

$$\begin{aligned} &(\forall d \in A)\, \widetilde{d}(\Diamond^d x, \alpha_d) < \varepsilon, \\ &(\forall d \in B)\, \widetilde{d}(\Diamond^d y, \alpha_d) < \delta. \end{aligned}$$

Choose $\theta \in \mathbb{Q}^{>0}$ such that

$$\begin{aligned} &(\forall d \in A)\, \widetilde{d}(\Diamond^d x, \alpha_d) + 2\theta < \varepsilon, \\ &(\forall d \in B)\, \widetilde{d}(\Diamond^d y, \alpha_d) + 2\theta < \delta. \end{aligned}$$

Since $(\alpha_d)_{d\in M} \in \widetilde{X}$, there exists $z \in X$ such that $\mathsf{b}_{A\cup B}(z,\theta) \in \alpha$. Then, for each $d \in A$,

$$\begin{aligned} d(z,x) + \theta &\le \widetilde{d}(\Diamond^d z, \alpha_d) + \widetilde{d}(\alpha_d, \Diamond^d x) + \theta \\ &< \widetilde{d}(\alpha_d, \Diamond^d x) + 2\theta < \varepsilon. \end{aligned}$$

Hence, $\mathsf{b}_{A\cup B}(z,\theta) <_X \mathsf{b}_A(x,\varepsilon)$. Similarly, we have $\mathsf{b}_{A\cup B}(z,\theta) <_X \mathsf{b}_B(y,\delta)$, from which (UP1) follows. For (UP2), let $\mathsf{b}_A(x,\varepsilon), \mathsf{b}_B(y,\delta) \in U_X$, and suppose that $\mathsf{b}_A(x,\varepsilon) \in \alpha$ and $\mathsf{b}_A(x,\varepsilon) \le_X \mathsf{b}_B(y,\delta)$. For each $d \in B$, since $d \in A$, we have $\mathsf{b}^d(x,\varepsilon) \in \alpha_d$. Since $\mathsf{b}^d(x,\varepsilon) \le_{X_d} \mathsf{b}^d(y,\delta)$, we have $\mathsf{b}^d(y,\delta) \in \alpha_d$. Hence $\mathsf{b}_B(y,\delta) \in \alpha$.

Conversely, given any $\alpha \in \mathcal{P}t(\mathcal{U}(X))$, define

$$\alpha_d \stackrel{\text{def}}{=} \left\{\mathsf{b}^d(x,\varepsilon) \in M_{X_d} \mid \mathsf{b}_d(x,\varepsilon) \in \alpha\right\}$$

for each $d \in M$. Clearly, we have $\alpha_d \in \mathcal{P}t(\mathcal{M}(X_d))$. Define a function $\Psi : \mathcal{P}t(\mathcal{U}(X)) \to \widetilde{X}$ by

$$\Psi(\alpha) \stackrel{\text{def}}{=} (\alpha_d)_{d\in M}.$$

By (UP3), we have $\Psi(\alpha) \in \widetilde{X}$ for each $\alpha \in \mathcal{P}t(\mathcal{U}(X))$.

We show that Φ and Ψ are mutual inverse. First, given $(\alpha_d)_{d\in M} \in \widetilde{X}$, by letting $(\beta_d)_{d\in M} = (\Psi \circ \Phi)\left((\alpha_d)_{d\in M}\right)$, we have $\alpha_d \subseteq \beta_d$ for each $d \in M$. Thus $\alpha_d = \beta_d$ for each

$d \in M$ by Corollary 2.4.10. Hence, $\Psi \circ \Phi = id_{\widetilde{X}}$. Next, for each $\alpha \in \mathcal{P}t(\mathcal{U}(X))$, we have $\alpha \subseteq (\Phi \circ \Psi)(\alpha)$, so $\alpha = (\Phi \circ \Psi)(\alpha)$. Hence, $\Phi \circ \Psi = id_{\mathcal{P}t(\mathcal{U}(X))}$.

For any $(\alpha_d)_{d \in M}, (\beta_d)_{d \in M} \in \widetilde{X}$, and $d \in M$, we have

$$(d, \widetilde{d}) \left((\alpha_d)_{d \in M}, (\beta_d)_{d \in M}\right) = \widetilde{d}\left(\Phi\left((\alpha_d)_{d \in M}\right), \Phi\left((\beta_d)_{d \in M}\right)\right),$$

so Φ is isometrical. Finally, we have

$$\begin{aligned}(\Phi \circ \mu_X)(x) &= \left\{\mathsf{b}_A(y,\varepsilon) \in U_X \mid (\forall d \in A)\, \mathsf{b}^d(y,\varepsilon) \in \Diamond^d x\right\} \\ &= \left\{\mathsf{b}_A(y,\varepsilon) \in U_X \mid \rho_A(x,y) < \varepsilon\right\} \\ &= \Diamond x = i_X(x)\end{aligned}$$

for each $x \in X$, so the diagram (3.8) commutes.

Theorem 3.1.31. *The embedding $i_X : (X, M) \to \left(\mathcal{P}t(\mathcal{U}(X)), \widetilde{M}\right)$ is a completion of X.*

Remark 3.1.32. By the uniqueness of completion, if X is complete, then the embedding $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ is a uniform isomorphism.

3.1.4 Compactness and local compactness

We show that the localic completion of a locally compact uniform space is locally compact and that a uniform space is total bounded iff its localic completion is compact. The results in this section extend the corresponding results for metric spaces [48, Section 4]. Moreover, we give an elementary characterisation of the cover of the localic completion of a locally compact uniform space.

Definition 3.1.33. A uniform space $X = (X, M)$ is *totally bounded* if for each $A \in \mathsf{Fin}^+(M)$, the pseudometric space (X, ρ_A) is totally bounded, i.e.

$$\left(\forall \varepsilon \in \mathbb{Q}^{>0}\right)(\exists Y_\varepsilon \in \mathsf{Fin}(X))\, X \subseteq \bigcup_{y \in Y_\varepsilon} B_A(y,\varepsilon).$$

The set Y_ε is called an *ε-net* to X with respect to ρ_A. A uniform space is *compact* if it is complete and totally bounded. A uniform space X is *locally compact* if for each open ball $B_A(x,\varepsilon)$ of X, there exists a compact subset $K \subseteq X$ such that $B_A(x,\varepsilon) \subseteq K$. Thus, every compact uniform space is locally compact.

Given a uniform space X, define a relation $\sqsubseteq$ on $\mathsf{Pow}(U_X)$ by

$$U \sqsubseteq V \overset{\text{def}}{\iff} (\exists (A,\varepsilon) \in \mathcal{E}_X)\, U \downarrow \mathcal{C}_A^\varepsilon \leq_X V \tag{3.9}$$

for all $U, V \subseteq U_X$. By (U2), we have

$$U \sqsubseteq V \implies U \lhd_X V.$$

Lemma 3.1.34. *Let X be a uniform space. Then, the following are equivalent for all $a \in U_X$ and $U \subseteq U_X$:*

1. $(\exists V \in \mathsf{Fin}(U_X))\, a_* \subseteq V_* \;\&\; V <_X U$;
2. $(\exists V \in \mathsf{Fin}(U_X))\, a \sqsubseteq V <_X U$;
3. $(\exists V \in \mathsf{Fin}(U_X))\, a \lhd_X V <_X U$.

Proof. $(1 \Rightarrow 2)$: Suppose that 1 holds. Then, there exists $V \in \mathsf{Fin}(U_X)$ such that $a_* \subseteq V_*$ and $V <_X U$. Write $V = \big\{\mathsf{b}_{A_0}(x_0, \varepsilon_0), \ldots, \mathsf{b}_{A_{n-1}}(x_{n-1}, \varepsilon_{n-1})\big\}$, and choose $\theta \in \mathbb{Q}^{>0}$ such that $V' = \{\mathsf{b}_{A_i}(x_i, \varepsilon_i + \theta) \mid i < n\} <_X U$. Let $A = \bigcup_{i<n} A_i$, and let $\mathsf{b}_D(z, \xi) \in a \downarrow \mathcal{C}^{\theta}_A$. Then, there exists $i < n$ such that $\rho_{A_i}(x_i, z) < \varepsilon_i$, so we have $\rho_{A_i}(x_i, z) + \xi < \varepsilon_i + \xi \leq \varepsilon_i + \theta$. Thus, $\mathsf{b}_D(z, \xi) <_X \mathsf{b}_{A_i}(x_i, \varepsilon_i + \theta)$, and hence $a \sqsubseteq V'$.

$(2 \Rightarrow 3)$: We have $a \sqsubseteq V \implies a \lhd_X V$.

$(3 \Rightarrow 1)$: We have $a \lhd_X V \implies a_* \subseteq V_*$. □

The cover of the localic completion of a locally compact uniform space admits an elementary characterisation.

Lemma 3.1.35. *Let X be a locally compact uniform space. Then, the following are equivalent for all $a \in U_X$ and $U \subseteq U_X$:*

1. $a \lhd_X U$;
2. $(\forall b <_X a)\, (\exists V \in \mathsf{Fin}(U_X))\, b_* \subseteq V_* \;\&\; V <_X U$;
3. $(\forall b <_X a)\, (\exists V \in \mathsf{Fin}(U_X))\, b \sqsubseteq V <_X U$;
4. $(\forall b <_X a)\, (\exists V \in \mathsf{Fin}(U_X))\, b \lhd_X V <_X U$.

Proof. By Lemma 3.1.34, it suffices to show that 1 implies 2. Given $U \subseteq U_X$, define a predicate Φ_U on U_X by

$$\Phi_U(a) \overset{\text{def}}{\iff} (\forall b <_X a)\, (\exists V \in \mathsf{Fin}(U_X))\, b_* \subseteq V_* \;\&\; V <_X U.$$

We show that

$$a \lhd_X U \implies \Phi_U(a)$$

for all $a \in U_X$ by induction on $\lhd_X$. We must check the conditions (ID1) – (ID3) for the localised axioms (U1') and (U2').

The conditions (ID1) and (ID2) are straightforward to check, using Lemma 3.1.10. For (ID3), we have two axioms to be checked.

(U1') $\dfrac{(\forall b <_X a)\, \Phi_U(b)}{\Phi_U(a)}$: Suppose that $\Phi_U(b)$ for all $b <_X a$. Let $b <_X a$. Then, there exists $c \in U_X$ such that $b <_X c <_X a$. Since $\Phi_U(c)$, there exists $V \in \mathsf{Fin}(U_X)$ such that $b_* \subseteq V_*$ and $V <_X U$. Hence, $\Phi_U(a)$.

(U2') $\dfrac{(\forall b \in \mathcal{C}^{\theta}_{C} \downarrow a)\ \Phi_U(b)}{\Phi_U(a)}$ for each $(C, \theta) \in \mathcal{E}_X$: Suppose that $\Phi_U(b)$ for all $b \in \mathcal{C}^{\theta}_{C} \downarrow a$. Let $b <_X a$, and write $a = \mathsf{b}_A(x, \varepsilon)$ and $b = \mathsf{b}_B(y, \delta)$. Since X is locally compact, there exists a compact subset $K \subseteq X$ such that $B_A(x, \varepsilon) \subseteq K$. Choose $\xi \in \mathbb{Q}^{>0}$ such that $2\xi < \theta$ and $\rho_A(x, y) + \delta + 4\xi < \varepsilon$. Let $Z = \{z_0, \ldots, z_{n-1}\}$ be a ξ-net to K with respect to $\rho_{C \cup A}$. Split Z into finite enumerable subsets Z^+ and Z^- such that $Z = Z^+ \cup Z^-$ and

- $z \in Z^+ \implies \rho_A(z, x) < \varepsilon - 2\xi$,
- $z \in Z^- \implies \rho_A(z, x) > \varepsilon - 3\xi$.

Let $z \in Z^+$. Since $\mathsf{b}_{C \cup A}(z, \xi) <_X \mathsf{b}_{C \cup A}(z, 2\xi) \in \mathcal{C}^{\theta}_{C} \downarrow a$, we have $\Phi_U(\mathsf{b}_{C \cup A}(z, 2\xi))$. Hence there exists $V_z \in \mathsf{Fin}(U_X)$ such that $B_{C \cup A}(z, \xi) \subseteq V_{z*}$ and $V_z <_X U$. Since Z^+ is finitely enumerable, there exists $V \in \mathsf{Fin}(U_X)$ such that $\bigcup_{z \in Z^+} B_{C \cup A}(z, \xi) \subseteq V_*$ and $V <_X U$. Now, it suffices to show that $b_* \subseteq \bigcup_{z \in Z^+} B_{C \cup A}(z, \xi)$. Let $y' \in b_*$. Then, there exists $i < n$ such that $\rho_{C \cup A}(y', z_i) < \xi$. Then

$$\begin{aligned} \rho_A(z_i, x) &\leq \rho_A(z_i, y') + \rho_A(y', y) + \rho_A(y, x) \\ &< \xi + \delta + \rho_A(y, x) < \varepsilon - 3\xi, \end{aligned}$$

and thus $z_i \in Z^+$. Hence, $y' \in \bigcup_{z \in Z^+} B_{C \cup A}(z, \xi)$, and therefore $\Phi_U(a)$. □

Remark 3.1.36. By Lemma 3.1.35, inductive generation of the cover of the localic completion of a locally compact uniform space does not require wREA in CZF.

Corollary 3.1.37. *For any locally compact uniform space X, we have*

$$a <_X b \implies a \ll b$$

for all $a, b \in U_X$.

By the axiom (U1), we obtain the following.

Theorem 3.1.38. *The localic completion of a locally compact uniform space is locally compact.*

Theorem 3.1.39. *A uniform space X is totally bounded iff $\mathcal{U}(X)$ is compact.*

Proof. Let $X = (X, M)$ be a uniform space. Suppose that X is totally bounded. Let $U \subseteq U_X$, and suppose that $U_X \lhd_X U$. Choose any $d \in M$ and $\varepsilon \in \mathbb{Q}^{>0}$, and let $\{x_0, \ldots, x_{n-1}\}$ be an ε-net to X with respect to d. By (U2), we have $U_X \lhd_X \mathcal{C}^{\varepsilon}_{\{d\}} \lhd_X \{\mathsf{b}_d(x_i, 2\varepsilon) \mid i < n\}$. Thus, there exists $\mathsf{b}_d(y, \delta) \in U_X$ such that $U_X \lhd_X \mathsf{b}_d(y, \delta)$. Since $\mathsf{b}_d(y, 2\delta) \lhd_X U$, there exists $V \in \mathsf{Fin}(U_X)$ such that $\mathsf{b}_d(y, \delta) \lhd_X V <_X U$ by Lemma 3.1.35. Then, there exists $U_0 \in \mathsf{Fin}(U)$ such that $U_X \lhd_X U_0$. Therefore, $\mathcal{U}(X)$ is compact.

Conversely, suppose that $\mathcal{U}(X)$ is compact. Let $A \in \mathsf{Fin}^+(M)$ and $\varepsilon \in \mathbb{Q}^{>0}$. Since $U_X \lhd_X \mathcal{C}^{\varepsilon}_{A}$, there exists $V = \{\mathsf{b}_A(x_0, \varepsilon), \ldots, \mathsf{b}_A(x_{n-1}, \varepsilon)\} \in \mathsf{Fin}(\mathcal{C}^{\varepsilon}_{A})$ such that $U_X \lhd_X V$. Then, $X = U_{X*} \subseteq V_* = \bigcup_{i<n} B_A(x_i, \varepsilon)$, and hence $\{x_0, \ldots, x_{n-1}\}$ is an ε-net to X with respect to ρ_A. Therefore, X is totally bounded. □

3.1.5 Functorial embedding I

We show that the category of locally compact uniform spaces can be embedded into that of formal topologies by extending the construction of a localic completion to a full and faithful functor. The results in this section extend the corresponding results for metric spaces [48, Section 5].

Definition 3.1.40. A function $f \colon X \to Y$ from a locally compact uniform space X to a uniform space Y is *continuous* if f is uniformly continuous on each open ball of X, or equivalently if f is uniformly continuous on each compact subset of X.

Since the image of a totally bounded uniform space under a uniformly continuous function is again totally bounded, continuous functions between locally compact uniform spaces are closed under composition. Thus, the locally compact uniform spaces and continuous functions between them form a category, which we denote by **LKUSpa**.

Lemma 3.1.41. *A locally compact uniform space is complete.*

Proof. Let $X = (X, M)$ be a locally compact uniform space. Let $\mathcal{F}$ be a Cauchy filter on X. Choose any $d \in M$ and $\varepsilon \in \mathbb{Q}^{>0}$. By (CF3) there exist $x \in X$ and $U_0 \in \mathcal{F}$ such that $U_0 \subseteq B_d(x, \varepsilon)$. Since X is locally compact, there exists a compact subset $K \subseteq X$ such that $B_d(x, \varepsilon) \subseteq K$. Let $\mathcal{G} = \{U \in \mathcal{F} \mid U \subseteq K\}$. Note that $\mathcal{G}$ is a filter on K. We show that $\mathcal{G}$ is a Cauchy filter on K, i.e. $\mathcal{G}$ satisfies (CF3). Let $A \in \mathsf{Fin}^+(M)$ and $\delta \in \mathbb{Q}^{>0}$. By (CF3), there exist $y \in X$ and $U \in \mathcal{F}$ such that $U \subseteq B_A(y, \delta/2)$. Then, by (CF2) there exists $W \in \mathcal{F}$ such that $W \subseteq U \cap U_0$. There exists $z \in W$ by (CF1) so that $W \subseteq B_A(z, \delta)$. Since $z \in W \subseteq U_0 \subseteq K$, we have $W \in \mathcal{G}$ and $z \in K$. Thus, $\mathcal{G}$ is a Cauchy filter on K. Since K is complete, $\mathcal{G}$ converges to some $w \in K$. Since $\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{F}$ also converges to w. Therefore, X is complete. □

Thus, for each locally compact uniform space X, the embedding $i_X : X \to \mathcal{P}t(\mathcal{U}(X))$ is a uniform isomorphism.

Given any function $f : X \to Y$ between uniform spaces (X, M) and (Y, N), define a relation $r_f \subseteq U_X \times U_Y$ by

$$a \; r_f \; b \overset{\text{def}}{\iff} (\exists b' <_Y b)\, f[a_*] \subseteq b'_* \tag{3.10}$$

for all $a \in U_X$ and $b \in U_Y$.

Lemma 3.1.42. *If $f : X \to Y$ is uniformly continuous on each open ball of X, then r_f is a formal topology map from $\mathcal{U}(X)$ to $\mathcal{U}(Y)$.*

Proof. (FTMi1): Let $a \in U_X$. Choose any $d \in N$ and $\varepsilon \in \mathbb{Q}^{>0}$. Since f is uniformly continuous on a_*, there exist $A \in \mathsf{Fin}^+(M)$ and $\delta \in \mathbb{Q}^{>0}$ such that

$$(\forall x, x' \in a_*)\, \rho_A(x, x') < \delta \implies d(f(x), f(x')) < \varepsilon.$$

Then by (U2), we have $a \lhd_X a \downarrow \mathcal{C}_A^{\delta} \subseteq r_f{}^{-} \mathcal{C}_{\{d\}}^{2\varepsilon} \subseteq r_f{}^{-} U_Y$.

(FTMi2): Let $b, c \in U_Y$ and $a \in r_f{}^- b \downarrow r_f{}^- c$. Then, there exist $b' <_Y b$ and $c' <_Y c$ such that $f[a_*] \subseteq b'_* \cap c'_*$. Write $b' = \mathsf{b}_B(y, \delta)$ and $c' = \mathsf{b}_C(z, \gamma)$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\mathsf{b}_B(y, \delta + 2\theta) <_Y b$ and $\mathsf{b}_C(z, \gamma + 2\theta) <_Y c$. Since f is uniformly continuous on a_*, there exist $A \in \mathsf{Fin}^+(M)$ and $\varepsilon \in \mathbb{Q}^{>0}$ such that

$$(\forall x, x' \in a_*)\, \rho_A(x, x') < \varepsilon \implies \rho_{B \cup C}(f(x), f(x')) < \theta.$$

Let $\mathsf{b}_{A'}(x', \varepsilon') \in a \downarrow \mathcal{C}^{\varepsilon}_A$. Then, $f[\mathsf{b}_{A'}(x', \varepsilon')_*] \subseteq \mathsf{b}_{B \cup C}(f(x'), \theta)_*$, and since $\mathsf{b}_{B \cup C}(f(x'), 2\theta) \in \mathsf{b}_B(y, \delta + 2\theta) \downarrow \mathsf{b}_C(z, \gamma + 2\theta) \subseteq b \downarrow c$, we have $\mathsf{b}_{A'}(x', \varepsilon') \in r_f{}^-(b \downarrow c)$. Hence by (U2), we have $a \lhd_X r_f{}^-(b \downarrow c)$.

(FTMi3): For (U1), we have $r_f{}^- b \lhd_X r_f{}^- \{b' \in U_Y \mid b' <_Y b\}$ for all $b \in U_Y$ by Lemma 3.1.10.2. For (U2), the argument is similar to the proof of (FTMi1) above.

(FTMi4): Obvious. $\square$

Lemma 3.1.43. *Let $X = (X, M)$ be a locally compact uniform space, and let $Y = (Y, N)$ be a complete uniform space. For any formal topology map $r : \mathcal{U}(X) \to \mathcal{U}(Y)$, the composition*

$$f = i_Y{}^{-1} \circ \mathcal{P}t(r) \circ i_X$$

is uniformly continuous on each open ball of X. Here, $\mathcal{P}t(r) : \mathcal{P}t(\mathcal{U}(X)) \to \mathcal{P}t(\mathcal{U}(Y))$ is a function defined by $\mathcal{P}t(r)(\alpha) = r\alpha$ for each $\alpha \in \mathcal{P}t(\mathcal{U}(X))$ (See (2.6)).

Proof. Let $B_A(x, \varepsilon)$ be an open ball of X, and let $d \in N$ and $\delta \in \mathbb{Q}^{>0}$. By (U2) and (FTMi3), we have $\mathsf{b}_A(x, 3\varepsilon) \lhd_X r^- \mathcal{C}^{\delta/2}_{\{d\}}$. Then by Lemma 3.1.35, there exists $V \in \mathsf{Fin}(U_X)$ such that $\mathsf{b}_A(x, 2\varepsilon) \sqsubseteq V <_X r^- \mathcal{C}^{\delta/2}_{\{d\}}$. Thus, there exists $(B, \gamma) \in \mathcal{E}_X$ such that $\mathsf{b}_A(x, 2\varepsilon) \downarrow \mathcal{C}^{\gamma}_B \leq_X V$. Let $\theta = \min\{\varepsilon, \gamma\}$ and $C = A \cup B$. Let $z, z' \in B_A(x, \varepsilon)$ such that $\rho_C(z, z') < \theta$. Since $\mathsf{b}_C(z, \theta) <_X \mathsf{b}_A(x, 2\varepsilon)$, there exists $b \in \mathcal{C}^{\delta/2}_{\{d\}}$ such that $\mathsf{b}_C(z, \theta) \lhd_X r^- b$. Since $\mathsf{b}_C(z, \theta) \in \Diamond z \cap \Diamond z'$, we have $b \in r\Diamond z \cap r\Diamond z'$. Hence, $d(f(z), f(z')) \leq \delta/2 + \delta/2 = \delta$. Therefore, f is uniformly continuous on $B_A(x, \varepsilon)$. $\square$

Lemma 3.1.44. *Let X and Y be complete uniform spaces, and let $f : X \to Y$ be a function which is uniformly continuous on each open ball of X. Then, the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \mathcal{P}t(\mathcal{U}(X)) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}t(r_f)} \\ Y & \xleftarrow[i_Y{}^{-1}]{} & \mathcal{P}t(\mathcal{U}(Y)) \end{array}$$

Proof. Since $\mathcal{U}(Y)$ is regular, it suffices to show that $\mathcal{P}t(r_f)(\Diamond x) \subseteq \Diamond f(x)$ for each $x \in X$. Let $x \in X$ and $b \in \mathcal{P}t(r_f)(\Diamond x)$. Then, there exist $a \in \Diamond x$ and $b' <_Y b$ such that $f[a_*] \subseteq b'_*$. Since $x \in a_*$, we have $f(x) \in b_*$, i.e. $b \in \Diamond f(x)$. $\square$

Lemma 3.1.45. *Let X be a locally compact uniform space, and let Y be a complete uniform space. Then, for any formal topology map $r : \mathcal{U}(X) \to \mathcal{U}(Y)$, we have $r_f = r$, where $f \stackrel{\text{def}}{=} i_Y{}^{-1} \circ \mathcal{P}t(r) \circ i_X$.*

Proof. Since $\mathcal{U}(Y)$ is regular, it suffices to show that $r \leq r_f$. Let $a \in U_X$ and $b \in U_Y$, and suppose that $a \; r \; b$. Then, $a \lhd_X r^- \{b' \in U_Y \mid b' <_Y b\}$. Let $a' \in r^- \{b' \in U_Y \mid b' <_Y b\}$, and let $b' <_Y b$ such that $a' \; r \; b'$. Then, we have $f[a'_*] \subseteq b'_*$, so that $a' \; r_f \; b$. Thus, $a \lhd_X {r_f}^- b$, and hence $r \leq r_f$. □

Lemma 3.1.46. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between locally compact uniform spaces. Then,*

$$r_{g \circ f} = r_g \circ r_f.$$

Proof. By Lemma 3.1.45, it suffices to show that

$${i_Z}^{-1} \circ \mathcal{P}t(r_{g \circ f}) \circ i_X = {i_Z}^{-1} \circ \mathcal{P}t(r_g \circ r_f) \circ i_X.$$

We have

$$\begin{aligned} {i_Z}^{-1} \circ \mathcal{P}t(r_{g \circ f}) \circ i_X &= g \circ f \\ &= {i_Z}^{-1} \circ \mathcal{P}t(r_g) \circ i_Y \circ {i_Y}^{-1} \circ \mathcal{P}t(r_f) \circ i_X \\ &= {i_Z}^{-1} \circ \mathcal{P}t(r_g) \circ \mathcal{P}t(r_f) \circ i_X \\ &= {i_Z}^{-1} \circ \mathcal{P}t(r_g \circ r_f) \circ i_X. \end{aligned}$$

□

Similarly, we can show that $r_{id_X} = id_{\mathcal{U}(X)}$ for any locally compact uniform space X. Hence, we conclude as follows.

Theorem 3.1.47. *The localic completion $\mathcal{U}$ extends to a full and faithful functor*

$$\mathcal{U} \colon \mathbf{LKUSpa} \to \mathbf{OLKReg}$$

from the category of locally compact uniform spaces **LKUSpa** *to that of overt locally compact regular formal topologies* **OLKReg**.

Proof. For each morphism $f : X \to Y$ of **LKUSpa**, define $\mathcal{U}(f) = r_f$. Then, by Lemma 3.1.46, $\mathcal{U}$ is a functor. By Lemma 3.1.44, $\mathcal{U}$ is faithful, and by Lemma 3.1.43 and Lemma 3.1.45, $\mathcal{U}$ is full. □

By an abuse of terminology, we call the functor $\mathcal{U} \colon \mathbf{LKUSpa} \to \mathbf{OLKReg}$ the localic completion.

3.1.6 Finite products

We show that the localic completion preserves finite products of locally compact uniform spaces.

Given any pair of uniform spaces $X = (X, M)$ and $Y = (Y, N)$, their binary product $X \times Y$ is a uniform space $(X \times Y, M \times N)$, where each pair $(d, \sigma) \in M \times N$ denotes a pseudometric on $X \times Y$ defined by

$$(d, \sigma)\left((x, y), (x', y')\right) \stackrel{\text{def}}{=} \max\left\{d(x, x'), \sigma(y, y')\right\}.$$

It is straightforward to check that the uniform space $X \times Y$ together with the projections to X and Y is a binary product of X and Y in **USpa**.

Lemma 3.1.48. *A binary product of locally compact uniform spaces is locally compact.*

Proof. Let $X = (X, M)$ and $Y = (Y, N)$ be locally compact uniform spaces. Let $A = \{(d_0, \sigma_0), \ldots, (d_n, \sigma_n)\} \in \mathsf{Fin}^+(M \times N)$, $\varepsilon \in \mathbb{Q}^{>0}$ and $(x, y) \in X \times Y$. We must find a compact subset $Z \subseteq X \times Y$ such that $B_A((x, y), \varepsilon) \subseteq Z$. Let $A_X = \{d_k \mid k \leq n\}$ and $A_Y = \{\sigma_k \mid k \leq n\}$. Since X and Y are locally compact, there exist compact subsets $K \subseteq X$ and $L \subseteq Y$ such that $B_{A_X}(x, \varepsilon) \subseteq K$ and $B_{A_Y}(y, \varepsilon) \subseteq L$. Then $B_A((x, y), \varepsilon) \subseteq K \times L$. Since inhabited compact uniform spaces are closed under finite product (See Corollary 3.1.52), the subset $K \times L$ is compact. □

Let X and Y be locally compact uniform spaces. By the localic completion $\mathcal{U} : \mathbf{LKUSpa} \to \mathbf{OLKReg}$, the projection $\pi_X : X \times Y \to X$ gives rise to a formal topology map $\mathcal{U}(\pi_X) : \mathcal{U}(X \times Y) \to \mathcal{U}(X)$ given by

$$\begin{aligned}\mathsf{b}_C((x, y), \varepsilon)\, \mathcal{U}(\pi_X)\, a \overset{\text{def}}{\iff} &(\exists\, \mathsf{b}_A(z, \delta) <_X a)\, (\forall (x', y') \in X \times Y) \\ &\rho_C((x, y), (x', y')) < \varepsilon \implies \rho_A(z, x') < \delta.\end{aligned}$$

Similarly, the projection π_Y determines a formal topology map $\mathcal{U}(\pi_Y) : \mathcal{U}(X \times Y) \to \mathcal{U}(Y)$. Let $r : \mathcal{U}(X \times Y) \to \mathcal{U}(X) \times \mathcal{U}(Y)$ denote the canonical map $\langle \mathcal{U}(\pi_X), \mathcal{U}(\pi_Y) \rangle : \mathcal{U}(X \times Y) \to \mathcal{U}(X) \times \mathcal{U}(Y)$.

Theorem 3.1.49. *For any locally compact uniform spaces $X = (X, M)$ and $Y = (Y, N)$,*

$$\mathcal{U}(X) \times \mathcal{U}(Y) \cong \mathcal{U}(X \times Y).$$

Proof. In the following, we write $\lhd$ for the cover of $\mathcal{U}(X) \times \mathcal{U}(Y)$. Define a relation $s \subseteq (U_X \times U_Y) \times U_{X \times Y}$ by

$$\begin{aligned}&(\mathsf{b}_A(x, \varepsilon), \mathsf{b}_B(y, \delta))\; s\; \mathsf{b}_C((x', y'), \gamma) \\ &\quad \overset{\text{def}}{\iff} \mathsf{b}_A(x, \varepsilon) <_X \mathsf{b}_{C_X}(x', \gamma) \;\&\; \mathsf{b}_B(y, \delta) <_Y \mathsf{b}_{C_Y}(y', \gamma),\end{aligned}$$

where $C_X \overset{\text{def}}{=} \{d \in M \mid (\exists \sigma \in N)\, (d, \sigma) \in C\}$, and C_Y is similarly defined.

We show that s is a formal topology map from $\mathcal{U}(X) \times \mathcal{U}(Y)$ to $\mathcal{U}(X \times Y)$, and that s is the inverse of r. We check (FTMi1) – (FTMi4).

(FTMi1): For any $(\mathsf{b}_A(x, \varepsilon), \mathsf{b}_B(y, \delta)) \in U_X \times U_Y$, we have

$$(\mathsf{b}_A(x, \varepsilon), \mathsf{b}_B(y, \delta))\; s\; \mathsf{b}_{A \times B}((x, y), \max\{\varepsilon, \delta\} + 1).$$

(FTMi2): Let $\mathsf{b}_C((u, v), \xi), \mathsf{b}_D((u', v'), \zeta) \in U_{X \times Y}$ and $(a, b) = (\mathsf{b}_A(x, \varepsilon), \mathsf{b}_B(y, \delta)) \in U_X \times U_Y$, and suppose that

$$(a, b) \in s^- \mathsf{b}_C((u, v), \xi) \downarrow s^- \mathsf{b}_D((u', v'), \zeta).$$

Choose $\theta \in \mathbb{Q}^{>0}$ such that

$$\begin{aligned}&\mathsf{b}_A(x, \varepsilon + 2\theta) <_X \mathsf{b}_{C_X}(u, \xi), && \mathsf{b}_A(x, \varepsilon + 2\theta) <_X \mathsf{b}_{D_X}(u', \zeta), \\ &\mathsf{b}_B(y, \delta + 2\theta) <_Y \mathsf{b}_{C_Y}(v, \xi), && \mathsf{b}_B(y, \delta + 2\theta) <_Y \mathsf{b}_{D_Y}(v', \zeta).\end{aligned}$$

By (U2), we have $(a,b) \lhd \left(\mathcal{C}_A^\theta \times \mathcal{C}_B^\theta\right) \downarrow (a,b)$. Let $(\mathsf{b}_{A'}(x',\varepsilon'), \mathsf{b}_{B'}(y',\delta')) \in \left(\mathcal{C}_A^\theta \times \mathcal{C}_B^\theta\right) \downarrow (a,b)$. Then,

$$\begin{aligned}\rho_{C_X}(x',u) + 2\theta &\leq \rho_{C_X}(x',x) + \rho_{C_X}(x,u) + 2\theta \\ &\leq \rho_A(x',x) + \rho_{C_X}(x,u) + 2\theta \\ &\leq \varepsilon + \rho_{C_X}(x,u) + 2\theta < \xi.\end{aligned}$$

Thus, $\mathsf{b}_{A'}(x',2\theta) <_X \mathsf{b}_{C_X}(u,\xi)$. Similarly, we have $\mathsf{b}_{B'}(y',2\theta) <_Y \mathsf{b}_{C_Y}(v,\xi)$. Hence

$$\mathsf{b}_{A'\times B'}\left((x',y'),2\theta\right) <_{X\times Y} \mathsf{b}_C\left((u,v),\xi\right).$$

By the similar argument, we have

$$\mathsf{b}_{A'\times B'}\left((x',y'),2\theta\right) <_{X\times Y} \mathsf{b}_D\left((u',v'),\zeta\right).$$

Since $(\mathsf{b}_{A'}(x',\varepsilon'), \mathsf{b}_{B'}(y',\delta'))\ s\ \mathsf{b}_{A'\times B'}\left((x',y'),2\theta\right)$, we have

$$(\mathsf{b}_A(x,\varepsilon), \mathsf{b}_B(y,\delta)) \lhd s^-\left(\mathsf{b}_C\left((u,v),\xi\right) \downarrow \mathsf{b}_D\left((u',v'),\zeta\right)\right).$$

(FTMi3): We must check this condition for (U1) and (U2). The case for (U1) is obvious from the definition of s. For (U2), given any $\varepsilon \in \mathbb{Q}^{>0}$ and $A \in \mathsf{Fin}^+(M \times N)$, choose $\delta \in \mathbb{Q}^{>0}$ such that $\delta < \varepsilon$. Then, we have

$$U_X \times U_Y \lhd \mathcal{C}_{A_X}^\delta \times \mathcal{C}_{A_Y}^\delta \subseteq s^-\mathcal{C}_A^\varepsilon$$

by (U2).

(FTMi4): This is obvious from the definition of s.

Next, we show that s is the inverse of r, i.e. $s \circ r = id_{\mathcal{U}(X\times Y)}$ and $r \circ s = id_{\mathcal{U}(X)\times\mathcal{U}(Y)}$ hold. Since these are maps between regular formal topologies, it suffices to show that $id_{\mathcal{U}(X\times Y)} \leq s \circ r$ and $id_{\mathcal{U}(X)\times\mathcal{U}(Y)} \leq r \circ s$.

$id_{\mathcal{U}(X\times Y)} \leq s \circ r$: Let $\mathsf{b}_A\left((x,y),\varepsilon\right) \in U_{X\times Y}$. By (U1), we have

$$\mathsf{b}_A\left((x,y),\varepsilon\right) \lhd_{X\times Y} \left\{\mathsf{b}_A\left((x,y),\delta\right) \in U_{X\times Y} \mid \delta \in \mathbb{Q}^{>0} \ \&\ \delta < \varepsilon\right\}.$$

Let $\delta \in \mathbb{Q}^{>0}$ such that $\delta < \varepsilon$, and choose $\gamma \in \mathbb{Q}^{>0}$ such that $\delta < \gamma < \varepsilon$. Then, we have

$$\begin{gathered}\mathsf{b}_A\left((x,y),\delta\right)\ r\ (\mathsf{b}_{A_X}(x,\gamma), \mathsf{b}_{A_Y}(y,\gamma)), \\ (\mathsf{b}_{A_X}(x,\gamma), \mathsf{b}_{A_Y}(y,\gamma))\ s\ \mathsf{b}_A\left((x,y),\varepsilon\right).\end{gathered}$$

Thus, the conclusion follows from transitivity.

$id_{\mathcal{U}(X)\times\mathcal{U}(Y)} \leq r \circ s$: Let $(a,b) \in U_X \times U_Y$. By (U1), we have

$$(a,b) \lhd \{(a',b') \in U_X \times U_Y \mid a' <_X a \ \&\ b' <_Y b\}.$$

Let $(a',b') = (\mathsf{b}_A(x,\varepsilon), \mathsf{b}_B(y,\delta)) \in U_X \times U_Y$ such that $a' <_X a$ and $b' <_Y b$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\mathsf{b}_A(x,\varepsilon+2\theta) <_X a$ and $\mathsf{b}_B(y,\delta+2\theta) <_Y b$. By (U2), we have $(a',b') \lhd \left(\mathcal{C}_A^\theta \times \mathcal{C}_B^\theta\right) \downarrow (a',b')$. Let $(\mathsf{b}_{A'}(x',\varepsilon'), \mathsf{b}_{B'}(y',\delta')) \in \left(\mathcal{C}_A^\theta \times \mathcal{C}_B^\theta\right) \downarrow (a',b')$. Then

$$\rho_A(x',x) + 2\theta \leq \varepsilon + 2\theta,$$

and thus, $\mathsf{b}_{A'}(x', 2\theta) \leq_X \mathsf{b}_A(x, \varepsilon + 2\theta) <_X a$. Similarly, we have $\mathsf{b}_{B'}(y', 2\theta) \leq_Y \mathsf{b}_B(y, \delta + 2\theta) <_Y b$. Hence, we have

$$(\mathsf{b}_{A'}(x', \varepsilon'), \mathsf{b}_{B'}(y', \delta')) \; s \; \mathsf{b}_{A' \times B'}((x', y'), 2\theta),$$
$$\mathsf{b}_{A' \times B'}((x', y'), 2\theta) \; r \; (a, b),$$

and the desired conclusion follows. □

Corollary 3.1.50. *The localic completion $\mathcal{U}$ preserves all finite products.*

Proof. It is easy to see that a one point set $X = \{*\}$ with the discrete metric is a terminal object in the category of locally compact uniform spaces. The base of the localic completion $\mathcal{U}(X)$ of X can be identified with the set $\mathbb{Q}^{>0}$ of positive rationals. We show that $\mathcal{U}(X)$ is isomorphic to $\mathbf{1} = (\{*\}, \in, =)$, the terminal object in **FTop** (See Section 2.1.3). There exists a canonical map $r : \mathcal{U}(X) \to \mathbf{1}$ given by $q \; r \; *$ for all $q \in \mathbb{Q}^{>0}$. Conversely, it is straightforward to check that a relation $s = \{*\} \times \mathbb{Q}^{>0}$ is a formal topology map $s : \mathbf{1} \to \mathcal{U}(X)$. Then, $s \circ r \leq id_{\mathcal{U}(X)}$ holds by (U2), so that $s \circ r = id_{\mathcal{U}(X)}$ by Lemma 2.4.9. Conversely, we always have $r \circ s = id_{\mathbf{1}}$. Hence, $\mathcal{U}(X)$ is isomorphic to $\mathbf{1}$. Therefore, by Theorem 3.1.49, $\mathcal{U}$ preserves all finite products. □

3.1.7 Countable products

We show that the localic completion preserves countable products of *inhabited* compact uniform spaces[4].

Proposition 3.1.51 ([9, Chapter 4, Problems 26][5]). *The product of a sequence of inhabited totally bounded uniform spaces is totally bounded.*

Proof. Let $((X_n, M_n))_{n\in\mathbb{N}}$ be a sequence of inhabited totally bounded uniform spaces, and let $(\prod_{n\in\mathbb{N}} X_n, M_\Pi)$ be the product of the sequence as defined by (3.7). Let $A \in \mathsf{Fin}^+(M_\Pi)$. Since the equality on $\mathbb{N}$ is decidable, we can write

$$A = (\{n_0\} \times A_0) \cup \cdots \cup (\{n_{N-1}\} \times A_{N-1})$$

such that $k \neq l \implies n_k \neq n_l$ for each $k, l < N$ and that $A_k \in \mathsf{Fin}^+(M_{n_k})$ for each $k < N$. Let $\varepsilon \in \mathbb{Q}^{>0}$. For each $k < N$, let $X^\varepsilon_{n_k} \in \mathsf{Fin}(X_{n_k})$ be an ε-net to X_{n_k} with respect to the pseudometric ρ_{A_k}. Let

$$X^\varepsilon \overset{\text{def}}{=} X^\varepsilon_{n_0} \times \cdots \times X^\varepsilon_{n_{N-1}}.$$

[4]We can decide whether each compact uniform space is inhabited or empty. Constructively, however, given a sequence of compact uniform spaces, we cannot in general decide whether all members of the sequence are inhabited or there exists an empty member in the sequence. Hence, the product of a sequence of compact uniform spaces may not be compact in general.

[5]In [8, Chapter 4, Problems 21], Bishop claimed that a product of any family of inhabited totally bounded uniform spaces is totally bounded. This seems to be false without the full form of the Axiom of Choice. In the later edition with Douglas Bridges, however, the claim was restricted to the case of countable products [9, Chapter 4, Problems 26].

Since X_n is inhabited for each $n \in \mathbb{N}$, the set $\prod_{n\in\mathbb{N}} X_n$ is inhabited by the Countable Choice. Take any $(y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$. For each $\overline{x} = \langle x_0, \dots, x_{N-1}\rangle \in X^\varepsilon$, define $f_{\overline{x}} \in \prod_{n\in\mathbb{N}} X_n$ by

$$f_{\overline{x}}(n) \stackrel{\text{def}}{=} \begin{cases} y_n & \text{if } n \neq n_k \text{ for all } k < N, \\ x_k & \text{if } n = n_k \text{ for some } k < N. \end{cases}$$

Let Y be the finitely enumerable subset of $\prod_{n\in\mathbb{N}} X_n$ given by

$$Y \stackrel{\text{def}}{=} \{ f_{\overline{x}} \mid \overline{x} \in X^\varepsilon \}.$$

We show that Y is an ε-net to $\prod_{n\in\mathbb{N}} X_n$ with respect ρ_A. Let $g \in \prod_{n\in\mathbb{N}} X_n$. For each $k < N$, there exists $x_k \in X^\varepsilon_{n_k}$ such that

$$\rho_{A_k}(g_{n_k}, x_k) < \varepsilon.$$

Then, $\overline{x} = \langle x_0, \dots, x_{N-1}\rangle \in X^\varepsilon$, and so $f_{\overline{x}} \in Y$. Then, $\rho_A(f_{\overline{x}}, g) < \varepsilon$. Therefore, $\prod_{n\in\mathbb{N}} X_n$ is totally bounded. □

Corollary 3.1.52. *A countable product of inhabited compact uniform spaces is compact.*

Proof. By Proposition 3.1.51 and Proposition 3.1.30.2. □

Let $((X_n, M_n))_{n\in\mathbb{N}}$ be a sequence of inhabited compact uniform spaces, and let $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n) = (S_\Pi, \lhd_\Pi, \leq_\Pi)$ be the product of the sequence $(\mathcal{U}(X_n))_{n\in\mathbb{N}}$ in **FTop**. Then, $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ is inductively generated by the following axiom-set (See Section 2.2.5):

(A1) $S_\Pi \lhd_\Pi \{\{(n,a)\} \in S_\Pi \mid a \in U_{X_n}\}$ for each $n \in \mathbb{N}$;

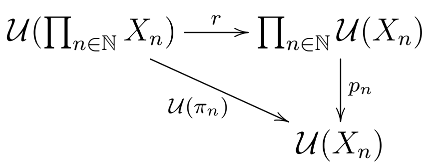
(A2) $\{(n,a),(n,b)\} \lhd_\Pi \{\{(n,c)\} \in S_\Pi \mid c \leq_{X_n} a \ \&\ c \leq_{X_n} b\}$ for each $n \in \mathbb{N}$ and $a, b \in U_{X_n}$;

(A3) $\{(n,a)\} \lhd_\Pi \{\{(n,b)\} \in S_\Pi \mid b <_{X_n} a\}$ for each $n \in \mathbb{N}$ and $a \in U_{X_n}$;

(A4) $S_\Pi \lhd_\Pi \{\{(n,a)\} \in S_\Pi \mid a \in {}_n\mathcal{C}^\varepsilon_A\}$ for each $n \in \mathbb{N}$ and $(A, \varepsilon) \in \mathcal{E}_{X_n}$.

