Title	動的認識論理のラベル付きシークエント計算の研究
Author(s)	野村，尚新
Citation	
Issue Date	2016-03
Type	Thesis or Dissertation
Text version	ETD
URL	http://hdl.handle.net/10119/13517
Rights	
Description	Supervisor:東条　敏，情報科学研究科，博士

Doctoral Dissertation

Labelled Sequent Calculi for Dynamic Epistemic Logics

Shoshin Nomura

Supervisor: Satoshi Tojo

School of Information Science
Japan Advanced Institute of Science and Technology

(Degree conferment March 2016)

Abstract

Dynamic Epistemic Logic (DEL) is a field of modern epistemic logic that aims at formally expressing the change of human knowledge through modifying Kripke models which represent the state of agents' knowledge. For example, if an agent called John does not know if it will rain tomorrow and he gets information from a weather forecast on TV which informs that it will rain tomorrow, then he is now not ignorant of the condition of tomorrow's weather (i.e., his knowledge-state was changed by the information on TV). This is a typical example of public announcement (public information), and *Public Announcement Logic* (PAL) by Plaza (1989) can formally express such a situation regarding the knowledge-change of agents. PAL became the basis of other DELs and we also started to investigate labelled sequent calculus from PAL. In addition to public announcements, information is not always shared among all agents (it is not always public) and it is totally possible to imagine that some information is for only a single agent (private announcement) or for a specific group of agents. PAL can cope with only public announcement (information), but *Logic of Epistemic Actions and Knowledge* (EAK) by Baltag et al. (1989) is a logic for the formal expression of such information flows which are more delicate than public announcements. EAK is a generalized and developed version of PAL and our second target is this epistemic logic. On the other hand, the term knowledge has philosophically profound meanings, and historically, the notion of knowledge contains evidence or verification to justify one's belief. Intuitionistic epistemic logics are candidates which express knowledge in a strict sense. Based on the intuitionistic modal logic IK by Fischer Servi (1984) and Simpson (1994), *Intuitionistic PAL* (IntPAL)—an intuitionistic version of PAL—is proposed by Ma et al. (2014), and this enables us to express the change of knowledge defined in a strict sense.

In this thesis, we provide three different cut-free labelled sequent calculi for PAL, EAK and IntPAL respectively. First, we investigate an existing labelled sequent calculus for PAL and this investigation becomes an important foundation for the three labelled sequent calculi of ours with respect to the soundness theorems, the completeness theorems and the cut-elimination theorems for other labelled systems. A labelled sequent calculus **G3PAL** for PAL is provided by Maffezioli and Negri (2011), but it in fact lacks inference rules for deriving an axiom of the Hilbert-system of PAL. So, we provide our revised calculus **GPAL**, and all the formulas derivable by Hilbert-system of PAL are also derivable in **GPAL** together with the cut rule. We also establish the cut elimination theorem. Moreover, we show the soundness of our calculus for Kripke semantics with the notion of surviveness of possible worlds in a restricted domain. Then we provide a direct proof of the semantic completeness of GPAL for the link-cutting semantics of PAL.

Secondly, we move onto EAK based on the study of labelled sequent calculus for PAL. We also provide a cut-free labelled sequent calculus (**GEAK**) on the background of existing studies of the Hilbert-system (we call it **HEAK**) and labelled calculi for PAL. Similar to the previous procedure, we first show that all the formulas derivable by the Hilbert-system of EAK are also derivable in **GEAK** with the cut rule, and we show

that the cut rule is eliminable in **GEAK**. Then we show **GEAK** is sound for Kripke semantics. After demonstrating that soundness, we derive the semantic completeness of **GIntPAL** as a corollary of these theorems

Thirdly and lastly, we introduce a labelled sequent calculus **GIntPAL** for IntPAL. Following the same manner of the construction of a labelled sequent calculus as the previous two, we show that all theorems of the Hilbert-system of IntPAL are also derivable in **GIntPAL** with the cut rule. Then we prove the cut-eliminability of **GIntPAL** and also its soundness for birelational Kripke semantics, and so its completeness for the semantics.

