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Author(s)	Murakami, Shota; Yamazaki, Takeshi; Yokoyama, Keita
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Japan Advanced Institute of Science and Technology

# On the Ramseyan factorization theorem

Shota Murakami<sup>1</sup>, Takeshi Yamazaki<sup>2</sup>, and Keita Yokoyama<sup>3,4</sup>

 Mathematical Institute, Tohoku University, Japan sb0m33@math.tohoku.ac.jp
 Mathematical Institute, Tohoku University, Japan yamazaki@math.tohoku.ac.jp
 Japan Advanced Institute of Science and Technology, Japan y-keita@jaist.ac.jp

**Abstract.** We study, in the context of reverse mathematics, the strength of Ramseyan factorization theorem  $(\mathrm{RF}_k^s)$ , a Ramsey-type theorem used in automata theory. We prove that  $\mathrm{RF}_k^s$  is equivalent to  $\mathrm{RT}_2^2$  for all  $s, k \geq 2, k \in \omega$  over  $\mathrm{RCA}_0$ . We also consider a weak version of Ramseyan factorization theorem and prove that it is in between ADS and CAC.

# 1 Introduction

In the current study of reverse mathematics, deciding the strength of Ramsey's theorem for pairs  $(RT_2^2)$  is one of the most important topics (see e.g., Cholak/Jockusch/Slaman[1] and Hirschfeldt[5], and for the study of reverse mathematics, Simpson[9] is the standard reference). In this paper, we study, in the context of reverse mathematics, the strength of a Ramsey-type theorem which is called Ramseyan factorization theorem. Ramseyan factorization theorem is used in the theory of automata (see, for example, [8]). We show that some kinds of Ramseyan factorization theorem are equivalent to  $RT_2^2$ . We also study a weak version of Ramseyan factorization theorem. We discuss it in section **3**, and show that a weak version is in between ADS and CAC. Note that ADS and CAC are just separated by Lerman/Solomon/Towsner[7]. Thus, it must be strictly stronger than ADS or strictly weaker than CAC. We also consider other variations of Ramseyan factorization theorem in section **5**.

### Notations and definitions

Let A be a set. Then  $A^{<\mathbb{N}}$  (resp.  $A^{\mathbb{N}}$ ) denotes the set of all finite (resp. infinite) sequences of elements from A. If  $u, v \in A^{<\mathbb{N}}$ ,  $u_i$  denotes the *i*-th element of u,  $u^{\frown}v$  (and uv for short) denotes the concatenation of u and v, and |u| denotes the length of u. The Ramseyan factorization theorem is the following statement.

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**Definition 1 (Ramseyan factorization theorem).** For any  $A \subseteq \mathbb{N}$  and finite  $B \subseteq \mathbb{N}$ , the following statement  $(\mathrm{RF}_B^A)$  holds:

For any  $u \in A^{\mathbb{N}}$  and  $f : A^{<\mathbb{N}} \to B$ , there exists  $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$  such that  $u = v_0^\frown v_1^\frown \cdots$  and for any  $j \ge i > 0$  and  $j' \ge i' > 0$ ,  $f(v_i^\frown v_{i+1}^\frown \cdots \frown v_j) = f(v_{i'}^\frown v_{i'+1}^\frown \cdots \frown v_{j'})$ .

If u, f and v satisfy the above condition, we call v a Ramseyan factorization for u and f. In this paper, we aim to study  $\mathrm{RF}_k^{\mathbb{N}}$  and  $\mathrm{RF}_k^s$  for  $s, k \in \mathbb{N}$ .

# 2 Ramseyan factorization theorem and Ramsey's theorem for pairs

In this section, we see the relation between Ramsey's theorem  $(\mathrm{RT}_{k}^{n})$  and Ramseyan factorization theorem  $(\mathrm{RF}_{k}^{s})$ .

**Proposition 2** (RCA<sub>0</sub>). For any  $k \in \mathbb{N}$ ,  $\mathrm{RF}_k^{\mathbb{N}} \Rightarrow \cdots \Rightarrow \mathrm{RF}_k^2 \Rightarrow \mathrm{RF}_k^1$ .

*Proof.* Trivial from the definition.