Here, ${}_n\mathcal{C}^\varepsilon_A$ is given by ${}_n\mathcal{C}^\varepsilon_A \stackrel{\text{def}}{=} \{\mathsf{b}_A(x,\varepsilon) \in U_{X_n} \mid x \in X_n\}$.

By applying the localic completion to the product diagram $\left(\pi_n : \prod_{n\in\mathbb{N}} X_n \to X_n\right)_{n\in\mathbb{N}}$, we obtain a diagram $\left(\mathcal{U}(\pi_n) : \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \mathcal{U}(X_n)\right)_{n\in\mathbb{N}}$ in **FTop**. Hence, there exists a canonical formal topology map $r : \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ which makes the following diagram commute for each $n \in \mathbb{N}$.



Lemma 3.1.53. *The formal topology map $r : \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ is an embedding.*

Proof. Put $\mathcal{U}(\prod_{n\in\mathbb{N}} X_n) = (U_X, \lhd_X, \leq_X)$, and write r_n for each $\mathcal{U}(\pi_n) : \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \mathcal{U}(X_n)$. Let $a = \mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon) \in U_X$. We must show that $a \lhd_X r^- r^{-*} \mathcal{A}_X \{a\}$. By (U1), we have

$$\mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon) \lhd_X \left\{\mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon') \in U_X \mid \varepsilon' \in \mathbb{Q}^{>0} \,\&\, \varepsilon' < \varepsilon\right\}.$$

Let $\varepsilon' \in \mathbb{Q}^{>0}$ such that $\varepsilon' < \varepsilon$, and choose $\gamma, \zeta \in \mathbb{Q}^{>0}$ such that $\varepsilon' + 2\gamma < \varepsilon$ and $\zeta < \gamma$. By (U2), we have

$$\mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon') \lhd_X \mathcal{C}_A^\zeta \downarrow \mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon').$$

Let $\mathsf{b}_B((y_n)_{n\in\mathbb{N}}, \theta) \in \mathcal{C}_A^\zeta \downarrow \mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon')$. We show that $\mathsf{b}_B((y_n)_{n\in\mathbb{N}}, \theta) \in r^- r^{-*} \mathcal{A}_X \{a\}$. Recall that A is an element of $\mathsf{Fin}^+\left(\sum_{n\in\mathbb{N}} M_n\right)$. Let $U = \{(k, \mathsf{b}_d(y_k, \gamma)) \mid (k, d) \in A\} \in S_\Pi$, where $\mathsf{b}_d(y_k, \gamma) \in U_{X_k}$ for each $(k, d) \in A$. Then, for each $(k, d) \in A$, we have

$$\pi_k \left[\mathsf{b}_B((y_n)_{n\in\mathbb{N}}, \theta)_*\right] \subseteq \mathsf{b}_d(y_k, \zeta)_*.$$

Thus, we have $\mathsf{b}_B((y_n)_{n\in\mathbb{N}}, \theta)\ r_k\ \mathsf{b}_d(y_k, \gamma)$ for each $(k, d) \in A$, and hence $\mathsf{b}_B((y_n)_{n\in\mathbb{N}}, \theta)\ r\ U$. It remains to be shown that $r^- \{U\} \lhd_X \{a\}$. Write $A = \{(i_0, d_0), \ldots, (i_N, d_N)\}$. By (U2) and the definition of r, it suffices to show that

$$r_{i_0}^- \{\mathsf{b}_{d_0}(y_{i_0}, \gamma)\} \downarrow \cdots \downarrow r_{i_N}^- \{\mathsf{b}_{d_N}(y_{i_N}, \gamma)\} \downarrow \mathcal{C}_A^\gamma \lhd_X a.$$

Let $\mathsf{b}_C((z_n)_{n\in\mathbb{N}}, \delta) \in r_{i_0}^- \{\mathsf{b}_{d_0}(y_{i_0}, \gamma)\} \downarrow \cdots \downarrow r_{i_N}^- \{\mathsf{b}_{d_N}(y_{i_N}, \gamma)\} \downarrow \mathcal{C}_A^\gamma$. Then, we have $\delta \leq \gamma$ and $d_k(z_{i_k}, y_{i_k}) < \gamma$ for all $k \leq N$. Thus

$$\begin{aligned}
\rho_A((z_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) + \delta &\leq \rho_A((z_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) + \rho_A((y_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) + \gamma \\
&< \gamma + \rho_A\left((y_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}\right) + \gamma \\
&< \varepsilon' + 2\gamma < \varepsilon.
\end{aligned}$$

Hence, $\mathsf{b}_C((z_n)_{n\in\mathbb{N}}, \delta) <_X \mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon)$, and thus $\mathsf{b}_C((z_n)_{n\in\mathbb{N}}, \delta) \lhd_X a$ as required. Therefore, by transitivity of $\lhd_X$, we obtain $a \lhd_X r^- r^{-*} \mathcal{A}_X \{a\}$. ☐

The image $\mathcal{S}_r$ of $\mathcal{U}(\prod_{n\in\mathbb{N}} X_n)$ under r is an overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ with the positivity $\mathsf{Pos}_X = rU_X$ by Lemma 2.3.4.

Lemma 3.1.54. Pos_X *is the largest splitting subset of* $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$.

Proof. Let $\mathsf{Pos} \subseteq S_\Pi$ be a splitting subset of $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$, and let $U \in \mathsf{Pos}$. By (A2) and (Loc), we can replace two elements $(n, a), (n, b) \in U$ with the same index $n \in \mathbb{N}$ with some (n, c) such that $c \in U_{X_n}$ and $c \in a \downarrow b$ in $\mathcal{U}(X_n)$. By applying this process finitely many times, we obtain $U' = \{(n_0, a_0), \ldots, (n_{N-1}, a_{N-1})\} \in \mathsf{Pos}$ such that $U' \leq_\Pi U$ and $n_i \neq n_j$ for all $0 \leq i < j < N$.

By (A1), for each $n \in \mathbb{N}$, there exists $a \in U_{X_n}$ such that $\{(n, a)\} \in \mathsf{Pos}$. By the Countable Choice, there exists a function $f \in \prod_{n\in\mathbb{N}} U_{X_n}$ such that $\{(n, f(n))\} \in \mathsf{Pos}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, write $f(n) = \mathsf{b}_{A_n}(y_n, \varepsilon_n)$. Define a sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ by

$$x_n \stackrel{\text{def}}{=} \begin{cases} y_n & \text{if } n \neq n_k \text{ for all } (n_k, a_k) \in U', \\ z & \text{if there exists } (n, \mathsf{b}_C(z, \delta)) \in U' \end{cases}$$

for each $n \in \mathbb{N}$. For each $i < N$, write $a_i = \mathsf{b}_{C_i}(z_i, \delta_i)$, and choose $\delta_i' \in \mathbb{Q}^{>0}$ such that $\delta_i' < \delta_i$. Let

$$A = \bigcup_{i<N} \{n_i\} \times C_i,$$
$$\varepsilon = \min\{\delta_i' \mid i < N\}.$$

Let $i < N$. Then, for any $(y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ such that $\rho_A((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) < \varepsilon$, we have $\rho_{C_i}(z_i, y_{n_i}) = \rho_{C_i}(x_{n_i}, y_{n_i}) < \delta_i'$. Thus, $\mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon)\ r_{n_i}\ \mathsf{b}_{C_i}(z_i, \delta_i)$ for each $i < N$. Hence $\mathsf{b}_A((x_n)_{n\in\mathbb{N}}, \varepsilon)\ r\ U'$, and so $U' \in \mathsf{Pos}_X$. Since Pos_X is upward closed, we have $U \in \mathsf{Pos}_X$. Therefore $\mathsf{Pos} \subseteq \mathsf{Pos}_X$. □

Proposition 3.1.55. *The image $\mathcal{S}_r$ is the largest overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$.*

Proof. Write $\mathcal{S} = \prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$. Then, $\mathcal{S}$ is regular by Proposition 2.4.11.2. Since $\prod_{n\in\mathbb{N}} X_n$ is compact by Corollary 3.1.52, $\mathcal{U}(\prod_{n\in\mathbb{N}} X_n)$ is compact by Theorem 3.1.39. Thus, $\mathcal{S}_r$ is an overt closed subtopology of $\mathcal{S}$ by Proposition 2.4.13.2. Hence, $\mathcal{S}_r$ is overt weakly closed subtopology with the positivity Pos_X by Corollary 2.3.23. Then, for any overt subtopology $\mathcal{S}'$ of $\mathcal{S}$ with a positivity Pos, we have $\mathcal{S}' \sqsubseteq \mathcal{S}^{\mathsf{Pos}} \sqsubseteq \mathcal{S}^{\mathsf{Pos}_X} = \mathcal{S}_r$ by Theorem 2.3.21. □

Let **OFTop** be the full subcategory of **FTop** consisting of overt formal topologies, and let $\mathcal{U}$: **LKUSpa** → **OFTop** be the composition of the embedding $\mathcal{U}$: **LKUSpa** → **OLKReg** followed by the inclusion **OLKReg** ↪ **OFTop**.

Theorem 3.1.56. *The functor $\mathcal{U}$:* **LKUSpa** → **OFTop** *preserves countable products of inhabited compact uniform spaces.*

Proof. Given a sequence $(X_n)_{n\in\mathbb{N}}$ of inhabited compact uniform spaces, the image $\mathcal{S}_r$ of $\mathcal{U}(\prod_{n\in\mathbb{N}} X_n)$ under the embedding $r\colon \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ described in Lemma 3.1.53 is the largest overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ by Proposition 3.1.55. Hence, by Lemma 2.3.4, any formal topology map $s\colon \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{U}(X_n)$ with an overt domain factors uniquely through $\mathcal{S}_r$. Hence, $\big(\mathcal{U}(\pi_n)\colon \mathcal{U}(\prod_{n\in\mathbb{N}} X_n) \to \mathcal{U}(X_n)\big)_{n\in\mathbb{N}}$ is a product of the sequence $(\mathcal{U}(X_n))_{n\in\mathbb{N}}$ in **OFTop**. □

3.2 Uniform spaces by covering uniformities

In this section, we extend the results obtained in Section 3.1 to the class of uniform spaces defined by covering uniformities. Classically, the two notions of uniformity, a set of pseudometrics and covering uniformity are equivalent. Constructively, the covering approach is more general than that of a set of pseudometrics since the former allows us to define the discrete uniformity on any set while the latter does not [7]. Covering uniformity is a natural extension of that of a set of pseudometrics in the sense that the category of uniform spaces defined by sets of pseudometrics can be embedded into that of uniform spaces defined by covering uniformities. Moreover, the notion of compactness is preserved and reflected by the embedding.

Let us mention that the notion of uniformity by a set of entourages is equivalent to our notion of covering uniformity. This is because we restrict our attention to those covering uniformities which are proper, and this makes the category of uniform spaces defined by sets of entourages and that of uniform spaces defined by covering uniformities equivalent (See Remark 3.2.10).

3.2.1 Uniform spaces by covering uniformities

We begin with a constructive definition of uniform space defined by a covering uniformity on a set [28]. The definition given below is classically called a base of a covering uniformity [62].

Definition 3.2.1. Let X be a set. A *cover* of X is a set $C \in \mathsf{Pow}(\mathsf{Pow}(X))$ such that $X = \bigcup C$. For any set $C \in \mathsf{Pow}(\mathsf{Pow}(X))$ and $Z \in \mathsf{Pow}(X)$, we define

$$\mathrm{St}_C(Z) \overset{\text{def}}{=} \bigcup \{A \in C \mid A \between Z\}.$$

For any $C, C' \in \mathsf{Pow}(\mathsf{Pow}(X))$, define

$$C' \leqslant C \overset{\text{def}}{\iff} (\forall A' \in C')\,(\exists A \in C)\, A' \subseteq A,$$
$$C' <^* C \overset{\text{def}}{\iff} (\forall A' \in C')\,(\exists A \in C)\, \mathrm{St}_{C'}(A') \subseteq A.$$

A *covering uniformity* on a set X is an inhabited[6] set $\mathcal{C}$ of covers of X such that

(CU1) $(\forall C_1, C_2 \in \mathcal{C})\,(\exists C_3 \in \mathcal{C})\, C_3 \leqslant C_1 \;\&\; C_3 \leqslant C_2$,

(CU2) $(\forall C \in \mathcal{C})\,(\exists C' \in \mathcal{C})\, C' <^* C$.

A covering uniformity $\mathcal{C}$ on a set X is

- *separating* if $[(\forall C \in \mathcal{C})\,(\exists A \in C)\, x, y \in A] \implies x = y$ for all $x, y \in X$,
- *proper* if $(\forall C \in \mathcal{C})\,(\exists C' \in \mathcal{C})\, C' \leqslant C^+$, where $C^+ \overset{\text{def}}{=} \{A \in C \mid A \between X\}$.

Remark 3.2.2. Not all covering uniformities are proper constructively [28]. If every covering uniformity were proper, it would imply $\neg\varphi \vee \neg\neg\varphi$ for any restricted formula φ[7]. In what follows, we are only interested in uniformities which are proper.

Definition 3.2.3. A *uniform space* is a pair $(X, \mathcal{C})$, where X is a set and $\mathcal{C}$ is a proper separating covering uniformity on X.

[6] Some authors omit the condition that the uniformity be inhabited. However, a uniform space with the empty covering uniformity is rather a strange object; for example, a one-point set $\{*\}$ with the discrete covering uniformity $\{\{*\}\}$ should be the terminal object in the category of uniform spaces, but there is no uniformly continuous function from a uniform space with the empty uniformity to $(\{*\}, \{\{*\}\})$.

[7] In particular, this implies non-constructive principle WLPO which contradicts Church's Thesis [10].

Notation 3.2.4. For a uniform space $(X, \mathcal{C})$, we use the same symbol as the underlying set to denote the uniform space, e.g. the above uniform space is denoted by X.

Remark 3.2.5. In Section 3.1, we defined a uniform space as a set equipped with a set of pseudometrics. In this section, however, unless explicitly mentioned, the term 'uniform space' always means a set with a covering uniformity. We also use the term 'uniformity' to mean a covering uniformity. Similar conventions apply to the other terms.

Definition 3.2.6. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be uniform spaces. A function $f : X \to Y$ is *uniformly continuous* if

$$(\forall D \in \mathcal{D})\,(\exists C \in \mathcal{C})\,(\forall A \in C)\,(\exists B \in D)\, f[A] \subseteq B.$$

A uniformly continuous function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is a *uniform isomorphism* if f has a uniformly continuous inverse $g : (Y, \mathcal{D}) \to (X, \mathcal{C})$.

Remark 3.2.7. Let $X = (X, \mathcal{C})$ be a uniform space. By letting $\mathcal{C}^+ = \{C^+ \mid C \in \mathcal{C}\}$, the condition of properness is equivalent to saying the identity function id_X is a uniform isomorphism $id_X : (X, \mathcal{C}) \to (X, \mathcal{C}^+)$. Since we are assuming that every covering uniformity of a uniform space is proper, we may assume that for any uniform space $(X, \mathcal{C})$, we have

$$(\forall C \in \mathcal{C})\,(\forall A \in C)\, A \between X. \tag{3.11}$$

In the following, we freely use this assumption.

Definition 3.2.8. Let $X = (X, \mathcal{C})$ be a uniform space. Given a subset Z of X, define

$$\begin{aligned} \mathcal{C}_Z &\stackrel{\text{def}}{=} \{C_Z \mid C \in \mathcal{C}\}, \\ C_Z &\stackrel{\text{def}}{=} \{A \cap Z \mid A \in C \;\&\; A \between Z\}. \end{aligned} \tag{3.12}$$

Then, $\mathcal{C}_Z$ is the largest uniformity on X for which the inclusion $i_Z : Z \to X$ is uniformly continuous, i.e. for any uniformity $\mathcal{D}$ on Z such that the inclusion $i_Z : (Z, \mathcal{D}) \to (X, \mathcal{C})$ is uniformity continuous, the identity function id_Z on Z is a uniformly continuous function $id_Z : (Z, \mathcal{D}) \to (Z, \mathcal{C}_Z)$.

A *subspace* of a uniform space $(X, \mathcal{C})$ is a subset Z of X together with the uniformity $\mathcal{C}_Z$ given by (3.12). In the following, a subset Z of a uniform space $(X, \mathcal{C})$ will be identified with the subspace $(Z, \mathcal{C}_Z)$.

A uniformly continuous function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is an *embedding* if the image $f[X]$ as a subspace of Y is uniformly isomorphic to X via the (co-)restriction of f to $f[X]$.

The uniform spaces and uniformly continuous functions between them form a category, which we denote by **CUSpa**. The category **USpa** of uniform spaces defined by sets of pseudometrics can be embedded into **CUSpa** by the functor $\mathcal{I}_U : \mathbf{USpa} \to \mathbf{CUSpa}$ given by

$$\begin{aligned} \mathcal{I}_U((X, M)) &\stackrel{\text{def}}{=} (X, \mathcal{C}_M), \\ \mathcal{C}_M &\stackrel{\text{def}}{=} \{C(A, \varepsilon) \mid (A, \varepsilon) \in \mathcal{E}_X\}, \\ C(A, \varepsilon) &\stackrel{\text{def}}{=} \{a_* \mid a \in \mathcal{C}_A^\varepsilon\} = \{B_A(x, \varepsilon) \mid x \in X\}. \end{aligned} \tag{3.13}$$

The fact that $\mathcal{I}_U$ is an embedding is expressed by the following.

Proposition 3.2.9. *Let $X = (X, M)$ and $Y = (Y, N)$ be uniform spaces defined by sets M, N of pseudometrics on X and Y respectively. Then, a function $f : X \to Y$ is a morphism $f : (X, M) \to (Y, N)$ of* **USpa** *iff f is a morphism $f : (X, \mathcal{C}_M) \to (Y, \mathcal{C}_N)$ of* **CUSpa**.

Proof. ($\Rightarrow$): Immediate from Lemma 3.1.7.

($\Leftarrow$): Suppose that f is a morphism of **CUSpa**. Let $d \in N$ and $\delta \in \mathbb{Q}^{>0}$. Then, there exist $A \in \mathsf{Fin}^+(M)$ and $\varepsilon \in \mathbb{Q}^{>0}$ such that for any $x \in X$, there exists $y \in Y$ such that $B_A(x, \varepsilon) \subseteq f^{-1}[B_d(y, \delta/2)]$. Let $x, x' \in X$, and suppose that $\rho_A(x, x') < \varepsilon$. Then, there exists $y \in Y$ such that $B_A(x, \varepsilon) \subseteq f^{-1}[B_d(y, \delta/2)]$. Thus $d(f(x), f(x')) \leq d(f(x), y) + d(y, f(x')) < \delta/2 + \delta/2 = \delta$. Therefore, f is a morphism of **USpa**. □

Remark 3.2.10. The category **CUSpa** is equivalent to the category of T_0 uniform spaces defined by entourage uniformities [28, 7]. Thus, one can use the notion of entourage uniformity instead of covering uniformity. However, the approach by covering uniformities seems to be more natural for the construction of a covering completion to be defined below.

3.2.2 Covering completions of uniform spaces

We introduce the notion of covering completion of a uniform space, which can be seen as the notion of localic completion for uniform spaces defined by covering uniformities.

Let $X = (X, \mathcal{C})$ be a uniform space, and let V_X be the set given by

$$V_X \stackrel{\text{def}}{=} \bigcup \mathcal{C} = \{A \mid A \in C \in \mathcal{C}\}.$$

An element $A \in V_X$ will be denoted by $\mathsf{c}(A)$. For each $\mathsf{c}(A) \in V_X$, define

$$\mathsf{c}(A)_* \stackrel{\text{def}}{=} A,$$

and for each $C \in \mathcal{C}$, let

$$V_X^C \stackrel{\text{def}}{=} \{a \in V_X \mid a_* \in C\}.$$

Define an order $\preceq_X$ and a transitive relation $\prec_X$ on V_X by

$$a \preceq_X b \stackrel{\text{def}}{\iff} a_* \subseteq b_*, \tag{3.14}$$

$$a \prec_X b \stackrel{\text{def}}{\iff} (\exists C \in \mathcal{C}) \left(\forall c \in V_X^C\right) c_* \between a_* \to c \preceq_X b. \tag{3.15}$$

For each $C \in \mathcal{C}$, define

$$a \prec_X^C b \stackrel{\text{def}}{\iff} \left(\forall c \in V_X^C\right) c_* \between a_* \to c \preceq_X b,$$

so $a \prec_X b \iff (\exists C \in \mathcal{C})\, a \prec_X^C b$. Note also that $a \prec_X^C b \iff \mathrm{St}_C(a_*) \subseteq b_*$.

The *covering completion* of a uniform space $X = (X, \mathcal{C})$ is a formal topology

$$\mathcal{V}(X) = (V_X, \blacktriangleleft_X, \preceq_X)$$

inductively generated by the axiom-set on $(V_X, \preceq_X)$ consisting of the following axioms:

(V1) $a \blacktriangleleft_X \{b \in V_X \mid b \prec_X a\}$;

(V2) $a \blacktriangleleft_X V_X^C$ for each $C \in \mathcal{C}$

for each $a \in V_X$. The axioms (V1) and (V2) are equivalent to the following localised axiom-set:

(V1') $a \blacktriangleleft_X \{b \in V_X \mid b \prec_X a\}$;

(V2') $a \blacktriangleleft_X V_X^C \downarrow a$ for each $C \in \mathcal{C}$.

Remark 3.2.11. The notion of covering completion is not new. Essentially the same constructions have been given by Fox [28] and Ishihara [32].

1. In [28], Fox defined the notion of uniform space by a constructive topological space together with a covering uniformly which induces a topology equivalent to that of the underlying topological space. He introduced the notion of uniform formal topology, a formal topology equipped with a covering uniformly, and established an adjunction between the category of uniform spaces and that of uniform formal topologies. He also defined the completion of a uniform formal topology, which is an inductively generated formal topology whose axioms are completely determined by the uniformity of the given uniform formal topology. Composition of the left adjoint of the adjunction and the completion of uniform formal topologies gives an equivalent construction as that of covering completion. However, in his construction, the underlying topological space of a given uniform space in his sense does not play any essential role.

2. Ishihara introduced the notion equivalent to covering completion in the setting of entourage uniformities [32]. He defined an inductively generated formal topology out of a given entourage uniformity. However, his axioms of the formal topology shows that he uses the representation of an entourage uniformity by the equivalent covering uniformity. Hence, the covering approach seems to be more natural.

Hence, we believe that our notion of covering completion is a natural presentation of the formal topologies considered by Fox and Ishihara. However, we owe many results to these authors. The results in Section 3.2.3 and Section 3.2.4 are essentially due to Fox, and the proof of Proposition 3.2.32 is based on the idea by Ishihara in his work. Although the results in Section 3.2.3 and Section 3.2.4 are essentially not new, they are important in obtaining an embedding of the category of compact uniform spaces into that of formal topologies, which the above two authors did not consider.

Lemma 3.2.12. *Let $X = (X, \mathcal{C})$ be a uniform space. Then,*

1. $a \prec_X b \implies a \preceq_X b$,

2. $a \preceq_X b \prec_X c \preceq_X d \implies a \prec_X d$,

3. $C \leqslant C' \ \& \ a \prec_X^{C'} b \implies a \prec_X^C b$

for all $a, b, c, d \in V_X$ *and* $C, C' \in \mathcal{C}$.

Proof. 1. Suppose that $a \prec_X b$. Then, there exists $C \in \mathcal{C}$ such that

$$\left(\forall c \in V_X^C\right) c_* \between a_* \implies c \preceq_X b.$$

Since C is a cover of X, we have $a_* \subseteq \bigcup_{c \in V_X^C} c_*$. Thus, $a_* \subseteq \bigcup_{c \in V_X^C} (c_* \cap a_*) \subseteq b_*$.

2. Suppose that $a \preceq_X b \prec_X c \preceq_X d$. Then, there exists $C \in \mathcal{C}$ such that $b \prec_X^C c$. Obviously, we have $a \prec_X^C d$, so $a \prec_X d$.

3. Obvious from the definition of $\leqslant$. □

Proposition 3.2.13. *For any uniform space* X*, its covering completion* $\mathcal{V}(X)$ *is 2-regular.*

Proof. Let $X = (X, \mathcal{C})$ be a uniform space. First, we show that $\mathcal{V}(X)$ is overt. Define

$$\mathsf{Pos}_X \stackrel{\text{def}}{=} \{a \in V_X \mid (\exists b \in V_X)\, b \prec_X a\}.$$

We show that Pos_X is the positivity of $\mathcal{V}(X)$. By (V1), we have

$$a \blacktriangleleft_X \mathsf{Pos}_X \cap \{a\}.$$

We use Proposition 2.2.17 to show that Pos_X is a splitting subset of $\mathcal{V}(X)$, using (V1') and (V2'). By Lemma 3.2.12.2, Pos_X is upward closed with respect to $\preceq_X$. It remains to be shown that Pos_X satisfies (Spl2') with respect to the axioms (V1') and (V2').

(V1'): Suppose that $\mathsf{Pos}_X(a)$. Then, there exists $b \in V_X$ such that $b \prec_X a$, so there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. Choose $C_1, C_2 \in \mathcal{C}$ such that $C_2 <^* C_1 <^* C$. Since b_* is inhabited and C_2 is a cover of X, there exists $a_2 \in V_X^{C_2}$ such that $a_{2*} \between b_*$. Then, there exist $a_1 \in V_X^{C_1}$ and $a' \in V_X^C$ such that $a_2 \prec_X a_1$ and $a_1 \prec_X a'$. Then, $a'_* \between b_*$, so that $a' \preceq_X a$. Thus, $\mathsf{Pos}_X(a_1)$ and $a_1 \prec_X a$.

(V2'): Let $C \in \mathcal{C}$, and suppose that $\mathsf{Pos}_X(a)$. Then, there exists $b \in V_X$ such that $b \prec_X a$, so there exists $C_1 \in \mathcal{C}$ such that $b \prec_X^{C_1} a$. Choose $C_2, C_3 \in \mathcal{C}$ such that $C_2 \leqslant C$, $C_2 \leqslant C_1$, and $C_3 <^* C_2$. By the similar argument as above, there exists $a_3 \in V_X^{C_3}$ such that $a_{3*} \between b_*$. Then, there exist $a_2 \in V_X^{C_2}$, $a' \in V_X^C$, and $a_1 \in V_X^{C_1}$ such that $a_3 \prec_X a_2$, $a_2 \preceq_X a'$, and $a_2 \preceq_X a_1$. Then, $a_{1*} \between b_*$ so that $a_1 \preceq_X a$. Thus, $\mathsf{Pos}_X(a_2)$, $a_2 \preceq_X a$, and $a_2 \preceq_X a'$. Therefore, Pos_X is the positivity of $\mathcal{V}(X)$.

Next, we show that $\mathcal{V}(X)$ is 2-regular, i.e. $a \blacktriangleleft_X \mathsf{wc}_2(a)$ for all $a \in V_X$. By (V1), it suffices to show that

$$a \prec_X b \implies a \in \mathsf{wc}_2(b)$$

for all $a, b \in V_X$. Let $a, b \in V_X$, and suppose that $a \prec_X b$. Then, there exists $C \in \mathcal{C}$ such that $a \prec_X^C b$. By (V2), we have

$$V_X \blacktriangleleft_X V_X^C \subseteq \{c \in V_X \mid \mathsf{Pos}_X(c \downarrow a) \to c \blacktriangleleft_X b\}.$$

Hence, $a \in \mathsf{wc}_2(b)$. □

3.2.3 The formal points of a covering completion

We define a uniformity on the formal points $\mathcal{P}t(\mathcal{V}(X))$ of the covering completion of a uniform space X, and show that $\mathcal{P}t(\mathcal{V}(X))$ is a completion of X.

First, we recall the characterisation of completeness of a uniform space in terms of the formal points of its covering completion [28] (See Proposition 3.2.24).

Definition 3.2.14. A *Cauchy point* of a uniform space $X = (X, \mathcal{C})$ is a formal point of $\mathcal{V}(X)$, namely a subset $\alpha \subseteq V_X$ such that

(CP1) $a, b \in \alpha \implies \alpha \between (a \downarrow b)$,

(CP2) $a \preceq_X b \;\&\; a \in \alpha \implies b \in \alpha$,

(CP3) $a \in \alpha \implies (\exists b \prec_X a)\, b \in \alpha$,

(CP4) $(\forall C \in \mathcal{C})\, \alpha \between V_X^C$.

Note that since $\mathcal{C}$ is inhabited, every Cauchy point is inhabited.

Completeness of a uniform space is usually defined in terms of Cauchy filters.

Definition 3.2.15. A *Cauchy filter* on a uniform space $X = (X, \mathcal{C})$ is a set $\mathcal{F}$ of subsets of X such that

(CF1) $(\forall A \in \mathcal{F})\, A \between X$,

(CF2) $(\forall A, B \in \mathcal{F})\, (\exists D \in \mathcal{F})\, D \subseteq A \cap B$,

(CF3) $(\forall C \in \mathcal{C})\, \left(\exists a \in V_X^C\right) (\exists A \in \mathcal{F})\, A \subseteq a_*$.

The class of Cauchy filters on a uniform space X is ordered by refinement:

$$\mathcal{F} \leqslant \mathcal{F}' \overset{\text{def}}{\iff} (\forall A \in \mathcal{F})\, (\exists B \in \mathcal{F}')\, B \subseteq A.$$

Cauchy filters $\mathcal{F}$ and $\mathcal{F}'$ on X are *equal*, denoted by $\mathcal{F} = \mathcal{F}'$, if

$$\mathcal{F} \leqslant \mathcal{F}' \;\&\; \mathcal{F}' \leqslant \mathcal{F}.$$

A Cauchy filter $\mathcal{F}$ on X is *minimal* if

$$\mathcal{F}' \leqslant \mathcal{F} \implies \mathcal{F} = \mathcal{F}'$$

for any Cauchy filter $\mathcal{F}'$ on X.

Remark 3.2.16. The term 'Cauchy filter' has already been introduced in Definition 3.1.29. Definition 3.2.15 and Definition 3.1.29 are compatible in the sense that for any object $X = (X, M)$ of **USpa**, a set $\mathcal{F} \subseteq \mathsf{Pow}(X)$ is a Cauchy filter on (X, M) iff $\mathcal{F}$ is a Cauchy filter on $\mathcal{I}_U(X)$. In what follows, however, unless explicitly mentioned, we use the term 'Cauchy filter' in the sense of Definition 3.2.15.

We show that the notion of minimal Cauchy filter and that of Cauchy point are equivalent. In the following, we fix a uniform space $X = (X, \mathcal{C})$.

Lemma 3.2.17. *For any Cauchy filter $\mathcal{F}$ on X, the set*

$$\alpha_{\mathcal{F}} \stackrel{\text{def}}{=} \{a \in V_X \mid (\exists b \prec_X a)\,(\exists A \in \mathcal{F})\, A \subseteq b_*\}$$

is a Cauchy point of X.

Proof. We check the conditions (CP1) – (CP4).

(CP4): Let $C \in \mathcal{C}$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. By (CF3), there exist $a' \in V_X^{C'}$ and $A \in \mathcal{F}$ such that $A \subseteq a'_*$. Then, there exists $a \in V_X^C$ such that $a' \prec_X a$. Thus, $a \in \alpha_{\mathcal{F}}$.

(CP1): Let $a, b \in \alpha_{\mathcal{F}}$. Then, there exist $a' \prec_X a$ and $b' \prec_X b$ and $A, B \in \mathcal{F}$ such that $A \subseteq a'_*$ and $B \subseteq b'_*$. Thus, there exists $C \in \mathcal{C}$ such that $a' \prec_X^C a$ and $b' \prec_X^C b$. By (CP4), there exists $c \in \alpha_{\mathcal{F}}$ such that $c \in V_X^C$. Then, $c_* \between a'_*$ and $c_* \between b'_*$ by (CF2), and so $c \in a \downarrow b$.

(CP2): Immediate from Lemma 3.2.12.2.

(CP3): Let $a \in \alpha_{\mathcal{F}}$. Then, there exist $b \prec_X a$ and $A \in \mathcal{F}$ such that $A \subseteq b_*$, so there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. By (CP4), there exists $c' \in \alpha_{\mathcal{F}}$ such that $c' \in V_X^{C'}$, so there exists $c \in V_X^C$ such that $c' \prec_X c$. Since $c'_* \between b_*$ by (CF2), we have $c_* \between b_*$. Thus, $c \preceq_X a$, and hence $c' \prec_X a$. □

Lemma 3.2.18. *For any Cauchy point α of X, the set*

$$\mathcal{F}_\alpha = \{a_* \mid a \in \alpha\}$$

is a minimal Cauchy filter on X.

Proof. The fact that $\mathcal{F}_\alpha$ is a Cauchy filter is obvious. To see that $\mathcal{F}_\alpha$ is minimal, let $\mathcal{F}$ be another Cauchy filter on X such that $\mathcal{F} \leqslant \mathcal{F}_\alpha$. Let $a \in \alpha$. By (CP3), there exists $b \prec_X a$ such that $b \in \alpha$, so there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. By (CF3) for $\mathcal{F}$, there exist $a' \in V_X^C$ and $A \in \mathcal{F}$ such that $A \subseteq a'_*$. Since $\mathcal{F} \leqslant \mathcal{F}_\alpha$, there exists $c \in \alpha$ such that $c_* \subseteq A$. By (CP1), we have $c_* \between b_*$, and so $a'_* \between b_*$. Thus, $a'_* \subseteq a_*$, and hence $A \subseteq a_*$. Therefore $\mathcal{F}_\alpha \leqslant \mathcal{F}$. □

Proposition 3.2.19. *The assignments*

$$\mathcal{F} \mapsto \alpha_{\mathcal{F}}, \qquad \alpha \mapsto \mathcal{F}_\alpha$$

define a bijection between the minimal Cauchy filters on X and the Cauchy points of X.

Proof. Since $\mathcal{F}_{\alpha_{\mathcal{F}}} \leqslant \mathcal{F}$ for any Cauchy filter $\mathcal{F}$ on X, we have $\mathcal{F}_{\alpha_{\mathcal{F}}} = \mathcal{F}$ for any minimal Cauchy filter $\mathcal{F}$ on X. Conversely, we have $\alpha \subseteq \alpha_{\mathcal{F}_\alpha}$ for any Cauchy point α of X by (CP3), and hence $\alpha = \alpha_{\mathcal{F}_\alpha}$ by Proposition 3.2.13 and Corollary 2.4.10. □

For each $x \in X$, the set

$$\mathcal{F}_x \stackrel{\text{def}}{=} \{\mathrm{St}_C(\{x\}) \mid C \in \mathcal{C}\}$$

is easily seen to be a Cauchy filter on X.

Definition 3.2.20. A Cauchy filter $\mathcal{F}$ on X *converges* to a point $x \in X$ if $\mathcal{F}_x \leqslant \mathcal{F}$. A uniform space X is *complete* if every Cauchy filter on X converges.

Remark 3.2.21. It is easy to see that Definition 3.2.20 is compatible with Definition 3.1.29 in the sense that the embedding $\mathcal{I}_U : \mathbf{USpa} \to \mathbf{CUSpa}$ preserves and reflects completeness.

For each $x \in X$, let

$$\alpha_x \stackrel{\text{def}}{=} \{a \in V_X \mid (\exists b \prec_X a)\, x \in b_*\}.$$

Lemma 3.2.22. *For each $x \in X$*

1. $\alpha_x = \alpha_{\mathcal{F}_x}$*,*
2. $\mathcal{F}_x = \mathcal{F}_{\alpha_x}$*.*

Proof. 1. The inclusion $\alpha_{\mathcal{F}_x} \subseteq \alpha_x$ is obvious. For the converse, let $a \in \alpha_x$. Then, there exists $b \prec_X a$ such that $x \in b_*$, so there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. By (CP4), there exists $c \in V_X^C \cap \alpha_{\mathcal{F}_x}$, and thus there exist $c' \prec_X c$ and $U \in \mathcal{F}_x$ such that $x \in U \subseteq c'_*$. Then, $x \in c_*$, and so $b_* \between c_*$. Thus, $c \preceq_X a$, and hence, $c' \prec_X a$. Therefore $a \in \alpha_{\mathcal{F}_x}$.

2. Since $\mathcal{F}_{\alpha_x} \leqslant \mathcal{F}_x$, it suffices to show that $\mathcal{F}_x \leqslant \mathcal{F}_{\alpha_x}$. Let $C \in \mathcal{C}$. By (CP4), there exists $a \in V_X^C$ such that $a \in \alpha_x$. Then, $a_* \subseteq \mathrm{St}_C(\{x\})$. Therefore, $\mathcal{F}_x \leqslant \mathcal{F}_{\alpha_x}$. □

Lemma 3.2.23. *A Cauchy filter $\mathcal{F}$ on X converges to $x \in X$ iff $\alpha_{\mathcal{F}} = \alpha_x$.*

Proof. $\mathcal{F}$ converges to $x \iff \mathcal{F}_x \leqslant \mathcal{F} \iff \mathcal{F}_{\alpha_x} \leqslant \mathcal{F} \iff \alpha_x \subseteq \alpha_{\mathcal{F}} \iff \alpha_x = \alpha_{\mathcal{F}}$. The last equivalence follows from Proposition 3.2.13 and Corollary 2.4.10. □

Hence, we have a characterisation of completeness in terms of Cauchy points.

Proposition 3.2.24. *A uniform space X is complete iff for any Cauchy point α of X there exists $x \in X$ such that $\alpha = \alpha_x$.*

Next, we show that the collection $\mathcal{P}t(\mathcal{V}(X))$ of Cauchy points of X admits a complete uniformity. We define a uniformity $\overline{\mathcal{C}}$ on $\mathcal{P}t(\mathcal{V}(X))$ as follows:

$$\overline{\mathcal{C}} \stackrel{\text{def}}{=} \left\{\overline{C} \mid C \in \mathcal{C}\right\};$$
$$\overline{C} \stackrel{\text{def}}{=} \left\{a^* \mid a \in V_X^C \cap \mathsf{Pos}_X\right\};$$
$$a^* \stackrel{\text{def}}{=} \{\alpha \in \mathcal{P}t(\mathcal{V}(X)) \mid a \in \alpha\}.$$

Lemma 3.2.25. *The set $\overline{\mathcal{C}}$ is a uniformity on $\mathcal{P}t(\mathcal{V}(X))$. Moreover, $\overline{\mathcal{C}}$ is proper and separating.*

Proof. First, for each $C \in \mathcal{C}$, $\overline{C}$ is a cover of $\mathcal{P}t(\mathcal{V}(X))$ by (CP4).

$\overline{\mathcal{C}}$ satisfies (CU1) and (CU2): It suffices to show that $C \leqslant C' \implies \overline{C} \leqslant \overline{C'}$ and $C <^* C' \implies \overline{C} <^* \overline{C'}$ for all $C, C' \in \mathcal{C}$. For the former, suppose that $C \leqslant C'$. Let $a \in V_X^C \cap \mathsf{Pos}_X$. Then, there exists $A \in C'$ such that $a_* \subseteq A$, and so $a \preceq_X \mathsf{c}(A)$. Then,

$a^* \subseteq \mathsf{c}(A)^*$ by (CP2). Since Pos_X is upward closed, we have $\mathsf{c}(A) \in \mathsf{Pos}_X$. For the latter, suppose that $C <^* C'$. Let $a \in V_X^C \cap \mathsf{Pos}_X$. Then, there exists $A \in C'$ such that $\mathrm{St}_C(a_*) \subseteq A$. Let $b \in V_X^C \cap \mathsf{Pos}_X$, and suppose that $b^* \between a^*$. Then, there exists $\alpha \in \mathcal{P}t(\mathcal{V}(X))$ such that $a, b \in \alpha$, so $a_* \between b_*$ by (CP1). Thus, $b_* \subseteq A$, i.e. $b \preceq_X \mathsf{c}(A)$, so that $b^* \subseteq \mathsf{c}(A)^*$. Since $a \prec_X \mathsf{c}(A)$, we have $\mathsf{c}(A) \in \mathsf{Pos}_X$. Hence, $\overline{C} <^* \overline{C'}$.

$\overline{\mathcal{C}}$ is proper: It suffices to show that $\overline{\mathcal{C}}$ satisfies (3.11). Let $C \in \mathcal{C}$ and $a \in V_X^C \cap \mathsf{Pos}_X$. Then, there exists $b \in V_X$ such that $b \prec_X a$. Since b_* is inhabited, there exists $x \in b_*$, and hence $a \in \alpha_x$, i.e. $\alpha_x \in a^*$. Hence, $\overline{\mathcal{C}}$ satisfies (3.11).

$\overline{\mathcal{C}}$ is separating: Let $\alpha, \beta \in \mathcal{P}t(\mathcal{V}(X))$, and suppose that for all $C \in \mathcal{C}$, there exists $a \in V_X^C \cap \mathsf{Pos}_X$ such that $\alpha, \beta \in a^*$. Let $b \in \alpha$. By (CP3), there exists $b' \in \alpha$ such that $b' \prec_X b$, so there exists $C \in \mathcal{C}$ such that $b' \prec_X^C b$. Thus, there exists $a \in V_X^C \cap \mathsf{Pos}_X$ such that $a \in \alpha \cap \beta$. By (CP1), we have $b'_* \between a_*$, so $a_* \subseteq b_*$, i.e. $a \preceq_X b$. Thus, $b \in \beta$ by (CP2), and hence $\alpha \subseteq \beta$. Therefore, $\alpha = \beta$ by Proposition 3.2.13 and Corollary 2.4.10, so $\overline{\mathcal{C}}$ is separating. □

Hence, the pair $(\mathcal{P}t(\mathcal{V}(X)), \overline{\mathcal{C}})$ is a uniform space, which we denote by $\overline{X}$.

Proposition 3.2.26. *The uniform space $\overline{X}$ is complete.*

Proof. Let Γ be a Cauchy point of $\overline{X}$. By Proposition 3.2.24, it suffices to find $\alpha \in \mathcal{P}t(\mathcal{V}(X))$ such that $\Gamma = \Gamma_\alpha$, where

$$\Gamma_\alpha \stackrel{\text{def}}{=} \{a \in V_{\overline{X}} \mid (\exists b \prec_{\overline{X}} a)\, \alpha \in b_*\}.$$

First, we introduce a notation. For each $a \in \mathsf{Pos}_X$, write $\overline{a}$ for the element $\mathsf{c}(a^*)$ of $V_{\overline{X}}$. Define a subset α_Γ of V_X by

$$\alpha_\Gamma \stackrel{\text{def}}{=} \left\{a \in V_X \mid (\exists b \prec_X a)\, b \in \mathsf{Pos}_X \;\&\; \overline{b} \in \Gamma\right\}.$$

We claim that α_Γ is a Cauchy point of X. We check the conditions (CP1) – (CP4).

(CP4): Let $C \in \mathcal{C}$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. By (CP4), there exists $b \in V_X^{C'}$ such that $b \in \mathsf{Pos}_X$ and $\overline{b} \in \Gamma$. Then, there exists $a \in V_X^C$ such that $b \prec_X a$. Hence, $a \in \alpha_\Gamma$.

