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Description	

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY

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LETTER

Simple Derivation of the Lifetime and the Distribution of Faces for a Binary Subdivision Model

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SUMMARY The iterative random subdivision of rectangles is used as a generation model of networks in physics, computer science, and urban planning. However, these researches were independent. We consider some relations in them, and derive fundamental properties for the average lifetime depending on birth-time and the balanced distribution of rectangle faces.

key words: *complex network science, iterative subdivision, random binary tree, road network, generation of dungeon*

1. Introduction

A mathematical model of networks is useful in physics, computer science, urban planning, etc. There are many methods for constructing spatial networks by applying a growing rule or optimization. One of the attractive methods is based on a recursive geometric growing rule for the division of a chosen triangle [1]–[4] or for the attachment which aims at a chosen edge [5]–[7] in a random or hierarchical selection. In particular, the fractal-like networks [8] generated by iterative subdivision of equilateral triangle or square faces are more efficient with shorter link lengths and more suitable with lower load for avoiding traffic congestion than the state-of-the-art complex networks. These typical complex networks are geometric growing models [1]–[7] and the spatially preferential attachment models [9]–[11] with various topological structures ranging from river to scale-free geographical networks [12]. By contrast, the advantages of the fractal-like networks are due to the bounded path lengths by the t-spanner property [13] and the small degrees of nodes without overloaded hubs [8]. The subdivision of squares [8] is generalized to the subdivision of rectangles into four or two smaller faces [14], [15]. Such a binary subdivision of rectangle faces is related not only to a self-organization of networks [14], [15] in complex network science but also to an object generation in computer graphics, e.g. the map L-system [20] for road network generation in urban modeling [21]–[23] and the space partitioning for dungeon generation in a role playing game (RPG) [24].

In addition, the hierarchical structure defined by inclusion relations of faces is equivalent to a binary tree. Binary tree [18] is a well-known date structure for sorting numbers in computer science. It is assumed that the input stream of query is a permutation of the integers $1, 2, \ldots, N$, whose orderings are at random in a general problem setting. For the search task, an integer of the input is inserted at a leaf as the terminal node that satisfies the ordering condition in any path starting at the root.

In spite of the above potential connections, these researches were independent. Moreover, the theoretical analysises for a random binary tree [16], [17] are a little difficult and probably unknown except in a community for mathematicians.

Thus, in this paper, we aim

- to make the derivation of fundamental properties more easily understandable
- to discuss some relations among the findings in the above different research fields

for the iteratively random subdivision.

2. Binary Subdivision Model

Let us consider the following subdivision model.

Step 0: Set an initial face of rectangle.
Step 1: At each discrete time $t = 1, 2, 3, \ldots$, chose a rectangle uniformly at random.
Step 2: The chosen rectangle face is divided by a line into smaller two ones which is called as twin faces.
Step 3: Until the break of a given condition, return to Step 1 at the next time.

In Step 3, for example, we consider a condition: the total number of faces is smaller than a given size.

To simplify the discussion without loss of the fundamental properties, we ignore the area ratio of the divided rectangle faces, therefore we do not care the edge lengths of the divided rectangle face by a bridge line over the chosen face. Conceptually, the stochastic subdivision of faces is equivalent to a random binary tree as shown in Fig. 1, although we do not discuss a search problem for random queries. The leaves in a random binary tree represent the rectangle faces, which are classified into adjacent twin faces generated at a same time and the other faces.

3. Average Lifetime of Faces

We denote $n_s(t)$ as the number of rectangle faces counted at time t for whose birth-time is s. In other words, the birth-time is the generation time of twin faces by subdivision. Although $n_s(t)$ is an integer 0, 1, or 2, in each sample of the

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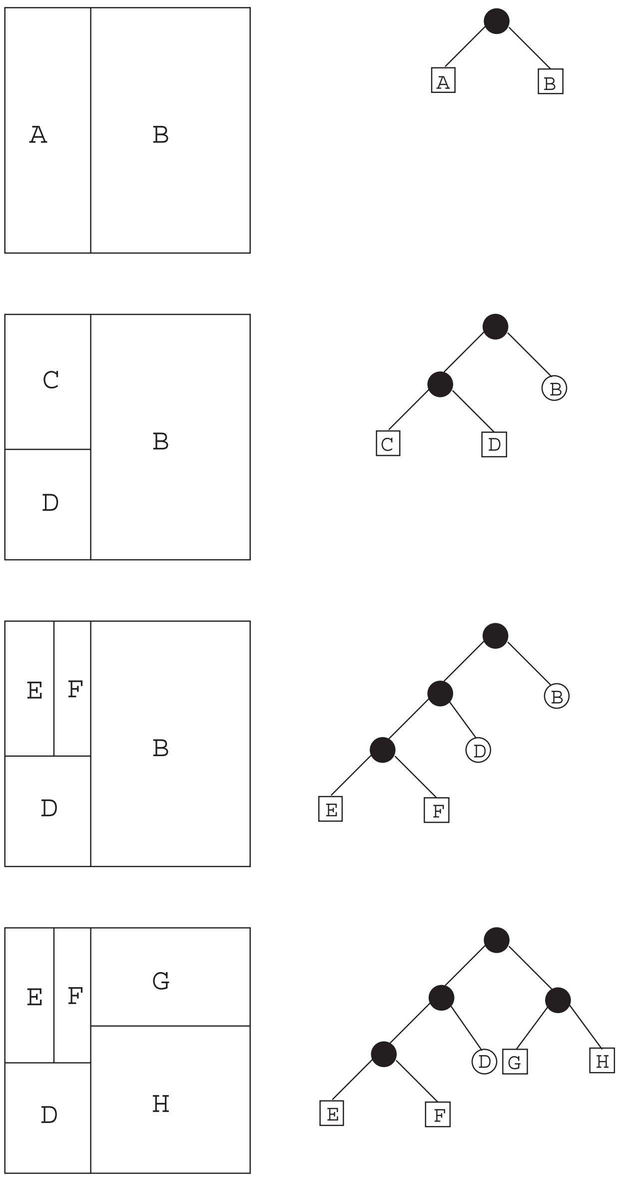