**Theorem 3** (RCA<sub>0</sub>). For any  $k \in \mathbb{N}$ ,  $\mathrm{RT}_k^2$  implies  $\mathrm{RF}_k^{\mathbb{N}}$ .

*Proof.* Let  $u \in \mathbb{N}^{\mathbb{N}}$  and  $f : \mathbb{N}^{<\mathbb{N}} \to k$ . Define  $P : [\mathbb{N}]^2 \to k$  as follows:

$$P(i,j) = f(u_i u_{i+1} \dots u_{j-1}).$$

Let X be an infinite homogeneous set for P. Define  $l \in \mathbb{N}^{\mathbb{N}}$  by setting  $l_i$  to be the *i*-th smallest element in X and define  $v \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$  by setting  $v_0 = u_0 \dots u_{l_0-1}$  and  $v_i = u_{l_{i-1}} \dots u_{l_i-1}$  for all  $i \geq 1$ . Then clearly v is a Ramseyan factorization for u and f.

**Theorem 4** (RCA<sub>0</sub>). For any  $k \in \mathbb{N}$ ,  $\operatorname{RF}_k^2$  implies  $\operatorname{RT}_k^2$ .

*Proof.* Let  $P : [\mathbb{N}]^2 \to k$ . We will find an infinite homogeneous set for P. Define  $u \in 2^{\mathbb{N}}$  and  $f : 2^{<\mathbb{N}} \to k$  as follows:

$$u = 1010010001 \dots 10^{n-1} 10^n 10^{n+1} 1 \dots$$
  
$$f(\sigma) = \begin{cases} P(m, n+2) & \text{if } \sigma = 0^k 10^m 1\tau 10^n 10^l \text{ for some } k, l, m, n \ge 0 \text{ and } \tau \in 2^{<\mathbb{N}} \\ 0 & \text{otherwise.} \end{cases}$$

Let v be a Ramseyan factorization for u and f. By combining  $v_i$ 's if necessary, we may assume that each  $v_i$  contains at least four 1's, *i.e.*,  $v_i$  is of the form  $0^k 10^m 1\tau 10^n 10^l$ . Let  $H = \{m \in \mathbb{N} \mid 1 \leq \exists i \leq m \ v_i = 0^k 10^m 1\tau 10^n 10^l\}$ . We can easily check that this H is an infinite homogeneous set for P.

From the above proposition and theorems, we can show that  $\mathrm{RF}_k^s$  is equivalent to  $\mathrm{RT}_2^2$  for all  $s, k \geq 2, k \in \omega$ .

**Corollary 5.** The following are equivalent over  $\mathsf{RCA}_0$ .

1.  $\operatorname{RT}_{2}^{2}$ . 2.  $\operatorname{RF}_{k}^{\mathbb{N}}$   $(k \ge 2, k \in \omega)$ . 3.  $\operatorname{RF}_{k}^{2}$   $(k \ge 2, k \in \omega)$ .

*Proof.* This is clear from the previous theorems and the fact that  $\mathsf{RCA}_0$  proves  $\mathrm{RT}_k^2 \Rightarrow \mathrm{RT}_{k+1}^2$  for all  $k \geq 2$ .

Corollary 6. The following are equivalent over  $\mathsf{RCA}_0$ .

1.  $\operatorname{RT}^2_{<\infty}$ . 2.  $\forall k \operatorname{RF}^{\mathbb{N}}_k$ . 3.  $\forall k \operatorname{RF}^2_k$ .

Next, we consider the remaining case, i.e. the strength of  $RF_k^1$ . In order to study  $RF_k^1$ , we consider the following version of Ramsey's theorem.

**Definition 7.** For a given function  $f : [\mathbb{N}]^n \to \mathbb{N}$ ,  $\mathrm{RT}_k^f$  is the following statement:

For any  $P : \mathbb{N} \to k$ , there exists an infinite set  $H \subseteq \mathbb{N}$  such that for any  $u, v \in [H]^n$ , P(f(u)) = P(f(v)).

If f is a bijection, we can prove the following.

**Proposition 8** (RCA<sub>0</sub>). For any  $n \in \mathbb{N}$  and any bijection  $f : [\mathbb{N}]^n \to \mathbb{N}$ ,  $\mathrm{RT}_k^f$  is equivalent to  $\mathrm{RT}_k^n$ .

