On the Ramseyan factorization theorem

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Abstract. We study, in the context of reverse mathematics, the strength of Ramseyan factorization theorem (RF^s_k), a Ramsey-type theorem used in automata theory. We prove that RF^s_k is equivalent to RT^2_2 for all $s, k \geq 2$, $k \in \omega$ over $\mathsf{RCA_0}$. We also consider a weak version of Ramseyan factorization theorem and prove that it is in between ADS and CAC.

1 Introduction

In the current study of reverse mathematics, deciding the strength of Ramsey's theorem for pairs (RT^2_2) is one of the most important topics (see e.g., Cholak/Jockusch/Slaman[1] and Hirschfeldt[5], and for the study of reverse mathematics, Simpson[9] is the standard reference). In this paper, we study, in the context of reverse mathematics, the strength of a Ramsey-type theorem which is called Ramseyan factorization theorem. Ramseyan factorization theorem is used in the theory of automata (see, for example, [8]). We show that some kinds of Ramseyan factorization theorem are equivalent to RT^2_2. We also study a weak version of Ramseyan factorization theorem. We discuss it in section 3, and show that a weak version is in between ADS and CAC. Note that ADS and CAC are just separated by Lerman/Solomon/Towsner[7]. Thus, it must be strictly stronger than ADS or strictly weaker than CAC. We also consider other variations of Ramseyan factorization theorem in section 5.

Notations and definitions

Let A be a set. Then $A^{<\mathbb{N}}$ (resp. $A^{\mathbb{N}}$) denotes the set of all finite (resp. infinite) sequences of elements from A. If $u, v \in A^{<\mathbb{N}}$, u_i denotes the i-th element of u, $u^\frown v$ (and uv for short) denotes the concatenation of u and v, and $|u|$ denotes the length of u. The Ramseyan factorization theorem is the following statement.

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Definition 1 (Ramseyan factorization theorem). *For any $A \subseteq \mathbb{N}$ and finite $B \subseteq \mathbb{N}$, the following statement (RF^A_B) holds:*

> *For any $u \in A^{\mathbb{N}}$ and $f : A^{<\mathbb{N}} \to B$, there exists $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$ such that $u = v_0 ^\frown v_1 ^\frown \cdots$ and for any $j \geq i > 0$ and $j' \geq i' > 0$, $f(v_i ^\frown v_{i+1} ^\frown \cdots ^\frown v_j) = f(v_{i'} ^\frown v_{i'+1} ^\frown \cdots ^\frown v_{j'})$.*

If u, f and v satisfy the above condition, we call v a Ramseyan factorization for u and f. In this paper, we aim to study $\mathrm{RF}^{\mathbb{N}}_k$ and RF^s_k for $s, k \in \mathbb{N}$.

2 Ramseyan factorization theorem and Ramsey's theorem for pairs

In this section, we see the relation between Ramsey's theorem (RT^n_k) and Ramseyan factorization theorem (RF^s_k).

Proposition 2 (RCA_0)**.** *For any $k \in \mathbb{N}$, $\mathrm{RF}^{\mathbb{N}}_k \Rightarrow \cdots \Rightarrow \mathrm{RF}^2_k \Rightarrow \mathrm{RF}^1_k$.*

Proof. Trivial from the definition.

Theorem 3 (RCA_0)**.** *For any $k \in \mathbb{N}$, RT^2_k implies $\mathrm{RF}^{\mathbb{N}}_k$.*

Proof. Let $u \in \mathbb{N}^{\mathbb{N}}$ and $f : \mathbb{N}^{<\mathbb{N}} \to k$. Define $P : [\mathbb{N}]^2 \to k$ as follows:

$$P(i, j) = f(u_i u_{i+1} \dots u_{j-1}).$$

Let X be an infinite homogeneous set for P. Define $l \in \mathbb{N}^{\mathbb{N}}$ by setting l_i to be the i-th smallest element in X and define $v \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ by setting $v_0 = u_0 \dots u_{l_0 - 1}$ and $v_i = u_{l_{i-1}} \dots u_{l_i - 1}$ for all $i \geq 1$. Then clearly v is a Ramseyan factorization for u and f.