Keywords— Dynamic Epistemic Logic, Public Announcement Logic, Intuitionistic Public Announcement Logic, Logic of Epistemic Actions and Knowledge, Labelled Sequent Calculus, Admissibility of Cut, Validity of Sequents

Contents

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List of Notations

Acronyms

ML	Multi-modal Logic
PAL	Public Announcement Logic
EAK	Logic of Epistemic Actions and Knowledge
IntPAL	Intuitionistic PAL

Set theoretic notations

$x \in X$	the membership relation of x and X	$\varnothing$	the emptyset
$X \subseteq Y$	X is a subset of Y	$\mathcal{P}(X)$	the power set of X
$X \cup Y$	the union of X and Y	$R \circ Q$	the composition of R and Q
$X \cap Y$	the intersection of X and Y	id_X	the identity relation on X
$X \times Y$	the Cartesian product of X and Y	$\mathbb{N}$	the set of all natural numbers
$X \setminus Y$	the relative complement X of Y		

Common notations

$a, b, ... \in \mathsf{Agt}$	agents	$R_a, R_b, ...$	accessibility relations
$p, q, ... \in \mathsf{Prop}$	propositional atoms	$\mathfrak{F}, \mathfrak{F}', ...$	Kripke frames
$A, B, ... \in \mathcal{L}$	formulas	$\mathfrak{M}, \mathfrak{M}', ...$	Kripke models
$x, y, ... \in \mathsf{Var}$	variables	$\mathbb{F}, \mathbb{F}', ...$	classes of Kripke frames
$w, v, ...$	worlds	$\mathbb{M}, \mathbb{M}', ...$	classes of Kripke models
$W, W', ...$	sets of worlds		

Notations for PAL and IntPAL

$\alpha, \beta, ...$	finite lists of formulas	$x\mathsf{R}_a^{\alpha}y$	relational atom
$x{:}^{\alpha}A$	labelled formula	$\mathfrak{A}, \mathfrak{B}, ...$	labelled expressions

Notations for EAK

$\mathsf{a}, \mathsf{b}, ... \in \mathsf{Act}$	actions	$\alpha, \beta, ...$	finite lists of pointed action models
$\mathsf{x}, \mathsf{y}, ... \in \mathsf{CVar} \subseteq \mathsf{Act}$	meta-variables for actions		
$\mathsf{S}, \mathsf{S}', ...$	finite sets of actions	$\langle x, \alpha\rangle{:}A$	labelled formula
$\sim_a, \sim_b, ...$	action relations	$\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$	relational atom
$\mathsf{M}, \mathsf{N}, ...$	action models	$\mathfrak{A}, \mathfrak{B}, ...$	labelled expressions
$\mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}, ... \in \mathsf{PAct}$	pointed action models		

Acknowledgement

This thesis would not be possible without the devoted support of my supervisors, Satoshi Tojo and Katsuhiko Sano. Their enthusiastic guidance enabled me to enter the field of knowledge-revision and modal logic and to realize the aim of my research in the Ph.D. program, and they taught me what a researcher should be. I wish, in particular, to thank assistant Prof. Sano for teaching modal logic from the basics and considering the essential parts, such as an adequate validity for a labelled sequent for DELs and the application of Simpson's lifting lemma in the proof of IntPAL's soundness. We used this information in our papers which became the basis of this thesis. I would like to thank my supervisor of minor research, Hiroakira Ono. He gives a valuable opportunity to develop and discuss the idea regarding the third paper of Dynamic Epistemic Logic, and I had unforgettable and fun times with him. I am immensely indebted to members of my Ph.D. defense, Makoto Kanazawa, Ryo Kashima, Hajime Ishihara and Norbert Preining. Their critical insight and suggestions improve my thesis and help to enhance my understanding of logic. I also thank Prof. Kanazawa for not only the defense but giving us valuable advice of the link-cutting semantics in the first paper of ours. I wish to thank to Hiroaki Suzuki, Shinya Hirose and Takafumi Kodama for supporting my coding, Sean Arn for helping to edit English documents, and all the other members of Tojo laboratory for encouraging me in the Ph.D. program. Without their help, this thesis would not be completed.