The full version of  $\mathrm{RT}_k^f$ , i.e.  $\forall f : [\mathbb{N}]^n \to \mathbb{N} \mathrm{RT}_k^f$ , is still equivalent to  $\mathrm{RT}_k^n$ .

**Proposition 9** (RCA<sub>0</sub>).  $\operatorname{RT}_{k}^{n}$  is equivalent to  $\forall f : [\mathbb{N}]^{n} \to \mathbb{N} \operatorname{RT}_{k}^{f}$ .

*Proof.* From left to right is trivial, because  $P \circ f$  is a function from  $[\mathbb{N}]^n$  to k when  $P : \mathbb{N} \to k$ . From right to left is proved from the above proposition.

If f is not a bijection,  $\operatorname{RT}_k^f$  may not be equivalent to  $\operatorname{RT}_k^n$ . In case f is the subtraction  $\operatorname{Subt}(a,b) = b - a$ ,  $\operatorname{RT}_k^f$  is equivalent to  $\operatorname{RF}_k^1$ . (The function Subt is considered as a function of  $[\mathbb{N}]^2$ .)

**Proposition 10** (RCA<sub>0</sub>). For any  $k \in \mathbb{N}$ ,  $RF_k^1$  is equivalent to  $RT_k^{Subt}$ .

*Proof.* We first prove  $\operatorname{RF}_k^1 \Rightarrow \operatorname{RT}_k^{\operatorname{Subt}}$ . Assume  $\operatorname{RF}_k^1$  and let  $P : \mathbb{N} \to k$ . Define  $f : 1^{\leq \mathbb{N}} \to k$  by  $f(0^n) = P(n)$  and let v be a Ramseyan factorization for  $0^{\mathbb{N}}$  and f. Let  $X = \{\sum_{j \leq i} |v_j| \mid i \in \mathbb{N}\}$ . Then X is an infinite homogeneous set for  $P \circ \operatorname{Subt}$ .

Next, we prove  $\operatorname{RT}_k^{\operatorname{Subt}} \Rightarrow \operatorname{RF}_k^1$ . Assume  $\operatorname{RT}_k^{\operatorname{Subt}}$  and let  $f: 1^{<\mathbb{N}} \to k$ . Define  $P: \mathbb{N} \to k$  by  $P(n) = f(0^n)$ . Then there exists an infinite homogeneous set  $H := \{l_0 < l_1 < \cdots\} \subseteq \mathbb{N}$  for P. Define  $v \in (1^{<\mathbb{N}})^{\mathbb{N}}$  by  $v_0 = 0^{l_0}$  and  $v_i = 0^{l_i - l_{i-1}}$  for all  $i \ge 1$ . Then v is a Ramseyan factorization for  $0^{\mathbb{N}}$  and f.

From the above, we can show that  $\forall k \mathbf{RF}_k^1$  is strong enough to prove the bounding principle for  $\Sigma_2^0$  formulas.

Corollary 11 (RCA<sub>0</sub>).  $\forall k \operatorname{RF}_k^1$  implies  $B\Sigma_2^0$ .

*Proof.* Because of the above and the equivalence of  $B\Sigma_2^0$  and  $\operatorname{RT}_{<\infty}^1$ , it's enough to prove  $\operatorname{RT}_k^{\operatorname{Subt}} \Rightarrow \operatorname{RT}_k^1$  for all  $k \in \mathbb{N}$ . Assume  $\operatorname{RT}_k^{\operatorname{Subt}}$  and let  $P : \mathbb{N} \to k$ . Then there exists an infinite set  $H \subseteq \mathbb{N}$  such that for any  $u, v \in [H]^2$ ,  $P(u_1 - u_0) = P(v_1 - v_0)$ . Then  $X = \{h - \min H \mid h \in H \setminus \{\min H\}\}$  is an infinite homogeneous set for P.

**Question 12.** Is  $RF_k^1$  equivalent to  $RT_2^2$  or  $RT_k^1$ ?

# 3 Weak factorization

In this section, we consider a weaker version of Ramseyan factorization theorem. For applications in automata theory, the following weaker version of Ramseyan factorization theorem is usually good enough.