(CP1): Let $a, b \in \alpha_\Gamma$. Then, there exist $a', b' \in V_X$ such that $a' \prec_X a$, $b' \prec_X b$, $a', b' \in \mathsf{Pos}_X$, and $\overline{a'}, \overline{b'} \in \Gamma$. Thus, there exists $C \in \mathcal{C}$ such that $a' \prec_X^C a$ and $b' \prec_X^C b$. By (CP4), there exists $c \in \alpha_\Gamma$ such that $c \in V_X^C$. So there exists $c' \prec_X c$ such that $c' \in \mathsf{Pos}_X$ and $\overline{c'} \in \Gamma$. By (CP1), we have $c'^* \between a'^*$ and $c'^* \between b'^*$, and so $c'_* \between a'_*$ and $c'_* \between b'_*$. Thus, $c_* \between a'_*$ and $c_* \between b'_*$, and hence $c \in a \downarrow b$.

(CP2): By Lemma 3.2.12.2.

(CP3): Let $a \in \alpha_\Gamma$. Then, there exists $b \in \mathsf{Pos}_X$ such that $b \prec_X a$ and $\overline{b} \in \Gamma$. Thus, there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. By (CP4), there exists $c' \in \alpha_\Gamma$ such that $c' \in V_X^{C'}$, so there exists $c \in V_X^C$ such that $c' \prec_X c$. Since $c' \in \alpha_\Gamma$, there exists $c'' \prec_X c'$ such that $c'' \in \mathsf{Pos}_X$ and $\overline{c''} \in \Gamma$. By (CP1), we have $c''^* \between b^*$, so $c''_* \between b_*$. Thus, $c_* \between b_*$, and hence $c \preceq_X a$. Therefore, $c' \prec_X a$.

Hence, α_Γ is a Cauchy point of X. We claim that $\Gamma_{\alpha_\Gamma} = \Gamma$. To see this, let $a \in \mathsf{Pos}_X$, and suppose that $\overline{a} \in \Gamma_{\alpha_\Gamma}$. Then, there exists $b \in \mathsf{Pos}_X$ such that $\overline{b} \prec_{\overline{X}} \overline{a}$ and $b \in \alpha_\Gamma$.

Thus, there exists $b' \in \mathsf{Pos}_X$ such that $b' \prec_X b$ and $\overline{b'} \in \Gamma$. Then, $\overline{b'} \prec_{\overline{X}} \overline{a}$, so that $\overline{a} \in \Gamma$ by (CP2). Hence $\Gamma_{\alpha_\Gamma} \subseteq \Gamma$. Therefore, $\Gamma_{\alpha_\Gamma} = \Gamma$ by Proposition 3.2.13 and Corollary 2.4.10. □

Definition 3.2.27. A subset Z of a uniform space $X = (X, \mathcal{C})$ is *dense* if

$$(\forall C \in \mathcal{C})\, (\forall x \in X)\, \mathrm{St}_C(\{x\}) \between Z.$$

Lemma 3.2.28. *A function $i_X : X \to \mathcal{P}t(\mathcal{V}(X))$ given by*

$$i_X(x) \stackrel{\text{def}}{=} \alpha_x \tag{3.16}$$

is a uniform embedding $i_X : X \to \overline{X}$, and the image $i_X[X]$ is dense in $\overline{X}$.

Proof. First, we show that the image $i_X[X]$ is dense in $\overline{X}$, i.e.

$$(\forall C \in \mathcal{C})\, (\forall \alpha \in \mathcal{P}t(\mathcal{V}(X)))\, (\exists x \in X)\, \alpha_x \in \mathrm{St}_{\overline{C}}(\{\alpha\}).$$

Let $C \in \mathcal{C}$ and $\alpha \in \mathcal{P}t(\mathcal{V}(X))$. By (CP4), there exists $a \in V_X^C$ such that $a \in \alpha$. Then, by (CP3), there exists $b \prec_X a$ such that $b \in \alpha$. Let $x \in b_*$. Then, $a \in \alpha_x$, so we have $\alpha_x, \alpha \in a^*$. Hence $\alpha_x \in \mathrm{St}_{\overline{C}}(\{\alpha\})$, and therefore $i_X[X]$ is dense in $\overline{X}$.

Next, we show that i_X is injective. Let $x, y \in X$, and suppose that $\alpha_x = \alpha_y$. Let $C \in \mathcal{C}$. By (CP4), there exists $a \in V_X^C$ such that $a \in \alpha_x$. Thus $x, y \in a_*$. Since $\mathcal{C}$ is separating, we have $x = y$.

Lastly, we show that i_X is a uniform embedding. Let $C \in \mathcal{C}$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. Let $A' \in C'$. Then, there exists $A \in C$ such that $\mathrm{St}_{C'}(A') \subseteq A$. Then, for any $x \in A'$, we have $\mathsf{c}(A) \in \alpha_x$, i.e. $\alpha_x \in \mathsf{c}(A)^*$. Thus, $i_X[A'] \subseteq \mathsf{c}(A)^*$. Therefore, i_X is uniformly continuous. Conversely, the inverse $\alpha_x \mapsto x$ is uniformly continuous on $i_X[X]$ since for any $C \in \mathcal{C}$, $A \in C$ and $x \in X$, we have $\alpha_x \in \mathsf{c}(A)^* \implies x \in A$. □

Proposition 3.2.29. *Let $X = (X, \mathcal{C})$ be a uniform space, and let Z be a dense subset of X. Then, for any uniformly continuous function $f : Z \to Y$ to a complete uniform space $Y = (Y, \mathcal{D})$, there exists a unique uniformly continuous function $g : X \to Y$ such that $g \circ i_Z = f$, where $i_Z : Z \to X$ is the inclusion of Z into X.*

Proof. For each $x \in X$, define

$$\alpha_x^Z \stackrel{\text{def}}{=} \{b \in V_Z \mid a \in \alpha_x \;\&\; a_* \cap Z \subseteq b_*\}.$$

Note that we have $a_* \between Z$ for any $a \in \alpha_x$. To see this, let $a \in \alpha_x$. Then, there exists $b \prec_X a$ such that $x \in b_*$. Thus, there exists $C \in \mathcal{C}$ such that $b \prec_X^C a$. Since Z is dense, there exists $b' \in V_X^C$ such that $x \in b'_*$ and $Z \between b'_*$. Thus $b'_* \between b_*$, and so $b'_* \subseteq a_*$. Hence $Z \between a_*$. Then, it is straightforward to show that α_x^Z is a Cauchy point of Z, using the corresponding properties of α_x. Let

$$\beta_x \stackrel{\text{def}}{=} \left\{b \in V_Y \mid (\exists b' \prec_Y b)\, \left(\exists a \in \alpha_x^Z\right) f[a_*] \subseteq b'_*\right\}.$$

By Lemma 3.2.37, we see that β_x is a Cauchy point of Y[8]. Since Y is complete, there exists a unique $y \in Y$ such that

$$\alpha_y = \beta_x.$$

Define $g : X \to Y$ by

$$g(x) \stackrel{\text{def}}{=} \text{unique } y \in Y \text{ such that } \alpha_y = \beta_x$$

for each $x \in X$. Then, it is easy to see that $\alpha_{f(z)} = \beta_z$ for all $z \in Z$, i.e. $g(z) = f(z)$ for all $z \in Z$.

We show that g is uniformly continuous. Let $D \in \mathcal{D}$. Choose $D' \in \mathcal{D}$ such that $D' <^* D$. Since f is uniformly continuous, there exists $C \in \mathcal{C}$ such that

$$(\forall A \in C)\, A \between Z \implies (\exists B \in D')\, f[A \cap Z] \subseteq B.$$

Choose $C' \in \mathcal{C}$ such that $C' <^* C$. Let $A' \in C'$. Then, there exists $A \in C$ such that $\mathrm{St}_{C'}(A') \subseteq A$. Since Z is dense in X, we have $A \between Z$, so there exists $B' \in D'$ such that $f[A \cap Z] \subseteq B'$. Moreover, there exists $B \in D$ such that $\mathrm{St}_{D'}(B') \subseteq B$. Let $x \in A'$. Then, $\mathsf{c}(A) \in \alpha_x$, so we have $\mathsf{c}(B) \in \beta_x$. Thus, $g(x) \in B$, and hence $g[A'] \subseteq B$. Therefore, g is uniformly continuous.

Lastly, we show that g is a unique extension of f. Let $h : X \to Y$ be another uniformly continuous function such that $h \circ i_Z = f$. It suffices to show that $\alpha_{h(x)} = \beta_x$ for all $x \in X$. Let $x \in X$ and $b \in \alpha_{h(x)}$. Then, there exists $b' \prec_Y b$ such that $h(x) \in b'_*$, so there exists $D \in \mathcal{D}$ such that $b' \prec_Y^D b$. Choose $D' \in \mathcal{D}$ such that $D' <^* D$. Since h is uniformly continuous, there exists $C \in \mathcal{C}$ such that $(\forall A \in C)\, (\exists B' \in D')\, h[A] \subseteq B'$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. Since C' covers X, there exists $A' \in C'$ such that $x \in A'$, so there exists $A \in C$ such that $\mathrm{St}_{C'}(A') \subseteq A$. Thus, $\mathsf{c}(A) \in \alpha_x$. Now, there exist $B' \in D'$ and $B \in D$ such that $h[A] \subseteq B'$ and $\mathrm{St}_{D'}(B') \subseteq B$. Then, we have $f[A \cap Z] = h[A \cap Z] \subseteq B'$, so that $b'' = \mathsf{c}(B) \in \beta_x$. Since $h(x) \in B$, we have $b'_* \between b''_*$. Thus $b'' \preceq_Y b$, so by (CP2), we have $b \in \beta_x$. Hence $\alpha_{h(x)} \subseteq \beta_x$. Therefore $\alpha_{h(x)} = \beta_x$ by Proposition 3.2.13 and Corollary 2.4.10. ☐

Theorem 3.2.30. *The embedding $i_X : X \to \overline{X}$ given by (3.16) is a completion of a uniform space X.*

Predicative justification of completions

Since the collection of formal points of a formal topology does not necessarily form a set, the definition of the uniform space $\mathcal{P}t(\mathcal{V}(X))$ is problematic from a constructive point of view. In CZF, however, we can show that the collection of Cauchy points of any uniform space forms a set. The argument is based the construction of completions of entourage uniformities by Berger et al. [7].

[8]Using the notation of Lemma 3.2.37, we have $\beta_x = r_f(\alpha_x^Z)$, which is a well-defined operation from $\mathcal{P}t(\mathcal{V}(Z))$ to $\mathcal{P}t(\mathcal{V}(Y))$ (See (2.6)).

Given a uniform space $X = (X, \mathcal{C})$, let $\mathsf{mv}(\mathcal{C}, V_X)$ denote the class of total relations from $\mathcal{C}$ to V_X. By Fullness, there exists a subset $R \subseteq \mathsf{mv}(\mathcal{C}, V_X)$ such that

$$(\forall s \in \mathsf{mv}(\mathcal{C}, V_X))\,(\exists r \in R)\, r \subseteq s. \tag{3.17}$$

Define a predicate φ on $\mathsf{mv}(\mathcal{C}, V_X)$ by

$$\varphi(r) \overset{\text{def}}{\iff} (\forall (C,a) \in r)\, a \in V_X^C \;\&\; (\forall (C,a),(C',a') \in r)\, a_* \between a'_*.$$

Let $R_\varphi \overset{\text{def}}{=} \{r \in R \mid \varphi(r)\}$. For each $r \in R_\varphi$, define

$$V_r \overset{\text{def}}{=} \{a \in V_X \mid (C,a) \in r\}.$$

Lemma 3.2.31. *For each $r \in R_\varphi$, the set*

$$\alpha_r \overset{\text{def}}{=} \{a \in V_X \mid (\exists b \in V_r)\, b \prec_X a\}$$

is a Cauchy point of X.

Proof. (CP4): Let $C \in \mathcal{C}$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. Since r is total, there exists $a \in V_X^{C'} \cap V_r$. So there exists $b \in V_X^C$ such that $a \prec_X b$. Then, $b \in \alpha_r$.

(CP1): Let $a, b \in \alpha_r$. Then, there exist $a', b' \in V_r$ such that $a' \prec_X a$ and $b' \prec_X b$. Then, there exists $C \in \mathcal{C}$ such that $a' \prec_X^C a$ and $b' \prec_X^C b$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. By (CP4), there exists $c' \in \alpha_r$ such that $c' \in V_X^{C'}$. Thus, there exists $c \in V_X^C$ such that $c' \prec_X c$. Since $c'_* \between a'_*$ and $c'_* \between b'_*$, we have $c_* \between a'_*$ and $c_* \between b'_*$. Hence $c \in a \downarrow b$. Since α_r is upward closed, we have $c \in \alpha_r$.

(CP2): By Lemma 3.2.12.2.

(CP3): The proof is contained in the proof of (CP1). $\square$

Conversely, given any Cauchy point α of X, define

$$s \overset{\text{def}}{=} \left\{(C,a) \in \mathcal{C} \times V_X \mid a \in V_X^C \cap \alpha\right\}.$$

By (CP4), we have $s \in \mathsf{mv}(\mathcal{C}, V_X)$. Since R satisfies (3.17), there exists $r \in R$ such that $r \subseteq s$, and by (CP1), we have $r \in R_\varphi$. Moreover, we have $\alpha_r \subseteq \alpha$. Thus $\alpha_r = \alpha$ by Proposition 3.2.13 and Corollary 2.4.10. Hence, the mapping

$$r \mapsto \alpha_r : R_\varphi \to \mathcal{P}t(\mathcal{V}(X))$$

is a surjection. Therefore, we conclude as follows.

Proposition 3.2.32. *For any uniform space X, the class $\mathcal{P}t(\mathcal{V}(X))$ of Cauchy points of X forms a set.*

3.2.4 Compactness

In this section, we extend Theorem 3.1.39 to covering completions.

The notion of compactness for uniform spaces defined by covering uniformities is a natural generalisation of the corresponding notion for uniform spaces defined by sets of pseudometrics (See Section 3.1.4).

Definition 3.2.33. A uniform space $(X, \mathcal{C})$ is *totally bounded* if

$$(\forall C \in \mathcal{C})\,(\exists\,\{A_0, \dots, A_{n-1}\} \in \mathsf{Fin}(C))\; X \subseteq \bigcup_{k<n} A_k.$$

Note that X is totally bounded iff

$$(\forall C \in \mathcal{C})\,\big(\exists\,\{a_0, \dots, a_{n-1}\} \in \mathsf{Fin}(V_X^C)\big)\; X \subseteq \bigcup_{k<n} a_{k*}.$$

A uniform space is *compact* if it is complete and totally bounded.

Remark 3.2.34. For any object X of **USpa**, X is totally bounded in **USpa** iff $\mathcal{I}_U(X)$ is totally bounded in **CUSpa**. Hence by Remark 3.2.21, X is compact in **USpa** iff $\mathcal{I}_U(X)$ is compact in **CUSpa**.

Lemma 3.2.35. *Let $X = (X, \mathcal{C})$ be a uniform space. Then, X is totally bounded iff*

$$(\forall C \in \mathcal{C})\,\big(\exists V_0 \in \mathsf{Fin}(V_X^C \cap \mathsf{Pos}_X)\big)\; V_X \blacktriangleleft_X V_0. \tag{3.18}$$

Proof. Suppose that X is totally bounded. Let $C \in \mathcal{C}$. Choose $C' \in \mathcal{C}$ such that $C' <^* C$. Since X is totally bounded, there exists $\{b_0, \dots, b_{n-1}\} \in \mathsf{Fin}(V_X^{C'})$ such that $X \subseteq \bigcup_{k<n} b_{k*}$. Then, there exists $\{a_0, \dots, a_{n-1}\} \in V_X^C$ such that $b_k \prec_X^{C'} a_k$ for each $k < n$. Thus, we have $\mathsf{Pos}_X(a_k)$ for each $k < n$. Let $b \in V_X^{C'}$. Then, there exists $k < n$ such that $b_* \between b_{k*}$, so that $b \preceq_X a_k$. Thus $V_X^{C'} \blacktriangleleft_X \{a_0, \dots, a_{n-1}\}$, and hence $V_X \blacktriangleleft_X \{a_0, \dots, a_{n-1}\}$ by (V2).

Conversely, suppose that (3.18) holds. Let $C \in \mathcal{C}$. Then, there exists $V_0 \in \mathsf{Fin}(V_X^C \cap \mathsf{Pos}_X)$ such that $V_X \blacktriangleleft_X V_0$. Let $x \in X$. Since α_x is a Cauchy point of X, there exists $a \in V_0$ such that $a \in \alpha_x$. Hence $x \in a_*$. Therefore $X \subseteq \bigcup_{a \in V_0} a_*$, so X is totally bounded. ∎

Theorem 3.2.36. *Let $X = (X, \mathcal{C})$ be a uniform space. Then, X is totally bounded iff $\mathcal{V}(X)$ is compact.*

Proof. The proof is based on the one given in [60] where Steven Vickers gave a characterisation of compactness for inductively generated formal topologies.

First, suppose that $\mathcal{V}(X)$ is compact. Then, the condition (3.18) holds, so X is totally bounded.

Conversely, suppose that X is totally bounded. Define

$$\theta \stackrel{\text{def}}{=} \{F \in \mathsf{Fin}(V_X) \mid (\exists G \in \mathsf{Fin}(V_X))\, V_X \blacktriangleleft_X G \prec_X F\},$$

where $G \prec_X F \overset{\text{def}}{\iff} (\forall b \in G)\,(\exists a \in F)\, b \prec_X a$.

Given any $U \subseteq V_X$, define a predicate Φ_U on $\mathsf{Fin}(V_X)$ by

$$\Phi_U(Z) \overset{\text{def}}{\iff} (\forall F \in \mathsf{Fin}(V_X))\,[F \cup Z \in \theta \to (\exists U_0 \in \mathsf{Fin}(U))\, F \cup U_0 \in \theta]\,.$$

Then, we have for any $Z \in \mathsf{Fin}(V_X)$

$$[(\forall a \in Z)\, \Phi_U(\{a\})] \implies \Phi_U(Z). \tag{3.19}$$

This is proved by induction on $\mathsf{Fin}(V_X)$. Indeed, if $Z = \emptyset$, then since $\Phi_U(\emptyset)$ holds, we have (3.19). For the inductive case, let $Z = Z_0 \cup \{a\}$, and suppose that (3.19) holds for Z_0. Suppose that $\Phi_U(\{a\})$ for all $a \in Z$, and let $F \in \mathsf{Fin}(V_X)$ such that $F \cup Z_0 \cup \{a\} \in \theta$. Since $\Phi_U(Z_0)$, there exists $U_0 \in \mathsf{Fin}(U)$ such that $F \cup \{a\} \cup U_0 \in \theta$. Since $\Phi_U(\{a\})$, there exists $U_1 \in \mathsf{Fin}(U)$ such that $F \cup U_1 \cup U_0 \in \theta$. Therefore $\Phi_U(Z)$.

Let Ψ be the predicate on V_X given by

$$\Psi(a) \overset{\text{def}}{\iff} \Phi_U(\{a\}).$$

We show that

$$a \blacktriangleleft_X U \implies \Psi(a) \tag{3.20}$$

for all $a \in V_X$ by induction on $\blacktriangleleft_X$. We must verify the conditions (ID1) – (ID3). The conditions (ID1) and (ID2) are straightforward. For (ID3), we must check the axioms (V1') and (V2').

(V1') $\dfrac{(\forall b \prec_X a)\, \Psi(b)}{\Psi(a)}$: Suppose that $\Psi(b)$ for all $b \prec_X a$. Let $F \in \mathsf{Fin}(V_X)$ such that $F \cup \{a\} \in \theta$. Then, there exists $G \in \mathsf{Fin}(V_X)$ such that $V_X \blacktriangleleft_X G \prec_X F \cup \{a\}$. Since G is finitely enumerable, there exists $C \in \mathcal{C}$ such that

$$(\forall u \in G)\,(\exists v \in F \cup \{a\})\, u \prec_X^C v.$$

Choose $C_1, C_2 \in \mathcal{C}$ such that $C_2 <^* C_1 <^* C$. By (3.18), there exists $V_2 \in \mathsf{Fin}(V_X^{C_2} \cap \mathsf{Pos}_X)$ such that $V_X \blacktriangleleft_X V_2$. Then, there exist $V_1 \in \mathsf{Fin}(V_X^{C_1} \cap \mathsf{Pos}_X)$ and $V_0 \in \mathsf{Fin}(V_X^{C} \cap \mathsf{Pos}_X)$ such that $V_2 \prec_X V_1 \prec_X V_0$. For each $u \in V_1$, there exists $w \in V_0$ such that $u \prec_X w$, and since $w \blacktriangleleft_X G \downarrow w$ and $\mathsf{Pos}_X(w)$, we have either $w \preceq_X v$ for some $v \in F$ or $w \preceq_X a$. Hence, we may split V_1 into finitely enumerable subsets ${V_1}^+$ and ${V_1}^-$ such that $V_1 = {V_1}^+ \cup {V_1}^-$ and that

- $b \in {V_1}^+ \implies b \prec_X a$,
- $b \in {V_1}^- \implies (\exists v \in F)\, b \prec_X v$.

Since ${V_1}^+ \subseteq \{b \in V_X \mid b \prec_X a\}$ and ${V_1}^+$ is finitely enumerable, we have $\Phi_U({V_1}^+)$ by (3.19). Since $V_X \blacktriangleleft_X V_2 \prec_X F \cup {V_1}^+$, we have $F \cup {V_1}^+ \in \theta$. Thus, there exists $U_0 \in \mathsf{Fin}(U)$ such that $F \cup U_0 \in \theta$. Hence, $\Psi(a)$.

(V2') $\dfrac{(\forall b \in V_X^C \downarrow a)\, \Psi(b)}{\Psi(a)}$ for each $C \in \mathcal{C}$: Let $C \in \mathcal{C}$, and suppose that $\Psi(b)$ for all $b \in V_X^C \downarrow a$. Let $F \in \mathsf{Fin}(V_X)$ such that $F \cup \{a\} \in \theta$. Then, there exists $G \in \mathsf{Fin}(V_X)$ such that $V_X \blacktriangleleft_X G \prec_X F \cup \{a\}$. Since G is finitely enumerable, there exists $C' \in \mathcal{C}$ such that $C' \leqslant C$ and

$$(\forall u \in G)\,(\exists v \in F \cup \{a\})\, u \prec_X^{C'} v.$$

Then, the proof proceeds just as in the case of (V1'). This completes the proof of (3.20).

Now, suppose that $V_X \blacktriangleleft_X U$. Since $\mathcal{C}$ is inhabited, there exists $C \in \mathcal{C}$. Then, there exists $C' \in \mathcal{C}$ such that $C' <^* C$. Since X is totally bounded, there exists $V' \in \mathsf{Fin}(V_X^{C'})$ such that $V_X \blacktriangleleft_X V'$. Moreover, there exists $V \in \mathsf{Fin}(V_X^{C})$ such that $V' \prec_X V$. Since $V \blacktriangleleft_X U$, we have $\Phi_U(V)$ by (3.20) and (3.19). Since $V \in \theta$, there exists $U_0 \in \mathsf{Fin}(U)$ such that $U_0 \in \theta$. Hence $V_X \blacktriangleleft_X U_0$. Therefore, $\mathcal{V}(X)$ is compact. □

3.2.5 Functorial embedding II

We show that the category of compact uniform spaces can be embedded into that of compact 2-regular formal topologies by extending the construction of a covering completion to a full and faithful functor. The functor generalises the embedding $\mathcal{U}\colon \mathbf{LKUSpa} \to \mathbf{OLKReg}$ in Theorem 3.1.47 to a wider category **CUSpa**, but only for the compact case.

Given any function $f : X \to Y$ between uniform spaces $X = (X, \mathcal{C})$ and $Y = (Y, \mathcal{D})$ define a relation $r_f \subseteq V_X \times V_Y$ by

$$a \; r_f \; b \overset{\text{def}}{\iff} (\exists b' \prec_Y b)\, f[a_*] \subseteq b'_*. \tag{3.21}$$

In the same setting as above, $f : X \to Y$ is said to be *locally uniformly continuous* if for each $C \in \mathcal{C}$ and $A \in C$, f is uniformly continuous on A, i.e.

$$(\forall D \in \mathcal{D})\,(\exists C' \in \mathcal{C})\,[(\forall A' \in C')\, A' \between A \implies (\exists B \in D)\, f\,[A' \cap A] \subseteq B]\,.$$

Lemma 3.2.37. *If $f : X \to Y$ is a locally uniformly continuous function, then r_f is a formal topology map from $\mathcal{V}(X)$ to $\mathcal{V}(Y)$.*

Proof. We must check (FTMi1) – (FTMi4).

(FTMi1): Let $a \in V_X$. Since $\mathcal{D}$ is inhabited, we can choose $D', D \in \mathcal{D}$ such that $D' <^* D$. Since f is uniformly continuous on a_*, there exists $C \in \mathcal{C}$ such that

$$\left(\forall a' \in V_X^C\right) a'_* \between a_* \implies \left(\exists d \in V_Y^{D'}\right) f[a'_* \cap a_*] \subseteq d_*.$$

Then, by (V2), we have

$$a \blacktriangleleft_X V_X^C \downarrow a \subseteq r_f^- V_Y^D \subseteq r_f^- V_Y,$$

and thus $V_X \blacktriangleleft_X r_f^- V_Y$.

(FTMi2): Let $b, c \in V_Y$, and let $a \in r_f^- b \downarrow r_f^- c$. Then, there exist $b' \prec_Y b$ and $c' \prec_Y c$ such that $f[a_*] \subseteq b'_* \cap c'_*$. Then, there exists $D \in \mathcal{D}$ such that $b' \prec_Y^D b$ and $c' \prec_Y^D c$. Choose $D_1, D_2 \in \mathcal{D}$ such that $D_2 <^* D_1 <^* D$. Since f is uniformly continuous on a_*, there exists $C \in \mathcal{C}$ such that

$$\left(\forall a' \in V_X^C\right) a'_* \between a_* \implies \left(\exists d \in V_Y^{D_2}\right) f[a'_* \cap a_*] \subseteq d_*.$$

Let $a' \in V_X^C \downarrow a$. Then, there exists $d_2 \in V_Y^{D_2}$ such that $f[a'_*] \subseteq d_{2*}$. Choose $d_1 \in V_Y^{D_1}$ and $d \in V_Y^D$ such that $d_2 \prec_Y d_1 \prec_Y d$. Since $f[a'_*] \subseteq f[a_*]$, we have $d_* \between (b'_* \cap c'_*)$, so that $d \preceq_Y b$ and $d \preceq_Y c$. Since $a' \; r_f \; d_1$, $d_1 \prec_X b$ and $d_1 \prec_X c$, we have $a' \in r_f^-(b \downarrow c)$. Thus, $a \blacktriangleleft_X r_f^-(b \downarrow c)$ by (V2).

(FTMi3): The proofs for the axioms (V1) and (V2) can be obtain by straightforward modifications of the proofs of (FTMi2) and (FTMi1) respectively.

(FTMi4): By Lemma 3.2.12.2. □

Lemma 3.2.38. *Let $X = (X, \mathcal{C})$ be a compact uniform space, and let $Y = (Y, \mathcal{D})$ be a complete uniform space. Then, for any formal topology map $r : \mathcal{V}(X) \to \mathcal{V}(Y)$, the composition*

$$f \stackrel{\text{def}}{=} {i_Y}^{-} \circ \mathcal{P}t(r) \circ i_X$$

is a uniformly continuous function from X to Y. Here, i_X and i_Y are uniform isomorphisms given by (3.16)*.*

Proof. Let $D \in \mathcal{D}$. By (V2), we have $V_X \blacktriangleleft_X r^- V_Y^D$, and by (V1), we have

$$V_X \blacktriangleleft_X \left\{ a \in V_X \mid \left(\exists a' \in r^- V_Y^D\right) a \prec_X a' \right\}.$$

Since $\mathcal{V}(X)$ is compact by Theorem 3.2.36, there exist $\{a_0, \dots, a_{n-1}\}, \{a'_0, \dots, a'_{n-1}\} \in \mathsf{Fin}(V_X)$ and $\{b_0, \dots, b_{n-1}\} \in \mathsf{Fin}(V_Y^D)$ such that $V_X \blacktriangleleft_X \{a_0, \dots, a_{n-1}\}$, and $a_k \prec_X a'_k$ and $a'_k \; r \; b_k$ for each $k < n$. Then, there exists $C \in \mathcal{C}$ such that $a_k \prec_X^C a'_k$ for each $k < n$. Let $C' \in \mathcal{C}$ such that $C' <^* C$. We show that

$$\left(\forall a \in V_X^{C'}\right) \left(\exists b \in V_Y^D\right) f[a_*] \subseteq b_*.$$

Let $a' \in V_X^{C'}$. Then, there exists $a \in V_X^C$ such that $a' \prec_X a$. Since $\mathsf{Pos}_X(a)$ and $a \blacktriangleleft_X \{a_0, \dots, a_{n-1}\}$, there exists $k < n$ such that $\mathsf{Pos}_X(a \downarrow a_k)$, so that $a \preceq_X a'_k$. Let $x \in a'_*$. Then, $a \in \alpha_x$, and so $a'_k \in \alpha_x$. Thus, $b_k \in \mathcal{P}t(r)(\alpha_x)$. Since $\mathcal{P}t(r)(\alpha_x) = \alpha_{f(x)}$, we have $f(x) \in b_{k*}$, and hence $f[a'_*] \subseteq b_{k*}$. Therefore, f is uniformly continuous. □

Lemma 3.2.39. *Let $X = (X, \mathcal{C})$ and $Y = (Y, \mathcal{D})$ be complete uniform spaces, and let $f : X \to Y$ be a locally uniformly continuous function. Then, the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\;i_X\;} & \mathcal{P}t(\mathcal{V}(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}t(r_f)} \\ Y & \xleftarrow[\;{i_Y}^{-1}\;]{} & \mathcal{P}t(\mathcal{V}(Y)) \end{array} \tag{3.22}$$

Proof. By Proposition 3.2.13 and Corollary 2.4.10, it suffices to show that $\mathcal{P}t(r_f)(\alpha_x) \subseteq \alpha_{f(x)}$ for each $x \in X$. Let $x \in X$, and let $b \in \mathcal{P}t(r_f)(\alpha_x)$. Then, there exists $a \in \alpha_x$ such that $a \; r_f \; b$, so there exists $b' \prec_Y b$ such that $f[a_*] \subseteq b'_*$. Since $x \in a_*$, we have $f(x) \in b'_*$. Hence $b \in \alpha_{f(x)}$. Therefore, $\mathcal{P}t(r_f)(\alpha_x) \subseteq \alpha_{f(x)}$. □

Lemma 3.2.40. *Let $X = (X, \mathcal{C})$ be a compact uniform space, and let $Y = (Y, \mathcal{D})$ be a complete uniform space. Then, for any formal topology map $r : \mathcal{V}(X) \to \mathcal{V}(Y)$, we have $r_f = r$, where $f \stackrel{\text{def}}{=} {i_Y}^{-1} \circ \mathcal{P}t(r) \circ i_X$.*

Proof. By Proposition 3.2.13 and Proposition 2.4.8, it suffices to show that $r \leq r_f$. Let $a \in V_X$ and $b \in V_Y$, and suppose that $a \; r \; b$. By (V1), we have

$$a \blacktriangleleft_X \{a' \in V_X \mid (\exists a'' \in V_X)\,(\exists b' \in V_Y)\, a' \prec_X a'' \;\&\; a'' \; r \; b' \;\&\; b' \prec_Y b\}\,.$$

Let $a' \in$ RHS. Then, there exist $a'' \in V_X$ and $b' \in V_Y$ such that $a' \prec_X a''$, $a'' \; r \; b'$, and $b' \prec_Y b$. Then, $f[a'_*] \subseteq b'_*$, and thus $a' \; r_f \; b$. Hence, $a \blacktriangleleft_X r_f^- b$. Therefore $r \leq r_f$. □

The following states that the assignment $f \mapsto r_f$ is functorial.

Lemma 3.2.41. *Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous functions between compact uniform spaces. Then, we have $r_{g \circ f} = r_g \circ r_f$. Moreover, we have $r_{id_X} = id_{\mathcal{V}(X)}$ for any compact uniform space X.*

Proof. Similar to the proof of Lemma 3.1.46. □

Theorem 3.2.42. *The covering completion $\mathcal{V}$ extends to a full and faithful functor $\mathcal{V}$:* **KCUSpa** $\to$ **KReg**$_2$ *from the category of compact uniform spaces* **KCUSpa** *to that of compact 2-regular formal topologies* **KReg**$_2$.

Proof. For each morphism $f : X \to Y$ of **KCUSpa**, define $\mathcal{V}(f) = r_f$. Then, by Lemma 3.2.41, $\mathcal{V}$ is a functor. By Lemma 3.2.39, $\mathcal{V}$ is faithful, and by Lemma 3.2.38 and Lemma 3.2.40, $\mathcal{V}$ is full. □

3.3 Localic completions vs. covering completions

In this section, we show that the notion of covering completion generalises that of localic completion. Given any uniform space X defined by a set of pseudometrics, we show that the covering completion of $\mathcal{I}_U(X)$ and the localic completion of X are isomorphic. Moreover, we show that the embedding $\mathcal{V}$: **KCUSpa** $\to$ **KReg**$_2$ in Theorem 3.2.42 generalises the embedding $\mathcal{U}$: **LKUSpa** $\to$ **OLKReg** in Theorem 3.1.47 in that the restrictions of the two functors to the full subcategory of compact uniform spaces defined by sets of pseudometrics are naturally isomorphic.

3.3.1 Comparison of topologies

Any uniform space $X = (X, M)$ defined by a set M of pseudometrics gives rise to two formal topologies, the localic completion $\mathcal{U}(X)$ and the covering completion $\mathcal{V}(\mathcal{I}_U(X))$ via the embedding $\mathcal{I}_U$: **USpa** $\to$ **CUSpa**. In this section, we show that the two topologies are isomorphic. In what follows, the term 'uniform space' always means an object of **USpa**.

If X is a uniform space, then the covering completion $\mathcal{V}(\mathcal{I}_U(X))$ of $\mathcal{I}_U(X)$ has the base $V_{\mathcal{I}_U(X)}$ given by

$$V_{\mathcal{I}_U(X)} \stackrel{\text{def}}{=} \{\mathsf{c}(a_*) \mid a \in U_X\},$$

where U_X is the base of the localic completion $\mathcal{U}(X)$. The orders $\preceq_{\mathcal{I}_U(X)}$ and $\prec_{\mathcal{I}_U(X)}$ on $V_{\mathcal{I}_U(X)}$ are given by (3.14) and (3.15) respectively. In the following, we use letters $u, v, w, \ldots$ for the elements of $V_{\mathcal{I}_U(X)}$ and $a, b, c, \ldots$ for the elements of U_X.

First, we study some connections between the orders on U_X and those on $V_{\mathcal{I}_U(X)}$.

Lemma 3.3.1. *Let X be a uniform space. Then,*

1. $a \leq_X b \implies \mathsf{c}(a_*) \preceq_{\mathcal{I}_U(X)} \mathsf{c}(b_*)$,
2. $a <_X b \implies \mathsf{c}(a_*) \prec_{\mathcal{I}_U(X)} \mathsf{c}(b_*)$

for all $a, b \in U_X$.

Proof. 1. This is equivalent to Lemma 3.1.10.3.

2. Let $a, b \in U_X$, and suppose that $a <_X b$. Write $a = \mathsf{b}_A(x, \varepsilon)$ and $b = \mathsf{b}_B(y, \delta)$, and choose $\gamma \in \mathbb{Q}^{>0}$ such that $\rho_B(x, y) + \varepsilon + 2\gamma < \delta$. We show that $\mathsf{c}(a_*) \prec^{C(A,\gamma)}_{\mathcal{I}_U(X)} \mathsf{c}(b_*)$. Let $c = \mathsf{b}_A(z, \gamma) \in \mathcal{C}^\gamma_A$, and suppose that $c_* \between a_*$. Then, $\rho_A(x, z) < \varepsilon + \gamma$, so

$$\begin{aligned} \rho_B(z, y) + \gamma &\leq \rho_B(z, x) + \rho_B(x, y) + \gamma \\ &< \rho_B(x, y) + \varepsilon + 2\gamma < \delta. \end{aligned}$$

Thus, $c \leq_X b$, and hence $\mathsf{c}(c_*) \preceq_{\mathcal{I}_U(X)} \mathsf{c}(b_*)$. Therefore, $\mathsf{c}(a_*) \prec^{C(A,\gamma)}_{\mathcal{I}_U(X)} \mathsf{c}(b_*)$. ☐

The following is a corollary of Lemma 3.1.34.

Lemma 3.3.2. *For any uniform space X, we have*

$$a_* \subseteq b_* \ \&\ b <_X c \implies a \lhd_X c$$

for all $a, b, c \in U_X$.

Now, define a relation $r_X \subseteq V_{\mathcal{I}_U(X)} \times U_X$ by

$$u \; r_X \; a \stackrel{\text{def}}{\iff} (\exists a' <_X a)\, u_* \subseteq a'_*. \tag{3.23}$$

Lemma 3.3.3. *The relation r_X is a formal topology map $r_X : \mathcal{V}(\mathcal{I}_U(X)) \to \mathcal{U}(X)$.*

Proof. We check the conditions (FTMi1) – (FTMi4).

(FTMi1): Let $u = \mathsf{c}(\mathsf{b}_A(x, \varepsilon)_*) \in V_{\mathcal{I}_U(X)}$. Then, we have $u \; r_X \; \mathsf{b}_A(x, 2\varepsilon)$, from which (FTMi1) follows.

(FTMi2): Let $a, b \in U_X$, and let $u \in r_X^- a \downarrow r_X^- b$. Then, there exist $a' <_X a$ and $b' <_X b$ such that $u_* \subseteq a'_* \cap b'_*$. Write $a' = \mathsf{b}_A(x, \varepsilon)$ and $b' = \mathsf{b}_A(y, \delta)$, and choose $\theta \in \mathbb{Q}^{>0}$ such that

$$\begin{aligned} \mathsf{b}_A(x, \varepsilon + 3\theta) &<_X a, \\ \mathsf{b}_B(y, \delta + 3\theta) &<_X b. \end{aligned}$$

Let $v \in V^{C(A\cup B,\theta)}_{\mathcal{I}_U(X)} \downarrow u$, and write $v = \mathsf{c}(\mathsf{b}_D(z,\gamma)_*)$. Then, $v_* \subseteq \mathsf{b}_{A\cup B}(z, 2\theta)_*$. Since $z \in u_*$, we have $\rho_A(z,x) + 3\theta < \varepsilon + 3\theta$, so that $\mathsf{b}_{A\cup B}(z,3\theta) <_X \mathsf{b}_A(x, \varepsilon + 3\theta)$. Similarly, we have $\mathsf{b}_{A\cup B}(z,3\theta) <_X \mathsf{b}_B(y,\delta+3\theta)$. Hence, $v\ r_X\ \mathsf{b}_{A\cup B}(z,3\theta)$ and $\mathsf{b}_{A\cup B}(z,3\theta) \in b \downarrow c$. Therefore, $u \blacktriangleleft_{\mathcal{I}_U(X)} r_X^-(b\downarrow c)$ by (V2).

(FTMi3): Preservation of the axiom (U1) follows from Lemma 3.1.10.2. For (U2), let $(A,\varepsilon) \in \mathcal{E}_X$. By letting $\delta = \varepsilon/2$, we have $V_{\mathcal{I}_U(X)} \blacktriangleleft_{\mathcal{I}_U(X)} V^{C(A,\delta)}_{\mathcal{I}_U(X)} \subseteq r_X^- \mathcal{C}^\varepsilon_A$ by (V2).

(FTMi4): By Lemma 3.1.10.1. □

Lemma 3.3.4. *The map $r_X : \mathcal{V}(\mathcal{I}_U(X)) \to \mathcal{U}(X)$ is an embedding.*

Proof. We must show that

$$u \blacktriangleleft_{\mathcal{I}_U(X)} r_X^- r_X^{-*} \mathcal{A}_{\mathcal{I}_U(X)} \{u\}$$

for all $u \in V_{\mathcal{I}_U(X)}$. Let $u \in V_{\mathcal{I}_U(X)}$, and let $v \in V_{\mathcal{I}_U(X)}$ such that $v \prec_{\mathcal{I}_U(X)} u$. Then, there exists $(A,\varepsilon) \in \mathcal{E}_X$ such that $v \prec^{C(A,\varepsilon)}_{\mathcal{I}_U(X)} u$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\theta < \varepsilon$, and let $w \in V^{C(A,\theta)}_{\mathcal{I}_U(X)} \downarrow v$. Then, there exists $\mathsf{b}_A(x,\theta) \in \mathcal{C}^\theta_A$ such that $w_* \subseteq \mathsf{b}_A(x,\theta)_*$, and thus $w\ r_X\ \mathsf{b}_A(x,\varepsilon)$. Let $w' \in r_X^- \mathsf{b}_A(x,\varepsilon)$. Then, we have $w'_* \subseteq \mathsf{b}_A(x,\varepsilon)_*$. Since $v_* \between \mathsf{b}_A(x,\varepsilon)_*$, we have $\mathsf{b}_A(x,\varepsilon)_* \subseteq u_*$. Hence $w' \preceq_{\mathcal{I}_U(X)} u$, so $\mathsf{b}_A(x,\varepsilon) \in r_X^{-*}\mathcal{A}_{\mathcal{I}_U(X)}\{u\}$. Therefore, by (V1) and (V2), we have $u \blacktriangleleft_{\mathcal{I}_U(X)} r_X^- r_X^{-*}\mathcal{A}_{\mathcal{I}_U(X)}\{u\}$, as required. □

Lemma 3.3.5. *The map $r_X : \mathcal{V}(\mathcal{I}_U(X)) \to \mathcal{U}(X)$ is a surjection.*

Proof. We must show that

$$r_X^- a \blacktriangleleft_{\mathcal{I}_U(X)} r_X^- U \implies a \lhd_X U$$

for all $a \in U_X$ and $U \subseteq U_X$. Since $b <_X a \implies \mathsf{c}(b_*)\ r_X\ a$ for all $a, b \in U_X$, it suffices to show that

$$\mathsf{c}(a_*) \blacktriangleleft_{\mathcal{I}_U(X)} r_X^- U \implies a \lhd_X U$$

for all $a \in U_X$ and $U \subseteq U_X$ by (U1). Given $U \subseteq U_X$, define a predicate Φ on $V_{\mathcal{I}_U(X)}$ by

$$\Phi(u) \overset{\text{def}}{\iff} (\forall a \in U_X)\, a_* \subseteq u_* \to a \lhd_X U.$$

Then, it suffices to show that

$$u \blacktriangleleft_{\mathcal{I}_U(X)} r_X^- U \implies \Phi(u)$$

for all $u \in V_{\mathcal{I}_U(X)}$. This is proved by induction on $\blacktriangleleft_{\mathcal{I}_U(X)}$. We must check the conditions (ID1) – (ID3).

(ID1): Suppose that $u \in r_X^- U$, and let $a \in U_X$ such that $a_* \subseteq u_*$. Then, there exist $b \in U$ and $b' <_X b$ such that $u_* \subseteq b'_*$. Thus $a_* \subseteq b'_*$, and hence, $a \lhd_X b$ by Lemma 3.3.2. Therefore $\Phi(u)$.

(ID2): Let $u, v \in V_{\mathcal{I}_U(X)}$, and suppose that $u \preceq_{\mathcal{I}_U(X)} v$ and $\Phi(v)$. Let $a \in U_X$ such that $a_* \subseteq u_*$. Then, $a_* \subseteq v_*$. Since $\Phi(v)$, we have $a \lhd_X U$, and so $\Phi(u)$.

(ID3): We need to check the axioms (V1') and (V2').

(V1') $\dfrac{(\forall v \prec_{\mathcal{I}_U(X)} u)\, \Phi(v)}{\Phi(u)}$: Suppose that $\Phi(v)$ holds for all $v \prec_{\mathcal{I}_U(X)} u$. Let $a \in U_X$ such that $a_* \subseteq u_*$, and let $b \in U_X$ such that $b <_X a$. Then, $\mathsf{c}(b_*) \prec_{\mathcal{I}_U(X)} \mathsf{c}(a_*)$ by Lemma 3.3.1, and so $\mathsf{c}(b_*) \prec_{\mathcal{I}_U(X)} u$. Thus, $\Phi(\mathsf{c}(b_*))$ holds, and so $b \lhd_X U$. Hence $a \lhd_X U$ by (U1), and therefore $\Phi(u)$.