Chapter 1

Introduction

The prelude to this thesis is modal logics. In the late 50s to the early 60s, Kripke [46, 47] provided *Kripke semantics* to modal logic.[1] This semantics provides an adequate interpretation for a formal expression of 'necessarily' and 'possibly.' In modal logic, 'it is possible that A,' may be expressed by formula $\Box A$, and in Kripke semantics, it is intuitively interpreted as 'for all possible worlds which are accessible from a specific world, A holds.' This semantics gives both formal and intuitive understandings to the once mysterious notion of modality. Since then, the field of modal logic has flourished and deepened studies of several kinds of modalities, such as temporal modalities [70], deontic modalities [43], doxastic modalities [36] and epistemic modalities. We are, in this thesis, concerned with epistemic logic which handles the last modalities, epistemic modalities.

1.1 Epistemic Logic and belief-revision

Epistemic logic is a logic which aims at formalizing knowledge, and has been developed by several logicians and philosophers, e.g., von Wright [86] and Hintikka [35]. The initial motivation for the study of this logic was to contribute to the field of philosophical epistemology since the concept of the formalization of knowledge (or belief) through formal languages suits the spirit of analytic philosophy in the early 20th century where Anglo-American philosophers discussed the reasonable measure for modern philosophy and aimed for expulsion of traditional metaphysics. Although epistemology in analytic philosophy gradually strengthened academic relationship with other natural sciences such as neuroscience and biology and left the formalization with modal logic, another movement of formalization of knowledge and belief was, instead, started in a different area, computer science. In the 80s, since the performance of computers became much more powerful than before, the study of artificial intelligence blossomed, and several works were submitted; for example, non-monotonic logics (default logic

[1] At the same period, similar semantics independently was given by Kanger [39], Hinttika [35], Montague [55] and Prior [70], and a closely related study had already been given by Jonsson and Tarski [37, 38] about ten years ago (see more detail in [32, 74]).

by Reiter [72], circumscription by McCarthy [52]) and belief-revision by Levi [48] and Gärdenfors and Makinson [29]. This area is called belief-revision (knowledge-revision). However, another tradition of formalization of knowledge, epistemic logic, was not popular in that field.

1.2 Dynamic Epistemic Logic (DEL)

In the last century, there were two movements of formalizing knowledge—philosophical epistemology and belief-revision— but these two had not deeply connected with each other. From the 90s, the movement of knowledge-revision (belief-revision) became conspicuous in epistemic logic. Specifically, *Public Announcement Logic* (PAL) by Plaza [68] and *Logic of Epistemic Actions and Knowledge* (EAK) by Baltag et al. [8] (elaborated in several papers, e.g., [8, 31, 81, 83]) [2] are outstanding studies, and they formed the field of Dynamic Epistemic Logic (DEL). Today, a number of followers have developed and refined this area for a formalization of knowledge, and have been specializing DEL to apply it to artificial intelligence, epistemology in philosophy, theoretical economics, formalizing law, and so on.

Public Announcement Logic Public Announcement Logic (PAL) was first presented by Plaza [68], and it has been the basis of Dynamic Epistemic Logic. PAL is a logic for formally expressing changes of human knowledge. When we obtain some information through communication with others, our state of knowledge may change. For example, if 'John does not know whether it will rain tomorrow or not' is true and he gets information from the weather forecast which says that 'it will not rain tomorrow,' then the state of John's knowledge changes, so 'John knows that it will not rain tomorrow' becomes true. While a Kripke model of the standard epistemic logic stands for the state of knowledge, the standard epistemic logic does not have any syntax for properly expressing changes of the state of knowledge. PAL was introduced for the purpose of dealing with the flexibility of human knowledge, and the change of knowledge formally realized by the announcement operator which can restrict possible worlds of Kripke semantics. A formula $[A]B$ of PAL reads 'after an announcement of A, B holds.'