**Definition 13.** For given sets  $A, B \subseteq \mathbb{N}$ , weak Ramseyan factorization theorem for A and B (WRF<sup>A</sup><sub>B</sub>) is the following statement:

For any  $u \in A^{\mathbb{N}}$  and  $f : A^{<\mathbb{N}} \to B$ , there exists  $v \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$  such that  $u = v_0^{\frown} v_1^{\frown} \dots$  and for any i, j > 0,  $f(v_i) = f(v_j)$ .

Here, such v is said to be a weak Ramseyan factorization for u and f.

Similarly, we consider a weaker version of Ramsey's theorem as follows.

**Definition 14.** Pseudo Ramsey's theorem  $psRT_k^n$  is the following statement:

For any coloring  $P : [\mathbb{N}]^n \to k$ , there exists an infinite set  $H = \{a_0 < a_1 < \dots\}$  such that for any  $i, j \in \mathbb{N}$ ,  $P(a_i, \dots, a_{i+n-1}) = P(a_j, \dots, a_{j+n-1})$ .

Such H is called pseudo homogeneous set for P.  $^5$ 

**Remark 15.** In general, a subset of a pseudo homogeneous set might not be pseudo homogeneous again.

**Question 16.** Does  $psRT_k^n$  imply  $psRT_{k+1}^n$  over  $RCA_0$ ?

**Proposition 17 (RCA<sub>0</sub>).** For any  $m \in \mathbb{N}$ ,  $\mathrm{WRF}_m^{\mathbb{N}} \Leftrightarrow \mathrm{psRT}_m^2$ . In particular,  $\mathrm{WRF}_2^{\mathbb{N}}$  is equivalent to  $\mathrm{psRT}_2^2$ .

<sup>&</sup>lt;sup>5</sup> In Friedman/Pelupessy[4], this set is called adjacent homogeneous.

*Proof.* We first show for a given  $m \in \mathbb{N}$  that  $\operatorname{WRF}_m^{\mathbb{N}} \Rightarrow \operatorname{psRT}_m^2$ . Fix  $u = \langle i \mid i \in \mathbb{N} \rangle \in \mathbb{N}^{\mathbb{N}}$ . For a given coloring  $P : [\mathbb{N}]^2 \to m$ , define  $f : \mathbb{N}^{<\mathbb{N}} \to m$  by  $f(\sigma) = P(a, a + k)$  if  $\sigma = \langle a + i \mid i < k \rangle$  for some  $a, k \in \mathbb{N}, k \ge 1$ , and  $f(\sigma) = 0$  otherwise. Now, let v be a weak Ramseyan factorization for u and f. Then, one can easily check that the set  $H = \{\sum_{j \le i} |v_j| \mid i \in \mathbb{N}\}$  is a pseudo homogeneous set for P.

Next, we show  $m \in \mathbb{N}$ ,  $\operatorname{psRT}_m^2 \Rightarrow \operatorname{WRF}_m^{\mathbb{N}}$ . Let  $u \in \mathbb{N}^{\mathbb{N}}$ , and let  $f : \mathbb{N}^{<\mathbb{N}} \to m$ . Then, define a coloring  $P : [\mathbb{N}]^2 \to m$  by  $P(a,b) = f(\langle u_i \mid a \leq i < b \rangle)$ . Let  $H = \{a_0 < a_1 < \ldots\}$  be an infinite weak homogeneous set for P. Define  $v_0 = \langle u_i \mid 0 \leq j < a_0 \rangle$  and  $v_{i+1} = \langle u_j \mid a_i \leq j < a_{i+1} \rangle$ . Then, v is a weak Ramseyan factorization for u and f.

How about the case  $\text{WRF}_B^A$  with A finite? We can apply a similar argument to that in Theorem 4, but this time, we have to add extra colors.

# **Proposition 18** (RCA<sub>0</sub>). For any $k \in \mathbb{N}$ , WRF<sup>2</sup><sub>k+5</sub> implies psRT<sup>2</sup><sub>k</sub>.