Theorem 4 (RCA_0)**.** *For any $k \in \mathbb{N}$, RF^2_k implies RT^2_k.*

Proof. Let $P : [\mathbb{N}]^2 \to k$. We will find an infinite homogeneous set for P. Define $u \in 2^{\mathbb{N}}$ and $f : 2^{<\mathbb{N}} \to k$ as follows:

$$u = 1010010001 \dots 10^{n-1} 10^n 10^{n+1} 1 \dots$$

$$f(\sigma) = \begin{cases} P(m, n+2) & \text{if } \sigma = 0^k 10^m 1 \tau 10^n 10^l \text{ for some } k, l, m, n \geq 0 \text{ and } \tau \in 2^{<\mathbb{N}}, \\ 0 & \text{otherwise.} \end{cases}$$

Let v be a Ramseyan factorization for u and f. By combining v_i's if necessary, we may assume that each v_i contains at least four 1's, *i.e.*, v_i is of the form $0^k 10^m 1 \tau 10^n 10^l$. Let $H = \{m \in \mathbb{N} \mid 1 \leq \exists i \leq m\ v_i = 0^k 10^m 1 \tau 10^n 10^l\}$. We can easily check that this H is an infinite homogeneous set for P.

From the above proposition and theorems, we can show that RF^s_k is equivalent to RT^2_2 for all $s, k \geq 2, k \in \omega$.

Corollary 5. *The following are equivalent over* RCA_0.

1. RT^2_2.
2. $\mathrm{RF}^{\mathbb{N}}_k$ *($k \geq 2$, $k \in \omega$).*
3. RF^2_k *($k \geq 2$, $k \in \omega$).*

Proof. This is clear from the previous theorems and the fact that RCA_0 proves $\mathrm{RT}^2_k \Rightarrow \mathrm{RT}^2_{k+1}$ for all $k \geq 2$.

Corollary 6. *The following are equivalent over* RCA_0.

1. $\mathrm{RT}^2_{<\infty}$.
2. $\forall k \mathrm{RF}^{\mathbb{N}}_k$.
3. $\forall k \mathrm{RF}^2_k$.

Next, we consider the remaining case, i.e. the strength of RF^1_k. In order to study RF^1_k, we consider the following version of Ramsey's theorem.

Definition 7. *For a given function $f : [\mathbb{N}]^n \to \mathbb{N}$, RT^f_k is the following statement:*

> *For any $P : \mathbb{N} \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that for any $u, v \in [H]^n$, $P(f(u)) = P(f(v))$.*

If f is a bijection, we can prove the following.

Proposition 8 (RCA_0)**.** *For any $n \in \mathbb{N}$ and any bijection $f : [\mathbb{N}]^n \to \mathbb{N}$, RT^f_k is equivalent to RT^n_k.*

The full version of RT^f_k, i.e. $\forall f : [\mathbb{N}]^n \to \mathbb{N}\ \mathrm{RT}^f_k$, is still equivalent to RT^n_k.

Proposition 9 (RCA_0)**.** RT^n_k *is equivalent to* $\forall f : [\mathbb{N}]^n \to \mathbb{N}\ \mathrm{RT}^f_k$.

Proof. From left to right is trivial, because $P \circ f$ is a function from $[\mathbb{N}]^n$ to k when $P : \mathbb{N} \to k$. From right to left is proved from the above proposition.

If f is not a bijection, RT^f_k may not be equivalent to RT^n_k. In case f is the subtraction $\mathrm{Subt}(a, b) = b - a$, RT^f_k is equivalent to RF^1_k. (The function Subt is considered as a function of $[\mathbb{N}]^2$.)

Proposition 10 (RCA_0)**.** *For any $k \in \mathbb{N}$, RF^1_k is equivalent to $\mathrm{RT}^{\mathrm{Subt}}_k$.*

Proof. We first prove $\mathrm{RF}^1_k \Rightarrow \mathrm{RT}^{\mathrm{Subt}}_k$. Assume RF^1_k and let $P : \mathbb{N} \to k$. Define $f : 1^{<\mathbb{N}} \to k$ by $f(0^n) = P(n)$ and let v be a Ramseyan factorization for $0^{\mathbb{N}}$ and f. Let $X = \{\sum_{j \leq i} |v_j| \mid i \in \mathbb{N}\}$. Then X is an infinite homogeneous set for $P \circ \mathrm{Subt}$.