Fig. 1 Generation process at $t = 1, 2, 3$ and 4 from top to bottom. (Left) random subdivision of face. (Right) the corresponding binary tree. The leaf nodes marked by square and open circle represent the twin faces and the other faces, called as feets and arms [16], respectively. A, B, . . ., F denotes an identifier of each face.

stochastic process, we consider the average behavior over many samples. The rate for choosing a face with the birth-time s is proportional to $n_s(t)$ because of the uniformly random selection. Thus, we obtain the expectation

$$n_s(t+1) - n_s(t) = -\frac{n_s(t)}{t+1},$$

and rewrite it to

$$n_s(t+1) = \left(1 - \frac{1}{t+1}\right) n_s(t), \quad 1 \le s \le t, \tag{1}$$

where the total number of faces at time t is exactly $t+1$. The initial configuration is one face, $t = 0$.

By recursively applying the difference Eq. (1) with the initial condition $n_s(s) = 2$, we derive

$$\begin{aligned} n_s(t) &= \Pi_{i=s}^{t-1} \left(1 - \frac{1}{i+1}\right) \times 2 \\ &= \frac{2s}{s+1} \frac{s+1}{s+2} \cdots \frac{t-2}{t-1} \frac{t-1}{t} = \frac{2s}{t}. \end{aligned} \tag{2}$$

For the average number of faces, our new result of Eq. (2) gives the time-course of decaying by $1/t$ with the dependency on the birth-time s.

We consider the expectation time t' when the average number of faces becomes one for the twin faces with a birth-time s. Since we obtain $t' = 2s$ from $n_s(t') = 1$ in Eq. (2), the average lifetime of more than one face after the births is

$$\Delta t_s = t' - s = s.$$

Thus, younger faces with a larger s have a longer lifetime. We emphasize that this iterative stochastic subdivision is not a Poisson process assumed in the analysis for a random quadtree model [19], because the selection probability of a face is decreasing as time passes even with uniform randomness in increasing the total number of faces. Therefore, younger faces have less chances for the selection. This is not independent and identically distributed.

We consider the case that both twin faces generated at time s remain at next time $s+1$. The probability for the unselection in the total $s+1$ faces is

$$1 - \frac{2}{s+1} = \frac{s-1}{s+1}. \tag{3}$$

Until time t, the remaining probability is given by the product of Eq. (3)

$$\begin{aligned} q_s(t) &= \Pi_{i=s+1}^{t} \frac{i-2}{i} \\ &= \frac{s-1}{s+1} \frac{s}{s+2} \frac{s+1}{s+3} \cdots \frac{t-4}{t-2} \frac{t-3}{t-1} \frac{t-2}{t} \\ &= \frac{s(s-1)}{t(t-1)}. \end{aligned}$$

By summing $q_s(t)$ for all twin faces with birth-times $s = 2, 3, \ldots, t-1$, we derive the rate of twin faces

$$\sum_{s=2}^{t-1} \frac{s(s-1)}{t(t-1)} = \frac{1}{t(t-1)} \left(\sum_{s=2}^{t-1} s^2 - \sum_{s=2}^{t-1} s \right) \approx \frac{t}{3}, \tag{4}$$

where we use the formulas $\sum_{s=2}^{t-1} s^2 = \frac{(t-1)t(2t-1)}{6} - 1$ and $\sum_{s=2}^{t-1} s = \frac{(t-1)t}{2} - 1$. The approximation is valid for a large t. Since each pair of the twin has two faces, the number of faces is $2t/3$ averagely. Thus, the rate of other faces is $t/3$. These rates are consistent with the asymptotical result [16] derived by a complicated analysis, and related to the existing rate $n/3$ of nodes with degrees 1, 2, and 3 [18] in a random binary tree. Here, $n = 2t + 1$ denotes the number of nodes including leaves with degree 1, non-terminal nodes with degrees 2 or 3, and a root with degree 2. From the rate of these degrees [18], the expected tree has a balanced shape without too deep layers by the dominant long chains of node degree 2. The balanced tree corresponds to a bell-shape of the Poisson distribution as mentioned in the next section.