Logic of Epistemic Actions and Knowledge (EAK) Another foundation of the field of DEL is Logic of Epistemic Actions and Knowledge (EAK) provided by [8]. EAK is a developed version of PAL; as the name PAL shows, it deals mainly with 'public announcements,' by which every agent shares the same information; however, the state of knowledge may be changed not only by public announcements but also announcements to a specific group in a community. A typical example is 'private announcements,' in which someone communicates something to only a single person (e.g., a personal letter). An extension of PAL, EAK is a logic which can express not only public announcements, but more delicate and complicated flows of information such as private

[2] The original EAK by [8] has reformed and improved until today, and this is sometimes called by different names, e.g., *Dynamic Epistemic Logic* (in a narrow sense) and *Action Model Logic* [83]. In fact, we basically follow the definitions of this logic introduced in [83] from the next chapter, but we employ the original name by [8].

announcement, and such a factor that causes a change of knowledge state is called an *action* (or *event*) as a whole. Technically, the notion of action a is defined with the action model which is almost the same as the Kripke model with a finite domain. Interestingly or oddly, an action model differs from a Kripke model and belongs to the syntax field of EAK. A formula $[\mathsf{a}]B$ of EAK reads 'after an action a occurs, B holds.'

1.3 Sequent calculi for modal logics

Sequent calculus is another principal of this thesis. We mainly refer to the survey paper of Negri [56] and Bednarska and Indrzejczak [9] for this section regarding sequent calculi of modal logics. Sequent calculus for propositional logic LK was given by Gentzen [30], and it has also applied to the proof theory of modal logics, and the reader can find sequent calculi for the modal logics K, T, S4, and S5 in the introduction of Ono [66]. The simple and standard sequent calculus for modal logic K includes the following additional inference rule to the sequent calculus LK for classical propositional logic,

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ (\Box).$$

Modal logics T, S4 and S5 include other additional rules respectively, and cut-elimination theorems of sequent calculi for some systems (in particularly S4) are established by Ohnishi and Matsumoto [62]. Also contraction-free calculi (called a G3-system, and we will see it in Section 2.1.3) for some systems are constructed respectively by Troelstra and Schwichtenberg [80]. However, in fact, the sequent calculus for modal logic S5 is problematic since Ohnishi and Matsumoto [63] also showed that the cut-elimination does not hold in this standard sequent calculus for S5. This crucial failure in S5 led several studies of the sequent calculus for modal logic. In this movement in the 70s and 80s, Mints [54] and Sato [75, 76] independently provided a cut-free calculus for S5, but they are fairly complicated and contain the problem with the subformula property. Shvarts [77] provided a cut-free sequent calculus for modal logic K45 and also showed that formula A is derivable in modal logic S5 iff $\Box A$ is derivable in modal logic K45; so it can be said that he gave an indirect solution of cut elimination of S5. In the same paper [77], he also provided a cut-free system for KD45. Moreover, many other researchers attempted to construct an adequate cut-free sequent calculus for modal logics including S5. For example, display calculus [10], nested sequent calculus [40, 79, 14], hypersequent calculus [69, 5], labelled sequent calculus and so on. In the following, we briefly introduce one of such new systems called labelled sequent calculus.