Next, we prove $\mathrm{RT}^{\mathrm{Subt}}_k \Rightarrow \mathrm{RF}^1_k$. Assume $\mathrm{RT}^{\mathrm{Subt}}_k$ and let $f : 1^{<\mathbb{N}} \to k$. Define $P : \mathbb{N} \to k$ by $P(n) = f(0^n)$. Then there exists an infinite homogeneous set $H := \{l_0 < l_1 < \cdots\} \subseteq \mathbb{N}$ for P. Define $v \in (1^{<\mathbb{N}})^{\mathbb{N}}$ by $v_0 = 0^{l_0}$ and $v_i = 0^{l_i - l_{i-1}}$ for all $i \geq 1$. Then v is a Ramseyan factorization for $0^{\mathbb{N}}$ and f.

From the above, we can show that $\forall k \mathrm{RF}^1_k$ is strong enough to prove the bounding principle for Σ^0_2 formulas.

Corollary 11 (RCA_0)**.** *$\forall k \mathrm{RF}^1_k$ implies $\mathrm{B}\Sigma^0_2$.*

Proof. Because of the above and the equivalence of $\mathrm{B}\Sigma^0_2$ and $\mathrm{RT}^1_{<\infty}$, it's enough to prove $\mathrm{RT}^{\mathrm{Subt}}_k \Rightarrow \mathrm{RT}^1_k$ for all $k \in \mathbb{N}$. Assume $\mathrm{RT}^{\mathrm{Subt}}_k$ and let $P : \mathbb{N} \to k$. Then there exists an infinite set $H \subseteq \mathbb{N}$ such that for any $u, v \in [H]^2$, $P(u_1 - u_0) = P(v_1 - v_0)$. Then $X = \{h - \min H \mid h \in H \setminus \{\min H\}\}$ is an infinite homogeneous set for P.

Question 12. *Is RF^1_k equivalent to RT^2_2 or RT^1_k ?*

3 Weak factorization

In this section, we consider a weaker version of Ramseyan factorization theorem. For applications in automata theory, the following weaker version of Ramseyan factorization theorem is usually good enough.

Definition 13. *For given sets $A, B \subseteq \mathbb{N}$, weak Ramseyan factorization theorem for A and B (WRF^A_B) is the following statement:*

> *For any $u \in A^{\mathbb{N}}$ and $f : A^{<\mathbb{N}} \to B$, there exists $v \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ such that $u = v_0 {}^\frown v_1 {}^\frown \dots$ and for any $i, j > 0$, $f(v_i) = f(v_j)$.*

Here, such v is said to be a weak Ramseyan factorization for u and f.

Similarly, we consider a weaker version of Ramsey's theorem as follows.

Definition 14. *Pseudo Ramsey's theorem psRT^n_k is the following statement:*

> *For any coloring $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H = \{a_0 < a_1 < \dots\}$ such that for any $i, j \in \mathbb{N}$, $P(a_i, \dots, a_{i+n-1}) = P(a_j, \dots, a_{j+n-1})$.*

Such H is called pseudo homogeneous set for P. [5]

Remark 15. *In general, a subset of a pseudo homogeneous set might not be pseudo homogeneous again.*

Question 16. *Does psRT^n_k imply psRT^n_{k+1} over RCA_0?*

Proposition 17 (RCA_0)**.** *For any $m \in \mathbb{N}$, $\mathrm{WRF}^{\mathbb{N}}_m \Leftrightarrow \mathrm{psRT}^2_m$. In particular, $\mathrm{WRF}^{\mathbb{N}}_2$ is equivalent to psRT^2_2.*

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[5] In Friedman/Pelupessy[4], this set is called adjacent homogeneous.