(V2') $\dfrac{\left(\forall v \in V^{C(A,\varepsilon)}_{\mathcal{I}_U(X)} \downarrow u\right) \Phi(v)}{\Phi(u)}$ for each $(A, \varepsilon) \in \mathcal{E}_X$: Let $(A, \varepsilon) \in \mathcal{E}_X$, and suppose that $\Phi(v)$ holds for all $v \in V^{C(A,\varepsilon)}_{\mathcal{I}_U(X)} \downarrow u$. Let $a \in U_X$ such that $a_* \subseteq u_*$. Let $b \in \mathcal{C}^{\varepsilon}_A \downarrow a$. Then, $\mathsf{c}(b_*) \in V^{C(A,\varepsilon)}_{\mathcal{I}_U(X)} \downarrow u$, so $\Phi(\mathsf{c}(b_*))$ holds. Thus, $b \lhd_X U$, and hence $a \lhd_X U$ by (U2). Therefore $\Phi(u)$. □

Theorem 3.3.6. *For any uniform space X, its localic completion and the covering completion are isomorphic, i.e. $\mathcal{U}(X) \cong \mathcal{V}(\mathcal{I}_U(X))$.*

3.3.2 Comparison of embeddings

If $f : X \to Y$ is a function between uniform spaces (X, M) and (Y, N) which is uniformly continuous on each open ball of X, then $\mathcal{I}_U(f)$ is a locally uniformly continuous function from $\mathcal{I}_U(X)$ to $\mathcal{I}_U(Y)$. In this case, let $r_f : \mathcal{U}(X) \to \mathcal{U}(Y)$ and $s_{\mathcal{I}_U(f)} : \mathcal{V}(\mathcal{I}_U(X)) \to \mathcal{V}(\mathcal{I}_U(Y))$ be the formal topology maps given by (3.10) and (3.21) respectively.

Lemma 3.3.7. *Let X and Y be uniform spaces. If $f : X \to Y$ is uniformly continuous on each open ball of X, then the following diagram in* **FTop** *commutes.*

$$\begin{array}{ccc} \mathcal{V}(\mathcal{I}_U(X)) & \xrightarrow{r_X} & \mathcal{U}(X) \\ {\scriptstyle s_{\mathcal{I}_U(f)}}\downarrow & & \downarrow{\scriptstyle r_f} \\ \mathcal{V}(\mathcal{I}_U(Y)) & \xrightarrow{r_Y} & \mathcal{U}(Y) \end{array}$$

Here, r_X and r_Y are the isomorphisms given by (3.23).

Proof. By Proposition 3.1.14 and Proposition 2.4.9, it suffices to show that

$$r_Y \circ s_{\mathcal{I}_U(f)} \leq r_f \circ r_X.$$

Let $b \in U_Y$, and let $u \in s_{\mathcal{I}_U(f)}{}^{-} r_Y{}^{-} b$. Then, there exist $v, v' \in V_{\mathcal{I}_U(Y)}$ and $b' \in U_Y$ such that

$$f[u_*] \subseteq v'_*, \quad v' \prec_{\mathcal{I}_U(Y)} v, \quad v_* \subseteq b'_*, \quad b' <_Y b.$$

Thus, there exists $(B, \delta) \in \mathcal{E}_Y$ such that $v' \prec^{C(B,\delta)}_{\mathcal{I}_U(Y)} v$. Since f is uniformly continuous on u_*, there exists $(A, \varepsilon) \in \mathcal{E}_X$ such that

$$(\forall x, x' \in u_*)\, \rho_A(x, x') < \varepsilon \implies \rho_B(f(x), f(x')) < \delta.$$

Let $u' \prec_{\mathcal{I}_U(X)} u$. Then, there exists $(D,\gamma) \in \mathcal{E}_X$ such that $u' \prec^{C(D,\gamma)}_{\mathcal{I}_U(X)} u$. Let $u'' \in V^{C(A\cup D,\theta)}_{\mathcal{I}_U(X)} \downarrow u'$, where $\theta = \min\{\varepsilon/3, \gamma/3\}$. Write $u'' = \mathsf{c}(\mathsf{b}_E(x,\zeta)_*)$. Then, $u''_* \subseteq \mathsf{b}_{A\cup D}(x, 2\theta)_*$, and thus $u'' \; r_X \; \mathsf{b}_{A\cup D}(x, 3\theta)$. Since $\mathsf{b}_D(x,\gamma)_* \between u'_*$, we have $\mathsf{b}_D(x,\gamma)_* \subseteq u_*$. Since $\mathsf{b}_{A\cup D}(x,3\theta)_* \subseteq \mathsf{b}_D(x,\gamma)_* \subseteq u_*$ and $\mathsf{b}_{A\cup D}(x,3\theta)_* \subseteq \mathsf{b}_A(x,\varepsilon)_*$, we have $f\left[\mathsf{b}_{A\cup D}(x,3\theta)_*\right] \subseteq v'_* \cap \mathsf{b}_B(f(x),\delta)_*$. Since $\mathsf{b}_B(f(x),\delta)_* \between v'_*$, we have $\mathsf{b}_B(f(x),\delta)_* \subseteq v_* \subseteq b'_*$. Thus, $\mathsf{b}_{A\cup D}(x,3\theta) \; r_f \; b$, and hence $u \blacktriangleleft_{\mathcal{I}_U(X)} r_X{}^- r_f{}^- b$ by (V1) and (V2). Therefore, $r_Y \circ s_{\mathcal{I}_U(f)} \leq r_f \circ r_X$. □

Remark 3.3.8. Since the assignment $f \mapsto r_f$ is the morphism part of the embedding $\mathcal{U} : \mathbf{LKUSpa} \to \mathbf{OLKReg}$, one could have defined an equivalent embedding of the category **LKUSpa** into **OLKReg** by first embedding **LKUSpa** into **CUSpa** via $\mathcal{I}_U$, and then applying the covering completion.

Let **KUSpa** be the full subcategory of **USpa** consisting of compact uniform spaces and uniformly continuous functions. Since $\mathcal{I}_U : \mathbf{USpa} \to \mathbf{CUSpa}$ preserves compactness by Remark 3.2.34, the restriction of $\mathcal{I}_U$ to **KUSpa** can be composed with the embedding $\mathcal{V} : \mathbf{KCUSpa} \to \mathbf{KReg}_2$. By post-composing with the inclusion $\mathbf{KReg}_2 \hookrightarrow \mathbf{FTop}$, it determines an embedding $\mathcal{V} \circ \mathcal{I}_U : \mathbf{KUSpa} \to \mathbf{FTop}$. Let $\mathcal{U} : \mathbf{KUSpa} \to \mathbf{FTop}$ be the restriction of the composition of $\mathcal{U} : \mathbf{LKUSpa} \to \mathbf{OLKReg}$ with the inclusion $\mathbf{OLKReg} \hookrightarrow \mathbf{FTop}$.

Since the assignments $f \mapsto r_f$ and $f \mapsto s_{\mathcal{I}_U(f)}$ in Lemma 3.3.7 are the morphism parts of $\mathcal{U}$ and $\mathcal{V} \circ \mathcal{I}_U$ respectively, we have the following.

Theorem 3.3.9. *The functors*

$$\mathcal{V} \circ \mathcal{I}_U,\ \mathcal{U} : \mathbf{KUSpa} \to \mathbf{FTop}$$

are naturally isomorphic, and the formal topology map $r_X : \mathcal{V}(\mathcal{I}_U(X)) \to \mathcal{U}(X)$ given by (3.23) is a component of a natural isomorphism from $\mathcal{V} \circ \mathcal{I}_U$ to $\mathcal{U}$.

Chapter 4

Point-free Characterisation of Compact Metric Spaces

In this chapter, we present the first main result of this thesis: a point-free characterisation of Bishop compact metric spaces (or just compact metric spaces). We show that the notion of compact overt enumerably completely regular formal topology characterises that of compact metric space up to isomorphism. Roughly speaking, a compact overt enumerably completely regular formal topology is a compact overt regular formal topology where the graph of its associated function wc is countable.

The adjunction described in Section 2.5.2 cannot be used for our purpose since, e.g. the lattice of open subsets of $[0, 1]$ cannot be compact. Instead, we use the localic completion of metric spaces [48], i.e. the restriction of the localic completion of uniform spaces described in Section 3.1.2 to the class of metric spaces. As a special case of Theorem 3.1.47, we have an embedding from the category of locally compact metric spaces to that of overt locally compact regular formal topologies. This is the embedding obtained in [48]. In what follows, we also call this embedding the localic completion. By Theorem 3.1.39, the localic completion restricts to an embedding from the category of compact metric spaces to that of compact overt regular formal topologies.

What we do in this chapter is to characterise the image of the category of compact metric spaces under the localic completion by the notion of compact overt enumerably completely regular formal topology in the following sense: the localic completion restricts to a full, faithful and essentially surjective functor from the category of compact metric spaces to that of the isomorphic closure of compact overt enumerably completely regular formal topologies[1]. Hence, the two categories can be regarded as essentially equivalent[2].

Our characterisation can be seen as a separable version of the well-known fact that the category of compact Hausdorff spaces and that of compact regular locales are equivalent [34]. In locale theory, this equivalence is shown by employing the Prime Ideal theorem

[1]The full subcategory of **FTop** consisting of formal topologies which are isomorphic to some compact overt enumerably completely regular formal topologies.

[2]Equivalent in the weaker sense that there exists a full, faithful and essentially surjective functor from one category to the other. Under the Axiom of Choice, this notion is equivalent to the usual notion of equivalence of categories [40].

and classical logic. Our result shows that in the separable case the equivalence, albeit in a weaker form, can be obtained constructively by replacing the notion of second countable compact Hausdorff space by that of compact metric space.

A crucial step in obtaining the characterisation is to relate the notion of compact subspace of a locally compact metric space X and that of compact overt subtopology of its localic completion $\mathcal{M}(X)$. This is done in Section 4.1, where we establish an equivalence between the two notions. Then, by exploiting the well-known arguments on completely regular locales, we show that the following three notions are equivalent:

1. compact overt enumerably completely regular formal topology;
2. compact overt subtopology of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$;
3. compact metric space.

Hence, our characterisation can also be thought as a special case of the famous Urysohn's metrisation theorem, which states that the notions of second countable regular space, separable metric space, and subspace of the countable product of $[0,1]$ are equivalent [38, 62].

As an application of the characterisation, we show that every inhabited compact enumerably completely regular formal topology is the image of the formal Cantor space under some formal topology map, a point-free version of the famous result due to Brouwer [55].

Remark 4.0.10. Except for the notion of located subset, all notions for metric spaces used in this chapter and Chapter 5 are special cases of those for uniform spaces defined in Section 3.1. This means that we identify a metric space (X,d) with a uniform space $(X,\{d\})$. See Appendix D for background on Bishop metric spaces.

Notation 4.0.11. In the following, we use the notations for localic completions of metric spaces introduced in Notation 3.1.27. For the localic completion of a metric space, we use different names for the axioms (U1) and (U2) as follows.

(M1) $a \lhd_X \{b \in M_X \mid b <_X a\}$,

(M2) $a \lhd_X \mathcal{C}_\varepsilon$ for each $\varepsilon \in \mathbb{Q}^{>0}$,

where $\mathcal{C}_\varepsilon \overset{\text{def}}{=} \{\mathsf{b}(x,\varepsilon) \in M_X \mid x \in X\}$. Moreover, we use a symbol $\mathcal{M} : \mathbf{LCM} \to \mathbf{OLKReg}$ for the restriction of the embedding $\mathcal{U} \colon \mathbf{LKUSpa} \to \mathbf{OLKReg}$ obtained in Theorem 3.1.47 to the full subcategory $\mathbf{LCM}$ of locally compact metric spaces. The category $\mathbf{LCM}$ has locally compact metric spaces as objects and continuous functions between them as morphisms (See Definition 3.1.33 and Definition 3.1.40).

4.1 Compact subspaces

In this section, we relate the compact subspaces of a locally compact metric space and the compact overt subtopologies of its localic completion by establishing a bijective correspondence between these classes of subspaces and subtopologies. The proof rests on the observation made by Spitters [53] that the notion of located subtopology corresponds to the metric notion of located subspace. The established correspondence leads to a preliminary point-free characterisation of compact metric spaces; we show that the image of the class of compact metric spaces under the localic completion can be characterised by compact overt subtopologies of localic completions of locally compact metric spaces.

We will also make a little digression into the abstract theory of located subtopologies. We extend the notion of located subtopology for compact regular formal topologies [53] to the class of locally compact formal topologies, and show that the notion of located subtopology is equivalent to that of overt closed subtopology.

Closed subsets of complete metric spaces

We establish a bijective correspondence between the closed subsets of a complete metric space X and the splitting subsets of its localic completion $\mathcal{M}(X)$. The correspondence will be refined to the one mentioned at the beginning of Section 4.1.

Definition 4.1.1. A subset A of a metric space X is *closed* if for each $x \in X$, we have

$$\left[\left(\forall \varepsilon \in \mathbb{Q}^{>0}\right) B(x, \varepsilon) \between A\right] \implies x \in A.$$

Stated in terms of localic completion, A is closed in X iff for each $x \in X$

$$\Diamond x \subseteq \Diamond A \implies x \in A.$$

The class of closed subsets of a metric space X is denoted by $Cl(X)$.

The following is a slightly more general version of [19, Lemma 3.2], which can be obtained as a corollary of Lemma 4.1.2 by applying Proposition 2.3.20. The proof requires the Dependent Choice.

Lemma 4.1.2. *Let X be a metric space, and let V be a splitting subset of $\mathcal{M}(X)$. Then, for each $a \in V$, there exists $\alpha \in \mathcal{P}t(\mathcal{M}(X)^V)$ such that $a \in \alpha$.*

Proof. Define a relation $R \subseteq V \times V$ by

$$\mathsf{b}(x, \varepsilon)\; R\; \mathsf{b}(y, \delta) \overset{\text{def}}{\iff} \mathsf{b}(y, \delta) <_X \mathsf{b}(x, \varepsilon) \;\&\; \delta \leq \varepsilon/2.$$

We show that R is a total relation on V. Let $a = \mathsf{b}(x, \varepsilon) \in V$. By (M1), (M2) and (Loc), we have

$$a \lhd_X \{b \in M_X \mid b <_X a\} \downarrow \mathcal{C}_{\varepsilon/2}.$$

Since $a \in V$ and V splits $\lhd_X$, there exists $b \in V$ such that $b <_X a$ and $b \in \downarrow \mathcal{C}_{\varepsilon/2}$. Clearly, we have $a \; R \; b$, and hence R is total.

Let $a_0 \in V$. By the Dependent Choice, there exists a function $f \colon \mathbb{N} \to V$ such that $f(0) = a_0$, and for all $n \in \mathbb{N}$, $f(n) \; R \; f(n+1)$. Define

$$\alpha \overset{\text{def}}{=} \{a \in M_X \mid (\exists n \in \mathbb{N})\, f(n) \leq_X a\}.$$

Trivially, we have $a_0 \in \alpha$. Since V is upward closed, we have $\alpha \subseteq V$. The fact that α is a formal point of $\mathcal{M}(X)$ follows from the definition of R. Therefore, $\alpha \in \mathcal{P}t(\mathcal{M}(X)^V)$. ☐

Recall that $\mathit{Red}(\mathcal{M}(X))$ denotes the class of splitting subsets of $\mathcal{M}(X)$ (See Definition 2.1.7).

Theorem 4.1.3. *Let X be a complete metric space. Then, there exists a bijective correspondence $\varphi\colon \mathit{Cl}(X) \to \mathit{Red}(\mathcal{M}(X))$ between the closed subsets of X and the splitting subsets of $\mathcal{M}(X)$ given by*

$$\varphi(A) \stackrel{\text{def}}{=} \Diamond A,$$
$$\varphi^{-1}(V) \stackrel{\text{def}}{=} \{x \in X \mid \Diamond x \subseteq V\}$$

for all $A \in \mathit{Cl}(X)$ and $V \in \mathit{Red}(\mathcal{M}(X))$.

Proof. First, we show that for any $A \in \mathit{Cl}(X)$, the set $\Diamond A$ is splitting. Let $a \in M_X$ and $U \subseteq M_X$, and suppose that $a \lhd_X U$ and $a \in \Diamond A$. Then, there exists $y \in A$ such that $a \in \Diamond y$. Since $\Diamond y \in \mathcal{P}t(\mathcal{M}(X))$, we have $\Diamond y \between U$, and hence $\Diamond A \between U$. Thus, $\Diamond A$ splits the cover $\lhd_X$.

Next, we show that for any $V \in \mathit{Red}(\mathcal{M}(X))$, the set $A = \{x \in X \mid \Diamond x \subseteq V\}$ is a closed subset of X. Let $x \in X$, and suppose that $\Diamond x \subseteq \Diamond A$. Let $a \in \Diamond x$. Then, $a \in \Diamond A$, so there exists $y \in A$ such that $a \in \Diamond y \subseteq V$. Hence, $\Diamond x \subseteq V$, that is $x \in A$. Therefore, A is a closed subset of X.

Finally, we show that φ is a bijection. First, we obviously have $A \subseteq (\varphi^{-1} \circ \varphi)(A)$. The converse inclusion is just the definition of closed subset in terms of localic completion. Hence, $A = (\varphi^{-1} \circ \varphi)(A)$. Next, we obviously have $(\varphi \circ \varphi^{-1})(V) \subseteq V$. For the converse, let $a \in V$. By Lemma 4.1.2, there exists $\alpha \in \mathcal{P}t(\mathcal{M}(X))$ such that $a \in \alpha \subseteq V$. Since X is complete, there exists $x \in X$ such that $\Diamond x = \alpha$. Thus, we have $a \in \Diamond x$ and $x \in \varphi^{-1}(V)$. Therefore, $a \in (\varphi \circ \varphi^{-1})(V)$. $\square$

Remark 4.1.4. For any metric space X and $V \in \mathit{Red}(\mathcal{M}(X))$, we have

$$\varphi^{-1}(V) = i_X^{-1}[\mathcal{P}t(\mathcal{M}(X)^V)],$$

where $i_X : X \to \mathcal{P}t(\mathcal{M}(X))$ is the dense embedding given by (3.6).

Located subsets of complete metric spaces

We refine the correspondence obtained above to a correspondence between the located subsets of a complete metric space X and the located subsets of its localic completion $\mathcal{M}(X)$.

Definition 4.1.5. A subset A of a metric space $X = (X, d)$ is *located* if for each $x \in X$ the distance

$$d(x, A) \stackrel{\text{def}}{=} \inf \{d(x, y) \mid y \in A\}$$

exists as a non-negative Dedekind real number, i.e. for each $x \in X$, the set

$$U_x = \left\{q \in \mathbb{Q}^{>0} \mid (\exists y \in A)\, d(x, y) < q\right\}$$

satisfies

1. $(\exists q \in \mathbb{Q}^{>0})\, q \in U_x$,
2. $(\forall p, q \in \mathbb{Q}^{>0})\, p < q \implies p \in \neg U_x \vee q \in U_x$.

Lemma 4.1.6. *Let $A \subseteq X$ be an inhabited subset of a metric space $X = (X, d)$. Then, the following are equivalent.*

1. *A is located.*
2. *$(\forall a, b \in M_X)\, a <_X b \implies a \in \neg\Diamond A \vee b \in \Diamond A$.*

Proof. First, note that since A is inhabited, A is located iff for all $x \in X$

$$\left(\forall p, q \in \mathbb{Q}^{>0}\right) p < q \implies p \in \neg U_x \vee q \in U_x.$$

(1 ⇒ 2): Suppose that A is located. Let $a, b \in M_X$, and suppose that $a <_X b$. Write $a = \mathsf{b}(x, \varepsilon)$ and $b = \mathsf{b}(y, \delta)$. Choose $\gamma \in \mathbb{Q}^{>0}$ such that $d(x, y) + \varepsilon + \gamma < \delta$. Then, either $\varepsilon \in \neg U_x$ or $\varepsilon + \gamma \in U_x$. In the former case, $\neg[B(x, \varepsilon) \between A]$. In the latter case, we have $B(x, \varepsilon + \gamma) \between A$, and hence $B(y, \delta) \between A$. Therefore $a \in \neg\Diamond A$ or $b \in \Diamond A$.

(2 ⇒ 1): Suppose that 2 holds. Let $x \in X$ and $p, q \in \mathbb{Q}^{>0}$ such that $p < q$. Then, $\mathsf{b}(x, p) <_X \mathsf{b}(x, q)$. Hence, $\mathsf{b}(x, p) \in \neg\Diamond A$ or $\mathsf{b}(x, q) \in \Diamond A$, that is, $p \in \neg U_x$ or $q \in U_x$. Therefore, A is located. □

Definition 4.1.7 (Spitters [53, Definition 44])**.** Let $\mathcal{M}(X)$ be the localic completion of a metric space X. A subset $V \subseteq M_X$ is *located* if V is a splitting subset of $\mathcal{M}(X)$ and moreover satisfies

$$a <_X b \implies a \in \neg V \vee b \in V$$

for all $a, b \in M_X$. A subtopology $\mathcal{S}'$ of $\mathcal{M}(X)$ is *located* if there exists a (necessarily unique) located subset V of $\mathcal{M}(X)$ such that $\mathcal{S}' = \mathcal{M}(X)^V$, where $\mathcal{M}(X)^V$ is the overt weakly closed subtopology of $\mathcal{M}(X)$ determined by V.

Spitters [53, Definition 44] defined a located subtopology of $\mathcal{M}(X)$ as a closed subtopology $\mathcal{M}(X)^{\mathcal{M}(X)-\neg V}$ of $\mathcal{M}(X)$ determined by some located subset V of $\mathcal{M}(X)$. However, the two definitions are equivalent.

Proposition 4.1.8 (Spitters [53, Proposition 51])**.** *Let X be a metric space, and let V be a located subset of $\mathcal{M}(X)$. Then, the overt weakly closed subtopology $\mathcal{M}(X)^V$ determined by V is closed, that is, $\mathcal{M}(X)^{\mathcal{M}(X)-\neg V} = \mathcal{M}(X)^V$.*

Proof. Since $\mathcal{M}(X)^{\mathcal{M}(X)-\neg V}$ is a closure of $\mathcal{M}(X)^V$ by Proposition 2.3.13, it suffices to show that for each $a \in M_X$, we have

$$a \lhd_X^{\mathcal{M}(X)-\neg V} \{a\} \cap V.$$

Let $a \in M_X$, and let $b <_X a$. Since V is located, either $b \in \neg V$ or $a \in V$. Since $b \lhd_X a$, we have $b \lhd_X \neg V \cup (\{a\} \cap V)$. Therefore, $a \lhd_X^{\mathcal{M}(X)-\neg V} \{a\} \cap V$ by (M1). □

Given a metric space X, let $LCl^+(X)$ denote the class of inhabited closed located subsets of X, and let $LRed^+(\mathcal{M}(X))$ denote the class of inhabited located subsets of $\mathcal{M}(X)$.

Theorem 4.1.9. *Let X be a complete metric space. Then, the bijection $\varphi\colon Cl(X) \to Red(\mathcal{M}(X))$ in Theorem 4.1.3 restricts to a bijection $\varphi\colon LCl^+(X) \to LRed^+(\mathcal{M}(X))$.*

Proof. First, let $A \in LCl^+(X)$. Then, $\varphi(A) = \Diamond A$ is a located subset of $\mathcal{M}(X)$ by Lemma 4.1.6. Clearly, $\varphi(A)$ is inhabited.

Conversely, let $V \in LRed^+(\mathcal{M}(X))$. Since V is inhabited, there exists $\alpha \in \mathcal{P}t(\mathcal{M}(X))$ such that $\alpha \subseteq V$ by Lemma 4.1.2. Since X is complete, there exists $x \in X$ such that $\Diamond x = \alpha$. Then, $x \in \varphi^{-1}(V)$, so $\varphi^{-1}(V)$ is inhabited. Next, let $a, b \in M_X$, and suppose that $a <_X b$. Since V is located, either $a \in \neg V$ or $b \in V$, that is, either $a \in \neg\Diamond\varphi^{-1}(V)$ or $b \in \Diamond\varphi^{-1}(V)$. Thus, $\varphi^{-1}(V)$ is located by Lemma 4.1.6. □

Located subsets of locally compact formal topologies

We make a short digression into the abstract theory of located subtopologies. We define the notion of located subtopology for locally compact formal topologies. Definition 4.1.10 extends the one given by Spitters [53] for compact regular formal topologies, and it enjoys the same characteristic property (See Theorem 4.1.15).

Definition 4.1.10. Let $\mathcal{S}$ be a locally compact formal topology. A subset $V \subseteq S$ is *located* if V is a splitting subset of $\mathcal{S}$ and moreover satisfies

$$a \ll b \implies a \in \neg V \vee b \in V$$

for all $a, b \in S$. A subtopology $\mathcal{S}'$ of a locally compact formal topology $\mathcal{S}$ is *located* if there exists a located subset V of $\mathcal{S}$ such that $\mathcal{S}' = \mathcal{S}^V$, where $\mathcal{S}^V$ is the overt weakly closed subtopology of $\mathcal{S}$ determined by V.

Note that since every locally compact formal topology is inductively generated (Remark 2.4.16), the notion of located subtopology makes sense.

Proposition 4.1.11. *A splitting subset V of a locally compact formal topology $\mathcal{S}$ is located iff*

$$a \in wb(b) \implies a \in \neg V \vee b \in V$$

for all $a, b \in S$, where $wb\colon S \to \mathsf{Pow}(S)$ is a base of the way-below relation on $\mathcal{S}$.

Proof. If V is located, then clearly V satisfies the condition.

Conversely, suppose that V satisfies the condition. Let $a, b \in S$, and suppose that $a \ll b$. Since $b \lhd wb(b)$, there exists $U \in \mathsf{Fin}(wb(b))$ such that $a \lhd U$. Then, either $U \subseteq \neg V$ or $b \in V$. In the former case, if $a \in V$, then $V \between \neg V$, a contradiction. Hence $a \in \neg V$. Therefore, V is located. □

By Corollary 3.1.37 and (M1), we see that Definition 4.1.10 is compatible with Definition 4.1.7.

Corollary 4.1.12. *Let X be a locally compact metric space, and let V be a splitting subset of $\mathcal{M}(X)$. Then, V is located in the sense of Definition 4.1.7 iff it is located in the sense of Definition 4.1.10.*

Next, we see that located subtopologies can be characterised as overt closed subtopologies.

Proposition 4.1.13. *Let $\mathcal{S}$ be a locally compact formal topology, and let V be a located subset of $\mathcal{S}$. Then, we have $\mathcal{S}^V = \mathcal{S}^{\mathcal{S}-\neg V}$. Hence, every located subtopology of a locally compact formal topology is overt and closed.*

Proof. Since $\mathcal{S}^{\mathcal{S}-\neg V}$ is the closure of $\mathcal{S}^V$ by Proposition 2.3.13, we have $\mathcal{S}^V \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg V}$.

Conversely, let $a \in S$. Then, for any $b \in wb(a)$, we have either $b \in \neg V$ or $a \in V$. Hence, we have $a \lhd wb(a) \lhd \neg V \cup (\{a\} \cap V)$, and thus $a \lhd^{\mathcal{S}-\neg V} \{a\} \cap V$. Since $\mathcal{S}^V$ is an overt weakly closed subtopology with the positivity V, we have $\mathcal{S}^{\mathcal{S}-\neg V} \sqsubseteq \mathcal{S}^V$. □

Proposition 4.1.14. *Let $\mathcal{S}$ be a locally compact formal topology, and let $V \subseteq S$. Then, the closed subtopology $\mathcal{S}^{\mathcal{S}-V}$ of $\mathcal{S}$ is overt iff it is located.*

Proof. Suppose that $\mathcal{S}^{\mathcal{S}-V}$ is overt with a positivity Pos. By Corollary 2.3.23, it suffices to show that Pos is a located subset of $\mathcal{S}$. Let $a, b \in S$, and suppose that $a \ll b$. Since Pos is the positivity of $\mathcal{S}^{\mathcal{S}-V}$, we have $b \lhd V \cup (\{b\} \cap \mathsf{Pos})$. Hence, there exists $B \in \mathsf{Fin}(\{b\} \cap \mathsf{Pos})$ such that $a \lhd V \cup B$. If B is inhabited, then $\mathsf{Pos}(b)$. If $B = \emptyset$, then $a \lhd^{\mathcal{S}-V} \emptyset$, and hence $a \in \neg \mathsf{Pos}$ because Pos splits $\lhd^{\mathcal{S}-V}$. Therefore, Pos is a located subset of $\mathcal{S}$.

The converse follows from the fact that every located subtopology is overt. □

Theorem 4.1.15. *Let $\mathcal{S}$ be a locally compact formal topology. Then, there exists an order isomorphism*

$$\begin{aligned} \Phi \colon \mathit{LRed}(\mathcal{S}) &\to \mathit{OCl}(\mathcal{S}) \\ V &\mapsto \mathcal{S}^{\mathcal{S}-\neg V} \end{aligned}$$

between the located subsets $\mathit{LRed}(\mathcal{S})$ of $\mathcal{S}$ and the overt closed subtopologies $\mathit{OCl}(\mathcal{S})$ of $\mathcal{S}$.

Proof. By Proposition 4.1.13 and Proposition 4.1.14, the order isomorphism in Theorem 2.3.21 restricts to the order isomorphism in the statement of this theorem. □

The following is also a corollary of Theorem 74 in [53].

Corollary 4.1.16. *Let $\mathcal{S}$ be a compact regular formal topology. Then, there exists an order isomorphism between the compact overt subtopologies of $\mathcal{S}$ and the located subsets of $\mathcal{S}$.*

Proof. By Proposition 2.4.23, Proposition 2.4.13 and Theorem 4.1.15. □

Compact subsets of locally compact metric spaces

We establish a bijective correspondence between the compact subspaces of a locally compact metric space X and the compact overt subtopologies of its localic completion $\mathcal{M}(X)$.

First, we review some connections between locatedness and compactness in Bishop metric space (See Appendix D.0.28). An inhabited locally compact metric space is compact iff it is bounded[3]. An inhabited subset of a locally compact metric space is locally compact iff it is closed and located. Thus, an inhabited subset of a locally compact metric space is compact iff it is closed, located and bounded. Since we can decide whether a given totally bounded subset is inhabited or empty, it follows that a subset $A \subseteq X$ of a locally compact metric space X is compact iff either $A = \emptyset$ or A is an inhabited, closed, located and bounded subset of X.

The following was shown by Coquand et al. [19, Theorem 3.5]. We give a different proof based on the above observation.

Lemma 4.1.17. *Let X be a locally compact metric space. Let $\mathcal{S}^{\mathsf{Pos}}$ be a compact overt subtopology of $\mathcal{M}(X)$ with a positivity Pos. Then, $A = \{x \in X \mid \Diamond x \subseteq \mathsf{Pos}\}$ is a compact subset of X.*

Proof. Since $\mathcal{S}^{\mathsf{Pos}}$ is compact overt with the positivity Pos, there exists $U \in \mathsf{Fin}(\mathsf{Pos})$ such that $S \lhd^{\mathsf{Pos}} U$. Then, either $U = \emptyset$ or U is inhabited. If $U = \emptyset$, then $A = \emptyset$, so A is compact. If U is inhabited, then Pos is inhabited. Since $\mathcal{M}(X)$ is locally compact regular, $\mathcal{S}^{\mathsf{Pos}}$ is an overt closed subtopology of $\mathcal{M}(X)$, and $\mathcal{S}^{\mathsf{Pos}}$ is uniquely determined by the located subset Pos. Then, by Theorem 4.1.9, A is an inhabited, located and closed subset of X. Hence, it suffices to show that A is bounded. Let $y \in A$. Then, $\Diamond y \in \mathcal{P}t(\mathcal{S}^{\mathsf{Pos}})$. Since $S \lhd^{\mathsf{Pos}} U$, we have $\Diamond y \between U$, that is, $y \in \bigcup_{a \in U} a_*$. Thus, $A \subseteq \bigcup_{a \in U} a_*$. Since U is finitely enumerable, there exists an open ball $B(x, \varepsilon)$ such that $\bigcup_{a \in U} a_* \subseteq B(x, \varepsilon)$. Therefore, A is bounded. □

Theorem 4.1.18. *Let $X = (X, d)$ be a locally compact metric space. Then, up to isomorphism, the localic completion $\mathcal{M} \colon$ **LCM** $\to$ **OLKReg** induces a bijection between the compact subspaces of X and the compact overt subtopologies of $\mathcal{M}(X)$.*

Proof. We will identify a compact subspace of X with a compact subset of X. We define a bijection Φ and its inverse Φ^{-1} between the compact subsets of X and the compact overt subtopologies of $\mathcal{M}(X)$ such that

$$\Phi(A) \cong \mathcal{M}(A),$$
$$\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}}) \cong \mathcal{P}t(\mathcal{S}^{\mathsf{Pos}})$$

for any compact subset $A \subseteq X$ and for any compact overt subtopology $\mathcal{S}^{\mathsf{Pos}} \sqsubseteq \mathcal{M}(X)$ with a positivity Pos.

First, given a compact subset $A \subseteq X$, let $i_A \colon A \to X$ be the inclusion. Let $\Phi(A)$ be the image of $\mathcal{M}(A)$ under the embedding $\mathcal{M}(i_A) \colon \mathcal{M}(A) \to \mathcal{M}(X)$. Note that $\mathcal{M}(i_A)$ is defined by

$$a \; \mathcal{M}(i_A) \; b \overset{\text{def}}{\iff} (\exists b' <_X b) \, i_A[a_*] \subseteq b'_*$$

[3] A metric space (X, d) is bounded if there exists $\varepsilon \in \mathbb{Q}^{>0}$ such that $d(x, y) < \varepsilon$ for all $x, y \in X$.

for all $a \in M_A$ and $b \in M_X$. Since $\mathcal{M}(A)$ is compact overt, $\Phi(A)$ is a compact overt subtopology of $\mathcal{M}(X)$. Clearly, $\Phi(A) \cong \mathcal{M}(A)$.

Conversely, given a compact overt subtopology $\mathcal{S}^{\mathsf{Pos}}$ of $\mathcal{M}(X)$ with a positivity Pos, let $\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}}) = i_X^{-1}[\mathcal{P}t(\mathcal{S}^{\mathsf{Pos}})] = \{x \in X \mid \Diamond x \subseteq \mathsf{Pos}\}$. By Lemma 4.1.17, $\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}})$ is a compact subset of X. Moreover, since X is complete, we have $\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}}) \cong \mathcal{P}t(\mathcal{S}^{\mathsf{Pos}})$.

To show that Φ and Φ^{-1} are mutual inverse, first, let $A \subseteq X$ be a compact subset of X. Since $\mathcal{M}(A)$ is compact overt with the positivity M_A, $\Phi(A)$ is overt with the positivity $\mathsf{Pos}_A = \mathcal{M}(i_A)M_A = \{a \in M_X \mid (\exists b \in M_A)\, b\, \mathcal{M}(i_A)\, a\}$ by Lemma 2.3.4. We show that $\mathsf{Pos}_A = \Diamond A$. Let $a \in \mathsf{Pos}_A$. Then, there exists $b \in M_A$ such that $b\ \mathcal{M}(i_A)\ a$. Clearly, we have $a \in \Diamond A$. Conversely, let $a = \mathsf{b}(x, \varepsilon) \in \Diamond A$. Then, there exists $y \in A$ such that $d(x, y) < \varepsilon$. Choose $\delta \in \mathbb{Q}^{>0}$ such that $d(x, y) + \delta < \varepsilon$. Then, $\mathsf{b}(y, \delta) <_X a$, so $\mathsf{b}(y, \delta)\ \mathcal{M}(i_A)\ a$. Hence, $a \in \mathsf{Pos}_A$, and thus $\mathsf{Pos}_A = \Diamond A$. Since A is a closed subset of X, we have $(\Phi^{-1} \circ \Phi)(A) = \{x \in X \mid \Diamond x \subseteq \Diamond A\} = A$.

Conversely, let $\mathcal{S}^{\mathsf{Pos}}$ be a compact overt subtopology of $\mathcal{M}(X)$ with a positivity Pos. Then, $\mathcal{S}^{\mathsf{Pos}}$ is uniquely determined by the located subset Pos. Since $(\Phi \circ \Phi^{-1})(\mathcal{S}^{\mathsf{Pos}})$ is uniquely determined by the positivity $\Diamond\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}})$, and we have $\Diamond\Phi^{-1}(\mathcal{S}^{\mathsf{Pos}}) = (\varphi \circ \varphi^{-1})(\mathsf{Pos}) = \mathsf{Pos}$, where φ is the bijection described in Theorem 4.1.3, it follows that $(\Phi \circ \Phi^{-1})(\mathcal{S}^{\mathsf{Pos}}) = \mathcal{S}^{\mathsf{Pos}}$. □

As a corollary, we obtain the following preliminary characterisation of the image of the class of compact metric spaces under the localic completion.

Corollary 4.1.19. *Let $\mathcal{S}$ be a formal topology. Then, the following are equivalent.*

1. *$\mathcal{S}$ is isomorphic to the localic completion of some compact metric space.*
2. *$\mathcal{S}$ is isomorphic to a compact overt subtopology of the localic completion of some locally compact metric space.*

Example 4.1.20. The formal unit interval $\mathcal{I}[0,1]$ is isomorphic to the localic completion of the unit interval $[0,1]$. To see this, we first show that the localic completion of the reals $\mathbb{R}$ is isomorphic to the formal reals $\mathcal{R}$ [48, Example 2.2]. Since the rationals $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we have $\mathcal{M}(\mathbb{R}) \cong \mathcal{M}(\mathbb{Q})$ by Proposition 3.1.16. Hence, it suffices to show that $\mathcal{M}(\mathbb{Q})$ is isomorphic to $\mathcal{R}$. To this end, let d be the standard metric on $\mathbb{Q}$ given by $d(p, q) \stackrel{\text{def}}{=} |p - q|$. Then, the base $M_{\mathbb{Q}}$ of $\mathcal{M}(\mathbb{Q})$ with relations $\leq_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ (defined analogously to those on $S_{\mathcal{R}}$) is isomorphic to the underlying order structure $(S_{\mathcal{R}}, \leq_{\mathcal{R}}, <_{\mathcal{R}})$ of $\mathcal{R}$. Then, $\mathcal{M}(\mathbb{Q})$ is defined by the following axiom-set on $(S_{\mathcal{R}}, \leq_{\mathcal{R}})$:

(Q1) $(p, q) \lhd_{\mathbb{Q}} \{(r, s) \in S_{\mathcal{R}} \mid (r, s) <_{\mathcal{R}} (p, q)\}$;

(Q2) $(p, q) \lhd_{\mathbb{Q}} \mathcal{C}_\varepsilon$ for each $\varepsilon \in \mathbb{Q}^{>0}$.

Since the axioms (Q1) and (R1) are the same, it suffices to show that the axioms (Q2) and (R2) are equivalent. But, (Q2) is clearly equivalent to the axiom (R2'), which is equivalent to (R2) (See Example 2.4.25). Hence, we have $\mathcal{M}(\mathbb{R}) \cong \mathcal{M}(\mathbb{Q}) \cong \mathcal{R}$.

Now, $\mathcal{I}[0,1]$ is a compact overt subtopology of $\mathcal{R}$ (See Example 2.3.24 and Example 2.4.26), and $\mathcal{R}$ is isomorphic to the localic completion of $\mathbb{R}$, which is a locally compact metric space. Thus, $\mathcal{I}[0,1]$ is isomorphic to the localic completion of some compact metric space. To see that $\mathcal{I}[0,1] \cong \mathcal{M}([0,1])$, note that $[0,1]$ is a compact subspace of $\mathbb{R}$. By the proof of Theorem 4.1.18, $\mathcal{M}([0,1])$ embeds into $\mathcal{R}$ through $\mathcal{M}(\mathbb{R})$ as a compact overt subtopology of $\mathcal{R}$. By Theorem 4.1.15, the image of $\mathcal{M}([0,1])$ in $\mathcal{R}$ is completely determined by its positivity. With some calculations, we can show that the positivity of the image is given by

$$\mathsf{Pos} = \{(p,q) \in S_{\mathcal{R}} \mid 0 < q \;\&\; p < 1\},$$

which is the positivity of $\mathcal{I}[0,1]$ (See (2.16)). Hence, we have $\mathcal{I}[0,1] \cong \mathcal{M}([0,1])$.

4.2 Enumerably complete regularity

In this section, we show that the class of enumerably completely regular formal topologies can be characterised by the subtopologies of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. The result is a special case of Tychonoff's embedding theorem for locales [34, Chapter IV, Theorem 1.7].

First, we recall the predicative notion of scale introduced by Curi [20].

Definition 4.2.1 (Curi [20, Section 2.1]). Let $\mathbb{I} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. Given a formal topology $\mathcal{S}$ and subsets $U, V \subseteq S$, a *scale* from U to V is a family $(U_q)_{q\in\mathbb{I}}$ of subsets of S such that $U \lhd U_0$, $U_1 \lhd V$, and for all $p, q \in \mathbb{I}$, $p < q$ implies $U_p \lll U_q$. Given any $U, V \subseteq S$, we say that U is *really covered* by V, denoted by $U \twoheadrightarrow\!\!\!\lll V$, if there exists a scale from U to V.

Note that while $\lll$ is a set relation on S, $\twoheadrightarrow\!\!\!\lll$ is not a set in general, as its definition involves a quantification over all the scales between two subsets of S.

Let $\mathcal{R}$ be the formal reals (See Example 2.2.22). For each $q \in \mathbb{Q}$, define

$$(q,\infty) \stackrel{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid r \geq q\},$$
$$(-\infty,q) \stackrel{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid s \leq q\}.$$

Proposition 4.2.2 ([34, Chapter IV, Proposition 1.4]). *Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. Then, the following are equivalent.*

1. *$U \twoheadrightarrow\!\!\!\lll V$.*
2. *There exists a formal topology map $r \colon \mathcal{S} \to \mathcal{R}$ such that*
   (a) *$r^-(0,\infty) \downarrow U \lhd \emptyset$,*
   (b) *$r^-(-\infty,1) \lhd V$.*
3. *There exists a formal topology map as in 2 which additionally factors through $\mathcal{I}[0,1]$.*

Proof. ($2 \Rightarrow 1$): Let $r \colon \mathcal{S} \to \mathcal{R}$ be a formal topology map which satisfies (2a) and (2b). Let $U_0 = U$. For each $q \in \mathbb{I} \cap \mathbb{Q}^{>0}$, let

$$U_q \stackrel{\text{def}}{=} r^-(-\infty,q).$$

Note that for any $p, q \in \mathbb{Q}$, we have $p < q \implies (-\infty, p) \lll (-\infty, q)$. To see this, suppose that $p < q$. For any $(r, s) \in S_{\mathcal{R}}$, we have either $s \leq q$ or $p \leq r$ or $r < p < q < s$. In the last case, we have $(r, s) \lhd_{\mathcal{R}} \{(r, q), (p, s)\} \lhd_{\mathcal{R}} (-\infty, p)^* \cup (-\infty, q)$. Thus, we have $p < q \implies r^-(-\infty, p) \lll r^-(-\infty, q) \iff U_p \lll U_q$. By (2b), we have $U_1 \lhd V$. Lastly, for any $q \in \mathbb{I} \cap \mathbb{Q}^{>0}$, since $S_{\mathcal{R}} \lhd_{\mathcal{R}} (-\infty, q) \cup (0, \infty)$, we have

$$S \lhd r^-(-\infty, q) \cup r^-(0, \infty) \lhd U^* \cup U_q$$

by (2a), and hence $U_0 \lll U_q$. Therefore, $(U_q)_{q \in \mathbb{I}}$ is a scale from U to V.