Labelled sequent calculus An original idea of labelled sequent calculus can be found in Kanger [39], where a sequent for S5 consists of formulas with natural numbers and this formula is called *spotted formula* $A^m (m \in \mathbb{N})$[3]. Modern labelled sequent calculus was explained in Negri and Plato [58] introduces a special syntactic object (it enriches the syntax) called *labelled formula*. The basic idea underlying this calculus

[3]The author is grateful to Hiroakira Ono who lent him Kanger's precious original reference [39] and told the origin of labelled formula.

is to internalize notations of the standard semantics of modal logic (Kripke semantics) into the enriched syntax. In other words, this enriched syntax includes a label consisting of variable x which corresponding to possible world w in Kripke semantics. A labelled formula $x{:}A$ which is a formula A with label x corresponds to a satisfaction relation $\mathfrak{M}, f(x) \Vdash A$ where $\mathfrak{M}$ is a Kripke model and f is a function which assigns a world to a variable. Moreover, this calculus includes another special syntax $x\mathsf{R}y$ called a *relational atom* (where x, y are labels). As one can imagine, the relational atom corresponds to accessibility relation $f(x)Rf(y)$ in Kripke semantics where f is the function as above. By importing special notations corresponding to semantic notations, inference rules of labelled sequent calculus are directly obtained from the definition of satisfaction relation. For example, given a Kripke model $\mathfrak{M} = (W, R, V)$, the definition

$$\mathfrak{M}, w \Vdash \Box A \text{ iff } wRv \text{ implies } \mathfrak{M}, v \Vdash \Box A \text{ for all } v \in W.$$

Since labelled sequent calculus has syntactic notations corresponding semantic notations, labelled formula $x{:}\Box A$ (corresponding to $\mathfrak{M}, f(x) \models A$) can be intuitively interpreted as an implication of $x\mathsf{R}y \to y{:}A$ for all y (corresponding to $f(x)Rf(y)$ implies $\mathfrak{M}, f(y) \Vdash A$); therefore, in such a labelled system, inference rules can be easily extracted from the definition of satisfaction relation in Kripke semantics. Let us look at inference rules for $\Box$ operator:

$$\frac{x\mathsf{R}y, \Gamma \Rightarrow y{:}\Box A, \Delta}{\Gamma \Rightarrow x{:}\Box A, \Delta}\ (R\Box)$$

where y does not appear in the lower sequent, and

$$\frac{y{:}A, x{:}\Box A, x\mathsf{R}y, \Gamma \Rightarrow \Delta}{x{:}\Box A, x\mathsf{R}y, \Gamma \Rightarrow \Delta}\ (L\Box).$$

As you can see, these rules are obtained from the idea that $x{:}\Box A$ is an implication of $x\mathsf{R}y \to y{:}A$ for all y.

Moreover, labelled sequent calculus can internalize frame properties such as reflexivity, symmetricity, transitiveness etc. quite easily as well. Assume we have a labelled calculus for modal logic K, and we can obtain a calculus for modal logic T by adding the following inference rules into the set of inference rules of it:

$$\frac{x\mathsf{R}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathit{ref}).$$

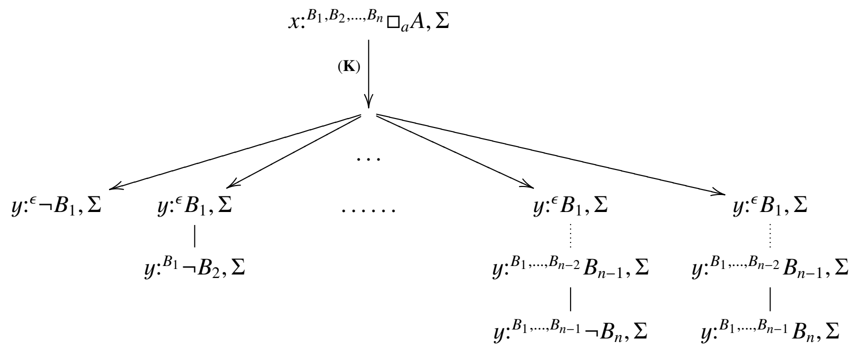
This calculus which enriches syntax and has syntactic notations corresponding to semantic notations can construct inference rules relating with frame properties and add them straightforwardly. Because of that, the construction of this type of calculus starts from K and then afterwards expand it to T, S4, S5 etc. by adding such inference rules. We also follow this method. In other words, we will construct our labelled calculi based on modal logic K at first in the following sections.