*Proof.* Let  $w_i = 10^i \in 2^{<\mathbb{N}}$ , and let  $u = w_0 w_1 \ldots$  For a given coloring  $P : [\mathbb{N}]^2 \to k$ , we define a function  $f : 2^{<\mathbb{N}} \to k + 5$  as follows:

$$f(\sigma) = \begin{cases} P(m, n+2) & \text{if } \sigma = 0^i \cap w_m \cap \cdots \cap w_n^- 10^j \text{ for some } i, j \ge 0 \text{ and } 1 \le m \le n, \\ k & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \ge 0 \text{ such that } i \text{ and } j \text{ are both even}, \\ k+1 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \ge 0 \text{ such that } i \text{ is odd and } j \text{ is even}, \\ k+2 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \ge 0 \text{ such that } i \text{ is even and } j \text{ is odd}, \\ k+3 & \text{if } \sigma = 0^i 10^j \text{ for some } i, j \ge 0 \text{ such that } i \text{ and } j \text{ are both odd}, \\ k+4 & \text{otherwise.} \end{cases}$$

Take a weak Ramseyan factorization v for u and f, and let  $f(v_i) = d$  for all  $i \ge 1$ . If  $v_i$  contains at least one '1', then  $f(v_i) \ne k+4$ . Thus,  $d \ne k+4$ . If  $k \le d < k+4$ , then each  $v_i$  contains only one '1'. However, one can easily check that this is impossible. Therefore, for any  $i \ge 1$ ,  $f(v_i) = d$  for some d < k. This means that  $H = \{m \in \mathbb{N} \mid v_l = 0^i \widehat{w}_m \cdots \widehat{w}_n \widehat{w}_1 0^j$  for some  $i, j \ge 0, 1 \le m \le n$ , and  $l \ge 1\}$  is a pseudo homogeneous set for P.

Question 19. Is it possible to reduce the number of colorings in the above proof?

One of the reviewers told us that if we change the color "k + 4" to "0", the above proof still works without changing the weak Ramseyan factorization v. Therefore, thank to him or her, we can prove the following.

**Proposition 20** (RCA<sub>0</sub>). For any  $k \in \mathbb{N}$ , WRF<sup>2</sup><sub>k+4</sub> implies psRT<sup>2</sup><sub>k</sub>.

The following question still remains.

**Question 21.** Is  $WRF_2^2$  equivalent to  $psRT_2^2$  over  $RCA_0$ ?

# 4 The strength of $\operatorname{WRF}_k^{\mathbb{N}}$ , or equivalently $\operatorname{psRT}_k^2$

Our main goal in this section is to prove that  $WRF_2^{\mathbb{N}}$ , or equivalently  $psRT_2^2$ , is in between CAC and ADS. In order to show it, we use the facts that ADS is equivalent to  $trRT_2^2$ , transitive Ramsey's theorem for pairs, and CAC is equivalent to  $strRT_2^2$ , semi-transitive Ramsey's theorem for pairs, which were both proved in Hirschfeldt/Shore[6].

### Definition 22 (Transitive and semi-transitive colorings [6]).

- 1. A k-coloring  $P : [\mathbb{N}]^2 \to k$  is said to be transitive if  $P(a, b) = P(b, c) = i \Rightarrow P(a, c) = i$ .
- 2. A k-coloring  $P : [\mathbb{N}]^2 \to k$  is said to be semi-transitive if  $P(a,b) = P(b,c) = i > 0 \Rightarrow P(a,c) = i$ .

Now, we consider the following variations of Ramsey's theorem for pairs.

**Definition 23.** 1. Transitive Ramsey's theorem  $\operatorname{trRT}_k^2$ : Any transitive k-coloring  $P : [\mathbb{N}]^2 \to k$  has an infinite homogeneous set.

- 2. Semi-transitive Ramsey's theorem  $\operatorname{str}\operatorname{RT}_k^2$ : Any semi-transitive k-coloring  $P:[\mathbb{N}]^2 \to k$  has an infinite homogeneous set.
- 3. Semi-pseudo Ramsey's theorem  $\operatorname{spsRT}_k^2$ : Any k-coloring  $P : [\mathbb{N}]^2 \to k$  has an infinite homogeneous set H such that  $P([H]^2) = \{0\}$  or an infinite pseudo homogeneous set  $H' = \{h_0 < h_1 < \dots\}$  such that  $P(h_i, h_{i+1}) > 0$ .