Proof. We first show for a given $m \in \mathbb{N}$ that $\mathrm{WRF}^{\mathbb{N}}_m \Rightarrow \mathrm{psRT}^2_m$. Fix $u = \langle i \mid i \in \mathbb{N}\rangle \in \mathbb{N}^{\mathbb{N}}$. For a given coloring $P : [\mathbb{N}]^2 \to m$, define $f : \mathbb{N}^{<\mathbb{N}} \to m$ by $f(\sigma) = P(a, a+k)$ if $\sigma = \langle a+i \mid i < k\rangle$ for some $a, k \in \mathbb{N}$, $k \geq 1$, and $f(\sigma) = 0$ otherwise. Now, let v be a weak Ramseyan factorization for u and f. Then, one can easily check that the set $H = \{\sum_{j \leq i} |v_j| \mid i \in \mathbb{N}\}$ is a pseudo homogeneous set for P.

Next, we show $m \in \mathbb{N}$, $\mathrm{psRT}^2_m \Rightarrow \mathrm{WRF}^{\mathbb{N}}_m$. Let $u \in \mathbb{N}^{\mathbb{N}}$, and let $f : \mathbb{N}^{<\mathbb{N}} \to m$. Then, define a coloring $P : [\mathbb{N}]^2 \to m$ by $P(a,b) = f(\langle u_i \mid a \leq i < b\rangle)$. Let $H = \{a_0 < a_1 < \dots\}$ be an infinite weak homogeneous set for P. Define $v_0 = \langle u_i \mid 0 \leq j < a_0\rangle$ and $v_{i+1} = \langle u_j \mid a_i \leq j < a_{i+1}\rangle$. Then, v is a weak Ramseyan factorization for u and f.

How about the case WRF^A_B with A finite? We can apply a similar argument to that in Theorem 4, but this time, we have to add extra colors.

Proposition 18 (RCA_0). *For any $k \in \mathbb{N}$, WRF^2_{k+5} implies psRT^2_k.*

Proof. Let $w_i = 10^i \in 2^{<\mathbb{N}}$, and let $u = w_0 ^\frown w_1 ^\frown \dots$. For a given coloring $P : [\mathbb{N}]^2 \to k$, we define a function $f : 2^{<\mathbb{N}} \to k+5$ as follows:

$$f(\sigma) = \begin{cases} P(m, n+2) & \text{if } \sigma = 0^i {}^\frown w_m ^\frown \dots ^\frown w_n ^\frown 10^j \text{ for some } i, j \geq 0 \text{ and } 1 \leq m \leq n, \\ k & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \geq 0 \text{ such that } i \text{ and } j \text{ are both even}, \\ k+1 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \geq 0 \text{ such that } i \text{ is odd and } j \text{ is even}, \\ k+2 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \geq 0 \text{ such that } i \text{ is even and } j \text{ is odd}, \\ k+3 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \geq 0 \text{ such that } i \text{ and } j \text{ are both odd}, \\ k+4 & \text{otherwise.} \end{cases}$$

Take a weak Ramseyan factorization v for u and f, and let $f(v_i) = d$ for all $i \geq 1$. If v_i contains at least one '1', then $f(v_i) \neq k+4$. Thus, $d \neq k+4$. If $k \leq d < k+4$, then each v_i contains only one '1'. However, one can easily check that this is impossible. Therefore, for any $i \geq 1$, $f(v_i) = d$ for some $d < k$. This means that $H = \{m \in \mathbb{N} \mid v_l = 0^i {}^\frown w_m ^\frown \cdots ^\frown w_n ^\frown 10^j \text{ for some } i, j \geq 0,\ 1 \leq m \leq n, \text{ and } l \geq 1\}$ is a pseudo homogeneous set for P.

Question 19. *Is it possible to reduce the number of colorings in the above proof?*

One of the reviewers told us that if we change the color "$k+4$" to "0", the above proof still works without changing the weak Ramseyan factorization v. Therefore, thank to him or her, we can prove the following.

Proposition 20 (RCA_0). *For any $k \in \mathbb{N}$, WRF^2_{k+4} implies psRT^2_k.*

The following question still remains.

Question 21. *Is WRF^2_2 equivalent to psRT^2_2 over RCA_0?*

4 The strength of $\mathbf{WRF}^{\mathbb{N}}_{\boldsymbol{k}}$, or equivalently $\mathbf{psRT}^{\mathbf{2}}_{\boldsymbol{k}}$

Our main goal in this section is to prove that $\mathrm{WRF}^{\mathbb{N}}_2$, or equivalently psRT^2_2, is in between CAC and ADS. In order to show it, we use the facts that ADS is equivalent to trRT^2_2, transitive Ramsey's theorem for pairs, and CAC is equivalent to strRT^2_2, semi-transitive Ramsey's theorem for pairs, which were both proved in Hirschfeldt/Shore[6].