Cut elimination for S5, which was the primary interest of new version of sequent calculus for modal logics, holds of course, and a contraction free calculus can also be constructed in this system for S5. The more specific and formal definition of this calculus will be given in the next section.

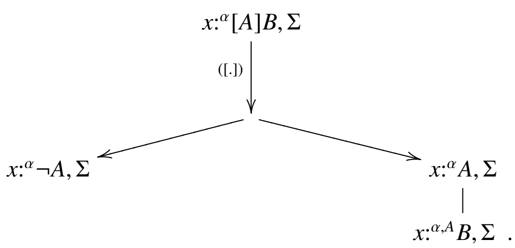
1.4 Proof-theoretic studies for DELs

Let us move onto the topic of proof-theoretic formalizations of DELs such as PAL and EAK. For each of the two DELs, there exists the Hilbert-system (c.f. [8, 68]) which is sound and complete with respect to Kripke semantics (we will discuss them in Chapter 2). Based on the Hilbert-systems, several proof-theoretic studies for PAL and EAK have been appeared. We discuss such related works below other than a labelled sequent calculus for PAL which will be introduced in Chapter 3 in detail.

Labelled tableaux method for PAL A tableaux method for PAL is introduced in Balbiani el al. [7]. Its calculation is carried out with *a labelled formula* $x{:}^{\alpha}A$[4] where α is a finite list of formulas of PAL, $x \in \mathsf{Var}$ (Var is a set of variables) and A is a formula of PAL. The labelled formula of this method is the same as that of labelled sequent calculus, it corresponds to the definition of the satisfaction relation in PAL's Kripke semantics, and it intuitively reads that after the sequence of announcements α, formula A still holds at world x. Added to that, this method included the ternary relation $\Sigma \subseteq \mathsf{Agt} \times \mathsf{Var} \times \mathsf{Var}$ (Agt is a set of agents, and we denote the triple (a, x, y) by $x\mathsf{R}_a y$) which represents the accessibility relation. The below is two examples, (**K**) and ([.]), of the inference rules of this calculus which are for the box (knowledge) operator and the announcement operator:



where $x\mathsf{R}_a y \in \Sigma$.



[4]The notation of labelled formula differs from the original paper [7]; however, since the labelled formula here also has the same type as that of a labelled sequent calculus in this thesis, we unify different notations of labelled formula into $x{:}^{\alpha}A$.

Further, a different tableaux method is given by Ma et al. [50] which is sound and complete for a non-normal modal logic characterized by neighborhood models, and this method does not include a syntactic expression of accessibility relation which differs from above. This system also contains a labelled formula like the one above, but they explicitly include the following labelled formula: $x{:}^{\alpha} \times$. This semantically means that a world corresponding to x does not *survive* after the sequence of public announcements α. To be specific, possible worlds in Kripke semantics can be restricted by an announcement, and this suggests that some worlds can be eliminated and the others survive. The notation $\times$ in the labelled formula is a sign of a world which does not survive. The idea of survival is also used in the semantics of labelled sequent calculi in this thesis (see Sections 3.3 and 3.4). The following is examples of the inference rules ($\Box_a^{\cap}$) and ([.]):

$$
\begin{array}{c}
x{:}^{\alpha}\Box_a A \\
| \\
x{:}^{\beta}\neg\Box_a B \\
\Big\downarrow{\scriptstyle (\Box_a^{\cap})} \\
\begin{array}{cc}
\swarrow & \searrow \\
\begin{array}{c} y{:}^{\alpha} A \\ | \\ y{:}^{\beta}\neg B \end{array} & \begin{array}{c} y{:}^{\alpha} \times \\ | \\ y{:}^{\beta}\neg B \end{array}
\end{array}
\end{array}
,\qquad
\begin{array}{c}
x{:}^{\alpha}[A]B \\
\Big\downarrow{\scriptstyle ([.])} \\
\begin{array}{cc}
\swarrow & \searrow \\
x{:}^{\alpha}\neg A & x{:}^{\alpha,A} B
\end{array}
\end{array}
$$