(1 $\Rightarrow$ 3): Let $(U_q)_{q \in \mathbb{I}}$ be a scale from U to V. Extend $(U_q)_{q \in \mathbb{I}}$ to $(U_q)_{q \in \mathbb{Q}}$ by defining $U_q = \emptyset$ if $q < 0$, and $U_q = S$ if $1 < q$. Then, we have $p < q \implies U_p \lll U_q$. Define a relation $r \subseteq S \times S_{\mathcal{R}}$ by

$$a \; r \; (p, q) \overset{\text{def}}{\iff} (\exists (p', q') \in S_{\mathcal{R}})\, p < p' < q' < q \;\&\; a \lhd U^*_{p'} \downarrow U_{q'} \tag{4.1}$$

for all $a \in S$ and $(p, q) \in S_{\mathcal{R}}$. We show that r is a formal topology map.

(FTMi1): For any $(p, q) \in S_{\mathcal{R}}$, such that $p < 0$ and $1 < q$ we have

$$U^*_p \downarrow U_q =_{\mathcal{S}} S \downarrow S = S.$$

Hence $a \; r \; (p, q)$ for any $a \in S$, and therefore $S \lhd r^- S_{\mathcal{R}}$.

(FTMi2): Let $(p, q), (u, v) \in S_{\mathcal{R}}$, and let $a \in r^-\{(p, q)\} \downarrow r^-\{(u, v)\}$. Then, there exist $(p', q'), (u', v') \in S_{\mathcal{R}}$ such that $p < p' < q' < q$ and $a \lhd U^*_{p'} \downarrow U_{q'}$, and $u < u' < v' < v$ and $a \lhd U^*_{u'} \downarrow U_{v'}$. Then

$$a \lhd U^*_{p'} \downarrow U^*_{u'} \downarrow U_{q'} \downarrow U_{v'} =_{\mathcal{S}} U^*_{\max\{p', u'\}} \downarrow U_{\min\{q', v'\}}.$$

If $\max\{p', u'\} < \min\{q', v'\}$, then we have $a \lhd r^-((p, q) \downarrow (u, v))$. Otherwise, we must have $a \lhd \emptyset$, because for any $p \leq q$, we have

$$U_p \downarrow U^*_q \lhd U^{**}_p \downarrow U^*_q \lhd U^{**}_q \downarrow U^*_q \lhd \emptyset.$$

Thus, in either case, we have $a \lhd r^-((p, q) \downarrow (u, v))$.

(FTMi3): It suffices to show that r preserves the axioms (R1) and (R2) of the formal reals. For (R2), let $(p, q), (u, v) \in S_{\mathcal{R}}$ such that $p < u < v < q$, and let $a \in S$ such that $a \; r \; (p, q)$. Then, there exists $(p', q') \in S_{\mathcal{R}}$ such that $p < p' < q' < q$ and $a \lhd U^*_{p'} \downarrow U_{q'}$. If $q' < v$ or $u < p'$, then we immediately have $a \lhd r^- \{(p, v), (u, q)\}$. So suppose that $p' \leq u < v \leq q'$. Let $u', v' \in \mathbb{Q}$ such that $u < u' < v' < v$. Then

$$\begin{aligned}
\left(U^*_{p'} \downarrow U_{v'}\right) \cup \left(U^*_{u'} \downarrow U_{q'}\right) &=_{\mathcal{S}} \left(U^*_{p'} \cup U^*_{u'}\right) \downarrow \left(U^*_{p'} \cup U_{q'}\right) \downarrow \left(U_{v'} \cup U^*_{u'}\right) \downarrow \left(U_{v'} \cup U_{q'}\right) \\
&=_{\mathcal{S}} \left(U^*_{p'} \cup U^*_{u'}\right) \downarrow S \downarrow S \downarrow \left(U_{v'} \cup U_{q'}\right) \\
&=_{\mathcal{S}} \left(U^*_{p'} \cup U^*_{u'}\right) \downarrow \left(U_{v'} \cup U_{q'}\right) \\
&=_{\mathcal{S}} U^*_{p'} \downarrow U_{q'}.
\end{aligned}$$

Thus, we have

$$a \lhd \left(U^*_{p'} \downarrow U_{v'}\right) \cup \left(U^*_{u'} \downarrow U_{q'}\right) \lhd r^-\{(p, v)\} \cup r^-\{(u, q)\} \lhd r^- \{(p, v), (u, q)\}.$$

Hence r preserves (R2). The fact that r preserves (R1) is immediate from the definition of r.

(FTMi4): Immediate from the definition of r.

Next, we show that r satisfies the conditions (2a) and (2b).

(2a): We have

$$\begin{aligned} r^{-}(0,\infty) \downarrow U \lhd \left(\bigcup_{q \in \mathbb{Q}^{>0}} (U_0^* \downarrow U_q) \right) \downarrow U_0 \\ =_{\mathcal{S}} \bigcup_{q \in \mathbb{Q}^{>0}} (U_0^* \downarrow U_q \downarrow U_0) \\ =_{\mathcal{S}} \bigcup_{q \in \mathbb{Q}^{>0}} (\emptyset \downarrow U_q) \lhd \emptyset. \end{aligned}$$

(2b): This is immediate from the fact that $q < 1$ implies $U_q \lhd V$.

Lastly, r factors through $\mathcal{I}[0,1]$ by Lemma 2.3.17, since we have

$$r^{-}((-\infty,0) \cup (1,\infty)) \lhd r^{-}(-\infty,0) \cup r^{-}(1,\infty) \lhd \emptyset$$

by the definition of $(U_q)_{q \in \mathbb{Q}}$ outside $\mathbb{I}$.

(3 $\Rightarrow$ 2): Trivial. □

Definition 4.2.3. Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. A scale $(U_q)_{q \in \mathbb{I}}$ from U to V is *finitary* if $U_q \in \mathsf{Fin}(S)$ for all $q \in \mathbb{I}$. For any $U, V \subseteq S$, the *set* of finitary scales from U to V is denoted by $\mathrm{Sc}_{\lll}(U,V)$. Explicitly, $\mathrm{Sc}_{\lll}(U,V)$ is the following set:

$$\{F \in \mathbb{I} \to \mathsf{Fin}(S) \mid U \lhd F(0) \;\&\; F(1) \lhd V \;\&\; (\forall p, q \in \mathbb{I})\, p < q \to F(p) \lll F(q)\}.$$

Lemma 4.2.4. *Let $\mathcal{S}$ be a compact regular formal topology. For any $U, V \subseteq S$ such that $U \lll V$, there exists $W \in \mathsf{Fin}(S)$ such that $U \lll W \lll V$.*

Proof. By Proposition 2.4.23 and Lemma 2.4.17. □

The following is a special case of Urysohn's lemma for locales [34, Chapter IV, Proposition 1.6] (See [20] for a proof in terms of formal topology[4]). The proof requires the Dependent Choice.

Proposition 4.2.5. *Let $\mathcal{S}$ be a compact regular formal topology. Then, for any $U, V \subseteq S$, if $U \lll V$, then there exists a finitary scale from U to V.*

Proof. Let $U, V \subseteq S$, and suppose that $U \lll V$. By Lemma 4.2.4, there exist $U_0, V_1 \in \mathsf{Fin}(S)$ such that $U \lll U_0 \lll V_1 \lll V$. Choose a bijection

$$q \colon \mathbb{N} \to \{p \in \mathbb{Q} \mid 0 < p < 1\}.$$

[4] The proof in [20] for normal formal topologies seems to require the Relativized Dependent Choice, since the class of scales between two subsets of the base may not be a set.

Define a subset A of $\mathsf{Fin}(S)^*$ by

$$A \overset{\text{def}}{=} \{l \in \mathsf{Fin}(S)^* \mid (\forall n < |l|)\, U_0 \lll l(n) \lll V_1 \,\&$$
$$(\forall n, m < |l|)\, q(n) < q(m) \to l(n) \lll l(m)\}.$$

Define a relation $R \subseteq A \times A$ by

$$l \; R \; k \overset{\text{def}}{\iff} (\exists W \in \mathsf{Fin}(S))\, l * \langle W \rangle = k$$

for all $l, k \in A$, i.e. $l \; R \; k$ iff k is an immediate successor of l. We show that R is a total relation on A. Let $l \in A$, and let $n = |l|$. Define subsets $W, W' \subseteq S$ by

$$W \overset{\text{def}}{=} \begin{cases} l(m) & \text{if } m = \max\{i < n \mid q(i) < q(n)\} \text{ exists,} \\ U_0 & \text{otherwise.} \end{cases}$$
$$W' \overset{\text{def}}{=} \begin{cases} l(m') & \text{if } m' = \min\{i < n \mid q(n) < q(i)\} \text{ exists,} \\ V_1 & \text{otherwise.} \end{cases}$$

By the definition of A, we have $W \lll W'$, so by Lemma 4.2.4, there exists $W'' \in \mathsf{Fin}(S)$ such that $W \lll W'' \lll W'$. Let $k = l * \langle W'' \rangle$. Then, we have $l \; R \; k$, and hence R is a total relation on A.

Since $\langle\rangle \in A$, by the Dependent Choice, there exists a function $f \colon \mathbb{N} \to A$ such that $f(0) = \langle\rangle$, and $(\forall n \in \mathbb{N})\, f(n) \; R \; f(n+1)$. Define a function $W_{(-)} \colon \mathbb{I} \to \mathsf{Fin}(S)$ by

$$W_p \overset{\text{def}}{=} \begin{cases} U_0 & \text{if } p = 0, \\ V_1 & \text{if } p = 1, \\ f(q^{-1}(p)+1)(q^{-1}(p)) & \text{otherwise} \end{cases}$$

for all $p \in \mathbb{I}$. We have $U \lhd W_0$ and $W_1 \lhd V$. Furthermore, let $r, s \in \mathbb{I}$, and suppose that $r < s$. If $r = 0$ or $s = 1$, then we have $W_r = U_0 \lll W_s$ or $W_r \lll V_1 = W_s$ by the definitions of $W_{(-)}$ and A. So suppose that $0 < r < s < 1$. Write $n = q^{-1}(r)$ and $m = q^{-1}(s)$, and assume without loss of generality that $n < m$. Then, $f(m+1)(n) \lll f(m+1)(m)$ by the definition of A. Since $f(n+1)$ is an initial segment of $f(m+1)$ by the definition of R, we have

$$\begin{aligned} W_r &= f(q^{-1}(r)+1)(q^{-1}(r)) \\ &= f(n+1)(n) \\ &= f(m+1)(n) \\ &\lll f(m+1)(m) \\ &= f(q^{-1}(s)+1)(q^{-1}(s)) = W_s. \end{aligned}$$

Therefore, $(W_p)_{p \in \mathbb{I}}$ is a finitary scale from U to V, as required. □

The following is a slight modification of the notion of enumerably completely regular formal topology due to Curi [20, Section 2.2].

Definition 4.2.6. A formal topology $\mathcal{S}$ is *enumerably completely regular* if there exists a function $wc\colon S \to \mathsf{Pow}(S)$ such that

1. $a \lhd wc(a)$,
2. the relation $\overline{wc} = \{(b, a) \in S \times S \mid b \in wc(a)\}$ is countable, i.e. there exists a surjection $f\colon \mathbb{N} \to \overline{wc}$,
3. there exists a function $sc \in \prod_{(b,a)\in\overline{wc}} \mathrm{Sc}_{\lll}(\{b\}, \{a\})$, called a *choice of scale* for wc.

Remark 4.2.7. Assuming the Dependent Choice, any compact regular formal topology $\mathcal{S}$ with a function $wc : S \to \mathsf{Pow}(S)$ which satisfies the conditions 1 and 2 of Definition 2.4.3 and whose corresponding relation $\overline{wc}$ is countable is enumerably completely regular by Proposition 4.2.5.

In Section 2.2.5, we gave a construction of the product of a set-indexed family of inductively generated formal topologies. Here, we are interested in the countable product $\prod_{n\in\mathbb{N}} \mathcal{I}[0, 1]$ of the formal unit interval. Explicitly, $\prod_{n\in\mathbb{N}} \mathcal{I}[0, 1]$ is defined as follows. The base is a preorder $(S_\Pi, \leq)$ given by

$$S_\Pi \overset{\text{def}}{=} \mathsf{Fin}(\mathbb{N} \times S_{\mathcal{R}}),$$
$$A \leq B \overset{\text{def}}{\iff} (\forall\, (n, b) \in B)\, (\exists (m, a) \in A)\, [m = n \;\&\; a \leq_{\mathcal{R}} b]\,,$$

where $S_{\mathcal{R}}$ is the base of the formal reals $\mathcal{R}$ ordered by $\leq_{\mathcal{R}}$ and $<_{\mathcal{R}}$ (See Example 2.2.22). The cover $\lhd_\Pi$ of $\prod_{n\in\mathbb{N}} \mathcal{I}[0, 1]$ is generated by the following axioms:

(H1) $S_\Pi \lhd_\Pi \{\{(n, a)\} \in S_\Pi \mid a \in S_{\mathcal{R}}\}$ for each $n \in \mathbb{N}$;

(H2) $\{(n, a), (n, b)\} \lhd_\Pi \{\{(n, c)\} \in S_\Pi \mid c \leq_{\mathcal{R}} a \;\&\; c \leq_{\mathcal{R}} b\}$ for each $n \in \mathbb{N}$ and $a, b \in S_{\mathcal{R}}$;

(H3) $\{(n, a)\} \lhd_\Pi \{\{(n, b)\} \in S_\Pi \mid b <_{\mathcal{R}} a\}$ for each $n \in \mathbb{N}$ and $a \in S_{\mathcal{R}}$;

(H4) $S_\Pi \lhd_\Pi \left\{\{(n, (p, q))\} \in S_\Pi \mid q - p = 2^{-k}\right\}$ for each $n, k \in \mathbb{N}$;

(H5) $\{(n, (p, q))\} \lhd_\Pi \{\{(n, (p, q))\} \in S_\Pi \mid p < 1 \;\&\; 0 < q\}$ for each $n \in \mathbb{N}$ and $(p, q) \in S_{\mathcal{R}}$.

The axiom (H4) is derived from the axiom (R2') instead of the axiom (R2) (See Example 2.4.25).

Recall that $\mathcal{I}[0, 1]$ is regular with the function $wc_{\mathcal{R}}$ given by (2.18) (See Example 2.4.26). Then, the proof of Proposition 2.4.11.2 shows that the product $\prod_{n\in\mathbb{N}} \mathcal{I}[0, 1]$ is regular with the function $wc_\Pi : S_\Pi \to \mathsf{Pow}(S_\Pi)$ given by

$$wc_\Pi(A) \overset{\text{def}}{=} \{\{(m_0, b_0), \ldots, (m_{n-1}, b_{n-1})\} \in S_\Pi \mid (\forall i < n)\, b_i <_{\mathcal{R}} a_i\} \tag{4.2}$$

for each $A = \{(m_0, a_0), \ldots, (m_{n-1}, a_{n-1})\} \in S_\Pi$.

Lemma 4.2.8. *The formal topology* $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ *is enumerably completely regular.*

Proof. We must show that $\overline{wc_\Pi}$ is countable and define a choice of scale for wc_Π.

First, the set S_Π is countable, since it is the set of finitely enumerable subsets of a countable set. Moreover, for each $A \in S_\Pi$ the set $wc_\Pi(A)$ is countable, since it is a finite product of countable sets, and hence the set $\overline{wc_\Pi} = \{(B,A) \in S_\Pi \times S_\Pi \mid B \in wc_\Pi(A)\}$ is countable.

Next, we define a choice of scale for wc_Π. Let $(B,A) \in \overline{wc_\Pi}$ so that A and B are of the forms

$$A = \{(m_0,(p_0,q_0)),\dots,(m_{n-1},(p_{n-1},q_{n-1}))\},$$
$$B = \{(m_0,(p'_0,q'_0)),\dots,(m_{n-1},(p'_{n-1},q'_{n-1}))\}$$

such that $(p'_i,q'_i) <_{\mathcal{R}} (p_i,q_i)$ for each $i<n$. Then, for each $i<n$, we can choose an order reversing bijection $\varphi_i : \mathbb{I} \to [p_i,p'_i]\cap\mathbb{Q}$ and an order preserving bijection $\psi_i : \mathbb{I} \to [q'_i,q_i]\cap\mathbb{Q}$. For each $q \in \mathbb{I}$, define

$$B_q \overset{\text{def}}{=} \{(m_0,(\varphi_0(q),\psi_0(q))),\dots,(m_{n-1},(\varphi_{n-1}(q),\psi_{n-1}(q)))\}.$$

Then, the family $(\{B_q\})_{q\in\mathbb{I}}$ is a finitary scale from $\{B\}$ to $\{A\}$. Thus, we can define a function $sc \in \prod_{(B,A)\in\overline{wc_\Pi}} \mathrm{Sc}_{\lll}(\{B\},\{A\})$ which assigns to each $(B,A) \in \overline{wc_\Pi}$ the finitary scale from $\{B\}$ to $\{A\}$ as described above. □

The following is a special case of Tychonoff's embedding theorem for locales [34, Chapter IV, Theorem 1.7].

Proposition 4.2.9. *A formal topology is isomorphic to an enumerably completely regular formal topology iff it can be embedded into* $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$.

Proof. ($\Rightarrow$): It suffices to show that any enumerably completely regular formal topology can be embedded into $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$. Let $\mathcal{S}$ be an enumerably completely regular formal topology. Then, there exists a function $wc\colon S \to \mathsf{Pow}(S)$ such that $a \lhd wc(a)$ for each $a \in S$ and that $\overline{wc} = \{(b,a) \in S\times S \mid b \in wc(a)\}$ is countable. Moreover, for each $n\in\mathbb{N}$ and the pair $(b_n,a_n) \in \overline{wc}$ indexed by $n\in\mathbb{N}$, there exists a finitary scale $(U_q)_{q\in\mathbb{I}}$ from $\{b_n\}$ to $\{a_n\}$ chosen by the choice of scale associated with $\mathcal{S}$. By Proposition 4.2.2, for each $n\in\mathbb{N}$, the scale $(U_q)_{q\in\mathbb{I}}$ determines a formal topology map $r_n : \mathcal{S} \to \mathcal{I}[0,1]$ such that

- $r_n^-(0,\infty) \downarrow b_n \lhd \emptyset$,
- $r_n^-(-\infty,1) \lhd a_n$.

Let $r\colon \mathcal{S} \to \prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ be the canonical formal topology map determined by the sequence $(r_n\colon \mathcal{S} \to \mathcal{I}[0,1])_{n\in\mathbb{N}}$. We show that r is an embedding, i.e. $a \lhd r^- r^{-*}\mathcal{A}\{a\}$ for each $a \in S$.

Let $a \in S$ and $b \in wc(a)$, and let $n \in \mathbb{N}$ be the index of the pair $(b,a) \in \overline{wc}$. Then, we have

$$\begin{aligned} b &\lhd \left(r_n^-(-\infty,1) \cup r_n^-(0,\infty)\right) \downarrow b \\ &\lhd \left(r_n^-(-\infty,1) \downarrow b\right) \cup \left(r_n^-(0,\infty) \downarrow b\right) \\ &\lhd \emptyset \cup r_n^-(-\infty,1) \\ &=_{\mathcal{S}} r^- \left\{\{(n,(p,q))\} \mid (p,q) \in (-\infty,1)\right\} \lhd a. \end{aligned}$$

Thus, $b \lhd r^- r^{-*} \mathcal{A}\{a\}$, and hence $a \lhd wc(a) \lhd r^- r^{-*} \mathcal{A}\{a\}$.

($\Leftarrow$): This follows from the fact that any subtopology of an enumerably completely regular formal topology is enumerably completely regular (See Proposition 2.4.11.1). □

4.3 Point-free characterisation

We show that the image of the class of compact metric spaces under the localic completion can be characterised by the class of compact overt enumerably completely regular formal topologies. The argument is analogous to the classical proof of Urysohn's metrisation theorem.

Lemma 4.3.1. *The localic completion $\mathcal{M}(X)$ of a compact metric space X is isomorphic to a compact overt enumerably completely regular formal topology.*

Note that any compact metric space is separable.

Proof. Let $X = (X,d)$ be a compact metric space, and let $Y \subseteq X$ be a countable dense subset of X. By Theorem 3.1.16, $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are isomorphic. Hence, without loss of generality, we may assume that X is countable. Since $M_X = \bigcup_{\varepsilon \in \mathbb{Q}^{>0}} \mathcal{C}_\varepsilon$, and the set $\mathcal{C}_\varepsilon$ is countable for each $\varepsilon \in \mathbb{Q}^{>0}$, M_X is countable. By (M1), we have

$$\mathsf{b}(x,\varepsilon) \lhd_X \left\{\mathsf{b}(x,\delta) \in M_X \mid \delta \in \mathbb{Q}^{>0} \ \&\ \delta < \varepsilon\right\}$$

for each $\mathsf{b}(x,\varepsilon) \in M_X$. By Lemma 3.1.13, we may define a function $wc\colon M_X \to \mathsf{Pow}(M_X)$ by

$$wc(\mathsf{b}(x,\varepsilon)) \overset{\text{def}}{=} \left\{\mathsf{b}(x,\delta) \in M_X \mid \delta \in \mathbb{Q}^{>0} \ \&\ \delta < \varepsilon\right\}.$$

The set $wc(\mathsf{b}(x,\varepsilon))$ is countable, and hence, the relation $\overline{wc}$ is countable.

Moreover, for each $\mathsf{b}(x,\delta) \in wc(\mathsf{b}(x,\varepsilon))$, we can choose an order preserving bijection $\varphi\colon \mathbb{I} \to [\delta,\varepsilon] \cap \mathbb{Q}$. Then, the family $(\{\mathsf{b}(x,\varphi(q))\})_{q \in \mathbb{I}}$ is a finitary scale from $\{\mathsf{b}(x,\delta)\}$ to $\{\mathsf{b}(x,\varepsilon)\}$. Thus, we can define a function $sc \in \prod_{(b,a) \in \overline{wc}} \mathrm{Sc}_{\lll}(\{b\},\{a\})$ which assigns to each $(b,a) \in \overline{wc}$ the finitary scale from $\{b\}$ to $\{a\}$ as described above.

Since X is compact, $\mathcal{M}(X)$ is compact by Theorem 3.1.39. Therefore, $\mathcal{M}(X)$ is a compact overt enumerably completely regular formal topology with the function $wc\colon M_X \to \mathsf{Pow}(M_X)$ and the choice of scale $sc \in \prod_{(b,a) \in \overline{wc}} \mathrm{Sc}_{\lll}(\{b\},\{a\})$ for wc. □

Theorem 4.3.2. *Let $\mathcal{S}$ be a formal topology. Then, the following are equivalent.*

1. *$\mathcal{S}$ is isomorphic to a compact overt enumerably completely regular formal topology.*
2. *$\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$.*
3. *$\mathcal{S}$ is isomorphic to the localic completion of some compact metric space.*

Proof. ($1 \Leftrightarrow 2$): This follows from Proposition 4.2.9 together with the fact that overtness and compactness are preserved by isomorphisms.

($2 \Rightarrow 3$): Suppose that $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Since $\mathcal{M}([0,1]) \cong \mathcal{I}[0,1]$ by Example 4.1.20 and $[0,1]$ is a compact metric space, the embedding of $\mathcal{S}$ into $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ factors through $\mathcal{M}(\prod_{n\in\mathbb{N}}[0,1])$ by Lemma 3.1.54. Since $\prod_{n\in\mathbb{N}}[0,1]$ is a compact metric space, $\mathcal{S}$ is isomorphic to the localic completion of some compact metric space by Corollary 4.1.19.

($3 \Leftarrow 1$): By Lemma 4.3.1. □

Remark 4.3.3. Let **OKReg** be the full subcategory of **FTop** consisting of compact overt regular formal topologies, and let **OKECReg** be the full subcategory of **OKReg** consisting of formal topologies which are isomorphic to some compact overt enumerably completely regular formal topologies. Then, Theorem 4.3.2 is equivalent to saying that the localic completion $\mathcal{M} : \mathbf{LCM} \to \mathbf{OLKReg}$ restricts to a full, faithful, and essentially surjective functor from the category of compact metric spaces (and uniformly continuous functions) to **OKECReg**. Hence, the category of compact metric spaces and **OKECReg** are essentially equivalent (See the footnote 2).

4.4 An application

In constructive mathematics, it is well-known that every compact metric space is a uniform quotient of the Cantor space [10, 55]. In this section, we prove a partial counterpart of this result in formal topology[5].

Theorem 4.4.1. *Any inhabited compact enumerably completely regular formal topology is the image of the formal Cantor space under some formal topology map.*

Notation 4.4.2. In the following, we use $\mathbb{N}^*$ and 2^* for the bases of the formal Baire space and the formal Cantor space respectively (See Example 2.2.21 and Example 2.2.20).

Since $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ is compact regular by Proposition 2.4.11.2 and Theorem 2.4.14, to prove Theorem 4.4.1, it suffices to show that any inhabited located subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ is the image of the formal Cantor space under some formal topology map (See Corollary 4.1.16 and Theorem 4.3.2).

Let $\mathcal{S}^{\mathsf{Pos}}$ be the located subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ determined by some inhabited located subset Pos of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. The formal topology $\mathcal{S}^{\mathsf{Pos}}$ is generated by the axioms (H1) - (H5) of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ (See Section 4.2), together with the following extra axiom:

[5] In locale theory, a more general result is presented in [26, Theorem 4.6]. However, we have been unable to convince ourselves that their result holds constructively. For example, they claim that every separable compact regular locale is in the image of the formal Cantor space. However, since the formal Cantor space is overt, overtness should be included in the condition.

(H6) $A \lhd^{\mathsf{Pos}} \{B \in S_\Pi \mid B = A \;\&\; B \in \mathsf{Pos}\}$.

For each $n, k \in \mathbb{N}$, define subsets $\mathcal{C}_k^n$, $\mathcal{C}_k^{\leq n}$, and $\mathcal{D}_n$ of S_Π by

$$\mathcal{C}_k^n \overset{\text{def}}{=} \left\{\{(n,(p,q))\} \in S_\Pi \mid q - p = 2^{-k}\right\}, \tag{4.3}$$

$$\mathcal{C}_k^{\leq n} \overset{\text{def}}{=} \left\{\{(0,(p_0,q_0)),\dots,(n,(p_n,q_n))\} \in S_\Pi \mid (\forall i \leq n)\, q_i - p_i = 2^{-k}\right\}, \tag{4.4}$$

$$\mathcal{D}_n \overset{\text{def}}{=} \mathcal{C}_n^{\leq n}.$$

For each $n \in \mathbb{N}$, since $\mathcal{D}_n =_{\prod \mathcal{I}[0,1]} \mathcal{C}_n^0 \downarrow \cdots \downarrow \mathcal{C}_n^n$ and Pos is the positivity of $\mathcal{S}^{\mathsf{Pos}}$, we have

$$S_\Pi \lhd^{\mathsf{Pos}} \mathcal{D}_n \cap \mathsf{Pos}$$

by (H4). Since $\mathcal{S}^{\mathsf{Pos}}$ is compact, we have

$$(\forall n \in \mathbb{N})\, (\exists F_n \in \mathsf{Fin}(\mathcal{D}_n \cap \mathsf{Pos}))\, S_\Pi \lhd^{\mathsf{Pos}} F_n.$$

By the Countable Choice, there exists a sequence $(F_n)_{n\in\mathbb{N}}$ such that

$$(\forall n \in \mathbb{N})\, F_n \in \mathsf{Fin}(\mathcal{D}_n \cap \mathsf{Pos}) \;\&\; S_\Pi \lhd^{\mathsf{Pos}} F_n.$$

Let $(A_n)_{n\in\mathbb{N}}$ be an enumeration of $\bigcup_{n\in\mathbb{N}} F_n$. Note that each A_n is of the form

$$A_n = \{(0,a_0),\dots,(m,a_m)\} \in \mathcal{D}_m$$

for some $m \in \mathbb{N}$.

Define a subtree T of $\mathbb{N}^*$ by

$$T_0 = \{\langle i \rangle \mid A_i \in F_0\},$$
$$T_{n+1} = \left\{l * \langle i \rangle \in \mathbb{N}^* \mid l \in T_n \;\&\; A_i \in F_{n+1} \;\&\; A_{l(|l|-1)} \between A_i\right\},$$
$$T = \bigcup_{n\in\mathbb{N}} T_n \cup \{\langle\rangle\},$$

where for each $A, B \in S_\Pi$, we define

$$A \between B \overset{\text{def}}{\iff} (\forall (i,(p,q)) \in A)\, (\forall (j,(s,t)) \in B)\, i = j \implies \max\{p,s\} < \min\{q,t\}. \tag{4.5}$$

Clearly, we have $(\forall n \in \mathbb{N})\, (\forall l \in T_n)\, |l| = n+1$. We show that T is a fan in the formal Baire space $\mathbf{B}$ (See Definition 2.5.10). First, since Pos is inhabited and $S_\Pi \lhd^{\mathsf{Pos}} F_0$, T is inhabited. Moreover, T is a decidable subset of $\mathbb{N}^*$ since each F_n is decidable and the relation $\between$ is decidable. By construction, T is upward closed (i.e. closed under predecessor). Since F_n is finite for each $n \in \mathbb{N}$, T is finitely branching. Finally, to see that T satisfies the second condition in Definition 2.5.10, let $n, k \in \mathbb{N}$. Since $A_k \lhd^{\mathsf{Pos}} F_n \downarrow A_k \lhd^{\mathsf{Pos}} \{B \cup A_k \mid B \in F_n\}$ and $\mathsf{Pos}(A_k)$ there exists $B \in F_n$ such that $\mathsf{Pos}(B \cup A_k)$ so that $B \between A_k$ by (H2). This, together with the fact that Pos is inhabited, implies that T satisfies the second condition. Hence, T is a fan in $\mathbf{B}$.

Since T is a splitting subset of $\mathbf{B}$, T determines an overt weakly closed subtopology $\mathbf{B}^T$ of $\mathbf{B}$ by Theorem 2.3.21. In the following, we denote the cover of $\mathbf{B}^T$ by $\lhd^T$.

Define a relation $r \subseteq \mathbb{N}^* \times S_\Pi$ by

$$l \; r \; A \overset{\text{def}}{\iff} l \in T \;\&\; \overline{A_l} < A$$

for all $l \in \mathbb{N}^*$ and $A \in S_\Pi$, where

$$\begin{aligned} A_l &\overset{\text{def}}{=} A_{l(|l|-1)}, \\ \overline{A_l} &\overset{\text{def}}{=} \left\{ (n, (p - 2^{-|l|+1}, q + 2^{-|l|+1})) \mid (n, (p, q)) \in A_l \right\}, \\ A < B &\overset{\text{def}}{\iff} (\forall (n, b) \in B)\,(\exists (m, a) \in A)\, n = m \;\&\; a <_{\mathcal{R}} b \end{aligned}$$

for all $l \in T$ and $A, B \in S_\Pi$. Here, we define $A_{\langle\rangle} \overset{\text{def}}{=} \emptyset$.

Lemma 4.4.3. *The relation r is a formal topology map from $\mathbf{B}^T$ to $\mathcal{S}^{\mathsf{Pos}}$.*

Proof. (FTMi1): Let $l \in \mathbb{N}^*$, and suppose that $l \in T$. Then, $l \in r^- S_\Pi$, and so $l \lhd^T r^- S_\Pi$.

(FTMi2): Let $A, B \in S_\Pi$, and let $l \in r^- A \downarrow r^- B$. Suppose that $l \in T$. Then, $\overline{A_l} < A$ and $\overline{A_l} < B$, and thus, $\overline{A_l} < A \cup B \in A \downarrow B$. Hence, $l \in r^-(A \downarrow B)$. Therefore, $r^- A \downarrow r^- B \lhd^T r^-(A \downarrow B)$.

(FTMi3): We must check the axioms (H1) – (H6).

(H1): Let $n \in \mathbb{N}$, and $l \in \mathbb{N}^*$. Suppose that $l \in T$. Then, either $|l| > n$ or $|l| \leq n$. If $|l| > n$, then there exists $a \in S_{\mathcal{R}}$ such that $\overline{A_l} < \{(n, a)\}$. If $|l| \leq n$, then $l \lhd^T \{l' \in \mathbb{N}^* \mid l' \leq l \;\&\; |l'| = n + 1\} \cap T$ by Lemma 2.5.12. Then, for any $l' \in \mathrm{RHS}$, we have $\overline{A_{l'}} < \{(n, a)\}$ for some $a \in S_{\mathcal{R}}$. Hence, $l \lhd^T r^- \{\{(n, a)\} \mid a \in S_{\mathcal{R}}\}$.

(H2): Let $l \in r^- \{(n, a), (n, b)\}$. Then, $l \in T$ and $\overline{A_l} < \{(n, a), (n, b)\}$. By the definition of A_l, we have $\overline{A_l} < \{(n, c)\}$ for some $c <_{\mathcal{R}} a$ and $c <_{\mathcal{R}} b$. Hence,

$$r^- \{(n, a), (n, b)\} \lhd^T r^- \{\{(n, c)\} \mid c \leq_{\mathcal{R}} a \;\&\; c \leq_{\mathcal{R}} b\}.$$

(H3): By the definition of r.

(H4): The proof is similar to the case of (H1). Instead of $n \in \mathbb{N}$, we take $N = \max\{n, k + 2\}$.

(H5): It suffices to show the case for (H6).

(H6): Let $l \in r^- A$. Then, $l \in T$ and $\overline{A_l} < A$. Since $A_l < \overline{A_l}$ and $\mathsf{Pos}(A_l)$, we have $\mathsf{Pos}(A)$. Hence, $l \in r^- \{B \in S_\Pi \mid B = A \;\&\; \mathsf{Pos}(B)\}$.

(FTMi4): Obvious. □

Lemma 4.4.4. *For any $A \in S_\Pi$, we have*

$$A \lhd^{\mathsf{Pos}} \{A_l \in S_\Pi \mid l \in T \;\&\; l \; r \; A\}.$$

Proof. Let $A \in S_\Pi$. Write $A = \{(i_0, (p_0, q_0)), \ldots, (i_{n-1}, (p_{n-1}, q_{n-1}))\}$. Without loss of generality, we may assume that $n > 0$. We use (H3). Choose $(p'_k, q'_k) <_{\mathcal{R}} (p_k, q_k)$ for each $k < n$, and put

$$A' \overset{\text{def}}{=} \left\{ (i_0, (p'_0, q'_0)), \ldots, (i_{n-1}, (p'_{n-1}, q'_{n-1})) \right\}.$$

Choose $m \geq \max\{i_k \mid k < n\}$ such that $p_k < p'_k - 2^{-m+1}$ and $q'_k + 2^{-m+1} < q_k$ for each $k < n$. Since $S_\Pi \lhd^{\mathsf{Pos}} F_m$, we have

$$A' \lhd^{\mathsf{Pos}} (F_m \downarrow A') \cap \mathsf{Pos} \lhd^{\mathsf{Pos}} \{B \cup A' \in \mathsf{Pos} \mid B \in F_m\}.$$

Let $B \in F_m$ such that $B \cup A' \in \mathsf{Pos}$, and write $B = \{(0,(r_0,s_0)),\dots,(m,(r_m,s_m))\}$. By (H2), we have $\max\{r_{i_k}, p'_k\} < \min\{s_{i_k}, q'_k\}$ for all $k < n$. For each $k < n$, since $s_{i_k} - r_{i_k} = 2^{-m}$, we have $(r_{i_k} - 2^{-m}, s_{i_k} + 2^{-m}) <_{\mathcal{R}} (p_k, q_k)$. Since $B \in \mathsf{Pos}$, there exist $B_0 \in F_0, \dots, B_{m-1} \in F_{m-1}$ and $l \in T$ such that $A_l = B$ and for all $k < m$, $A_{l(k)} = B_k$. Then, $l \; r \; A$ and $B \cup A' \leq B = A_l$, so

$$A' \lhd^{\mathsf{Pos}} \{A_l \in S_\Pi \mid l \in T \;\&\; l \; r \; A\}.$$

Therefore, $A \lhd^{\mathsf{Pos}} \{A_l \in S_\Pi \mid l \in T \;\&\; l \; r \; A\}$ as required. □

Proposition 4.4.5. *The map $r : \mathbf{B}^T \to \mathcal{S}^{\mathsf{Pos}}$ is surjective, i.e.*

$$r^- A \lhd^T r^- \mathcal{U} \implies A \lhd^{\mathsf{Pos}} \mathcal{U}$$

for all $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$.

Proof. First, we show that

$$l \in T \;\&\; l \lhd^T r^- \mathcal{U} \implies A_l \lhd^{\mathsf{Pos}} \mathcal{U}. \tag{4.6}$$

Given $\mathcal{U} \subseteq S_\Pi$, define a predicate Φ on $\mathbb{N}^*$ by

$$\Phi(l) \overset{\text{def}}{\iff} (\forall l' \leq l)\, l' \in T \to A_{l'} \lhd^{\mathsf{Pos}} \mathcal{U}.$$

We show that $l \lhd^T r^- \mathcal{U} \implies \Phi(l)$ for all $l \in \mathbb{N}^*$ by induction on $\lhd^T$.

(ID1): Suppose that $l \in r^- \mathcal{U}$. Let $l' \leq l$ such that $l' \in T$. Since $l \in r^- \mathcal{U}$, there exists $B \in \mathcal{U}$ such that $\overline{A_l} < B$. Since $l' \leq l$, we have a $A_{l'} < \overline{A_l}$. Hence, $A_{l'} \lhd^{\mathsf{Pos}} \mathcal{U}$. Therefore $\Phi(l)$.

(ID2): Obvious from the definition of Φ.

(ID3): We have two axioms to be checked. First, we check the axiom of the formal Baire space, namely

$$l \lhd_{\mathbf{B}} \{l * \langle n \rangle \mid n \in \mathbb{N}\}.$$

Let $l \in \mathbb{N}^*$, and suppose that $\Phi(l * \langle n \rangle)$ for all $n \in \mathbb{N}$. Let $l' \leq l$ such that $l' \in T$. Then, we have

$$A_{l'} \lhd^{\mathsf{Pos}} \left(F_{|l'|} \downarrow A_{l'}\right) \cap \mathsf{Pos} \lhd^{\mathsf{Pos}} \left\{A_{l' * \langle n \rangle} \mid n \in \mathbb{N} \;\&\; l' * \langle n \rangle \in T\right\}.$$

For each $n \in \mathbb{N}$ such that $l' * \langle n \rangle \in T$, there exists $m \in \mathbb{N}$ such that $l' * \langle n \rangle \leq l * \langle m \rangle$. Hence, $\left\{A_{l' * \langle n \rangle} \mid n \in \mathbb{N} \;\&\; l' * \langle n \rangle \in T\right\} \lhd^{\mathsf{Pos}} \mathcal{U}$. Therefore $\Phi(l)$.

Next, we check the axiom of $\mathbf{B}^T$, namely

$$l \lhd^T \{l\} \cap T.$$

Suppose that $l \in T \implies \Phi(l)$. Let $l' \leq l$ such that $l' \in T$. Since T is upward closed, we have $l \in T$. Thus, $\Phi(l)$, and hence $A_{l'} \lhd^{\mathsf{Pos}} \mathcal{U}$. Therefore $\Phi(l)$.

Now, suppose that $r^- A \lhd^T r^- \mathcal{U}$. By Lemma 4.4.4, we have $A \lhd^{\mathsf{Pos}} \{A_l \mid l \in T \;\&\; l \; r \; A\}$. Then, for any $l \in T$ such that $l \; r \; A$, since $l \lhd^T r^- \mathcal{U}$, we have $A_l \lhd^{\mathsf{Pos}} \mathcal{U}$ by (4.6). Hence, $A \lhd^{\mathsf{Pos}} \mathcal{U}$. Therefore, r is surjective. □

The following is well-known in the point-set case (See [54, Chapter 4. Proposition 7.5]).

Lemma 4.4.6. *Let T be a splitting subset of $\mathbf{B}$, and let $\mathbf{B}^T$ be the overt weakly closed subtopology of $\mathbf{B}$ determined by T. Then, there exists an embedding $r : \mathbf{B}^T \hookrightarrow \mathbf{C}$ into the formal Cantor space.*

Proof. Define r by

$$l \; r \; a \overset{\text{def}}{\iff} l \in T \;\&\; a \preccurlyeq a_l,$$
$$a_l \overset{\text{def}}{=} \langle 0, \underbrace{1, \ldots, 1}_{l(0)} \rangle * \cdots * \langle 0, \underbrace{1, \ldots, 1}_{l(|l|-1)} \rangle$$

for all $l \in \mathbb{N}^*$ and $a \in 2^*$. We show that r is a formal topology map. The conditions (FTMi1) and (FTMi2) trivially hold.

(FTMi3): We must show that $r^- a \lhd^T r^- \{a * \langle 0 \rangle, a * \langle 1 \rangle\}$, where $\lhd^T$ denotes the cover of $\mathbf{B}^T$. Let $l \in \mathbb{N}^*$, and suppose that $l \; r \; a$. Then $l \in T$ and $a \preccurlyeq a_l$, and thus $l \lhd^T \{l * \langle n \rangle \mid n \in \mathbb{N}\} \cap T \subseteq r^- \{a * \langle 0 \rangle, a * \langle 1 \rangle\}$.

(FTMi4): By the definition of the underlying preorder of $\mathbf{C}$.

To see that r is an embedding, let $l \in \mathbb{N}^*$, and let $n \in \mathbb{N}$ such that $l * \langle n \rangle \in T$. Then, $l * \langle n \rangle \; r \; a_{l*\langle n \rangle}$ and $r^- a_{l*\langle n \rangle} \lhd^T l$. Therefore, $l \lhd^T r^- r^{-*} \mathcal{A}^T \{l\}$. ☐

If T is a fan in $\mathbf{B}$, then $\mathbf{B}^T$ is a finitary formal topology by Lemma 2.5.14, and thus $\mathbf{B}^T$ is compact overt (cf. See Example 2.4.24). In this case, the image of the embedding described in Lemma 4.4.6 is a compact overt subtopology of $\mathbf{C}$. Since $\mathbf{C}$ is compact regular, the image is uniquely determined by some located subset of $\mathbf{C}$. Note that since $\mathbf{C}$ is finitary, every located subset of $\mathbf{C}$ is decidable[6].

Let $\mathsf{Pos} \subseteq 2^*$ be an inhabited located subset of $\mathbf{C}$. For each $a \in 2^*$, define $\widetilde{a} \in 2^*$ by recursion:

$$\widetilde{\langle\rangle} \overset{\text{def}}{=} \langle\rangle, \qquad \widetilde{a * \langle i \rangle} \overset{\text{def}}{=} \begin{cases} \widetilde{a} * \langle i \rangle & \text{if } \widetilde{a} * \langle i \rangle \in \mathsf{Pos}, \\ \widetilde{a} * \langle j \rangle & j \equiv i + 1 \pmod 2 \text{ otherwise.} \end{cases}$$

Since Pos is located, $\widetilde{(\cdot)}$ is a well-defined function from 2^* to Pos. We have

1. $\mathsf{Pos}(\widetilde{a})$,
2. $\mathsf{Pos}(a) \implies a = \widetilde{a}$,
3. $a \leq b \implies \widetilde{a} \leq \widetilde{b}$

for all $a, b \in 2^*$. Let $\mathbf{C}^{\mathsf{Pos}}$ be the overt weakly closed subtopology of $\mathbf{C}$ determined by Pos. In the following, we denote the cover of $\mathbf{C}^{\mathsf{Pos}}$ by $\lhd_{\mathbf{C}}^{\mathsf{Pos}}$.

[6] Note that for any finitary formal topology $\mathcal{S}$, we have $a \ll a$ for all $a \in S$.

Lemma 4.4.7. *The relation* $r \subseteq 2^* \times 2^*$ *defined by*

$$a \; r \; b \overset{\text{def}}{\iff} \widetilde{a} = b$$

is a formal topology map $r : \mathbf{C} \to \mathbf{C}^{\mathsf{Pos}}$*. Moreover,* r *is surjective.*

Proof. (FTMi1) is trivial since r is a function.

(FTMi2): Let $a \in r^- b \downarrow r^- c$. Then, there exist $a_0, a_1 \in 2^*$ such that $a \in a_0 \downarrow a_1$, $\widetilde{a_0} = b$, and $\widetilde{a_1} = c$. Assume without loss of generality that $a \leq a_0 \leq a_1$. Then $\widetilde{a} \leq \widetilde{a_0} \leq \widetilde{a_1}$. Thus $\widetilde{a} \in b \downarrow c$, and hence $a \in r^- (b \downarrow c)$.