Definition 22 (Transitive and semi-transitive colorings [6]).

1. *A k-coloring $P : [\mathbb{N}]^2 \to k$ is said to be transitive if $P(a, b) = P(b, c) = i \Rightarrow P(a, c) = i$.*
2. *A k-coloring $P : [\mathbb{N}]^2 \to k$ is said to be semi-transitive if $P(a, b) = P(b, c) = i > 0 \Rightarrow P(a, c) = i$.*

Now, we consider the following variations of Ramsey's theorem for pairs.

Definition 23.
1. *Transitive Ramsey's theorem* trRT^2_k*: Any transitive k-coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.*
2. *Semi-transitive Ramsey's theorem* strRT^2_k*: Any semi-transitive k-coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set.*
3. *Semi-pseudo Ramsey's theorem* spsRT^2_k*: Any k-coloring $P : [\mathbb{N}]^2 \to k$ has an infinite homogeneous set H such that $P([H]^2) = \{0\}$ or an infinite pseudo homogeneous set $H' = \{h_0 < h_1 < \dots\}$ such that $P(h_i, h_{i+1}) > 0$.*

Clearly, spsRT^2_k is a stronger version of psRT^2_k. First, we show the lower bound for psRT^2_2.

Theorem 24 (RCA_0)**.** *For any $m \in \mathbb{N}$,* psRT^2_m *implies* trRT^2_m.

Proof. If P is a transitive coloring, a pseudo homogeneous set for P is actually a homogeneous set for P.

Next, we consider the upper bound for psRT^2_2

Lemma 25 (RCA_0)**.** *For any $m \in \mathbb{N}$,* spsRT^2_m *implies* strRT^2_m.

Proof. If P is a semi-transitive coloring, a pseudo homogeneous set H for P with $P([H]^2) \neq \{0\}$ is actually a homogeneous set for P.

The converse is true for the case $m = 2$.

Lemma 26 (RCA_0)**.** strRT^2_2 *implies* spsRT^2_2.

Proof. Let $P : [\mathbb{N}]^2 \to 2$. We want to find a homogeneous set for 0, or a pseudo homogeneous set for 1. Define $\bar{P} : [\mathbb{N}]^2 \to 2$ as follows: $\bar{P}(a, b) = 1$ if there exists a sequence $a = a_0 < \dots < a_l = b$ such that $P(a_i, a_{i+1}) = 1$ for any $i < l$, and $\bar{P}(a, b) = 0$ otherwise. Then, $\bar{P}$ is a semi-transitive coloring. Thus, by strRT^2_2, take an infinite homogeneous set H for $\bar{P}$. If $\bar{P}([H]^2) = \{0\}$, then we have $P([H]^2) = \{0\}$ and we have done. If $\bar{P}([H]^2) = \{1\}$, then for any $a, b \in H$, we can (effectively) find a sequence $a = a_0 < \dots < a_l = b$ such that $P(a_i, a_{i+1}) = 1$ for every $i < l$. Thus, we can construct a set $H' \supseteq H$ which is a pseudo homogeneous set for P with the value 1.

Question 27. *Over* RCA_0*, does* $\mathrm{strRT}^2_{<\infty}$ *imply* $\mathrm{spsRT}^2_{<\infty}$ *or* $\mathrm{psRT}^2_{<\infty}$*?*

Although psRT^2_k might not prove psRT^2_{k+1}, we can show the following.

Lemma 28 (RCA_0)**.** *For any* $m \geq 2$*,* spsRT^2_m *implies* spsRT^2_{m+1}*.*

Proof. Let $P : [\mathbb{N}]^2 \to m+1$. Define $\bar{P} : [\mathbb{N}]^2 \to m$ by $\bar{P}(a,b) = 0$ if $P(a,b) \in \{0,1\}$ and $\bar{P}(a,b) = P(a,b) - 1$ if $P(a,b) \geq 2$. If $\bar{P}$ has a pseudo homogeneous set with the value $d \geq 1$, then it is a pseudo homogeneous set for P. Otherwise, $\bar{P}$ has a homogeneous set H with the value 0. Then, $P \restriction [H]^2$ is a 2-coloring, thus we can apply spsRT^2_2 again, and we have done. [6]

Combining the above, we have the following.