4. Distribution of Layered Faces

Next, we consider the distribution of layered faces. Faces on the l-th layer are one-to-one corresponding to the leaves at the depth l in the binary tree. The number of faces on the l-th layer can increase until 2^l. Figure 2 shows an example of the layer.

We denote $N_l(\tau)$ as the number of faces that belong to the l-th layer generated after the selections of l times on the descendant from the initial face. As mentioned in the appendix of Ref. [14], we consider the random process by subdivision in the following continuous-time approximation

$$\frac{dN_l(\tau)}{d\tau} = 2 \times N_{l-1}(\tau) - N_l(\tau), \quad l \geq 1, \tag{5}$$

$$\frac{dN_0(\tau)}{d\tau} = -N_0(\tau). \tag{6}$$

The self-similarity in the iterative subdivision of squares [14] does not affect the analysis of the distribution, because we treat only the number of faces on each layer without dependence on the shapes.

The solutions of Eqs. (5), (6) are $N_0 = e^{-\tau}$ and

$$N_l(\tau) = 2^l \frac{\tau^{l-1}}{(l-1)!} e^{-\tau}, \quad l \geq 1.$$

The total number of faces is given by

$$\mathcal{N}(\tau) = \sum_l N_l(\tau) = 2e^{\tau},$$

where we use the Taylor series expansion $\sum_l \frac{(2\tau)^{l-1}}{(l-1)!} = e^{2\tau}$. Therefore, from $p_l(\tau) = N_l(\tau)/\mathcal{N}(\tau)$, we obtain the Poisson distribution with a parameter 2τ

$$p_l(\tau) = \frac{(2\tau)^{l-1}}{(l-1)!} e^{-2\tau}, \quad l \geq 1. \tag{7}$$

Note that p_l is a function of variable l, and τ is a auxiliary variable to take a temporal snapshot. The mean and the variance of p_l follow $2\tau \propto \log t$ by the variable transformation between a linear time variable t and a logarithmic time variable τ from the relation for the total number of faces $1 + t = 2e^{\tau}$ [14]. Of course they show the asymptotic behavior for a large t.

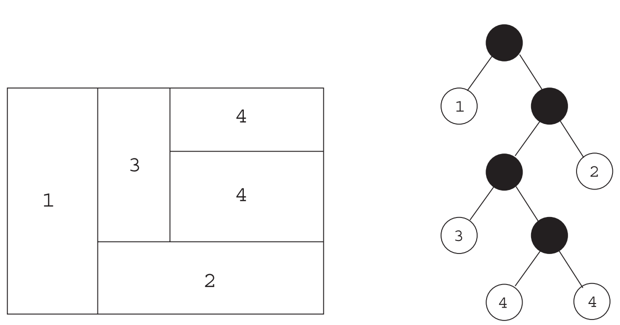

Fig. 2 Example of the layer numbered by $l = 1, 2, 3$ and 4. (Left) layered faces. (Right) the corresponding binary tree.

Thus, in the bell-shaped Poisson distribution of layers, the peak position for the most majority of layers shifts to be deeper, and the width becomes wider as the divisions are iterated. Even if the expanding property can be qualitatively predicted, the logarithmic time-course is not trivial. This simply analyzed property of $\log t$ is also related to the deepest level in a random binary tree [17]. However, the Poisson approximation of distribution was not derived.

5. Conclusion and Discussion

By using analytical approaches of difference and differential equations, we have more easily derived the fundamental properties for the average lifetime of faces and the distribution of layered faces in the iteratively random binary subdivision of rectangles which is usually treated as a discrete mathematical problem. We remark that Eqs. (2), (4), (7) hold in more general case divided by non-vertical and non-horizontal lines with any angles because of no relation to area and shape of faces.

Our obtained results will be useful for generating road networks in virtual cities [21] and dungeons in a RPG [24], automatically. In a dungeon generation, the layered faces is applicable to a design of corridor placement [24]. In particular, rooms assigned to twin faces are connected with a corridor. Eq. (4) suggests that such rooms exist averagely in 2/3 of the whole rooms for the uniformly random divisions, and a game player can directly wander back and forth between them. When we consider a preferential selection of face according to the depth of layer [19] instead of the uniformly random selection, we can control the rate of twin faces. The rate becomes larger as a shallow face is chosen for the division, then balanced similar depths appear. In contrast, it becomes smaller as a deeper face is chosen, then unbalanced various depths appear. So, the rate of 2/3 gives a baseline. On the other hand, the layered faces represent a historical trace of the construction in a road network. Generally, an area of face is smaller as the layer becomes deeper. Thus, long-range access roads tend to be constructed at first, thereafter short-range lanes tend to be added by little and little. A bridge lane that produces twin faces by subdivision may be related to increasing the efficiency of traffic through bypaths on the road network. In the modeling of road networks, we can generate both T-shaped and +-shaped intersections by using a probabilistic selection with a constant mixing rate for quartered and binary divisions of faces, instead of the uniformly random selection. Conversely, the mixing rate of quartered and binary divisions may be estimated from real data of road networks. These base line, historical trace, and mixing rate can be also discussed in complex network science.

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