(FTMi3): There are two axioms to be checked. We first show that

$$r^- a \lhd_{\mathbf{C}} r^- \{a * \langle 0 \rangle, a * \langle 1 \rangle\} .$$

Let $b \in r^- a$. Then $\widetilde{b} = a$, and so $b \lhd_{\mathbf{C}} \{b * \langle 0 \rangle, b * \langle 1 \rangle\} \subseteq r^- \{a * \langle 0 \rangle, a * \langle 1 \rangle\}$. Next, we show that

$$r^- a \lhd_{\mathbf{C}} r^- (\{a\} \cap \mathsf{Pos}) .$$

Let $b \in r^- a$. Then, $\widetilde{b} = a$, and so $\mathsf{Pos}(a)$. Hence, $b \in r^- (\{a\} \cap \mathsf{Pos})$.

(FTMi4): Let $a \leq b$, and suppose that $c \; r \; a$. Then $\widetilde{c} = a$, so there exists $c' \preccurlyeq c$ such that $\widetilde{c'} = b$. Then $c \leq c' \in r^- \{b\}$.

Lastly, to see that r is surjective, let $a \in 2^*$ and $U \subseteq 2^*$, and suppose that $r^- a \lhd_{\mathbf{C}} r^- U$. We must show that $a \lhd_{\mathbf{C}}^{\mathsf{Pos}} U$. Suppose that $\mathsf{Pos}(a)$. Then, $\widetilde{a} = a$, i.e. $a \; r \; a$. Thus, $a \lhd_{\mathbf{C}} r^- U$, and hence $a \lhd_{\mathbf{C}}^{\mathsf{Pos}} r^- U \cap \mathsf{Pos} \subseteq U$. Therefore, r is surjective. □

This completes the proof of Theorem 4.4.1. As a corollary, we obtain the following well-known fact for compact metric spaces.

Corollary 4.4.8. *The Fan theorem implies that every compact metric space is topologically compact.*

Proof. Assume that the Fan theorem holds. Then, by Lemma 2.5.6, Theorem 2.5.15, Theorem 4.3.2 and Theorem 4.4.1, every localic completion of a compact metric space is spatial. Let X be a compact metric space. Since $X \cong \mathcal{P}t(\mathcal{M}(X))$, we have $\Omega(X) \cong \mathcal{M}(X)$ by spatiality. Since $\mathcal{M}(X)$ is compact by Theorem 3.1.39, we have that $\Omega(X)$ is compact. □

Chapter 5

Point-free Characterisation of Bishop Locally Compact Metric Spaces

In this chapter, we extend the point-free characterisation of compact metric spaces obtained in Chapter 4 to the class of Bishop locally compact metric spaces (See Definition 5.0.9). The characterisation is a natural generalisation of the one obtained for compact metric spaces; we show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space up to isomorphism. Roughly speaking, an inhabited enumerably locally compact regular formal topology is an inhabited locally compact regular formal topology where the graph of the base of the way-below relation is countable.

As in Chapter 4, we obtain this characterisation by showing that the localic completion of metric spaces restricts to a full, faithful and essentially surjective functor from the category of Bishop locally compact metric spaces to that of the isomorphic closure of inhabited enumerably locally compact regular formal topologies. Hence, we conclude that the two categories are essentially equivalent.

The category of inhabited enumerably locally compact regular formal topologies is classically equivalent to that of inhabited separable locally compact regular locales, and the latter is classically equivalent to that of inhabited topologically locally compact metric spaces[1]. These equivalences may not have been expected since the notion of Bishop locally compact metric space is incompatible with the topological notion of local compactness, e.g. the open interval $(0, 1)$ is not Bishop locally compact. But the fact is that the *category* of Bishop locally compact metric spaces is classically equivalent to that of inhabited topologically locally compact metric spaces.

The key to obtaining the above mentioned characterisation is the notion of one-point compactification of a Bishop locally compact metric space [8]. It allows us to present every Bishop locally compact metric space as a pair $(X, X - A)$ of a compact metric space X and the inhabited metric complement $X - A$ of a compact subset A of X. Such a pair is isomorphic to the given Bishop locally compact metric space in a category called **OLCM** which contains the category of Bishop locally compact metric spaces as a full subcategory.

[1]A metric space is topologically locally compact if every point has a compact neighbourhood.

The category **OLCM** was introduced by Palmgren [49], and he also constructed a full and faithful functor $\mathcal{OM} : \mathbf{OLCM} \to \mathbf{FTop}$ which extends the localic completion of locally compact metric spaces[2].

Based on these facts, we sketch our plan of obtaining a point-free characterisation of Bishop locally compact metric spaces. First, we characterise the image of such pairs $(X, X - A)$ mentioned above under the embedding $\mathcal{OM}$ by the notion of the open complement of a located subtopology (Section 5.1). Then, we show that every inhabited enumerably locally compact regular formal topology is in the image of the pairs of the form $(X, X - A)$ mentioned above under the embedding $\mathcal{OM}$ (Section 5.2). Finally, we complete the point-free characterisation by using the fact that a pair of the form $(X, X - A)$ is isomorphic to some Bishop locally compact metric space in **OLCM** (Section 5.1) and that the localic completion of a Bishop locally compact metric space is always inhabited enumerably locally compact regular (Section 5.3).

We end this introductory remark by introducing the notion of Bishop locally compact metric space, as well as recalling the notion of locally compact metric space.

Definition 5.0.9. A metric space X is *locally compact* if for each open ball $B(x, \varepsilon)$, there exists a compact subset $K \subseteq X$ such that $B(x, \varepsilon) \subseteq K$. A locally compact metric space is *Bishop locally compact* if it is inhabited.

Note that every locally compact metric space is complete, and that a locally compact metric space is Bishop locally compact iff it is separable (See Proposition D.0.24).

Throughout this chapter, we use the notational conventions for metric spaces introduced in Notation 4.0.11.

5.1 Open complements of located subtopologies

In this section, we introduce the notion of the open complement of a located subtopology for the class of locally compact formal topologies. Then, we show that every inhabited open complement of a located subtopology of the localic completion of a compact metric space is isomorphic to the localic completion of some Bishop locally compact metric space. This gives a sufficient condition for a formal topology to be in the image of the class of Bishop locally compact metric spaces under the localic completion.

We exploit the category **OLCM** of open complements of locally compact metric spaces due to Palmgren [49].

Definition 5.1.1. The category **OLCM** has as objects the pairs (X, U) where $X = (X, d)$ is a locally compact metric space and U is an open subset of X. The morphisms $f : (X, U) \to (Y, V)$ of **OLCM** are functions $f : U \to V$ such that for each inhabited compact subset $K \Subset U$

1. f is uniformly continuous on K,

[2] Palmgren's aim in [49] was to characterise the morphisms between open subtopologies of localic completions of locally compact metric spaces.

2. $f[K] \Subset V$,

where we define

$$K \Subset U \overset{\text{def}}{\iff} \left(\exists r \in \mathbb{Q}^{>0}\right) K_r \subseteq U, \tag{5.1}$$
$$K_r \overset{\text{def}}{=} \{x \in X \mid d(x, K) \leq r\}$$

for each located subset K.

Remark 5.1.2. Since every inhabited totally bounded subset of a metric space is located (Lemma D.0.19), and since the image of a totally bounded subset under a uniformly continuous function is totally bounded, the second condition of morphisms is well-defined, i.e. the distance $d(y, f[K])$ exists for all $y \in Y$ (See Definition 4.1.5).

The category **LCM** of locally compact metric spaces can be see as a full subcategory of **OLCM** via an inclusion $X \mapsto (X, X)$. Palmgren showed that **OLCM** can be embedded into **FTop** via a full and faithful functor. We denote this functor by $\mathcal{OM}$: **OLCM** $\to$ **FTop**. The functor $\mathcal{OM}$ assigns to each object (X, U) of **OLCM** the open subtopology $\mathcal{M}(X)_{H(U)}$ of $\mathcal{M}(X)$ determined by the set

$$H(U) \overset{\text{def}}{=} \{\mathsf{b}(x, \varepsilon) \in M_X \mid B(x, \varepsilon) \subseteq U\}.$$

Given a morphism $f : (X, U) \to (Y, V)$ of **OLCM**, the formal topology map $\mathcal{OM}(f)$: $\mathcal{M}(X)_{H(U)} \to \mathcal{M}(Y)_{H(V)}$ is given by

$$a \; \mathcal{OM}(f) \; b \overset{\text{def}}{\iff} (\forall a' <_X a \downarrow H(U)) \, (\exists b' <_Y b \downarrow H(V)) \, f[a'_*] \subseteq b'_*$$

for all $a \in M_X$ and $b \in M_Y$. Note that the restriction of $\mathcal{OM}$ to **LCM** agrees with the embedding $\mathcal{M}$: **LCM** $\to$ **OLKReg** of locally compact metric spaces since we have $H(X) = M_X$. Thus, we have $\mathcal{OM}((X, X)) = \mathcal{M}(X)$ for any locally compact metric space X, and $\mathcal{OM}(f) = \mathcal{M}(f)$ for any continuous function $f : X \to Y$ between locally compact metric spaces. Hence, the functor $\mathcal{OM}$ is an extension of $\mathcal{M}$.

For interested readers, we give a proof of the fact that $\mathcal{OM}$ is full and faithful in Appendix C. However, the details of the proof is not necessary in the rest of this chapter.

Using the functor $\mathcal{OM}$, we relate the notion of metric complement and its point-free counterpart, the open complement of a located subtopology.

Definition 5.1.3. Let $X = (X, d)$ be a metric space, and let $A \subseteq X$ be a located subset of X. The *metric complement* of A is the open subset $X - A$ of X given by

$$X - A \overset{\text{def}}{=} \{x \in X \mid d(x, A) > 0\}.$$

The metric complement of a located subset A will be denoted by $X - A$.

Definition 5.1.4. Let $\mathcal{S}$ be a locally compact formal topology, and let $V \subseteq S$ be a located subset of $\mathcal{S}$. The *open complement* of the located subtopology $\mathcal{S}^V$ determined by V is the open subtopology $\mathcal{S}_{\neg V}$ determined by $\neg V$.

Let X be a locally compact metric space. Let $LCl^+(X)$ be the class of inhabited closed located subsets of X and let $LRed^+(\mathcal{M}(X))$ be the class of inhabited located subsets of $\mathcal{M}(X)$. By Theorem 4.1.9 there exists a bijective correspondence $\varphi\colon LCl^+(X) \to LRed^+(\mathcal{M}(X))$ given by

$$\varphi(A) = \Diamond A,$$
$$\varphi^{-1}(V) = \{x \in X \mid \Diamond x \subseteq V\}.$$

Proposition 5.1.5. *Let $X = (X, d)$ be a locally compact metric space. For any inhabited closed located subset A of X, we have*

$$H(X - A) = \neg\varphi(A). \tag{5.2}$$

Dually, for any inhabited located subset V of $\mathcal{M}(X)$, we have

$$(\neg V)_* = X - \varphi^{-1}(V). \tag{5.3}$$

The assignments $U \mapsto H(U)$ and $W \mapsto W_*$ restrict to a bijective correspondence between the metric complements of inhabited closed located subsets of X and the open complements of inhabited located subtopologies of $\mathcal{M}(X)$.

Proof. (5.2): Let A be an inhabited closed located subset of X. Let $\mathsf{b}(x,\varepsilon) \in H(X - A)$, and suppose that $\mathsf{b}(x,\varepsilon) \in \Diamond A$. Then, there exists $y \in A$ such that $y \in B(x,\varepsilon)$, and thus $d(y, A) > 0$, a contradiction. Hence $\mathsf{b}(x,\varepsilon) \in \neg\varphi(A)$.

Conversely, let $\mathsf{b}(x,\varepsilon) \in \neg\varphi(A)$, and let $x' \in B(x,\varepsilon)$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,x')+\theta < \varepsilon$. Suppose that $d(x',A) < \theta$. Then, there exists $y \in A$ such that $d(x',y) < \theta$, and so $d(x,y) < \varepsilon$. Thus, $\mathsf{b}(x,\varepsilon) \in \varphi(A)$, a contradiction. Hence, $d(x',A) \geq \theta$, and therefore $\mathsf{b}(x,\varepsilon) \in H(X - A)$.

(5.3): Let V be an inhabited located subset of $\mathcal{M}(X)$. Let $x \in (\neg V)_*$. Then, there exists $\mathsf{b}(y,\delta) \in \neg V$ such that $d(x,y) < \delta$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,y) + \theta < \delta$. Suppose that $d(x,\varphi^{-1}(V)) < \theta$. Then, there exists $x' \in \varphi^{-1}(V)$ such that $d(x,x') < \theta$, so $\mathsf{b}(x,\theta) \in \Diamond x' \subseteq V$. Since $\mathsf{b}(x,\theta) <_X \mathsf{b}(y,\delta)$, we have $\mathsf{b}(y,\delta) \in V$, a contradiction. Hence $d(x,\varphi^{-1}(V)) \geq \theta$. Therefore $x \in X - \varphi^{-1}(V)$.

Conversely let $x \in X - \varphi^{-1}(V)$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,\varphi^{-1}(V)) > \theta$. Suppose that $\mathsf{b}(x,\theta) \in V$. By Lemma 4.1.2, there exists $\alpha \in \mathcal{P}t(\mathcal{M}(X))$ such that $\mathsf{b}(x,\theta) \in \alpha \subseteq V$. Since X is complete, the function $i_X : X \to \mathcal{P}t(\mathcal{M}(X))$ given by (3.6) is an isomorphism. Thus, there exists $x' \in X$ such that $\Diamond x' = \alpha$. Hence, we have $d(x,x') < \theta$ and $x' \in \varphi^{-1}(V)$, contradicting $d(x,\varphi^{-1}(V)) > \theta$. Therefore, $\mathsf{b}(x,\theta) \in \neg V$, and so $x \in (\neg V)_*$.

Lastly, for any inhabited closed located subset A of X, we have

$$X - A = X - \varphi^{-1}(\varphi(A)) = (\neg\varphi(A))_* = (H(X - A))_*.$$

Conversely, for any inhabited located subset V of $\mathcal{M}(X)$, we have

$$\neg V = \neg\left(\varphi(\varphi^{-1}(V))\right) = H(X - \varphi^{-1}(V)) = H\left((\neg V)_*\right). \qquad \square$$

Let X be a metric space. For any compact subset A of X, we extend the notion of the metric complement $X - A$ of A as follows.

$$X - A \stackrel{\text{def}}{=} \begin{cases} X & \text{if } A = \emptyset, \\ \{x \in X \mid d(x, A) > 0\} & \text{if } A \text{ is inhabited.} \end{cases}$$

Note that since a compact metric space is totally bounded, we can decide whether it is empty or inhabited.

Since a subset of a compact metric space is compact iff it is empty or inhabited, closed, and located (See Lemma D.0.23), we have the following.

Corollary 5.1.6. *Let X be a compact metric space. For any located subset V of $\mathcal{M}(X)$, there exists a unique compact subset $A \subseteq X$ such that $\mathcal{OM}((X, X - A)) = \mathcal{M}(X)_{\neg V}$.*

Proof. Let V be a located subset of $\mathcal{M}(X)$. Since the located subtopology $\mathcal{M}(X)^V$ is compact overt and V is its positivity, V is either empty or inhabited. In the former case, we can take $A = \emptyset$. Then, $\mathcal{OM}((X, X - A)) = \mathcal{M}_{H(X-A)} = \mathcal{M}_{H(X)} = \mathcal{M}_{M_X} = \mathcal{M}_{\neg\emptyset}$. In the latter case, the desired conclusion follows from Proposition 5.1.5. □

Lemma 5.1.7. *Let X be a compact metric space, and let V be a located subset of $\mathcal{M}(X)$. Then, the open complement $\mathcal{M}(X)_{\neg V}$ is inhabited iff $(\neg V)_*$ is inhabited.*

Proof. Since a_* is inhabited for all $a \in M_X$, we have that $\mathcal{M}(X)_{\neg V}$ is inhabited iff $(\neg V)_*$ is inhabited. □

Corollary 5.1.8. *Let X be a compact metric space, and let V be a located subset of $\mathcal{M}(X)$ such that $\mathcal{M}(X)_{\neg V}$ is inhabited. Then, there exists a unique compact subset $A \subseteq X$ such that its metric complement $X - A$ is inhabited and that $\mathcal{OM}((X, X - A)) = \mathcal{M}(X)_{\neg V}$.*

For Proposition 5.1.10, we need the following characterisation of locally compact metric spaces.

Lemma 5.1.9 (Palmgren [49, Lemma 2.2][3]). *Let X be a locally compact metric space. Then, for any $x \in X$ and $\varepsilon, \delta \in \mathbb{Q}^{>0}$ such that $\varepsilon < \delta$, there exists a compact subset $K \subseteq X$ such that*

$$B(x, \varepsilon) \subseteq K \subseteq B(x, \delta).$$

Proof. Since X is locally compact we have

$$a <_X b \implies a \ll b$$

for all $a, b \in M_X$ by Lemma 3.1.37. Let $x \in X$ and $\varepsilon, \delta \in \mathbb{Q}^{>0}$ such that $\varepsilon < \delta$. Choose $N \in \mathbb{N}$ such that $\varepsilon + 2^{-N} < \delta$. For each $n \in \mathbb{N}$, let $a_n \stackrel{\text{def}}{=} \mathsf{b}(x, \varepsilon + 2^{-(N+n)})$. Then, for each $n \in \mathbb{N}$, since $a_{n+1} <_X a_n$, there exists $V_n \in \mathsf{Fin}(a_n \downarrow \mathcal{C}_{2^{-n}})$ such that

$$a_{n+1} \lhd_X V_n.$$

By the Countable Choice, we obtain a sequence $(V_n)_{n \in \mathbb{N}} : \mathbb{N} \to \mathsf{Fin}(M_X)$ such that

[3] The proof given in [49, Lemma 2.2] seems to be incomplete. The proof of [48, Proposition 4.8] by the same author contains a correct proof for the lemma.

1. $a_{n+1} \lhd_X V_n \leq_X a_n$,

2. $(\forall \mathsf{b}(z,\gamma) \in V_n)\, \gamma \leq 2^{-n}$.

Let $A = \{y \in X \mid (\exists n \in \mathbb{N})\, (\exists \gamma \in \mathbb{Q}^{>0})\, \mathsf{b}(y,\gamma) \in V_n\}$. The set A is clearly totally bounded. Then, the closure $\mathsf{cl}(A)$ of A is compact, and we have $B(x,\varepsilon) \subseteq \mathsf{cl}(A) \subseteq B(x,\delta)$. Hence, $\mathsf{cl}(A)$ is a desired compact subset. □

Proposition 5.1.10. *Let $X = (X,d)$ be a compact metric space, and let $A \subseteq X$ be a compact subset of X. Then, there exists a locally compact metric space Y which is isomorphic to $(X, X - A)$ in the category* **OLCM**.

Moreover, if $X - A$ is inhabited, then there exists a Bishop locally compact metric space Y which is isomorphic to $(X, X - A)$ in **OLCM**.

Proof. If $A = \emptyset$, we put $Y = X$. Suppose that A is inhabited. Let $Y = X - A$, and define a new metric d^* on Y by

$$d^*(x,y) \stackrel{\text{def}}{=} d(x,y) + \left| \frac{1}{d(x,A)} - \frac{1}{d(y,A)} \right|$$

for all $x, y \in Y$. It is straightforward to show that d^* is a metric on Y. We show that the metric space $Y = (Y, d^*)$ has the required properties. Since $d(x,y) \leq d^*(x,y)$ for all $x, y \in Y$, the inclusion $i_Y : Y \hookrightarrow (X - A)$ is uniformly continuous. Let $K \subseteq Y$ be an inhabited compact subset. Then, K is contained in some open ball $B^*(y,\varepsilon) = \{y' \in Y \mid d^*(y',y) < \varepsilon\}$ of Y, so the proof of local compactness of Y (see below) shows that there exists a d-compact subset L of X such that $B^*(y,\varepsilon) \subseteq L \Subset X - A$. Hence, i_Y is a morphism from (Y,Y) to $(X, X - A)$ in **OLCM**. Moreover, i_Y is injective; for suppose that $d^*(x,y) > 0$. Choose $r \in \mathbb{Q}^{>0}$ such that $d^*(x,y) > r$. Let $c = \min\{d(x,A), d(y,A)\}$. Since $d^*(x,y) \leq (1 + 1/c^2)\, d(x,y)$, we have $d(x,y) \geq r/(1 + 1/c^2)$. Hence, i_Y is injective.

Next, we show that the inverse $j : (X - A) \to Y$ of i_Y is uniformly continuous on each inhabited compact subset K of $X - A$ such that $K \Subset X - A$. Let $K \Subset X - A$ be an inhabited compact subset. Then, there exists $r \in \mathbb{Q}^{>0}$ such that $K_r \subseteq X - A$, so we have $d(x,A) \geq r$ for all $x \in K$. Hence, $d^*(x,y) \leq (1 + 1/r^2)\, d(x,y)$ for all $x, y \in K$. Uniform continuity of $j : K \to Y$ now follows.

It remains to be shown that Y is locally compact with respect to d^*. Let $y \in Y$ and $\varepsilon \in \mathbb{Q}^{>0}$. We must find a d^*-compact subset $K \subseteq Y$ such that $B^*(y,\varepsilon) \subseteq K$. To this end, it suffices to show that there exists a d-compact subset $K \Subset X - A$ such that $B^*(y,\varepsilon) \subseteq K$, for if such K exists, then $i_Y : Y \to (X - A)$ and $j : (X - A) \to Y$ restrict to uniform isomorphisms on K. To find such K, we note that for any $x \in B^*(y,\varepsilon)$, since $d(x,y) + |1/d(x,A) - 1/d(y,A)| < \varepsilon$, we have $d(x,A) > 1/\left(\varepsilon + 1/d(y,A)\right)$. Putting $r = 1/\left(\varepsilon + 1/d(y,A)\right)$, we have $B^*(y,\varepsilon) \subseteq U_{A,r} = \{x \in X \mid d(x,A) \geq r\}$. Then, by Lemma D.0.26, there exists a d-compact subset $K \subseteq X$ such that $U_{A,r} \subseteq K \Subset X - A$, as required.

The second statement is obvious. □

Corollary 5.1.11. *Let X be a compact metric space, and let V be a located subset of $\mathcal{M}(X)$. Then, there exists a locally compact metric space Y such that its localic completion $\mathcal{M}(Y)$ is isomorphic to the open complement $\mathcal{M}(X)_{\neg V}$.*

Proof. By Lemma 5.1.6, there exists a unique compact subset A of X such that

$$\mathcal{OM}((X, X - A)) = \mathcal{M}(X)_{\neg V}.$$

Then, there exists a locally compact metric space Y such that

$$(Y, Y) \cong (X, X - A)$$

in **OLCM** by Proposition 5.1.10. Since every functor preserves isomorphisms, we have

$$\mathcal{M}(Y) = \mathcal{OM}((Y, Y)) \cong \mathcal{OM}((X, X - A)) = \mathcal{M}(X)_{\neg V}.$$ □

Corollary 5.1.12. *Let X be a compact metric space, and let V be a located subset of $\mathcal{M}(X)$ such that the open complement $\mathcal{M}(X)_{\neg V}$ is inhabited. Then, there exists a Bishop locally compact metric space Y such that its localic completion $\mathcal{M}(Y)$ is isomorphic to $\mathcal{M}(X)_{\neg V}$.*

5.2 Point-free one-point compactification

In this section, we lift the construction of a one-point compactification of a Bishop locally compact metric space to the setting of formal topologies. We introduce the notion of enumerably locally compact formal topology, and show that every overt enumerably locally compact regular formal topology can be embedded into a compact overt enumerably completely regular formal topology as the open complement of a formal point. This allows us to represent every such locally compact formal topology as a pair of a compact overt enumerably completely regular formal topology and its formal point.

Definition 5.2.1 (cf. Definition 4.2.1)**.** Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. Let $\mathbb{I} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. A *wb-scale* from U to V is a family $(U_q)_{q \in \mathbb{I}}$ of subsets of S such that $U \lhd U_0$, $U_1 \lhd V$, and for all $p, q \in \mathbb{I}$, $p < q \implies U_p \ll U_q$. The notion of *finitary wb-scale* is similarly defined as that of finitary scale (See Definition 4.2.3).

The following is a direct consequence of Lemma 2.4.17 and the Dependent Choice. The proof is identical to that of Proposition 4.2.5 except that the well-covered relation $\lll$ is replaced by the way-below relation $\ll$.

Proposition 5.2.2. *Let $\mathcal{S}$ be a locally compact formal topology. Then, for any $U, V \subseteq S$, if $U \ll V$, then there exists a finitary wb-scale from U to V.*

Definition 5.2.3. A formal topology $\mathcal{S}$ is *enumerably locally compact* if there exists a function $wb \colon S \to \mathsf{Pow}(S)$ such that

1. $(\forall b \in wb(a))\, b \ll a$,
2. $a \lhd wb(a)$,
3. the relation $\overline{wb} = \{(b, a) \in S \times S \mid b \in wb(a)\}$ is countable, i.e. there exists a surjection $f \colon \mathbb{N} \to \overline{wb}$,

4. there exists a function $sc \in \prod_{(b,a)\in\overline{wb}} \mathrm{Sc}_{\ll}(\{b\},\{a\})$,

where $\mathrm{Sc}_{\ll}(\{b\},\{a\})$ is the set of finitary wb-scales from $\{b\}$ to $\{a\}$. The function sc is called a *choice of wb-scale* for wb.

Remark 5.2.4. Assuming the Dependent Choice, any locally compact formal topology $\mathcal{S}$ with a base $wb : S \to \mathsf{Pow}(S)$ such that the corresponding relation $\overline{wb}$ is countable is enumerably locally compact by Proposition 5.2.2.

The following is a direct consequence of Lemma 2.4.22.

Lemma 5.2.5. *Any enumerably locally compact regular formal topology is enumerably completely regular.*

Definition 5.2.6. Let $\mathcal{S}$ be an overt enumerably locally compact regular formal topology. A *one-point compactification* of $\mathcal{S}$ is a triple $(\mathcal{T}, \omega, r)$ consisting of a compact overt enumerably completely regular formal topology $\mathcal{T}$, a formal point $\omega \in \mathcal{P}t(\mathcal{T})$, and an embedding $r : \mathcal{S} \to \mathcal{T}$ such that the image of $\mathcal{S}$ under r is isomorphic to the open complement $\mathcal{T}_{\neg\omega}$ of the located subtopology determined by ω.

Note that every formal point of a locally compact regular formal topology is located. This follows from Lemma 2.4.22. Thus, every formal point of a locally compact regular formal topology determines a located subtopology.

Theorem 5.2.7. *Any overt enumerably locally compact regular formal topology has a one-point compactification.*

The rest of this section is devoted to the proof of the theorem.

In the following, we fix an overt enumerably locally compact regular formal topology $\mathcal{S}$. We write Pos for its positivity, $wb : S \to \mathsf{Pow}(S)$ for the function which satisfies the conditions in Definition 5.2.3, and $(b_n, a_n)_{n\in\mathbb{N}}$ for an enumeration of $\overline{wb} = \{(b,a) \in S \times S \mid b \in wb(a)\}$. Moreover, for each $n \in \mathbb{N}$, we write $r_n : \mathcal{S} \to \mathcal{I}[0,1]$ for the formal topology map determined by the wb-scale from $\{b_n\}$ to $\{a_n\}$ which is chosen by the choice of wb-scale for wb associated with $\mathcal{S}$. Note that each r_n is given by (4.1) and satisfies

1. $r_n^-(0,\infty) \downarrow \{b_n\} \lhd \emptyset$,

2. $r_n^-(-\infty,1) \lhd \{a_n\}$.

Then, we write $r : \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ for the canonical embedding determined by the sequence $(r_n : \mathcal{S} \to \mathcal{I}[0,1])_{n\in\mathbb{N}}$.

We recall some notations introduced in Section 4.4. For each $n, k \in \mathbb{N}$, we defined subsets $\mathcal{C}_k^n$ and $\mathcal{C}_k^{\leq n}$ of the base S_Π of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ by (4.3) and (4.4) respectively. Since $\mathcal{C}_k^{\leq n} =_{\prod \mathcal{I}[0,1]} \mathcal{C}_k^0 \downarrow \cdots \downarrow \mathcal{C}_k^n$, we have $S_\Pi \lhd_\Pi \mathcal{C}_k^{\leq n}$ by (H4), and hence $S \lhd r^- \mathcal{C}_k^{\leq n}$ for each $n, k \in \mathbb{N}$.

Lemma 5.2.8. *For any $N \in \mathbb{N}$ such that $a_N \ll S$, there exists a compact overt subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ such that $\mathcal{S}_{b_N} \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}_{a_N}$, where $\mathcal{S}_{b_N}$ and $\mathcal{S}_{a_N}$ are the open subtopologies of $\mathcal{S}$ determined by $\{b_N\}$ and $\{a_N\}$ respectively.*

Proof. Let $N \in \mathbb{N}$ such that $a_N \ll S$. For each $n \in \mathbb{N}$, there exists $\mathcal{E}_n \in \mathsf{Fin}\left(\mathcal{C}^{\leq n}_{n+3}\right)$ such that $a_N \lhd r^- \mathcal{E}_n$ and $\mathcal{E}_n \subseteq r\,\mathsf{Pos}$. Thus, by the Countable Choice, there exists a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $\mathcal{E}_n \in \mathsf{Fin}\left(C^{\leq n}_{n+3}\right)$, $\mathcal{E}_n \subseteq r\,\mathsf{Pos}$, and $a_N \lhd r^- \mathcal{E}_n$. For each $n \in \mathbb{N}$, write $\mathcal{E}_n = \left\{A^n_0, \dots, A^n_{N_n-1}\right\}$, and for each $i < N_n$ write $A^n_i = \left\{(0, (p^{n,i}_0, q^{n,i}_0)), \dots, (n, (p^{n,i}_n, q^{n,i}_n))\right\}$.

Split $\mathcal{E}_N$ into finitely enumerable subsets $\mathcal{E}^+_N$ and $\mathcal{E}^-_N$ such that $\mathcal{E}_N = \mathcal{E}^+_N \cup \mathcal{E}^-_N$, and

- $A^N_i \in \mathcal{E}^+_N \implies (p^{N,i}_N, q^{N,i}_N) \in (-\infty, 1/2)$,
- $A^N_i \in \mathcal{E}^-_N \implies (p^{N,i}_N, q^{N,i}_N) \in (1/4, \infty)$.

Define a subset T of S^*_Π by

$$\begin{aligned} T_0 &= \left\{\langle A\rangle \mid A \in \mathcal{E}^+_N\right\}, \\ T_{n+1} &= \left\{l * \langle A\rangle \mid l \in T_n \;\&\; l = l' * \langle A'\rangle \;\&\; A \in \mathcal{E}_{N+n+1} \;\&\; A' \asymp A\right\}, \\ T &= \bigcup_{n\in\mathbb{N}} T_n, \end{aligned}$$

where the notation $A \asymp A'$ has been defined by (4.5). Note that T_n is finitely enumerably for each $n \in \mathbb{N}$.

Define subsets U_T and K of S by

$$\begin{aligned} U_T &\stackrel{\text{def}}{=} \bigcup \left\{r^- A_l \mid l \in T\right\}, \\ K &\stackrel{\text{def}}{=} \left\{a \in S \mid \mathsf{Pos}(U_T \downarrow a)\right\}, \end{aligned}$$

where for each $l \in T$, A_l denotes the last element of l. We show that K is a located subset of $\mathcal{S}$.

Since K is the positivity of the open subtopology $\mathcal{S}_{U_T}$ determined by U_T (See Lemma 2.3.6.3), K is a splitting subset of $\mathcal{S}$. Thus, it remains to be shown that for each $L \in \mathbb{N}$, either $b_L \in \neg K$ or $a_L \in K$. Let $L \in \mathbb{N}$, and define $n_L \in \mathbb{N}$ by

$$n_L \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } L \leq N, \\ L - N & \text{if } L > N. \end{cases}$$

We have the following two cases.

1. $(\exists l \in T_{n_L})\,(\forall\,(i, (p, q)) \in A_l)\, i = L \implies (p, q) \in (-\infty, 3/4)$.
2. $(\forall l \in T_{n_L})\,(\forall\,(i, (p, q)) \in A_l)\, i = L \implies (p, q) \in (1/2, \infty)$.

In the first case, there exist $l \in T_{n_L}$ and $(L,(p,q)) \in A_l$ such that $(p,q) \in (-\infty, 3/4)$. Thus

$$\begin{aligned} r^- A_l &\lhd r^- \{(L,(p,q))\} \\ &\lhd r_L^-(p,q) \lhd r_L^-(-\infty, 3/4) \lhd a_L, \end{aligned}$$

and since $A_l \in r\,\mathsf{Pos}$, we have $\mathsf{Pos}(r^- A_l \downarrow a_L)$. Hence $a_L \in K$.

In the second case, suppose that $b_L \in K$. Then, there exist $n \in \mathbb{N}$ and $l \in T_n$ such that $\mathsf{Pos}(r^- A_l \downarrow b_L)$. If $n > n_L$, then by writing $l = \langle A_0, \ldots, A_n \rangle$ so that $A_l = A_n$, we have $A_{n_L} \gtrless A_{n_L+1}, \ldots, A_{n-1} \gtrless A_n$. By the assumption we have $(p,q) \in (1/2, \infty)$ for $(L,(p,q)) \in A_{n_L}$, so we have $(s,t) \in (0,\infty)$ for $(L,(s,t)) \in A_l$. Thus,

$$r^- A_l \downarrow b_L \lhd r^- \{(L,(s,t))\} \downarrow b_L \lhd r_L^-(0,\infty) \downarrow b_L \lhd \emptyset.$$

Since $\mathsf{Pos}\,(r^- A_l \downarrow b_L)$, we have $\mathsf{Pos} \between \emptyset$, a contradiction. If $n \leq n_L$, then since

$$r^- A_l \lhd r_N^-(\infty, 3/4) \lhd a_N,$$

we have

$$r^- A_l \lhd r^-(\mathcal{E}_{N+n+1} \downarrow \cdots \downarrow \mathcal{E}_{N+n_L} \downarrow A_l).$$

Since $\mathsf{Pos}\,(r^- A_l \downarrow b_L)$, there exist $A_{n+1} \in \mathcal{E}_{N+n+1}, \ldots, A_{n_L} \in \mathcal{E}_{N+n_L}$ such that

$$\mathsf{Pos} \between \left(r^-(A_l \downarrow A_{n+1} \downarrow \cdots \downarrow A_{n_L}) \downarrow b_L\right).$$

Then, $l * \langle A_{n+1}, \ldots, A_{n_L} \rangle \in T_{n_L}$, and thus by the assumption

$$r^- A_{n_L} \downarrow b_L \lhd r_L^-(1/2, \infty) \downarrow b_L \lhd \emptyset.$$

Since $\mathsf{Pos}(r^- A_{n_L} \downarrow b_L)$, we have $\mathsf{Pos} \between \emptyset$, a contradiction. Hence, $b_L \in \neg K$. Therefore K is located.

Next, we show that $\mathcal{S}_{b_N} \sqsubseteq \mathcal{S}^K \sqsubseteq \mathcal{S}_{a_N}$, where $\mathcal{S}^K$ is the located subtopology of $\mathcal{S}$ determined by K. Since $\mathcal{S}^K$ is the closure $\mathcal{S}^{\mathcal{S}-\neg K}$ of $\mathcal{S}_{U_T}$ by Proposition 4.1.13 and Proposition 2.3.13, it suffices to show that $b_N \lhd U_T \ll a_N$ by Lemma 2.4.2.1 and Lemma 2.4.22. Since $b_N \lhd r^- \mathcal{E}_N$, we have $b_N \lhd (r^- \mathcal{E}_N \downarrow b_N) \cap \mathsf{Pos}$. Let $c \in r^- \mathcal{E}_N \downarrow b_N$ such that $\mathsf{Pos}(c)$. Then, there exists $A \in \mathcal{E}_N$ such that $c \in r^- A \downarrow b_N$. If $A \in \mathcal{E}_N^-$, then we have

$$c \lhd r^- A \downarrow b_N \lhd r_N^-(1/4, \infty) \downarrow b_N \lhd \emptyset,$$

and thus $\mathsf{Pos} \between \emptyset$, a contradiction. Hence, $A \in \mathcal{E}_N^+$, and so $c \lhd r^- \mathcal{E}_N^+$. Therefore

$$b_N \lhd r^- \mathcal{E}_N^+ \lhd U_T.$$

Now, let $n \in \mathbb{N}$ and $l \in T_n$. Write $l = \langle A_0, \ldots, A_n \rangle$. Since $A_i \gtrless A_{i+1}$ for all $i < n$ and $(p,q) \in (-\infty, 1/2)$ for $(N,(p,q)) \in A_0$, we have

$$r^- A_l \lhd r_N^-(-\infty, 3/4) \ll a_N.$$

Hence, $U_T \lhd r_N^-(-\infty, 3/4) \ll a_N$.

Lastly, since $\mathcal{S}^K$ is closed and bounded by $\{a_N\}$, it is compact by Proposition 2.4.19. $\square$

The following is a point-free version of Lemma 5.1.9.

Proposition 5.2.9. *For any $U, V \subseteq S$ such that $U \ll V$, there exists a compact overt subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ such that $\mathcal{S}_U \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}_V$.*

Proof. Let $U, V \subseteq S$, and suppose that $U \ll V$. Then, by the similar argument as in the proof Lemma 2.4.17, there exists $\{(b_0, a_0), \dots, (b_{n-1}, a_{n-1})\} \in \mathsf{Fin}(\overline{wb})$ such that $U \lhd \{b_0, \dots, b_{n-1}\}$ and $\{a_0, \dots, a_{n-1}\} \ll V$. By Lemma 5.2.8, for each $i < n$, there exists a located subset K_i such that

$$\mathcal{S}_{b_i} \sqsubseteq \mathcal{S}^{K_i} \sqsubseteq \mathcal{S}_{a_i}.$$

Let $K = \bigcup_{i<n} K_i$. Since a finite union of located subsets is located, K is located. Moreover, we have

$$\mathcal{S}_U \sqsubseteq \mathcal{S}_{\{b_0,\dots,b_{n-1}\}} \sqsubseteq \mathcal{S}^K \sqsubseteq \mathcal{S}_{\{a_0,\dots,a_{n-1}\}} \sqsubseteq \mathcal{S}_V.$$

Since the set $\{a_0, \dots, a_{n-1}\}$ is bounded, $\mathcal{S}^K$ is compact overt by Proposition 2.4.19. ☐

Let $\mathcal{S}_r$ be the image of $\mathcal{S}$ under the embedding $r : \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Then, $\mathcal{S}_r$ is overt with the positivity $r\,\mathsf{Pos}$. Define $\omega \in \mathcal{P}t\left(\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]\right)$ by

$$\omega \stackrel{\text{def}}{=} \{A \in S_\Pi \mid (\forall (n, (p, q)) \in A)\, p < 1 < q\}.$$

Note that ω is a decidable subset of S_Π. Let

$$\overline{\mathsf{Pos}} \stackrel{\text{def}}{=} r\,\mathsf{Pos} \cup \omega.$$

Note that $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ is compact regular by Proposition 2.4.11.2 and Theorem 2.4.14, and hence it is locally compact by Proposition 2.4.23.

Lemma 5.2.10. *$\overline{\mathsf{Pos}}$ is a located subset of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$.*

Proof. Since $\overline{\mathsf{Pos}}$ is a union of the splitting subsets $r\,\mathsf{Pos}$ and ω, it is splitting. Let wc_Π be the function defined by (4.2). Let $A = \{(m_0, (p_0, q_0)), \dots, (m_{n-1}, (p_{n-1}, q_{n-1}))\} \in S_\Pi$ and $A' = \left\{(m_0, (p'_0, q'_0)), \dots, (m_{n-1}, (p'_{n-1}, q'_{n-1}))\right\} \in wc_\Pi(A)$, so that for each $i < n$ we have $p_i < p'_i < q'_i < q_i$. By Proposition 2.4.23, it suffices to show that either $A' \in \neg\overline{\mathsf{Pos}}$ or $A \in \overline{\mathsf{Pos}}$.

Since ω is decidable, we have either $A \in \omega$ or $A \in \neg\omega$. In the former case, we have $A \in \overline{\mathsf{Pos}}$. In the latter case, there exists $i_* < n$ such that either $1 \leq p_{i_*}$ or $q_{i_*} \leq 1$. Suppose that $1 \leq p_{i_*}$, and suppose further that $A' \in r\,\mathsf{Pos}$. Then, there exists $a \in \mathsf{Pos}$ such that $a\ r\ A'$, and thus

$$a \lhd r^-_{m_{i_*}}(p_{i_*}, q_{i_*}) \lhd r^-_{m_{i_*}} \{(p_{i_*}, q_{i_*}) \mid p_{i_*} < 1 \;\&\; 0 < q_{i_*}\} \lhd \emptyset$$

by the positivity of $\mathcal{I}[0,1]$ (See Example 2.3.24). Since $\mathsf{Pos}(a)$, we have $\mathsf{Pos} \between \emptyset$, a contradiction. Since $A \in \neg\omega \implies A' \in \neg\omega$, we have $A' \in \neg\overline{\mathsf{Pos}}$.

Now, suppose that $q_{i_*} \leq 1$. Then, we have

$$r^-_{m_{i_*}}(p'_{i_*}, q'_{i_*}) \ll a_{m_{i_*}},$$

where $a_{m_{i_*}}$ is the second component of the pair $(b_{m_{i_*}}, a_{m_{i_*}}) \in \overline{wb}$ indexed by m_{i_*}. Let

$$U_* \stackrel{\text{def}}{=} r^-_{m_{i_*}}(p'_{i_*}, q'_{i_*}).$$

By Proposition 5.2.9, there exists a compact overt subtopology $\mathcal{S}^K$ of $\mathcal{S}$ determined by a located subset $K \subseteq S$ such that

$$\mathcal{S}_{U_*} \sqsubseteq \mathcal{S}^K.$$

Choose $k \in \mathbb{N}$ and $\theta \in \mathbb{Q}^{>0}$ such that $2^{-k} < \theta$ and $p_i < p'_i - 2\theta < q'_i + 2\theta < q_i$ for each $i < n$. By (H4), we have

$$\begin{aligned} S &\lhd r^- \left(\mathcal{C}^{m_0}_k \downarrow \cdots \downarrow \mathcal{C}^{m_{n-1}}_k\right) \\ &\lhd r^- \left\{\{(m_0,(s_0,t_0)),\ldots,(m_{n-1},(s_{n-1},t_{n-1}))\} \mid (\forall i<n)\, t_i - s_i = 2^{-k}\right\}. \end{aligned}$$

Let $\mathcal{C}_A = \left\{\{(m_0,(s_0,t_0)),\ldots,(m_{n-1},(s_{n-1},t_{n-1}))\} \mid (\forall i<n)\, t_i - s_i = 2^{-k}\right\}$. Since $\mathcal{S}^K$ is compact, there exist $B_0,\ldots,B_{N-1} \in \mathcal{C}_A$ such that $B_j \in rK$ for each $j < N$ and $S \lhd^K r^- \{B_0,\ldots,B_{N-1}\}$. For each $j < N$, write

$$B_j = \left\{\left(m_0,(s^j_0,t^j_0)\right),\ldots,\left(m_{n-1},(s^j_{n-1},t^j_{n-1})\right)\right\}.$$

Then, either $(\forall i<n)\,(s^j_i,t^j_i) \leq_{\mathcal{R}} (p'_i - 2\theta, q'_i + 2\theta)$ or $(\exists i<n)\,(s^j_i,t^j_i) \in (-\infty,p'_i) \cup (q'_i,\infty)$. Hence, we have the following two cases.

1. $(\exists j<N)\,(\forall i<n)\,(s^j_i,t^j_i) \leq_{\mathcal{R}} (p'_i - 2\theta, q'_i + 2\theta)$.
2. $(\forall j<N)\,(\exists i<n)\,(s^j_i,t^j_i) \in (-\infty,p'_i) \cup (q'_i,\infty)$.