Clearly,  $\text{spsRT}_k^2$  is a stronger version of  $\text{psRT}_k^2$ . First, we show the lower bound for  $\text{psRT}_2^2$ .

**Theorem 24** (RCA<sub>0</sub>). For any  $m \in \mathbb{N}$ ,  $psRT_m^2$  implies  $trRT_m^2$ .

*Proof.* If P is a transitive coloring, a pseudo homogeneous set for P is actually a homogeneous set for P.

Next, we consider the upper bound for  $psRT_2^2$ 

**Lemma 25** (RCA<sub>0</sub>). For any  $m \in \mathbb{N}$ , spsRT<sup>2</sup><sub>m</sub> implies strRT<sup>2</sup><sub>m</sub>.

*Proof.* If P is a semi-transitive coloring, a pseudo homogeneous set H for P with  $P([H]^2) \neq \{0\}$  is actually a homogeneous set for P.

The converse is true for the case m = 2.

Lemma 26 (RCA<sub>0</sub>). strRT<sub>2</sub><sup>2</sup> *implies* spsRT<sub>2</sub><sup>2</sup>.

Proof. Let  $P: [\mathbb{N}]^2 \to 2$ . We want to find a homogeneous set for 0, or a pseudo homogeneous set for 1. Define  $\overline{P}: [\mathbb{N}]^2 \to 2$  as follows:  $\overline{P}(a,b) = 1$  if there exists a sequence  $a = a_0 < \cdots < a_l = b$  such that  $P(a_i, a_{i+1}) = 1$  for any i < l, and  $\overline{P}(a,b) = 0$  otherwise. Then,  $\overline{P}$  is a semi-transitive coloring. Thus, by strRT<sub>2</sub><sup>2</sup>, take an infinite homogeneous set H for  $\overline{P}$ . If  $\overline{P}([H]^2) = \{0\}$ , then we have  $P([H]^2) = \{0\}$  and we have done. If  $\overline{P}([H]^2) = \{1\}$ , then for any  $a, b \in H$ , we can (effectively) find a sequence  $a = a_0 < \cdots < a_l = b$  such that  $P(a_i, a_{i+1}) = 1$  for every i < l. Thus, we can construct a set  $H' \supseteq H$  which is a pseudo homogeneous set for P with the value 1. Question 27. Over RCA<sub>0</sub>, does strRT<sup>2</sup><sub> $<\infty$ </sub> imply spsRT<sup>2</sup><sub> $<\infty$ </sub> or psRT<sup>2</sup><sub> $<\infty$ </sub>?

Although  $psRT_k^2$  might not prove  $psRT_{k+1}^2$ , we can show the following.

**Lemma 28** (RCA<sub>0</sub>). For any  $m \ge 2$ ,  $\operatorname{spsRT}_m^2$  implies  $\operatorname{spsRT}_{m+1}^2$ .

*Proof.* Let  $P : [\mathbb{N}]^2 \to m + 1$ . Define  $\overline{P} : [\mathbb{N}]^2 \to m$  by  $\overline{P}(a, b) = 0$  if  $P(a, b) \in \{0, 1\}$  and  $\overline{P}(a, b) = P(a, b) - 1$  if  $P(a, b) \ge 2$ . If  $\overline{P}$  has a pseudo homogeneous set with the value  $d \ge 1$ , then it is a pseudo homogeneous set for P. Otherwise,  $\overline{P}$  has a homogeneous set H with the value 0. Then,  $P \upharpoonright [H]^2$  is a 2-coloring, thus we can apply  $\operatorname{spsRT}_2^2$  again, and we have done.<sup>6</sup>

Combining the above, we have the following.

**Theorem 29.** The following are equivalent over  $RCA_0$ .

 $\begin{array}{ll} 1. & \mathrm{spsRT}_2^2. \\ 2. & \mathrm{strRT}_2^2. \\ 3. & \mathrm{spsRT}_k^2 \ for \ any \ k \in \omega, \ k \geq 2. \\ 4. & \mathrm{strRT}_k^2 \ for \ any \ k \in \omega, \ k \geq 2. \end{array}$ 

Thus, within  $RCA_0$ ,  $psRT_2^2$  is provable from any one of the above.