Theorem 29. *The following are equivalent over* RCA_0*.*

1. spsRT^2_2.
2. strRT^2_2.
3. spsRT^2_k *for any* $k \in \omega$*,* $k \geq 2$*.*
4. strRT^2_k *for any* $k \in \omega$*,* $k \geq 2$*.*

Thus, within RCA_0*,* psRT^2_2 *is provable from any one of the above.*

Corollary 30 (RCA_0)**.** psRT^2_2 *is stronger than* ADS *and weaker than* CAC*.*

Proof. By Hirschfeldt/Shore[6], ADS is equivalent to trRT^2_2 and CAC is equivalent to strRT^2_2.

Question 31. *Is* psRT^2_2 *equivalent to* ADS *or* CAC *over* RCA_0*?*

Corollary 32 (RCA_0)**.** SRT^2_2 *does not imply* psRT^2_2*.*

Proof. By Chong/Slaman/Yang[2], SRT^2_2 does not imply COH. On the other hand, by Hirschfeldt/Shore[6], ADS implies COH, and thus psRT^2_2 implies COH.

Corollary 33 (RCA_0)**.** psRT^2_2 *does not imply* DNR*.*

Proof. By Hirschfeldt/Shore[6], CAC does not imply DNR, thus psRT^2_2 does not, either.

Question 34. *Does* P^2_2 *or* $\mathrm{RWKL}^{0'}$ *imply* psRT^2_2*? (See, e.g., Flood[3] for the definitions of these statements. Note that* $\mathrm{RWKL}^{0'}$ *is introduced as* $\mathrm{RKL}^{(1)}$ *in [3].)*

5 Other topics

In this section, we focus on some other versions of Ramseyan factorization theorem.

[6] Note that this argument still works for any n-tuples.

5.1 Stable versions

We can consider stable versions of RF or WRF. For given $u \in \mathbb{N}^{<\mathbb{N}}$ and $f : \mathbb{N}^{<\mathbb{N}} \to k$, f is said to be *stable* on u if for any $m \in \mathbb{N}$, there exists $n > m$ such that for any $l > n$, $f(\langle u_i \mid m \leq i < n \rangle) = f(\langle u_i \mid m \leq i < l \rangle)$. Then, SRF^A_k and SWRF^A_k are the following statements:

Definition 35. 1. SRF^A_k*: For any* $u \in A^{\mathbb{N}}$ *and* $f : A^{<\mathbb{N}} \to k$ *such that* f *is stable on* u*, there exists a Ramseyan factorization for* u *and* f*.*

2. SWRF^A_k*: For any* $u \in A^{<\mathbb{N}}$ *and* $f : A^{<\mathbb{N}} \to k$ *such that* f *is stable on* u*, there exists a weak Ramseyan factorization for* u *and* f*.*

As in Theorems 3 and 4, we can show the following.

Theorem 36. *Within* RCA_0*, the following are equivalent for any* $m \in \mathbb{N}$*.*

1. SRT^2_m.
2. $\mathrm{SRF}^{\mathbb{N}}_m$.
3. SRF^2_m.

Theorem 37. *Within* RCA_0*, the following are equivalent for any* $m \in \mathbb{N}$*.*

1. SWRT^2_m*: Any stable coloring* $P : [\mathbb{N}]^2 \to m$ *has an infinite pseudo homogeneous set.*
2. $\mathrm{SWRF}^{\mathbb{N}}_m$.

5.2 Tree versions

In this subsection, we consider a slightly stronger version of RF^2_m. For given two trees $T, S \subsetneq 2^{<\mathbb{N}}$, a tree embedding is an injective function $\pi : S \to T$ such that for any $\sigma, \tau \in S$, $\pi(\sigma) \cap \pi(\tau) = \pi(\sigma \cap \tau)$. For a given tree embedding $\pi : S \to T$, and for any $\sigma, \tau \in S$ such that $\sigma \subsetneq \tau$, the edge between $\pi(\sigma)$ and $\pi(\tau)$, denoted by $E_\pi(\sigma, \tau)$, is the sequence $\rho \in 2^{<\mathbb{N}}$ such that $\pi(\sigma)^\frown\rho = \pi(\tau)$. Then, we consider the following tree version of Ramseyan factorization theorem.