In the first case, there exists $j < N$ such that $B_j \leq A$, and hence $r^- B_j \lhd r^- A$. Since $B_j \in rK$ and K is splitting, we have

$$A \in rK \subseteq r\,\mathsf{Pos} \subseteq \overline{\mathsf{Pos}}.$$

In the second case, suppose that $A' \in r\,\mathsf{Pos}$, and let $a \in \mathsf{Pos}$ such that $a\ r\ A'$. Let $\mathcal{S}_{\neg K}$ be the open complement of $\mathcal{S}^K$, and let $\mathsf{Pos}_{\neg K}$ be the positivity of $\mathcal{S}_{\neg K}$. Then, $\mathsf{Pos} = \mathsf{Pos}_{\neg K} \cup K$, so either $a \in \mathsf{Pos}_{\neg K}$ or $a \in K$. If $a \in \mathsf{Pos}_{\neg K}$, then we have $\mathsf{Pos}(\neg K \downarrow a)$. Since $\mathcal{S}_{U_*} \sqsubseteq \mathcal{S}^K = \mathcal{S}^{\mathcal{S}-\neg K}$, we have

$$\neg K \downarrow a \lhd \neg K \downarrow r^- A' \lhd \neg K \downarrow U_* \lhd \emptyset.$$

Thus, $\mathsf{Pos} \between \emptyset$, a contradiction. If $a \in K$, then since

$$a \lhd^K \left(r^- \{B_0,\ldots,B_{N-1}\}\right) \downarrow a$$

and K is splitting, there exists $j < N$ such that $K(r^- B_j \downarrow a)$. Thus, by the assumption there exists $i < n$ such that $(s^j_i,t^j_i) \in (-\infty,p'_i) \cup (q'_i,\infty)$. If $(s^j_i,t^j_i) \in (-\infty,p'_i)$, then

$$\begin{aligned} r^- B_j \downarrow a &\lhd r^-_{m_i}(-\infty,p'_i) \downarrow r^-_{m_i}(p'_i,q'_i) \\ &\lhd r^-_{m_i}\left((-\infty,p'_i) \downarrow (p'_i,q'_i)\right) \lhd \emptyset, \end{aligned}$$

so we have $K \between \emptyset$, a contradiction. If $(s^j_i,t^j_i) \in (q'_i,\infty)$, we similarly obtain a contradiction. Thus $A' \in \neg\,(r\,\mathsf{Pos})$, and hence $A' \in \neg\overline{\mathsf{Pos}}$. Therefore, $\overline{\mathsf{Pos}}$ is located. $\square$

Thus, $\overline{\mathsf{Pos}}$ determines a compact overt subtopology of $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ by Corollary 4.1.16, namely the closed subtopology of $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ determined by $\neg\overline{\mathsf{Pos}}$. Write $\overline{\mathcal{S}} = (S_\Pi, \overline{\lhd}, \leq)$ for this subtopology. Since ω is a formal point of $\overline{\mathcal{S}}$, it is a located subset of $\overline{\mathcal{S}}$. Let $\overline{\mathcal{S}}_{\neg\omega}$ be the open complement of the located subtopology determined by ω in $\overline{\mathcal{S}}$. Then, the cover $\overline{\lhd}_{\neg\omega}$ of $\overline{\mathcal{S}}_{\neg\omega}$ is given by

$$A \overline{\lhd}_{\neg w} \mathcal{U} \overset{\text{def}}{\iff} A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}$$

for all $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$.

Lemma 5.2.11. *The embedding $r : \mathcal{S} \to \prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ satisfies $S \lhd r^- \neg\omega$.*

Proof. Let $a \in S$ and $b \in wb(a)$, and let $n \in \mathbb{N}$ be the index of the pair $(b,a) \in \overline{wb}$. Then,

$$\begin{aligned} b &\lhd r_n^- \left((-\infty,1) \cup (0,\infty)\right) \downarrow b \\ &\lhd \left(r_n^-(-\infty,1) \downarrow b\right) \cup \left(r_n^-(0,\infty) \downarrow b\right) \\ &\lhd r_n^-(-\infty,1) \lhd r^- \neg\omega. \end{aligned}$$

Hence, $a \lhd wb(a) \lhd r^- \neg\omega$. Therefore $S \lhd r^- \neg\omega$. □

Lemma 5.2.12. *We have $\mathcal{S}_r = \overline{\mathcal{S}}_{\neg\omega}$ in the lattice of subtopologies of $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$, i.e.*

$$r^- A \lhd r^- \mathcal{U} \iff A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}$$

for all $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$.

Proof. Let $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$. First, suppose that $A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathsf{Pos}} \downarrow \mathcal{U}$. By Lemma 5.2.11, we have

$$\begin{aligned} r^- A &\lhd r^- A \downarrow r^- \neg\omega \\ &\lhd r^- \left(A \downarrow \neg\omega\right) \\ &\lhd r^- \left(\neg\overline{\mathsf{Pos}} \cup \mathcal{U}\right) \\ &\lhd \left(r^- \left(\neg r\,\mathsf{Pos} \downarrow \neg\omega\right) \cup r^- \mathcal{U}\right) \cap \mathsf{Pos} \\ &\lhd \left(\left(r^- \neg r\,\mathsf{Pos}\right) \cap \mathsf{Pos}\right) \cup \left(r^- \mathcal{U} \cap \mathsf{Pos}\right) \\ &\lhd r^- \mathcal{U} \cap \mathsf{Pos} \lhd r^- \mathcal{U}. \end{aligned}$$

Conversely, suppose that $r^- A \lhd r^- \mathcal{U}$. Let $B \in A \downarrow \neg\omega$. We must show that $B \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}$. Write $B = \{(m_0,(p_0,q_0)),\ldots,(m_{n_B-1},(p_{n_B-1},q_{n_B-1}))\}$. Since $B \in \neg\omega$, there exists $i_* < n_B$ such that either $1 \leq p_{i_*}$ or $q_{i_*} \leq 1$. If $1 \leq p_{i_*}$, then

$$B \lhd_\Pi \{(m_{i_*}(p_{i_*}, q_{i_*}))\} \lhd_\Pi \neg\overline{\mathsf{Pos}} \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}.$$

Now, suppose that $q_{i_*} \leq 1$. Let $B' \in wc_\Pi(B)$ so that B' is of the form

$$B' = \left\{(m_0,(p'_0,q'_0)),\ldots,(m_{n_B-1},(p'_{n_B-1},q'_{n_B-1}))\right\}$$

such that $p_i < p'_i < q'_i < q_i$ for each $i < n_B$. Since $q'_{i_*} < 1$, we have $r^- B' \lhd r^-_{m_{i_*}}(p'_{i_*}, q'_{i_*}) \ll a_{m_{i_*}}$, and since $B' \lll A$ in $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$, we have $r^- B' \lll r^- A$ in $\mathcal{S}$ by Lemma 2.4.2.4. Hence $r^- B' \ll r^- A$ by Lemma 2.4.20. Moreover, since $\mathcal{U} \lhd_\Pi \mathcal{U}_<$ where

$$\mathcal{U}_< \stackrel{\text{def}}{=} \{C' \in S_\Pi \mid (\exists C \in \mathcal{U})\, C' \in wc_\Pi(C)\}\,,$$

we have $r^- A \lhd r^- \mathcal{U}_<$. Thus, there exist $\{C_0, \dots, C_{n_\mathcal{U}-1}\} \in \mathsf{Fin}(\mathcal{U})$ and $\{C'_0, \dots, C'_{n_\mathcal{U}-1}\} \in \mathsf{Fin}(S_\Pi)$ such that $r^- B' \lhd r^- \{C'_0, \dots, C'_{n_\mathcal{U}-1}\}$ and for each $j < n_\mathcal{U}$, the sets C_j and C'_j are of the forms

$$C_j = \{(l_{j,0}, (s_{j,0}, t_{j,0})), \dots, (l_{j,n_j-1}, (s_{j,n_j-1}, t_{j,n_j-1}))\}\,,$$
$$C'_j = \left\{(l_{j,0}, (s'_{j,0}, t'_{j,0})), \dots, (l_{j,n_j-1}, (s'_{j,n_j-1}, t'_{j,n_j-1}))\right\}$$

such that $s_{j,i} < s'_{j,i} < t'_{j,i} < t_{j,i}$ for each $i < n_j$. Let

$$M \stackrel{\text{def}}{=} \max\{l_{j,i} \mid j < n_\mathcal{U} \;\&\; i < n_j\}\,,$$

and choose $k \in \mathbb{N}$ and $\theta \in \mathbb{Q}^{>0}$ such that $2^{-k} < \theta$ and

$$(\forall j < n_\mathcal{U})\, (\forall i < n_j)\, s_{j,i} < s'_{j,i} - \theta \;\&\; t'_{j,i} + \theta < t_{j,i}.$$

Then, we have $B' \overline{\lhd} \left(B' \downarrow \mathcal{C}_k^{\le M}\right) \cap \overline{\mathsf{Pos}}$. Let $B'' \in$ RHS. Since $B'' \in \overline{\mathsf{Pos}}$, we have either $B'' \in r\,\mathsf{Pos}$ or $B'' \in \omega$. Since $B' \in \neg\omega$, we have $B'' \in \neg\omega$ as well, so in the latter case, we have a contradiction. Thus $B'' \in r\,\mathsf{Pos}$. Since

$$r^- B'' \lhd r^- \{C'_0, \dots, C'_{n_\mathcal{U}-1}\} \downarrow r^- B'' \lhd r^- \left(\{C'_0, \dots, C'_{n_\mathcal{U}-1}\} \downarrow B''\right),$$

there exists $j < n_\mathcal{U}$ such that $r\,\mathsf{Pos}\left(C'_j \downarrow B''\right)$. Hence $C'_j \between B''$, so that $B'' \le C_j \overline{\lhd}\, \mathcal{U}$ by the choice of θ. Then, $B' \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}$, and therefore $B \lhd_\Pi wc_\Pi(B) \lhd_\Pi \neg\overline{\mathsf{Pos}} \cup \mathcal{U}$. □

Finally, since $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ is enumerably completely regular by Lemma 4.2.8 and $\overline{\mathcal{S}}$ is its subtopology, $\overline{\mathcal{S}}$ is a compact overt enumerably completely regular formal topology.

5.3 Point-free characterisation

We show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space.

The following is a direct consequence of Theorem 4.3.2, Corollary 5.1.11, and Theorem 5.2.7.

Proposition 5.3.1. *For any overt enumerably locally compact regular formal topology $\mathcal{S}$, there exists a locally compact metric space X such that $\mathcal{M}(X) \cong \mathcal{S}$.*

The image of any inhabited formal topology under a formal topology map is inhabited. This follows from (FTM1). Hence, by Corollary 5.1.12, we obtain the following.

Corollary 5.3.2. *For any inhabited enumerably locally compact regular formal topology $\mathcal{S}$, there exists a Bishop locally compact metric space X such that $\mathcal{M}(X) \cong \mathcal{S}$.*

The converse follows from the following lemma.

Lemma 5.3.3. *The localic completion of a Bishop locally compact metric space is isomorphic to an inhabited enumerably locally compact regular formal topology.*

Proof. The proof is almost identical to that Lemma 4.3.1 noting that any Bishop locally compact metric space is separable (Lemma D.0.24.2), and that we have $a <_X b \implies a \ll b$ for any locally compact metric space X (Lemma 3.1.37). □

In summary, we have obtained a point-free characterisation of Bishop locally compact metric spaces.

Theorem 5.3.4. *Let $\mathcal{S}$ be a formal topology. Then, $\mathcal{S}$ is isomorphic to an inhabited enumerably locally compact regular formal topology iff it is isomorphic to the localic completion of some Bishop locally compact metric space.*

Remark 5.3.5. Let **BLCM** be the full subcategory of **LCM** consisting of Bishop locally compact metric spaces, and let **iELKReg** be the full subcategory of **OLKReg** consisting of formal topologies which are isomorphic to some inhabited enumerably locally compact regular formal topologies. Then, Theorem 5.3.4 is equivalent to saying that the localic completion $\mathcal{M} : \mathbf{LCM} \to \mathbf{OLKReg}$ restricts to an essentially surjective functor from **BLCM** to **iELKReg**. Since $\mathcal{M}$ is full and faithful, the categories **BLCM** and **iELKReg** are essentially equivalent (See the footnote 2 of Chapter 4).

Chapter 6

Conclusions

6.1 Summary of results

6.1.1 Point-free characterisations of compact metric spaces and Bishop locally compact metric spaces

As we noted in Section 1.4, the main aim of this thesis is to give point-free characterisations of compact metric spaces and Bishop locally compact metric spaces respectively. We have achieved this aim by the following results.

- In Chapter 4, we showed that the notion of compact overt enumerably completely regular formal topology characterises that of compact metric space (Theorem 4.3.2).
- In Chapter 5, we showed the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space (Theorem 5.3.4).

These are the first topological characterisations of compact metric spaces and Bishop locally compact metric spaces in terms of covering compactness.

As an application of the point-free characterisations, we proved the following result in formal topology.

- Every overt enumerably locally compact regular formal topology has a one-point compactification (Theorem 5.2.7).

By the properties of point-free characterisations, the result gives another construction of a one-point compactification of a Bishop locally compact metric space.

Finally, our characterisations show how formal topology generalises the notions of compact metric space and Bishop locally compact metric space. First, note that the notion of compact overt enumerably completely regular formal topology is essentially equivalent to that of compact overt regular formal topology whose associated function wc has a countable graph (See Remark 4.2.7). Hence, we conclude that the notion of compact overt regular formal topology generalises that of compact metric space by dropping the requirement that the graph of the function wc be countable. Similar conclusion can be drawn for the notion of inhabited locally compact regular formal topology. In both cases, characteristic features of compact metric spaces and Bishop locally compact metric spaces are expressed by countability of bases of way-below relations.

6.1.2 Localic completions and covering completions of uniform spaces

In Chapter 3, we considered two extensions of the notion of localic completion of a metric space to the setting of uniform spaces.

For the class of uniform spaces defined by sets of pseudometrics, we defined the notion of localic completion of a uniform space, and extended most of the results on localic completions of metric spaces obtained in [48] to the setting of uniform spaces. A notable result is Theorem 3.1.56, where we showed that the localic completion preserves countable products of inhabited compact uniform spaces. The corresponding result for metric spaces has not been known.

For the class of uniform spaces defined by covering uniformities, we defined the notion of covering completion of a uniform space. Essentially the same notions have already been given by Fox [28] and Ishihara [32]. Our original result here is Theorem 3.2.42, where we extended the construction of a covering completion to a full and faithful functor from the category of compact uniform spaces to that of compact 2-regular formal topologies.

Finally, in Section 3.3, we showed that the notion of covering completion generalises that of localic completion (Theorem 3.3.6 and Theorem 3.3.9). This result is surprising in that in the previous works on localic completions of metric spaces, the difference between the orders $\leq_X$ and $\preceq_X$ was emphasised [48, 19] (See also Remark 3.1.11). Our result shows that the difference is inessential.

6.2 Further work

6.2.1 Overt closed subtopologies

At various points in this thesis (e.g. Corollary 4.1.19), the notion of located subtopology for locally compact formal topologies played important roles in obtaining the point-free characterisations of compact metric spaces and Bishop locally compact metric spaces. The characterisation of located subtopologies of a locally compact formal topology (Theorem 4.1.15) suggests that the notion of overt closed subtopology merits further study.

However, apart from [53, 19] and this thesis, little seems to be known about connections between overtness and closedness. In particular, just changing the statement of a theorem about closed subtopologies to the corresponding statement about overt closed subtopologies seems to pose significant challenge. For example, Proposition 5.2.9 is classically trivial since to construct such compact subtopology, one just takes the closure of the smaller subtopology and applies Lemma 2.4.19. But, without the countability condition on a base of the way-below relation, we do not know how to construct such a compact overt subtopology mentioned in the proposition.

Another example is the Stone-Čech compactification of a formal topology [34, 23]. The usual construction of a Stone-Čech compactification is to take a closure in a product of $\mathcal{I}[0,1]$ and apply Proposition 2.4.13.1. That construction, however, only yields a compact regular formal topology which is not necessarily overt. Can we construct a compact overt (completely) regular reflection of an overt locally compact regular formal topology?

Similarly, changing a statement about overt weakly closed subtopologies to the corresponding statement about overt closed subtopologies poses an interesting problem. The theory of powerlocales is a prime example [57]. Given an inductively generated formal topology $\mathcal{S}$, we can defined a formal topology $\mathcal{L}(\mathcal{S})$, called the lower powerlocale of $\mathcal{S}$, whose point corresponds to an overt weakly closed subtopology of $\mathcal{S}$ [11]. Since every overt closed subtopology is overt weakly closed by Proposition 2.3.20, it is interesting to see whether we can construct a formal topology which is a subtopology of $\mathcal{L}(\mathcal{S})$, and whose point corresponds to an overt closed subtopology of $\mathcal{S}$, and which satisfies a suitable universal property.

Since we believe that overtness is a meaningful distinction to be kept (cf. Example 2.3.10), we propose a systematic study of overt closed subtopologies as one direction of further research.

6.2.2 Avoiding spatiality

One of the motivations behind this thesis was the desire to obtain a point-free characterisation of compact metric spaces by avoiding the issues relating to spatiality, in particular the Fan theorem (cf. Section 2.5.3). In our case, the solution to this problem is provided by inductively generating a cover relation, i.e. we used the localic completion of a metric space instead of the formal topology determined by the associated concrete space of a metric space[1] (cf. Section 1.2.2).

A similar idea may be useful in other settings where spatiality of formal topologies are assumed. Various sheaf models for constructive (or intuitionistic) formal systems constitute prime examples. Sheaf models for constructive formal systems which satisfy the Fan theorem, or even the monotone bar induction, have long been known [27]. However, the existing constructions of these models relay on the Fan theorem (resp. the monotone bar induction), i.e. the spatiality of the formal Cantor space (resp. the formal Baire space). Hence, these constructions of models are not constructive. It is interesting to see if we can obtain a model of the Fan theorem without assuming the spatiality of the formal Cantor space by inductively generating some of the constructions of the models in [27].

[1] Given a metric space (X, d), its associated concrete space $(X, \Vdash, M_X)$ is given by

$$x \Vdash \mathsf{b}(y, \varepsilon) \overset{\text{def}}{\iff} x \in B(y, \varepsilon),$$

where M_X is the base of the localic completion of X (See Notation 4.0.11).

Appendix A

Constructive Zermelo-Fraenkel Set Theory (CZF)

The axioms of CZF

The set theory CZF is formulated in intuitionistic predicate logic with equality. The language of CZF contains only one binary relation symbol $\in$ as a non-logical symbol. The axioms of CZF are as follows.

Extensionality: $(\forall a)(\forall b)\,[(\forall x)\,(x \in a \leftrightarrow x \in b) \rightarrow a = b]$

Paring: $(\forall a)(\forall b)(\exists y)(\forall u)\,[u \in y \leftrightarrow u = a \vee u = b]$

Union: $(\forall a)(\exists y)(\forall x)\,[x \in y \leftrightarrow (\exists u \in a)\,(x \in u)]$

Restricted Separation:

$$(\forall a)\,(\exists b)\,(\forall x)\,[x \in b \leftrightarrow x \in a \wedge \varphi(x)]$$

for all *restricted* formulae $\varphi(x)$. A formula is restricted if all quantifiers in the formula occur in the forms $\forall x \in a$ or $\exists x \in a$.

Strong Collection:

$$(\forall a)\,[(\forall x \in a)\,(\exists y)\,\varphi(x, y) \rightarrow (\exists b)\,[(\forall x \in a)\,(\exists y \in b)\,\varphi(x, y) \wedge (\forall y \in b)\,(\exists x \in a)\,\varphi(x, y)]]$$

for all formulae $\varphi(x, y)$.

Subset Collection:

$$\begin{aligned}(\forall a)(\forall b)(\exists c)(\forall u)[&(\forall x \in a)\,(\exists y \in b)\,\varphi(x, y, u) \rightarrow \\ &(\exists d \in c)\,[(\forall x \in a)\,(\exists y \in d)\,\varphi(x, y, u) \wedge (\forall y \in d)\,(\exists x \in a)\,\varphi(x, y, u)]\,]\end{aligned}$$

for all formulae $\varphi(x, y, u)$.

Infinity: $(\exists a)\,[(\exists x)\, x \in a \;\&\; (\forall x \in a)\,(\exists y \in a)\, x \in y]$

Set Induction:

$$(\forall a)\,[(\forall x \in a)\,\varphi(x) \to \varphi(a)] \to (\forall a)\,\varphi(a)$$

for all formulae $\varphi(x)$.

In CZF, the following principle of Fullness is valid.

Fullness:

$$(\forall a)\,(\forall b)\,(\exists c)\,[c \subseteq \mathsf{mv}(a,b) \;\&\; (\forall s \in \mathsf{mv}(a,b))\,(\exists r \in c)\, r \subseteq s]$$

where $\mathsf{mv}(a,b)$ denotes the class of total relations from a to b.

Choice principles

The following choice principles are valid in the type-theoretic interpretation of CZF.

Countable Choice: For any formula φ, if

$$(\forall n \in \mathbb{N})\,(\exists x)\,\varphi(n,x),$$

then there exists a function f with domain $\mathbb{N}$ such that

$$(\forall n \in \mathbb{N})\,\varphi(n, f(n)).$$

Dependent Choice: For any set a and formula φ, if

$$(\forall x \in a)\,(\exists y \in a)\,\varphi(x,y),$$

then for any $a_0 \in a$ there exists a function $f : \mathbb{N} \to a$ such that

$$f(0) = a_0 \wedge (\forall n \in \mathbb{N})\,\varphi(f(n), f(n+1)).$$

Relativized Dependent Choice: For any formulae φ and ψ, if

$$(\forall x)\,[\varphi(x) \to (\exists y)\,[\varphi(y) \wedge \psi(x,y)]]\,,$$

then for any set a_0 such that $\varphi(a_0)$ there exists a function f with domain $\mathbb{N}$ such that

$$f(0) = a_0 \wedge (\forall n \in \mathbb{N})\,[\varphi(f(n)) \wedge \psi(f(n), f(n+1))]\,.$$

Regular extension axioms

The *Regular Extension Axiom* was introduced in CZF to accommodate inductive definitions [3].

The Regular Extension Axiom

A set A is *transitive* if $(\forall a \in A)\, a \subseteq A$, and it is *regular* if it is transitive and for any $a \in A$ and $R \in \mathsf{mv}(a, A)$, there exists $b \in A$ such that

$$(\forall x \in a)\,(\exists y \in b)\,(x, y) \in R \wedge (\forall y \in b)\,(\exists x \in a)\,(x, y) \in R.$$

The Regular Extension Axiom (REA) asserts that

REA: Every set is a subset of a regular set.

The Weak Regular Extension Axiom

A transitive inhabited set A is *weakly regular* if for any $a \in A$ and $R \in \mathsf{mv}(a, A)$, there exists $b \in A$ such that

$$(\forall x \in a)\,(\exists y \in b)\,(x, y) \in R.$$

The Weak Regular Extension Axiom (wREA) asserts that

wREA: Every set is a subset of a weakly regular set.

Inductive definitions

An inductive definition is a class Φ of ordered pairs. A class Y is Φ-*closed* if

$$(\forall X)\,(\forall a)\,[(X, a) \in \Phi \rightarrow (X \subseteq Y \rightarrow a \in Y)]\,.$$

Theorem A.0.1. *For any inductive definition Φ, there is a smallest Φ-closed class $I(\Phi)$.*

The class $I(\Phi)$ is called *the class inductively defined by* Φ.
Let Φ be an inductive definition. A class B is a *bound* for Φ if for any $(X, a) \in \Phi$, there is $b \in B$ and a surjection $f : b \to X$. An inductive definition Φ is *bounded* if

- there is a bound for Φ that is a set,
- $\{a \mid (X, a) \in \Phi\}$ is a set for each set X.

Theorem A.0.2 (CZF + wREA)**.** *If Φ is a bounded inductive definition, then the smallest Φ-closed class $I(\Phi)$ is a set.*

In particular, if Φ is a set, then $I(\Phi)$ is a set.

Appendix B

The Tychonoff Theorem for Formal Topologies

We give a proof of the Tychonoff theorem for formal topologies. The proof is based on that of Vickers [59] with some ideas taken from [12]. Coquand [16] gave another proof of the theorem; however, the techniques used in the two proofs are quite similar.

First, we recall the statement of the theorem.

Theorem B.0.3. *Let $(\mathcal{S}_i)_{i \in I}$ be a set-indexed family of inductively generated formal topologies such that $\mathcal{S}_i$ is compact for each $i \in I$. Then, the product $\prod_{i \in I} \mathcal{S}_i$ is compact.*

Let $(\mathcal{S}_i)_{i \in I}$ be a set-indexed family of inductively generated formal topologies, each of the form $\mathcal{S}_i = (S_i, \lhd_i, \leq_i)$, and let (K_i, C_i) be the axiom-set which generates $\mathcal{S}_i$. In the following, we use the same symbols for the base and the axiom-set of the product described in Section 2.2.5.

First, we define a function $(-)^* : \mathsf{Fin}(S_\Pi) \to \mathsf{Fin}(S_\Pi)$ by

$$\emptyset^* = \{\emptyset\} \,,$$
$$(\mathcal{U} \cup \{A\})^* = \left\{ U \cup V \in S_\Pi \mid U \in \mathcal{U}^* \;\&\; V \in \mathsf{Fin}^+(A) \right\}$$

for all $\mathcal{U} \in \mathsf{Fin}(S_\Pi)$ and $A \in S_\Pi$. Note that $(-)^*$ is well-defined on $\mathsf{Fin}(S_\Pi)$ since $\{A_0, A_1\}^* = \{A_1, A_0\}^*$, and $\{B_0, B_1\}^* = \{B\}^*$ whenever $B = B_0 = B_1$. Note also that $\{\emptyset\}^* = \emptyset$, and so $\emptyset \in \mathcal{U}$ implies $\mathcal{U}^* = \emptyset$.

Define a subset θ of $\mathsf{Fin}(S_\Pi)$ by

$$\theta \overset{\text{def}}{=} \{\mathcal{U} \in \mathsf{Fin}(S_\Pi) \mid (\forall A \in \mathcal{U}^*)\, (\exists i \in I)\, (\exists U \in \mathsf{FCov}(\mathcal{S}_i))\, \{i\} \times U \subseteq A\} \,,$$

where for each $i \in I$, $\mathsf{FCov}(\mathcal{S}_i)$ is given by

$$\mathsf{FCov}(\mathcal{S}_i) \overset{\text{def}}{=} \{U \in \mathsf{Fin}(S_i) \mid S_i \lhd_i U\} \,.$$

The following lemma is crucial.

Lemma B.0.4. *For any* $\mathcal{U} \in \mathsf{Fin}(S_\Pi)$*, we have*

$$\mathcal{U} \in \theta \implies S_\Pi \lhd_\Pi \mathcal{U}.$$

Proof. Let $\mathcal{U} \in \theta$. Since $\mathcal{U}^*$ is finitely enumerable, there exists $\mathcal{V} \in \mathsf{Fin}(S_\Pi)$ such that

1. $(\forall C \in \mathcal{V})\,(\exists i \in I)\,(\exists U \in \mathsf{FCov}(\mathcal{S}_i))\, C = \{i\} \times U$,
2. $(\forall B \in \mathcal{U}^*)\,(\exists C \in \mathcal{V})\, C \subseteq B$,
3. $(\forall C \in \mathcal{V})\,(\exists B \in \mathcal{U}^*)\, C \subseteq B$.

We show that

$$(\forall D \in \mathcal{V}^*)\,(\exists A \in \mathcal{U})\, A \subseteq D. \tag{B.1}$$

First, if $C = \emptyset$ for some $C \in \mathcal{V}$, then $\mathcal{V}^* = \emptyset$, so (B.1) trivially holds. Hence, we may assume that each $C \in \mathcal{V}$ is inhabited. Let $D \in \mathcal{V}^*$. We show by induction on $\mathsf{Fin}(S_\Pi)$ that

$$[(\forall B \in \mathcal{U}^*)\, D \between B] \implies (\exists A \in \mathcal{U})\, A \subseteq D. \tag{B.2}$$

If $\mathcal{U} = \emptyset$, then $(\forall B \in \mathcal{U}^*)\, D \between B$ is a contradiction. Let $\mathcal{U} = \mathcal{W} \cup \{A\}$, and suppose that (B.2) holds for $\mathcal{W}$. Suppose that $B \between D$ for all $B \in \mathcal{U}^*$. Let $(i, a) \in A$. Then, for each $B \in \mathcal{W}^*$, since $B \cup \{(i, a)\} \in \mathcal{U}^*$, either $(i, a) \in D$ or $B \between D$. Thus, either $(i, a) \in D$ or $(\forall B \in \mathcal{W}^*)\, B \between D$. In the latter case, there exists $A' \in \mathcal{W}$ such that $A' \subseteq D$. Hence, we obtain

$$A \subseteq D \vee (\exists A' \in \mathcal{W})\, A' \subseteq D,$$

that is, $(\exists A \in \mathcal{U})\, A \subseteq D$. This completes the induction step. Now, for any $B \in \mathcal{U}^*$, there exists $C \in \mathcal{V}$ such that $C \subseteq B$, and since $C \between D$, we have $D \between B$. Thus, there exists $A \in \mathcal{U}$ such that $A \subseteq D$ by (B.2). Thus, we have $\mathcal{V}^* \lhd_\Pi \mathcal{U}$.

It suffices to show that $S_\Pi \lhd_\Pi \mathcal{V}^*$. We show by induction on $\mathsf{Fin}(S_\Pi)$ that

$$[(\forall A \in \mathcal{V})\,(\exists i \in I)\,(\exists U \in \mathsf{FCov}(\mathcal{S}_i))\, A = \{i\} \times U] \implies S_\Pi \lhd_\Pi \mathcal{V}^*. \tag{B.3}$$

If $\mathcal{V} = \emptyset$, then $\mathcal{V}^* = \{\emptyset\}$, so the conclusion is obvious. Let $\mathcal{V} = \mathcal{W} \cup \{C\}$, and suppose that $\mathcal{W}$ satisfies (B.3). Suppose further that $\mathcal{V}$ satisfies the antecedent of (B.3). Then, C is of the form $\{i\} \times U$ for some $i \in I$ and $U \in \mathsf{FCov}(\mathcal{S}_i)$. By Lemma 2.2.24 and (S1), we have $S_\Pi \lhd_\Pi \{\{(i, a)\} \mid a \in U\}$. Since $S_\Pi \lhd_\Pi \mathcal{W}^*$, we have

$$S_\Pi \lhd_\Pi \{D \cup \{(i, a)\} \mid D \in \mathcal{W}^* \ \&\ a \in U\} \subseteq \mathcal{V}^*,$$

as required. ∎

The rest of the proof is similar to that of Theorem 3.2.36. To simplify the proof, we first prove the following technical lemma.

Lemma B.0.5. *Let $i \in I$, $V \in \mathsf{Fin}(S_i)$, and $U \subseteq S_i$ such that for any $W \in \mathsf{Fin}(S_i)$*

$$[(\forall a \in V)\, S_i \lhd_i \{a\} \cup W] \implies (\exists U_0 \in \mathsf{Fin}(U))\, S_i \lhd_i U_0 \cup W.$$

Then, for any $A \in S_\Pi$ and $\mathcal{V} \in \mathsf{Fin}(S_\Pi)$,

$$\{\{(i,a) \mid a \in V\} \cup A\} \cup \mathcal{V} \in \theta \implies (\exists U_0 \in \mathsf{Fin}(U))\, \{\{(i,b)\} \cup A \mid b \in U_0\} \cup \mathcal{V} \in \theta.$$

Proof. Let $A \in S_\Pi$ and $\mathcal{V} \in \mathsf{Fin}(S_\Pi)$, and suppose that

$$\{\{(i,a) \mid a \in V\} \cup A\} \cup \mathcal{V} \in \theta.$$

Let $B \in \mathcal{V}^*$. Then, for each $a \in V$, we have either

1. $(\exists j \in I)\, (\exists U_j \in \mathsf{FCov}(\mathcal{S}_j))\, \{j\} \times U_j \subseteq B$, or
2. $(\exists W_a \in \mathsf{Fin}(S_i))\, S_i \lhd_i \{a\} \cup W_a \;\&\; \{i\} \times W_a \subseteq B$.

Hence, either

1. $(\exists j \in I)\, (\exists U_j \in \mathsf{FCov}(\mathcal{S}_j))\, \{j\} \times U_j \subseteq B$, or
2. $(\forall a \in V)\, (\exists W_a \in \mathsf{Fin}(S_i))\, S_i \lhd_i \{a\} \cup W_a \;\&\; \{i\} \times W_a \subseteq B$.

In latter case, (by letting $W_B = \bigcup_{a \in V} W_a$) we have

$$(\exists W_B \in \mathsf{Fin}(S_i))\, (\forall a \in V)\, S_i \lhd_i \{a\} \cup W_B \;\&\; \{i\} \times W_B \subseteq B,$$

and hence by the assumption

$$(\exists W_B \in \mathsf{Fin}(S_i))\, (\exists U_B \in \mathsf{Fin}(U))\, S_i \lhd_i U_B \cup W_B \;\&\; \{i\} \times W_B \subseteq B.$$

Since $\mathcal{V}^*$ is finitely enumerable, we can split $\mathcal{V}^*$ into finitely enumerable subsets $\mathcal{V}^*_-$ and $\mathcal{V}^*_+$ such that $\mathcal{V}^* = \mathcal{V}^*_- \cup \mathcal{V}^*_+$ and

- $B \in \mathcal{V}^*_- \implies (\exists j \in I)\, (\exists U_j \in \mathsf{FCov}(\mathcal{S}_j))\, \{j\} \times U_j \subseteq B$,
- $B \in \mathcal{V}^*_+ \implies (\exists W_B \in \mathsf{Fin}(S_i))\, (\exists U_B \in \mathsf{Fin}(U))\, S_i \lhd_i U_B \cup W_B \;\&\; \{i\} \times W_B \subseteq B$.

Write $\mathcal{V}^*_+ = \{B_0, \ldots, B_{n-1}\}$, and let $\{(U_k, W_k) \in \mathsf{Fin}(U) \times \mathsf{Fin}(S_i) \mid k < n\}$ be a set such that $S_i \lhd_i U_k \cup W_k \;\&\; \{i\} \times W_k \subseteq B_k$ for each $k < n$. Let $U_* = \bigcup_{k<n} U_k$. We show that

$$\{\{(i,a)\} \cup A \mid a \in U_*\} \cup \mathcal{V} \in \theta.$$

Let $D \in (\{\{(i,a)\} \cup A \mid a \in U_*\} \cup \mathcal{V})^*$. Then, D is of the form $D = B \cup C$ for some $B \in \mathcal{V}^*$ and $C \in \{\{(i,a)\} \cup A \mid a \in U_*\}^*$. Then, either $C \between A$ or $C = \{(i,a) \mid a \in U_*\}$. In the former case, since $\{A\} \cup \mathcal{V} \in \theta$, there exist $j \in I$ and $U_j \in \mathsf{FCov}(\mathcal{S}_j)$ such that $\{j\} \times U_j \subseteq B \cup C$. In the latter case, we have either $B \in \mathcal{V}^*_-$ or $B \in \mathcal{V}^*_+$. If $B \in \mathcal{V}^*_-$, then the conclusion is immediate. If $B \in \mathcal{V}^*_+$, then there exists $k < n$ such that $S_i \lhd_i U_k \cup W_k \;\&\; \{i\} \times W_k \subseteq B$. Since $U_k \subseteq U_*$, we have $\{i\} \times (U_k \cup W_k) \subseteq D$. Therefore, $\{\{(i,a)\} \cup A \mid a \in U_*\} \cup \mathcal{V} \in \theta$. □

Now, define a predicate $\Phi_{\mathcal{U}}$ on $\mathsf{Fin}(S_\Pi)$ with a parameter $\mathcal{U} \subseteq S_\Pi$ by

$$\Phi_{\mathcal{U}}(\mathcal{W}) \overset{\text{def}}{\iff} (\forall \mathcal{V} \in \mathsf{Fin}(S_\Pi)) \left[\mathcal{V} \cup \mathcal{W} \in \theta \to (\exists \mathcal{U}_0 \in \mathsf{Fin}(\mathcal{U}) \mathcal{V} \cup \mathcal{U}_0 \in \theta)\right]. \tag{B.4}$$

Then, by induction on $\mathsf{Fin}(S_\Pi)$, we can show that

$$[(\forall A \in \mathcal{W})\, \Phi_{\mathcal{U}}(\{A\})] \implies \Phi_{\mathcal{U}}(\mathcal{W}) \tag{B.5}$$

for all $\mathcal{W} \in \mathsf{Fin}(S_\Pi)$. Define a predicate Ψ on S_Π by $\Psi(A) \overset{\text{def}}{\iff} \Phi_{\mathcal{U}}(\{A\})$. We show that

$$A \lhd_\Pi \mathcal{U} \implies \Psi(A)$$

for all $A \in S_\Pi$ by induction on $\lhd_\Pi$.

(ID1): Let $A \in \mathcal{U}$. Then, $\Psi(A)$ holds by letting $\mathcal{U}_0 = \{A\}$ in (B.4).

(ID2): Let $A, A' \in S_\Pi$, and suppose that $A' \leq_\Pi A$ and $\Psi(A)$. Let $\mathcal{V} \in \mathsf{Fin}(S_\Pi)$ such that $\mathcal{V} \cup \{A'\} \in \theta$. Since $A' \leq_\Pi A$, we have

$$(\forall B \in (\mathcal{V} \cup \{A\})^*) \left(\exists B' \in (\mathcal{V} \cup \{A'\})^*\right) B' \leq_\Pi B.$$

Thus, we have $\mathcal{V} \cup \{A\} \in \theta$, and hence there exists $\mathcal{U}_0 \in \mathsf{Fin}(\mathcal{U})$ such that $\mathcal{V} \cup \mathcal{U}_0 \in \theta$. Therefore $\Psi(A')$.

(ID3): We check the localised axioms of $\prod_{i \in I} \mathcal{S}_i$, namely (S1') – (S3').

(S1'): Let $A \in S_\Pi$ and $i \in I$, and suppose that $\Psi(A \cup \{(i,a)\})$ for all $a \in S_i$. We must show that $\Psi(A)$. Let $\mathcal{V} \in \mathsf{Fin}(S_\Pi)$, and suppose that $\mathcal{V} \cup \{A\} \in \theta$. Put $V = \emptyset$ and $U = S_i$ in Lemma B.0.5. Since $\mathcal{S}_i$ is compact, the assumption of the lemma is fulfilled. Hence, there exists $U_0 \in \mathsf{Fin}(S_i)$ such that $\{\{(i,a)\} \cup A \mid a \in U_0\} \cup \mathcal{V} \in \theta$. Since $\Psi(A \cup \{(i,a)\})$ for each $a \in U_0$, we have $\Phi_{\mathcal{U}}(\{\{(i,a)\} \cup A \mid a \in U_0\})$ by (B.5). Hence, there exists $\mathcal{U}_0 \in \mathsf{Fin}(\mathcal{U})$ such that $\mathcal{V} \cup \mathcal{U}_0 \in \theta$. Therefore $\Psi(A)$.

(S2'): Put $V = \{a, b\}$ and $U = a' \downarrow b'$ for $i \in I$, $a \leq_i a'$ and $b \leq_i b'$ in Lemma B.0.5. Then, the proof proceeds as in the case (S1').

(S3'): Put $V = \{a\}$ and $U = C_i(a', k)$ for $i \in I$, $a \leq_i a'$ and $k \in K_i(a')$ in Lemma B.0.5. This completes the inductive proof.

The compactness of $\prod_{i \in I} \mathcal{S}_i$ now follows. Indeed, let $\mathcal{U} \subseteq S_\Pi$ such that $S_\Pi \lhd_\Pi \mathcal{U}$. Then, $\emptyset \lhd_\Pi \mathcal{U}$, and thus $\Phi_{\mathcal{U}}(\{\emptyset\})$. Since $\{\emptyset\}^* = \emptyset$, we have $\{\emptyset\} \in \theta$. Thus, there exists $\mathcal{U}_0 \in \mathsf{Fin}(\mathcal{U})$ such that $\mathcal{U}_0 \in \theta$. Hence, $S_\Pi \lhd_\Pi \mathcal{U}_0$ by lemma B.0.4. Therefore $\prod_{i \in I} \mathcal{S}_i$ is compact.

Appendix C

Open Subspaces of Locally Compact Metric Spaces

In this appendix, we show that the functor $\mathcal{OM} : \mathbf{OLCM} \to \mathbf{FTop}$ defined in Section 5.1 is full and faithful. The proof is based on that of Palmgren [49]. See Section 5.1 for the definition of the category **OLCM** of open complements of locally compact metric spaces. For notations for localic completions of metric spaces, see Section 3.1.2 and Section 3.1.27.

The first step of the proof is to obtain another characterisation of morphisms of **OLCM** (Lemma C.0.7). To this end, we give another characterisation of the relation $\Subset$ (See (5.1) for the definition of $\Subset$).

Let X be a metric space, and let M_X be the base of the localic completion $\mathcal{M}(X)$. For subsets $A \subseteq X$ and $W \subseteq M_X$, define

$$A <_* W \overset{\text{def}}{\iff} (\exists F \in \mathsf{Fin}(M_X))\, A \subseteq F_* \;\&\; F <_X W.$$

Lemma C.0.6 (Palmgren [49, Corollary 2.4])**.** *Let X be a metric space. Let A be an inhabited totally bounded subset of X and let U be an open subset of X. Then,*

$$A \Subset U \iff A <_* H(U).$$

Proof. ($\Rightarrow$): Suppose that $A \Subset U$. Then, there exists $\varepsilon \in \mathbb{Q}^{>0}$ such that $A_{2\varepsilon} \subseteq U$. Let $\{x_0, \ldots, x_{n-1}\}$ be an ε-net to A. For each $i < n$, since $B(x_i, 2\varepsilon) \subseteq A_{2\varepsilon} \subseteq U$, we have $\mathsf{b}(x_i, 2\varepsilon) \in H(U)$. Let $F = \{\mathsf{b}(x_i, \varepsilon) \mid i < n\}$. Then, we have $A \subseteq F_*$ and $F <_X H(U)$, and thus $A <_* H(U)$.

($\Leftarrow$): Suppose that $A <_* H(U)$. Then, there exists $F \in \mathsf{Fin}(M_X)$ such that $A \subseteq F_*$ and $F <_X H(U)$. Write $F = \{\mathsf{b}(x_0, \varepsilon_0), \ldots, \mathsf{b}(x_{n-1}, \varepsilon_{n-1})\}$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $F' = \{\mathsf{b}(x_0, \varepsilon_0 + 2\theta), \ldots, \mathsf{b}(x_{n-1}, \varepsilon_{n-1} + 2\theta)\} <_X H(U)$. Let $x \in A_\theta$. Then, there exists $y \in A$ such that $d(x, y) < 2\theta$. Let $i < n$ such that $y \in B(x_i, \varepsilon_i)$. Then, $d(x, x_i) < \varepsilon_i + 2\theta$. Since $B(x_i, \varepsilon_i + 2\theta) \subseteq U$, we have $x \in U$. Hence $A_\theta \subseteq U$. Therefore $A \Subset U$. ☐

Lemma C.0.7. *Let $(X, U), (Y, V)$ be objects of* **OLCM***. Then, a function $f : U \to V$ is a morphism of* **OLCM** *iff for each $F \in \mathsf{Fin}^+(M_X)$ such that $F <_X H(U)$ we have*

1. *$f : F_* \to V$ is uniformly continuous,*

2. $f[F_*] <_* H(V)$.

Proof. (⇒): Suppose that f is a morphism of **OLCM**. Let $F \in \mathsf{Fin}^+(M_X)$ such that $F <_X H(U)$. Then, there exists $F' \in \mathsf{Fin}^+(M_X)$ such that $F <_X F' <_X H(U)$. Since F is finitely enumerable, there exists an inhabited totally bounded subset $A \subseteq X$ such that $F_* \subseteq A \subseteq F'_*$ by Lemma 5.1.9. Then, we have $A \Subset U$ by Lemma C.0.6. Let K be the closure of A. Then, we have $K \Subset U$ by Lemma D.0.17. Since f is uniformly continuous on K, f is uniformly continuous on F_*. Moreover, since $f[K] \Subset V$, we have $f[K] <_* H(V)$ by Lemma C.0.6. Hence $f[F_*] <_* H(V)$.