**Corollary 30** ( $RCA_0$ ).  $psRT_2^2$  is stronger than ADS and weaker than CAC.

*Proof.* By Hirschfeldt/Shore[6], ADS is equivalent to  $trRT_2^2$  and CAC is equivalent to  $strRT_2^2$ .

**Question 31.** Is  $psRT_2^2$  equivalent to ADS or CAC over  $RCA_0$ ?

Corollary 32 (RCA<sub>0</sub>).  $SRT_2^2$  does not imply  $psRT_2^2$ .

*Proof.* By Chong/Slaman/Yang[2],  $\text{SRT}_2^2$  does not imply COH. On the other hand, by Hirschfeldt/Shore[6], ADS implies COH, and thus  $\text{psRT}_2^2$  implies COH.

Corollary 33 (RCA<sub>0</sub>).  $psRT_2^2$  does not imply DNR.

*Proof.* By Hirschfeldt/Shore[6], CAC does not imply DNR, thus  $psRT_2^2$  does not, either.

**Question 34.** Does  $P_2^2$  or  $RWKL^{0'}$  imply  $psRT_2^2$ ? (See, e.g., Flood[3] for the definitions of these statements. Note that  $RWKL^{0'}$  is introduced as  $RKL^{(1)}$  in [3].)

## 5 Other topics

In this section, we focus on some other versions of Ramseyan factorization theorem.

 $<sup>^{6}</sup>$  Note that this argument still works for any *n*-tuples.

#### 5.1Stable versions

We can consider stable versions of RF or WRF. For given  $u \in \mathbb{N}^{<\mathbb{N}}$  and f:  $\mathbb{N}^{<\mathbb{N}} \to k$ , f is said to be *stable* on u if for any  $m \in \mathbb{N}$ , there exists n > m such that for any l > n,  $f(\langle u_i \mid m \leq i < n \rangle) = f(\langle u_i \mid m \leq i < l \rangle)$ . Then,  $\text{SRF}_k^A$  and  $SWRF_k^A$  are the following statements:

- Definition 35. 1. SRF<sup>A</sup><sub>k</sub>: For any u ∈ A<sup>N</sup> and f : A<sup><N</sup> → k such that f is stable on u, there exists a Ramseyan factorization for u and f.
  2. SWRF<sup>A</sup><sub>k</sub>: For any u ∈ A<sup><N</sup> and f : A<sup><N</sup> → k such that f is stable on u,
- there exists a weak Ramseyan factorization for u and f.

As in Theorems 3 and 4, we can show the following.

**Theorem 36.** Within  $\mathsf{RCA}_0$ , the following are equivalent for any  $m \in \mathbb{N}$ .

- $\begin{array}{ccc} 1. & \mathrm{SRT}_m^2.\\ 2. & \mathrm{SRF}_m^{\mathbb{N}}.\\ 3. & \mathrm{SRF}_m^2. \end{array}$

**Theorem 37.** Within  $\mathsf{RCA}_0$ , the following are equivalent for any  $m \in \mathbb{N}$ .

- 1. SWRT<sup>2</sup><sub>m</sub>: Any stable coloring  $P : [\mathbb{N}]^2 \to m$  has an infinite pseudo homogeneous set.
- 2. SWRF<sup> $\mathbb{N}$ </sup><sub>m</sub>.

#### 5.2Tree versions

In this subsection, we consider a slightly stronger version of  $RF_m^2$ . For given two trees  $T, S \subseteq 2^{<\mathbb{N}}$ , a tree embedding is an injective function  $\pi : S \to T$ such that for any  $\sigma, \tau \in S$ ,  $\pi(\sigma) \cap \pi(\tau) = \pi(\sigma \cap \tau)$ . For a given tree embedding  $\pi: S \to T$ , and for any  $\sigma, \tau \in S$  such that  $\sigma \subsetneq \tau$ , the edge between  $\pi(\sigma)$  and  $\pi(\tau)$ , denoted by  $E_{\pi}(\sigma,\tau)$ , is the sequence  $\rho \in 2^{<\mathbb{N}}$  such that  $\pi(\sigma) \cap \rho = \pi(\tau)$ . Then, we consider the following tree version of Ramsevan factorization theorem.