Definition 38. *Ramseyan factorization theorem for trees* TRF^2_k *is the following statement:*

> *For any infinite tree* $T \subsetneq 2^{<\mathbb{N}}$ *and a coloring* $f : 2^{<\mathbb{N}} \to k$*, there exists an infinite tree* $S \subsetneq 2^{<\mathbb{N}}$ *and a tree embedding* $\pi : S \to T$ *such that for any* $\sigma \subsetneq \tau \in S$ *and* $\sigma' \subsetneq \tau' \in S$*,* $f(E_\pi(\sigma, \tau)) = f(E_\pi(\sigma', \tau'))$*.*

Proposition 39 (RCA_0)**.** TRF^2_k *implies* RF^2_k *for all* $k \in \mathbb{N}$*. In particular,* TRF^2_2 *implies* RF^2_2 *(and, equivalently,* RT^2_2*).*

Proof. Assume TRF^2_k and let $u \in 2^{\mathbb{N}}$ and $f : 2^{<\mathbb{N}} \to k$. Define a tree $T \subsetneq 2^{<\mathbb{N}}$ by $T = \{u_0 u_1 \dots u_{i-1} \mid i \in \mathbb{N}\}$. By TRF^2_k, there exist $S = \{s_0 < s_1 < \cdots\} \subsetneq 2^{<\mathbb{N}}$ and an embedding $\pi : S \to T$ such that for all $\sigma \subsetneq \tau \in S$ and $\sigma' \subsetneq \tau' \in S$, $f(E_\pi(\sigma, \tau)) = f(E_\pi(\sigma', \tau'))$. Define $v \in (2^{<\mathbb{N}})^{\mathbb{N}}$ by setting $v_0 = \pi(s_0)$ and $v_i = E_\pi(s_{i-1}, s_i)$ for all $i \geq 1$. Then, v is a Ramseyan factorization for u and f.

We can also show that TRF^2_2 is weaker than $\mathsf{WKL_0} + \mathrm{RT}^2_2$.

Proposition 40. $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *implies* TRF^2_2*.*

Proof. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree and $f : 2^{<\mathbb{N}} \to 2$. By $\mathsf{WKL_0}$, there is an infinite path $u \in 2^{\mathbb{N}}$ through T. By RF^2_2, which is equivalent to RT^2_2, there is a Ramseyan factorization $v \in (2^{<\mathbb{N}})^{\mathbb{N}}$ for u and f. Define $S \subseteq 2^{<\mathbb{N}}$ and $\pi : S \to T$ by $S = \{0^i \mid i \in \mathbb{N}\}$ and $\pi(0^i) = v_0 ^\frown v_1 ^\frown \cdots ^\frown v_i$ for all $i \in \mathbb{N}$. Then S and π satisfy the condition.

Therefore, TRF^2_2 is in between $\mathsf{WKL_0} + \mathrm{RT}^2_2$ and RT^2_2.

Question 41. *Does* TRF^2_2 *imply* $\mathsf{WKL_0}$ *over* $\mathsf{RCA_0}$*?*

Remark 42. TRF^2_2 *may be equivalent to the following stronger version of* RT^2_2*:*

> RT^{2+}_2 : *If* $\mathcal{P}$ *be a class of colorings* $P : [F_P]^2 \to 2$ *where* $F_P = \{0, 1, \ldots, l\}$ *for some* $l \in \mathbb{N}$*, then there exists an infinite set* $H \subseteq \mathbb{N}$ *such that there exist infinitely many* $P \in \mathcal{P}$ *such that* P *is constant on* $[H \cap F_P]^2$*.*

We think that the equivalence should hold, but we do not know either $\mathrm{TRF}^2_2 \Rightarrow \mathrm{RT}^{2+}_2$ *or* $\mathrm{RT}^{2+}_2 \Rightarrow \mathrm{TRF}^2_2$*. This kind of strengthened Ramsey's theorem is studied in [10].*

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