(⇐): Suppose that f satisfies the conditions stated in the lemma. Let K be an inhabited compact subset of X, and suppose that $K \Subset U$. By Lemma C.0.6, there exists $F \in \mathsf{Fin}(M_X)$ such that $K \subseteq F_*$ and $F <_X H(U)$. Since f is uniformly continuous on F_*, f is uniformly continuous on K. Moreover, since $f[K] \subseteq f[F_*] <_* H(V)$, we have $f[K] \Subset V$ by Lemma C.0.6. □

With the above characterisation of morphisms of **OLCM**, we prove that the mapping $\mathcal{OM}$ is a full and faithful functor. We recall the definition of $\mathcal{OM}$:

$$\mathcal{OM}((X,U)) \overset{\text{def}}{=} \mathcal{M}(X)_{H(U)},$$
$$\mathcal{OM}(f) \overset{\text{def}}{=} r_f$$

for each object (X,U) and each morphism $f : (X,U) \to (Y,V)$ of **OLCM**, where $\mathcal{M}(X)_{H(U)}$ is the open subtopology of $\mathcal{M}(X)$ determined by $H(U)$, and the relation $r_f \subseteq M_X \times M_Y$ is given by

$$a \; r_f \; b \overset{\text{def}}{\iff} (\forall a' <_X a \downarrow H(U)) \, (\exists b' <_Y b \downarrow H(V)) \, f[a'_*] \subseteq b'_*$$

for all $a \in M_X$ and $b \in M_Y$.

In the following, for each object (X,U) of **OLCM**, we denote the cover of $\mathcal{M}(X)$ and $\mathcal{M}(X)_{H(U)}$ by $\lhd_X$ and $\lhd_{H(U)}$ respectively.

Lemma C.0.8. *For any morphism $f : (X,U) \to (Y,V)$ of* **OLCM***, the relation r_f is a formal topology map from $\mathcal{M}(X)_{H(U)}$ to $\mathcal{M}(Y)_{H(V)}$.*

Proof. We check the conditions (FTMi1) – (FTMi4).

(FTMi1): We show that for any $\varepsilon \in \mathbb{Q}^{>0}$

$$H(U) \lhd_X r_f^- \mathcal{C}_\varepsilon.$$

This suffices to prove (FTMi1). By (M1), we have

$$H(U) \lhd_X \{a \in M_X \mid a <_X H(U)\}.$$

Let $a \in M_X$ such that $a <_X H(U)$. By Lemma C.0.7, we have

$$f[a_*] <_* H(V).$$

Thus, there exists $W = \{\mathsf{b}(y_i, \delta_i), \dots, \mathsf{b}(y_{n-1}, \delta_{n-1})\} \in \mathsf{Fin}(M_Y)$ such that $f[a_*] \subseteq W_*$ and $W <_Y H(V)$. Choose $\gamma \in \mathbb{Q}^{>0}$ such that $2\gamma < \varepsilon$ and

$$\{\mathsf{b}(y_i, \delta_i + 2\gamma) \in M_Y \mid i < n\} <_Y H(V).$$

Since f is uniformly continuous on a_*, there exists $\theta \in \mathbb{Q}^{>0}$ such that

$$(\forall x, x' \in a_*)\, d(x, x') < \theta \implies d(f(x), f(x')) < \gamma.$$

Let $a' \in a \downarrow \mathcal{C}_\theta$, and write $a' = \mathsf{b}(x, \xi)$. Then, there exists $i < n$ such that $d(f(x), y_i) < \delta_i$. Thus, $d(f(x), y_i) + 2\gamma < \delta_i + 2\gamma$, so that

$$\mathsf{b}(f(x), \gamma) <_Y \mathsf{b}(f(x), 2\gamma) <_Y \mathsf{b}(y_i, \delta_i + 2\gamma).$$

Hence, $a' \in r_f^- \mathcal{C}_\varepsilon$. Therefore $H(U) \lhd_X r_f^- \mathcal{C}_\varepsilon$.

(FTMi2): Let $b, c \in M_Y$ and let $a \in r_f^- b \downarrow r_f^- c$. Let $a' \in M_X$ such that $a' <_X a \downarrow H(U)$. Then, there exist $b' <_Y b \downarrow H(V)$ and $c' <_Y c \downarrow H(V)$ such that $f[a'_*] \subseteq b'_* \cap c'_*$. Write $b' = \mathsf{b}(y, \delta)$ and $c' = \mathsf{b}(z, \gamma)$. Choose $\xi \in \mathbb{Q}^{>0}$ such that

$$\mathsf{b}(y, \delta + 2\xi) <_Y b \downarrow H(V),$$
$$\mathsf{b}(z, \gamma + 2\xi) <_Y c \downarrow H(V).$$

Since f is uniformly continuous on a'_*, there exists $\theta \in \mathbb{Q}^{>0}$ such that

$$(\forall x, x' \in a'_*)\, d(x, x') < \theta \implies d(f(x), f(x')) < \xi.$$

Let $\mathsf{b}(x, \theta') \in a' \downarrow \mathcal{C}_\theta$. Then $\mathsf{b}(f(x), 2\xi) <_Y \mathsf{b}(y, \delta + 2\xi)$ and $\mathsf{b}(f(x), 2\xi) <_Y \mathsf{b}(z, \gamma + 2\xi)$. Thus, $\mathsf{b}(x, \theta') \in r_f^- (b \downarrow c)$. Therefore

$$a \downarrow H(U) \lhd_X r_f^- (b \downarrow c)$$

by (M1) and (M2), from which (FTMi2) follows.

(FTMi3): We must check each axiom of $\mathcal{M}(Y)_{H(V)}$. For (M1), let $b \in M_Y$ and $a \in r_f^- b$. Let $a' \in M_X$ such that $a' <_X a \downarrow H(U)$. Then, there exists $b' <_Y b \downarrow H(V)$ such that $f[a'_*] \subseteq b'_*$. Choose $b'' \in M_Y$ such that $b' <_Y b'' <_Y b$. Then, $a' \; r_f \; b''$, and hence $r_f^- b \lhd_{H(U)} r_f^- \{b' \in M_Y \mid b' <_Y b\}$. The proofs for (M2) and the axiom of the open subtopology $\mathcal{M}(Y)_{H(V)}$ are already contained in the proof of (FTMi1).

(FTMi4): Obvious from the definition of r_f. □

Let (X, U) be an object of **OLCM**. Since X is complete, the embedding $i_X : X \to \mathcal{P}t(\mathcal{M}(X))$ given by (3.6) is a metric isomorphism. Then, i_X restricts to a metric isomorphism between U and $\mathcal{P}t(\mathcal{M}(X)_{H(U)})$, which we denote by $i_{X,U} : U \to \mathcal{P}t(\mathcal{M}(X)_{H(U)})$. Indeed, since U is an open subset of X, we have for any $x \in X$,

$$\begin{aligned} x \in U &\iff (\exists a \in M_X)\, x \in a_* \subseteq U \\ &\iff i_X(x) \between H(U) \\ &\iff i_X(x) \in \mathcal{P}t(\mathcal{M}(X)_{H(U)}). \end{aligned}$$

The relation $\sqsubseteq$ on the base U_X of the localic completion of a uniform space has been introduced in (3.9). For a metric space X, we have

$$U \sqsubseteq V \iff \left(\exists \varepsilon \in \mathbb{Q}^{>0}\right) U \downarrow \mathcal{C}_\varepsilon \leq_X V$$

for any $U, V \subseteq M_X$.

Lemma C.0.9. *Let $(X, U), (Y, V)$ be objects of* **OLCM**, *and let $r : \mathcal{M}(X)_{H(U)} \to \mathcal{M}(Y)_{H(V)}$ be a formal topology map. Then, the composition*

$$f = {i_{Y,V}}^{-1} \circ \mathcal{P}t(r) \circ i_{X,U}$$

is a morphism $f : (X, U) \to (Y, V)$ of **OLCM**.

Proof. We use Lemma C.0.7. Let $A \in \mathsf{Fin}^+(M_X)$ such that $A <_X H(U)$.

First, we show that f is uniformly continuous on A_*. Let $\varepsilon \in \mathbb{Q}^{>0}$. By (FTM1) and the axioms of $\mathcal{M}(Y)_{H(V)}$, we have

$$H(U) \lhd_X r^- \left\{b \in M_Y \mid b <_Y \mathcal{C}_{\varepsilon/2} \downarrow H(V)\right\}.$$

Write $A = \{\mathsf{b}(x_0, \delta_0), \dots, \mathsf{b}(x_{n-1}, \delta_{n-1})\}$, and choose $\gamma \in \mathbb{Q}^{>0}$ such that

$$\{\mathsf{b}(x_0, \delta_0 + \gamma), \dots, \mathsf{b}(x_{n-1}, \delta_{n-1} + \gamma)\} <_X H(U).$$

By Lemma 3.1.35, for each $i < n$, there exists $W_i \in \mathsf{Fin}(M_X)$ such that $\mathsf{b}(x_i, \delta_i + \gamma) \sqsubseteq W_i$ and $W_i \subseteq r^- \left\{b \in M_Y \mid b <_Y \mathcal{C}_{\varepsilon/2} \downarrow H(V)\right\}$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $\theta < \gamma$ and $\mathsf{b}(x_i, \delta_i + \gamma) \downarrow \mathcal{C}_\theta \leq_X W_i$ for each $i < n$. Let $x, x' \in A_*$, and suppose that $d(x, x') < \theta$. Then, there exists $i < n$ such that $d(x, x_i) < \delta_i$. Thus, $d(x, x_i) + \theta < \delta_i + \gamma$ so that $\mathsf{b}(x, \theta) \leq_X W_i$. Hence, there exists $b \in M_Y$ such that $\mathsf{b}(x, \theta) \lhd_X r^- b$ and $b <_Y \mathcal{C}_{\varepsilon/2} \downarrow H(V)$, from which we have

$$f[\mathsf{b}(x, \theta)_*] \subseteq b_*.$$

Thus, $d(f(x), f(x')) < \varepsilon$. Therefore, f is uniformly continuous on A_*.

The condition $f[A_*] <_* H(V)$ follows from the fact that $A \lhd_X \bigcup_{i<n} W_i$ and $\bigcup_{i<n} W_i$ is finitely enumerable. □

Lemma C.0.10. *For any morphism $f : (X, U) \to (Y, V)$ of* **OLCM**, *the following diagram commutes.*

$$\begin{array}{ccc} U & \xrightarrow{\;i_{X,U}\;} & \mathcal{P}t(\mathcal{M}(X)_{H(U)}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}t(r_f)} \\ V & \xleftarrow[{i_{Y,V}}^{-1}]{} & \mathcal{P}t(\mathcal{M}(Y)_{H(V)}) \end{array} \tag{C.1}$$

Proof. Let $x \in U$. We must show that

$$\Diamond f(x) = \mathcal{P}t(r_f)(\Diamond x).$$

Since $\mathcal{M}(Y)_{H(V)}$ is regular by Proposition 2.4.11.1, it suffices to show that $\mathcal{P}t(r_f)(\Diamond x) \subseteq \Diamond f(x)$ by Corollary 2.4.10. Let $b \in \mathcal{P}t(r_f)(\Diamond x)$. Then, there exists $a \in M_X$ such that $x \in a_*$ and $a \; r_f \; b$. Since U is open, there exists $a' \in M_X$ such that $x \in a'$ and $a' <_X a \downarrow H(U)$. Thus, there exists $b' <_Y b \downarrow H(V)$ such that $f[a'_*] \subseteq b'_*$. Hence $f(x) \in b_*$, that is $b \in \Diamond f(x)$. □

Lemma C.0.11. *Let $(X,U),(Y,V)$ be objects of* **OLCM**. *Then, for any formal topology map $r : \mathcal{M}(X)_{H(U)} \to \mathcal{M}(Y)_{H(V)}$, we have $r_f = r$, where $f \stackrel{\text{def}}{=} {i_{Y,V}}^{-1} \circ \mathcal{P}t(r) \circ i_{X,U}$.*

Proof. Since $\mathcal{M}(Y)_{H(V)}$ is regular, it suffices to show that $r \leq r_f$ by Proposition 2.4.9. Let $a \in M_X$ and $b \in M_Y$, and suppose that $a \; r \; b$. Since $b \lhd_{H(V)} \{b' \in M_Y \mid b' <_Y H(V) \downarrow b\}$, we have

$$a \lhd_{H(U)} r^{-}\{b' \in M_Y \mid b' <_Y H(V) \downarrow b\} \downarrow H(U).$$

Let $a' \in$ RHS. Then, there exists $b' <_Y H(V) \downarrow b$ such that $a' \lhd_{H(U)} r^{-} b'$, and thus $f[a'_*] \subseteq b'_*$. Hence $a' \; r_f \; b$, and so $a \lhd_{H(U)} r_f^{-} b$. Therefore $r \leq r_f$. □

Lemma C.0.12. *Let $f : (X,U) \to (Y,V)$ and $g : (Y,V) \to (Z,W)$ be morphisms of* **OLCM**. *Then, we have*

$$r_{g \circ f} = r_g \circ r_f.$$

Proof. See the proof of Lemma 3.1.46. □

Similarly, we have $r_{id_U} = id_{\mathcal{M}(X)_{H(U)}}$ for any object (X,U) of **OLCM**. In summary, we have the following.

Theorem C.0.13. *The mapping $\mathcal{OM}$ is a full and faithful functor from* **OLCM** *to* **FTop**.

Proof. By Lemma C.0.12, $\mathcal{OM}$ is functorial. By Lemma C.0.10, $\mathcal{OM}$ is faithful, and by Lemma C.0.9 and Lemma C.0.11, $\mathcal{OM}$ is full. □

Appendix D

Metric Spaces

In this appendix, we give some background on the theory of metric spaces in Bishop constructive mathematics. See [8, 9] for further details.

Except for the notion of located subset, all notions for metric spaces used in this thesis are special cases of the corresponding notions for uniform spaces defined by sets of pseudometrics (See Section 3.1). However, we will make explicit the connection between countable products of uniform spaces and those of metric spaces.

Notation D.0.14. In the following, unless explicitly mentioned, a metric space is identified with its underlying set, and it is assumed to have a metric named $d(-,-)$.

First, we repeat the definition of located subset (cf. Definition 4.1.5).

Definition D.0.15. A subset Y of a metric space X is *located* if for each $x \in X$ the distance

$$d(x, Y) \stackrel{\text{def}}{=} \inf \{d(x, y) \mid y \in Y\}$$

exists as a non-negative Dedekind real number, i.e. for each $x \in X$, the set

$$U_x = \left\{q \in \mathbb{Q}^{>0} \mid (\exists y \in Y)\, d(x, y) < q\right\}$$

satisfies

1. $(\exists q \in \mathbb{Q}^{>0})\, q \in U_x$,
2. $(\forall p, q \in \mathbb{Q}^{>0})\, p < q \implies p \in \neg U_x \vee q \in U_x$.

Classically, every inhabited subset of a metric space is located. Constructively, however, we cannot show that every inhabited subset is located.

Example D.0.16. Let $\alpha : \mathbb{N} \to \{0, 1\}$ be a binary sequence. Define a subset A of the reals $\mathbb{R}$ by

$$A = \{x \in \mathbb{R} \mid (\exists n \in \mathbb{N})\, \alpha(n) < x\}.$$

If A is located, then by putting $x = 1/2$ in the definition of located subset above, we have either $1/4 \notin U_{1/2}$ or $1/2 \in U_{1/2}$, and hence

$$(\forall n \in \mathbb{N})\, \alpha(n) = 1 \vee (\exists n \in \mathbb{N})\, \alpha(n) = 0.$$

Thus, if every inhabited subset of $\mathbb{R}$ were located, then the Limited Principle of Omniscience (LPO) would result. LPO is known to contradict Church's Thesis [10].

Importance of located subsets comes from Proposition D.0.23 and Proposition D.0.28. First, we note the following.

Lemma D.0.17. *A subset Y of a metric space X is located iff the closure of Y is located.*

Proof. Let $\mathsf{cl}(Y)$ be the closure of Y in X. For each $x \in X$, we have

$$\left\{q \in \mathbb{Q}^{>0} \mid (\exists y \in Y)\, d(x,y) < q\right\} = \left\{q \in \mathbb{Q}^{>0} \mid (\exists y \in \mathsf{cl}(Y))\, d(x,y) < q\right\}.$$

Hence, Y is located iff $\mathsf{cl}(Y)$ is located. □

Next, we recall the connection between compactness and locatedness. In the following, we make liberal use of Lemma 4.1.6.

Lemma D.0.18. *A located subset of a totally bounded metric space is totally bounded.*

Proof. Let X be a totally bounded metric space, and let $Y \subseteq X$ be a located subset. Let $\varepsilon \in \mathbb{Q}^{>0}$. Choose $\delta \in \mathbb{Q}^{>0}$ such that $3\delta < \varepsilon$. Let $X_0 = \{x_0, \dots, x_{n-1}\}$ be a δ-net to X. Since Y is located, we have either $B(x_i, \delta) \subseteq \neg Y$ or $B(x_i, 2\delta) \between Y$ for each $i < n$. Split X_0 into finitely enumerable subsets X_0^+ and X_0^- such that $X_0 = X_0^+ \cup X_0^-$ and

- $x \in X_0^+ \implies B(x, 2\delta) \between Y$,
- $x \in X_0^- \implies B(x, \delta) \subseteq \neg Y$.

Write $X_0^+ = \{z_0, \dots, z_{m-1}\}$. For each $i < m$, choose $y_i \in Y$ such that $d(z_i, y_i) < 2\delta$, and put $Y_0 = \{y_0, \dots, y_{m-1}\}$. We show that Y_0 is an ε-net to Y. Let $y \in Y$. Then, there exists $x \in X_0$ such that $d(x,y) < \delta$. Then, $x \in X_0^+$, and thus there exists $i < m$ such that $d(x, y_i) < 2\delta$. Hence, $d(y, y_i) < 3\delta < \varepsilon$. Therefore Y_0 is an ε-net to Y, so Y is totally bounded. □

Lemma D.0.19. *An inhabited totally bounded subset of a metric space is located.*

Proof. Let X be a metric space, and let $Y \subseteq X$ be an inhabited totally bounded subset. Let $x \in X$, and let $p, q \in \mathbb{Q}^{>0}$ such that $p < q$. Choose $r \in \mathbb{Q}^{>0}$ such that $p + r < q$, and let $\{y_0, \dots, y_{n-1}\}$ be an r-net to Y. Then, either

1. $(\exists i < n)\, d(x, y_i) < q$, or
2. $(\forall i < n)\, d(x, y_i) > p + r$.

In the former case, we have $B(x, q) \between Y$. In the latter case, if $B(x, p) \between Y$, then there exist $y \in Y$ and $i < n$ such that $d(x, y) < p$ and $d(y, y_i) < r$. Hence, $d(x, y_i) < p + r$, a contradiction. Thus, $B(x, p) \subseteq \neg Y$. Therefore, Y is located. □

Completeness of a metric space is usually defined in terms of Cauchy sequences. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a metric space X is a *Cauchy sequence* if

$$(\forall k \in \mathbb{N})\,(\exists N_k \in \mathbb{N})\,(\forall m, n \geq N_k)\, d(x_m, x_n) < 2^{-k}.$$

A metric space X is *complete* if every Cauchy sequence converges.

It is well-known that assuming the Countable Choice, the notion of completeness for metric spaces is compatible with the corresponding notion for uniform spaces. The definition of completeness for uniform spaces is given in Definition 3.1.29.

Proposition D.0.20. *Let $X = (X, d)$ be a metric space. Then, X is complete iff $(X, \{d\})$ is complete as a uniform space.*

Proof. Suppose that every Cauchy sequence converges. Let $\mathcal{F}$ be a Cauchy filter on X. Then,

$$(\forall k \in \mathbb{N})\,(\exists U \in \mathcal{F})\,(\exists x \in X)\, U \subseteq B(x, 2^{-k}).$$

By the Countable Choice, there exist sequences $(U_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ and X, respectively, such that

$$(\forall k \in \mathbb{N})\, U_k \subseteq B(x_k, 2^{-k}).$$

Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and hence it converges to some $x \in X$. Given any $k \in \mathbb{N}$, let $N \in \mathbb{N}$ such that $N > k$ and $d(x, x_m) < 2^{-(k+1)}$ for all $m \geq N$. Then, $U_N \subseteq B(x_N, 2^{-N}) \subseteq B(x, 2^{-k})$, and thus $\mathcal{F}$ converges to x.

Conversely, suppose that every Cauchy filter on X converges. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then, the set $\mathcal{F}$ given by

$$\begin{aligned} \mathcal{F} &\stackrel{\text{def}}{=} \{U_n \mid n \in \mathbb{N}\}, \\ U_n &\stackrel{\text{def}}{=} \{x_k \in X \mid k \geq n\} && (n \in \mathbb{N}) \end{aligned}$$

is a Cauchy filter on X, and thus $\mathcal{F}$ converges to some $x \in X$. Then, clearly $(x_n)_{n\in\mathbb{N}}$ converges to x as well. □

Some well-known facts about complete metric spaces carry over to our constructive setting.

Lemma D.0.21 (cf. Proposition 3.1.30)**.** *A closed subset of a complete metric space is complete.*

Proof. Let X be a complete metric space, and let $Y \subseteq X$ be a closed subset. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. Since X is complete, $(y_n)_{n\in\mathbb{N}}$ converges to some $x \in X$. Then, for any $\varepsilon \in \mathbb{Q}^{>0}$, we have $B(x, \varepsilon) \between \{y_n \mid n \in \mathbb{N}\}$. Since Y is closed, we have $x \in Y$. Therefore, Y is complete. □

Lemma D.0.22. *A complete subset of a metric space is closed.*

Proof. Let $Y \subseteq X$ be a complete subset of a metric space X. Let $x \in X$ such that $B(x, 2^{-n}) \between Y$ for all $n \in \mathbb{N}$. By the Countable Choice, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in Y such that $d(x, y_n) < 2^{-n}$ for all $n \in \mathbb{N}$. Since $(y_n)_{n\in\mathbb{N}}$ is clearly a Cauchy sequence in Y, and since Y is complete and x is the limit of $(y_n)_{n\in\mathbb{N}}$, we have $x \in Y$. Therefore, Y is closed. □

Hence, by the definition of compact metric space, we obtain the following characterisation.

Proposition D.0.23. *An inhabited subset Y of a compact metric space is compact iff Y is closed and located.*

We prove basic facts about locally compact metric spaces. The definitions of locally compact metric space and Bishop locally compact metric space are given in Definition 5.0.9.

Proposition D.0.24.

1. *A locally compact metric space is complete.*

2. *A Bishop locally compact metric space is separable.*

Proof. 1. Let X be a locally compact metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then, there exists $N_1 \in \mathbb{N}$ such that $d(x_{N_1}, x_m) < 1$ for all $m \geq N_1$. Since X is locally compact, there exists a compact subset $K \subseteq X$ such that $B(x_{N_1}, 1) \subseteq K$. Since K is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in K$. Hence X is complete.

2. Let X be a Bishop locally compact metric space. Let $x_0 \in X$. For each $n \in \mathbb{N}$, there exists a compact subset $K_n \subseteq X$ such that $B(x_0, n) \subseteq K_n$. By the Countable Choice, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $B(x_0, n) \subseteq K_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, since K_n is separable, there exists a sequence $(x^n_m)_{m\in\mathbb{N}}$ which is dense in K_n. Thus, by the Countable Choice, we have a sequence $((x^n_m))_{n\in\mathbb{N}}$ such that $(x^n_m)_{m\in\mathbb{N}}$ is dense in K_n for each $n \in \mathbb{N}$. Then, the subset $\{x^n_m \in X \mid n, m \in \mathbb{N}\}$ is countable and dense in X. Hence, X is separable. □

Then, we recall the connection between local compactness and locatedness.

Lemma D.0.25. *An inhabited locally compact subset of a metric space is located.*

Proof. Let $Y \subseteq X$ be an inhabited locally compact subset of a metric space X. Since Y is inhabited, we can choose $y_0 \in Y$. Let $x \in X$, and let $p, q \in \mathbb{Q}^{>0}$ such that $p < q$. Let $r \in \mathbb{Q}^{>0}$ such that $d(x, y_0) < r$. Since Y is locally compact, there exists a compact subset $K \subseteq Y$ such that $B(y_0, p + r) \subseteq K$. Choose $\varepsilon \in \mathbb{Q}^{>0}$ such that $p + \varepsilon < q$, and let $\{z_0, \dots, z_{n-1}\}$ be an ε-net to K. Then, either

1. $(\forall i < n)\, d(x, z_i) > p + \varepsilon$, or

2. $(\exists i < n)\, d(x, z_i) < q$.

In the latter case, we have $B(x, q) \between Y$. In the former case, suppose that $B(x, p) \between Y$. Then, there exists $y \in Y$ such that $d(x, y) < p$, and so $d(y_0, y) < r + p$. Thus, $y \in K$, so that $d(y, z_i) < \varepsilon$ for some $i < n$. Hence $d(x, z_i) \leq d(x, y) + d(y, z_i) < p + \varepsilon$, a contradiction. Thus, $B(x, p) \subseteq \neg Y$. Therefore Y is located. ☐

In the following lemma, we use Lemma 5.1.9. The relation $\Subset$ is defined by (5.1) in Chapter 5.

Lemma D.0.26. *Let X be a compact metric space, and let A be a located subset of X. Then, for any $\varepsilon \in \mathbb{Q}^{>0}$, if the set*

$$U_{A,\varepsilon} \stackrel{\text{def}}{=} \{x \in X \mid d(x, A) \geq \varepsilon\}$$

is inhabited, then there exists a compact subset $K \subseteq X$ such that

$$U_{A,\varepsilon} \subseteq K \Subset X - A.$$

Proof. Let $\varepsilon \in \mathbb{Q}^{>0}$, and suppose that $U_{A,\varepsilon}$ is inhabited. Choose $\theta \in \mathbb{Q}^{>0}$ such that $7\theta < \varepsilon$, and let $X_\theta = \{x_0, \ldots, x_{n-1}\}$ be a θ-net to X. Since A is located, we have either $B(x_i, 5\theta) \subseteq \neg A$ or $B(x_i, 6\theta) \between A$ for each $i < n$. Split X_θ into finitely enumerable subsets X_θ^+ and X_θ^- such that $X_\theta = X_\theta^+ \cup X_\theta^-$, and that

- $x \in X_\theta^+ \implies B(x, 5\theta) \subseteq \neg A$,
- $x \in X_\theta^- \implies B(x, 6\theta) \between A$.

Write $X_\theta^+ = \{z_0, \ldots, z_{m-1}\}$. Let $x \in U_{A,\varepsilon}$, and choose $i < n$ such that $d(x, x_i) < \theta$. If $B(x_i, 6\theta) \between A$, then we have $d(x, A) \leq 7\theta < \varepsilon$, contradicting $x \in U_{A,\varepsilon}$. Thus, $x_i \in X_\theta^+$, and hence $U_{A,\varepsilon} \subseteq \bigcup_{j<m} B(z_j, \theta)$. For each $j < m$, there exists a compact subset $K_j \subseteq X$ such that $B(z_j, \theta) \subseteq K_j \subseteq B(z_j, 2\theta)$ by Lemma 5.1.9. Let $K = \bigcup_{j<m} K_j$. Since $U_{A,\varepsilon} \subseteq K$, K is inhabited and totally bounded, and hence K is located by Lemma D.0.19. Let $x \in K_\theta = \{x' \in X \mid d(x', K) \leq \theta\}$, and suppose that $d(x, A) < \theta$. Then, there exist $y \in A$ and $w \in K$ such that $d(x, y) < \theta$ and $d(x, w) < 2\theta$. Thus, there exists $j < m$ such that $d(w, z_j) < 2\theta$, so

$$d(y, z_i) \leq d(y, x) + d(x, w) + d(w, z_i) \leq \theta + 2\theta + 2\theta \leq 5\theta.$$

Hence, $y \in B(z_j, 5\theta) \subseteq \neg A$, a contradiction, so we must have $d(x, A) \geq \theta$. Thus, $K_\theta \subseteq X - A$, and so $K \Subset X - A$. Hence $\mathsf{cl}(K) \Subset X - A$, and $\mathsf{cl}(K)$ is compact. ☐

Lemma D.0.27. *A closed and located subset of a locally compact metric space is locally compact.*

Proof. Let X be a locally compact metric space, and let $Y \subseteq X$ be a closed and located subset. Let $B(y_0, \varepsilon)$ be an open ball of Y. Since X is locally compact, there exists a compact subset $K \subseteq X$ such that $B(y_0, 4\varepsilon) \subseteq K$. Let $\{x_0, \ldots, x_{n-1}\}$ be an $\varepsilon/2$-net to K. Then, either

1. $(\forall i < n)\, d(y_0, x_i) < (3/2)\varepsilon$, or
2. $(\exists i < n)\, d(y_0, x_i) > \varepsilon$.

In the case 1, let $x \in K$, $y \in Y$, and $\delta \in \mathbb{Q}^{>0}$, and suppose that $d(y,x) < \delta$. Since $K \subseteq B(y_0, 2\varepsilon)$, there exists $\theta \in \mathbb{Q}^{>0}$ such that $d(y,x) + \theta < \delta$ and $d(y_0,x) + \theta < 2\varepsilon$. Since Y is located, there exists $y' \in Y$ such that $d(x,y') < d(x,Y) + \theta$, so that $d(x,y') < \delta$. Moreover,

$$\begin{aligned} d(y', y_0) &\leq d(y', x) + d(x, y_0) < d(x, Y) + \theta + 2\varepsilon \\ &< d(x, y_0) + \theta + 2\varepsilon < 4\varepsilon, \end{aligned}$$

and hence $y' \in K \cap Y$. Therefore $d(x, K \cap Y)$ exists and equals $d(x,Y)$. Thus, $K \cap Y$ is a closed located subset of K, and hence it is compact by Proposition D.0.23. Trivially, we have $B(y_0, \varepsilon) \subseteq K \cap Y$.

In the case 2, let $i < n$ such that $d(y_0, x_i) > \varepsilon$. Choose $\gamma, \delta, \theta \in \mathbb{Q}^{>0}$ such that $\varepsilon < \gamma < \delta - \theta$ and $\delta < \min\{d(y_0, x_i), 2\varepsilon\}$. By Lemma 5.1.9 and Lemma D.0.26, there exist compact subsets $A, B \subseteq K$ such that

1. $\{x \in K \mid d(x, A) \geq \theta\} \subseteq B \Subset K - A$,
2. $B(y_0, \gamma) \subseteq A \subseteq B(y_0, \delta - \theta)$.

Let $C = B \cup (K \cap Y)$. We show that C is located in K. Let $x \in K$. Then, either $d(x, y_0) > \delta$ or $d(x, y_0) < 2\varepsilon$. If $d(x, y_0) > \delta$, then $x \in B$, so $d(x, B) = 0$, and thus $d(x, C) = 0$. If $d(x, y_0) < 2\varepsilon$, then $d(x, K \cap Y)$ exists as in the case 1, and hence

$$d(x, C) = \min\{d(x, B), d(x, K \cap Y)\}.$$

Thus, C is located in K, and so the closure $\mathsf{cl}(C)$ in K is a compact subset of K.

Since $\mathsf{cl}(C)$ is locally compact, there exists a compact subset $D \subseteq \mathsf{cl}(C)$ such that $B(y_0, \varepsilon) \subseteq D \subseteq B(y_0, \gamma)$. Now, it suffices to show that $D \subseteq Y$. Let $z \in D$ and $\zeta \in \mathbb{Q}^{>0}$. Since $B \Subset K - A$, there exists $\zeta' < \zeta$ such that $B_{\zeta'} \subseteq K - A$. Since $z \in \mathsf{cl}(C)$, there exists $z' \in B \cup (K \cap Y)$ such that $d(z, z') < \zeta'$. If $z' \in B$, then $z \in K - A$. But since $D \subseteq A$, we have a contradiction. So we have $z' \in K \cap Y$. Since Y is closed, we conclude that $z \in Y$. Therefore $D \subseteq Y$, as required. □

Proposition D.0.28. *An inhabited subset Y of a locally compact metric space is locally compact iff Y is closed and located.*

Proof. ($\Leftarrow$): By Lemma D.0.27.

($\Rightarrow$): By Lemma D.0.25, Lemma D.0.22 and the fact that a locally compact metric space is complete (Lemma D.0.24.1). □

Lastly, we show that the embedding

$$(X, d) \mapsto (X, \{d\})$$

of the category of metric spaces into that of uniform spaces defined by sets of pseudo-metrics preserves countable products. The product of a sequence $((X_n, d_n))_{n\in\mathbb{N}}$ of metric spaces is a cartesian product $\prod_{n\in\mathbb{N}} X_n$ together with the metric d given by

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 2^{-n} d'_n(x_n, y_n),$$

where for each $n \in \mathbb{N}$, d'_n is the metric on X_n given by

$$d'_n(x, y) \stackrel{\text{def}}{=} \min\{d_n(x, y), 1\}.$$

Since d'_n is metrically equivalent to d_n, we may assume that d_n is bounded by 1.

Then, the family of projections $\pi_n \colon \prod_{n\in\mathbb{N}} X_n \to X_n$ forms a product of $((X_n, d_n))_{n\in\mathbb{N}}$ in the category of metric spaces.

Proposition D.0.29. *Let $((X_n, d_n))_{n\in\mathbb{N}}$ be a sequence of metric spaces. Let $X = (\prod_{n\in\mathbb{N}} X_n, d)$ be the product of $(X_n)_{n\in\mathbb{N}}$ in the category of metric spaces, and let $Y = (\prod_{n\in\mathbb{N}} X_n, M_\Pi)$ be the product of $(X_n)_{n\in\mathbb{N}}$ in the category of uniform spaces, where the uniformity M_Π is given by (3.7). Then, X and Y are isomorphic as uniform spaces.*

Proof. We assume that each d_n is bounded by 1.

By the universal property of Y, the identity function i_Π on $\prod_{n\in\mathbb{N}} X_n$ is a uniformly continuous function from X to Y.

Conversely, let $\varepsilon \in \mathbb{Q}^{>0}$, and choose $k \in \mathbb{N}$ such that $2^{-k+1} < \varepsilon$. Define $A \in M_\Pi$ by

$$A \stackrel{\text{def}}{=} \{(0, d_0), \dots, (k, d_k)\}.$$

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, and suppose that $\rho_A((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) < 2^{-(k+1)}$. Then

$$\begin{aligned} d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &= \sum_{n=0}^{n=k} 2^{-n} d_n(x_n, y_n) + \sum_{n=k+1}^{\infty} 2^{-n} d_n(x_n, y_n) \\ &< 2^{-k} + 2^{-k} < \varepsilon. \end{aligned}$$

Therefore, i_Π is a uniformly continuous function from Y to X. □

Hence, by the corresponding facts about uniform spaces (Proposition 3.1.30 and Proposition 3.1.51), we have the following.

Corollary D.0.30.

1. *A countable product of complete metric spaces is complete.*

2. *A countable product of inhabited totally bounded metric spaces is (inhabited and) totally bounded.*

In particular, a countable product of inhabited compact metric spaces is (inhabited and) compact.

Bibliography

[1] P. Aczel. The type theoretic interpretation of constructive set theory. In A. MacIntyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium '77*, pages 55–66. North Holland, Amsterdam, 1978.

[2] P. Aczel. The type theoretic interpretation of constructive set theory: Choice principles. In A. S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 1–40. North Holland, Amsterdam, 1982.

[3] P. Aczel. The type theoretic interpretation of constructive set theory: Inductive definitions. In R. B. Marcus, G. J. Dorn, and P. Weingartner, editors, *Logic, methodology and philosophy of science VII*, pages 17–49. North Holland, Amsterdam, 1986.

[4] P. Aczel. Aspects of general topology in constructive set theory. *Ann. Pure Appl. Logic*, 137(1-3):3–29, 2006.

[5] P. Aczel. Equalisers of frames in constructive set theory. *The University of Western Ontario Series in Philosophy of Science*, 75:221–228, 2011.

[6] P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report 40, Institut Mittag-Leffler, 2000/2001.

[7] J. Berger, H. Ishihara, E. Palmgren, and P. Schuster. A predicative completion of a uniform space. *Ann. Pure Appl. Logic*, 163(8):975–980, 2012.

[8] E. Bishop. *Foundations of Constructive Analysis.* McGraw-Hill, New York, 1967.

[9] E. Bishop and D. Bridges. *Constructive Analysis.* Springer, Berlin, 1985.

[10] D. S. Bridges and F. Richman. *Varieties of constructive mathematics*, volume 97 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1987.

[11] M. Bunge and J. Funk. Constructive theory of the lower power locale. *Math. Structures Comput. Sci.*, 6(1):69–83, 1996.

[12] J. Cederquist and T. Coquand. Entailment relations and distributive lattices. In S. Buss, P. Hajek, and P. Pudlak, editors, *Logic Colloquium '98*, Lecture Notes in Logic 13, pages 127–139. Association of Symbolic Logic, 2000.

[13] J. Cederquist and S. Negri. A constructive proof of the Heine-Borel covering theorem for formal reals. In S. Berardi and M. Coppo, editors, *Types for Proofs and Programs*, volume 1158 of *Lecture Notes in Computer Science*, pages 62–75. Springer Berlin Heidelberg, 1996.

[14] F. Ciraulo and G. Sambin. A constructive galois connection between closure and interior. *J. Symbolic Logic*, 77(4):1308–1324, 2012.

[15] F. Ciraulo, M. E. Maietti, and G. Sambin. Convergence in formal topology: a unifying notion. *Journal of Logic and Analysis*, 5, 2013.

[16] T. Coquand. Compact spaces and distributive lattices. *J. Pure Appl. Algebra*, 184 (1):1–6, 2003.

[17] T. Coquand, S. Sadocco, G. Sambin, and J. M. Smith. Formal topologies on the set of first-order formulae. *J. Symbolic Logic*, 65(3):1183–1192, 2000.

[18] T. Coquand, G. Sambin, J. Smith, and S. Valentini. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, 124(1-3):71–106, 2003.

[19] T. Coquand, E. Palmgren, and B. Spitters. Metric complements of overt closed sets. *MLQ Math. Log. Q.*, 57(4):373–378, 2011.

[20] G. Curi. Constructive metrisability in point-free topology. *Theoret. Comput. Sci.*, 305(1-3):85–109, 2003.

[21] G. Curi. A footnote on local compactness. preprint, May 2003.

[22] G. Curi. On the collection of points of a formal space. *Ann. Pure Appl. Logic*, 137 (1-3):126–146, 2006.

[23] G. Curi. Exact approximations to Stone-Čech compactification. *Ann. Pure Appl. Logic*, 146(2-3):103–123, 2007.

[24] G. Curi. On the existence of Stone-Čech compactification. *J. Symbolic Logic*, 75(4): 1137–1146, 2010.

[25] H. Diener and A. Hedin. The Vitali covering theorem in constructive mathematics. *Journal of Logic and Analysis*, 4, 2012.

[26] M. Fourman and R. Grayson. Formal spaces. In A. S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 107–122. North Holland, Amsterdam, 1982.

[27] M. Fourman and J. Hyland. Sheaf models for analysis. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of Sheaves*, volume 753 of *Lecture Notes in Mathematics*, pages 280–301. Springer Berlin Heidelberg, 1979.

[28] C. Fox. *Point-set and Point-free Topology in Constructive Set Theory.* PhD thesis, University of Manchester, 2005.

[29] N. Gambino and P. Schuster. Spatiality for formal topologies. *Math. Structures Comput. Sci.*, 17(01):65–80, 2007.

[30] S. Gebellato and G. Sambin. A preview of the Basic Picture: A new perspective on formal topology. In T. Altenkirch, B. Reus, and W. Naraschewski, editors, *Types for Proofs and Programs*, volume 1657 of *Lecture Notes in Computer Science*, pages 194–208. Springer, Berlin, 1999.

[31] A. Heyting. *Intuitionism : An introduction.* Studies in logic and the foundations of mathematics. North-Holland, 3rd revised edition, 1971.

[32] H. Ishihara. Localic completion of uniform spaces. preprint, February 2010.

[33] P. Johnstone. Tychonoff's theorem without the axiom of choice. *Fundamenta Mathematicae*, 113(1):21–35, 1981.

[34] P. T. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[35] P. T. Johnstone. The point of pointless topology. *Bulletin (New Series) of the American Mathematical Society*, 8(1):41–53, 1983.

[36] P. T. Johnstone. The art of pointless thinking: a student's guide to the category of locales. In *Category theory at work (Bremen, 1990)*, volume 18 of *Res. Exp. Math.*, pages 85–107. Heldermann, Berlin, 1991.

[37] J. Kelley. The Tychonoff product theorem implies the axiom of choice. *Fundamenta Mathematicae*, 37(1):75–76, 1950.

[38] J. Kelley. *General topology.* Springer-Verlag, 1975.

[39] I. Loeb. Equivalents of the (Weak) Fan Theorem. *Ann. Pure Appl. Logic*, 132(1): 51–66, 2005.

[40] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[41] M. E. Maietti. Predicative exponentiation of locally compact formal topologies over inductively generated formal topologies. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, number 48 in Oxford Logic Guides, pages 202–222. Oxford University Press, 2005.

[42] M. E. Maietti and S. Valentini. A structural investigation on formal topology: Coreflection of formal covers and exponentiability. *J. Symbolic Logic*, 69(4):967–1005, 2004.

[43] P. Martin-Löf. *Intuitionistic type theory.* Bibliopolis, Napoli, 1984.

[44] S. Negri and D. Sorovia. The continuum as a formal space. *Arch. Math. Logic*, 38: 423–447, 1999.

[45] S. Negri and S. Valentini. Tychonoff's theorem in the framework of formal topologies. *J. Symbolic Logic*, 62:1315–1332, 1997.

[46] E. Palmgren. Quotient spaces and coequalisers in formal topology. *J.UCS*, 11(12): 1996–2007, 2005.

[47] E. Palmgren. Predicativity problems in point-free topology. In V. Stoltenberg-Hansen and J. Väänänen, editors, *Logic Colloquium '03*, Lecture Notes in Logic 24. Association for Symbolic Logic, 2006.

[48] E. Palmgren. A constructive and functorial embedding of locally compact metric spaces into locales. *Topology Appl.*, 154:1854–1880, 2007.

[49] E. Palmgren. Open sublocales of localic completions. *Journal of Logic and Analysis*, 2:1–22, 2010.

[50] G. Sambin. Intuitionistic formal spaces — a first communication. In D. Skordev, editor, *Mathematical Logic and its Applications*, volume 305, pages 187–204. Plenum Press, 1987.

[51] G. Sambin. Some points in formal topology. *Theoret. Comput. Sci.*, 305(1-3):347–408, 2003.

[52] G. Sambin. *The Basic Picture and Positive Topology. New structures for Constructive Mathematics.* Oxford University Press, to appear.

[53] B. Spitters. Locatedness and overt sublocales. *Ann. Pure Appl. Logic*, 162(1):36–54, 2010.

[54] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics, An Introduction*, volume I. North-Holland, 1988.

[55] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics, An Introduction*, volume II. North-Holland, 1988.

[56] J. Vermeulen. *Constructive techniques in functional analysis.* PhD thesis, University of Sussex, 1986.

[57] S. Vickers. Constructive points of powerlocales. *Math. Proc. Cambridge Philos. Soc*, 122:207–222, 1995.

[58] S. Vickers. Localic completion of generalized metric spaces I. *Theory Appl. Categ.*, 14(15):328–356, 2005.

[59] S. Vickers. Some constructive roads to Tychonoff. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, number 48 in Oxford Logic Guides, pages 223–238. Oxford University Press, 2005.

[60] S. Vickers. Compactness in locales and in formal topology. *Ann. Pure Appl. Logic*, 137(1-3):413–437, 2006.

[61] S. Vickers. Sublocales in formal topology. *J. Symbolic Logic*, 72(2):463–482, 2007.

[62] S. Willard. *General topology.* Addison-Wesley series in mathematics. Addison-Wesley, 1970.

Publications

[1] Hajime Ishihara and Tatsuji Kawai, Completeness and cocompleteness of the categories of basic pairs and concrete spaces, *Mathematical Structures in Computer Science*, FirstView:1–23, 11 2014.

[2] Tatsuji Kawai, Point-free characterisation of Bishop compact metric spaces, submitted.