**Definition 38.** Ramseyan factorization theorem for trees  $\text{TRF}_k^2$  is the following statement:

For any infinite tree  $T \subseteq 2^{<\mathbb{N}}$  and a coloring  $f: 2^{<\mathbb{N}} \to k$ , there exists an infinite tree  $S \subseteq 2^{<\mathbb{N}}$  and a tree embedding  $\pi : S \to T$  such that for any  $\sigma \subsetneq \tau \in S \text{ and } \sigma' \subsetneq \tau' \in S, f(E_{\pi}(\sigma, \tau)) = f(E_{\pi}(\sigma', \tau')).$ 

**Proposition 39** (RCA<sub>0</sub>).  $\text{TRF}_k^2$  implies  $\text{RF}_k^2$  for all  $k \in \mathbb{N}$ . In particular,  $\text{TRF}_2^2$ implies  $RF_2^2$  (and, equivalently,  $RT_2^2$ ).

*Proof.* Assume  $\operatorname{TRF}_k^2$  and let  $u \in 2^{\mathbb{N}}$  and  $f: 2^{<\mathbb{N}} \to k$ . Define a tree  $T \subseteq 2^{<\mathbb{N}}$  by  $T = \{u_0 u_1 \dots u_{i-1} \mid i \in \mathbb{N}\}$ . By  $\operatorname{TRF}_k^2$ , there exist  $S = \{s_0 < s_1 < \dots\} \subseteq 2^{<\mathbb{N}}$  and an embedding  $\pi: S \to T$  such that for all  $\sigma \subsetneq \tau \in S$  and  $\sigma' \subsetneq \tau' \in S$ ,  $f(E_{\pi}(\sigma,\tau)) = f(E_{\pi}(\sigma',\tau'))$ . Define  $v \in (2^{<\mathbb{N}})^{\mathbb{N}}$  by setting  $v_0 = \pi(s_0)$  and  $v_i =$  $E_{\pi}(s_{i-1}, s_i)$  for all  $i \geq 1$ . Then, v is a Ramseyan factorization for u and f.

We can also show that  $\mathrm{TRF}_2^2$  is weaker than  $\mathsf{WKL}_0 + \mathrm{RT}_2^2$ .

**Proposition 40.**  $WKL_0 + RT_2^2$  implies  $TRF_2^2$ .

*Proof.* Let  $T \subseteq 2^{<\mathbb{N}}$  be an infinite tree and  $f: 2^{<\mathbb{N}} \to 2$ . By WKL<sub>0</sub>, there is an infinite path  $u \in 2^{\mathbb{N}}$  through T. By  $\operatorname{RF}_2^2$ , which is equivalent to  $\operatorname{RT}_2^2$ , there is a Ramseyan factorization  $v \in (2^{<\mathbb{N}})^{\mathbb{N}}$  for u and f. Define  $S \subseteq 2^{<\mathbb{N}}$  and  $\pi: S \to T$  by  $S = \{0^i \mid i \in \mathbb{N}\}$  and  $\pi(0^i) = v_0^\frown v_1^\frown \cdots \frown v_i$  for all  $i \in \mathbb{N}$ . Then S and  $\pi$  satisfy the condition.

Therefore,  $\text{TRF}_2^2$  is in between  $\text{WKL}_0 + \text{RT}_2^2$  and  $\text{RT}_2^2$ .

**Question 41.** Does  $\mathrm{TRF}_2^2$  imply WKL<sub>0</sub> over RCA<sub>0</sub>?

**Remark 42.**  $\text{TRF}_2^2$  may be equivalent to the following stronger version of  $\text{RT}_2^2$ :

 $\operatorname{RT}_{2}^{2+}$ : If  $\mathcal{P}$  be a class of colorings  $P: [F_P]^2 \to 2$  where  $F_P = \{0, 1, \ldots, l\}$ for some  $l \in \mathbb{N}$ , then there exists an infinite set  $H \subseteq \mathbb{N}$  such that there exist infinitely many  $P \in \mathcal{P}$  such that P is constant on  $[H \cap F_P]^2$ .

We think that the equivalence should hold, but we do not know either  $\text{TRF}_2^2 \Rightarrow$  $\text{RT}_2^{2+}$  or  $\text{RT}_2^{2+} \Rightarrow \text{TRF}_2^2$ . This kind of strengthened Ramsey's theorem is studied in [10].

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