Labelled tableaux method for EAK Tableau method for EAK is introduced by Aucher et al. [3], and Aucher and Schwarzentruber [4] in the context of the study regarding computational complexity of EAK. This method also contains *a labelled formula* like the tableaux methods for PAL. A labelled formula has the following form, $x{:}^{\alpha} A$ where $x \in \mathbb{N}$, α is a finite list of *actions* of EAK and A is a formula of EAK, and it intuitively reads that after the sequence of actions α, formula A still holds at world x. Moreover, this system includes a syntactic notation $x\mathsf{R}_a y$ of an accessibility relation as the initial example of tableau method for PAL (and sequent calculi in this thesis). $x{:}^{\alpha}\checkmark$ means that list of actions is *executable* in the world corresponding to x, and $x{:}^{\alpha}\otimes$ means that list α of actions is *not* executable in the world corresponding to x. The meaning of executable is that the precondintions of x which belong to actions in the list hold. The reader may notice that this method resembles the tableaux method for PAL from the following examples of the inference rules ($\Box_a$) and ([.]):

$$
\begin{array}{c}
x{:}^{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_n}\Box_a A, \Sigma \\
| \\
x\mathsf{R}_a y \\
\Big\downarrow{\scriptstyle (\Box_a)} \\
\begin{array}{cc}
\swarrow & \searrow \\
\begin{array}{c} y{:}^{\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_n}\checkmark \\ | \\ y{:}^{\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_n} A \end{array} & y{:}^{\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_n}\otimes
\end{array}
\end{array}
,\qquad
\begin{array}{c}
x{:}^{\alpha}[\mathsf{a}]A \\
\Big\downarrow{\scriptstyle [.]} \\
\begin{array}{cc}
\swarrow & \searrow \\
x{:}^{\alpha,\mathsf{a}}\otimes & \begin{array}{c} x{:}^{\alpha,\mathsf{a}}\checkmark \\ | \\ x{:}^{\alpha,\mathsf{a}} A \end{array}
\end{array}
\end{array}
,
$$

where every $\mathbf{b}_i$ is a state which is accessible from $\mathbf{a}_i$.

Display calculus for EAK Display calculus was first introduced in Belnap [10]. This calculus consists of an enriched syntax and introduces new *structural connectives* (,). While the above labelled systems are semantic-dependent systems (a labelled formula correspond to the semantic notion, the satisfaction relation), display calculus is a semantic-independent system. A sequent $X \Rightarrow Y$ is a pair of X and Y which are *structures* consisting of formulas and structure constants using the structural connectives. An outstanding feature of display calculi is a general cut-elimination theorem which for all display calculi satisfying eight syntactic conditions.

This powerful proof-theoretic framework is widely applied to several logics including EAK. Display calculus for EAK is given by Greco et al. [33]. It also provides an enriched syntactics than the normal version of EAK, in which all logical operations have adjoints. This calculus is sound and complete for semantics called the final coalgebra semantic (c.f. [1, 18]). The reason to employ such a non-standard semantics is that the standard Kripke semantics may not provide a natural interpretation to the extended logical operators such as $\boxdot$ and $\Diamond\!\!\!\cdot\;$ which are adjoint to $[\cdot]$ and $\langle\cdot\rangle$ (the action operator and the dual of this) respectively, but the final coalgebra semantics can do this. The following is an example of inference rule of this calculus:

$$\frac{A \Rightarrow \{\mathsf{a}\}X}{A \Rightarrow [\mathsf{a}]X}\ ([\mathsf{a}]_R),$$

where a is an action and $\{\mathsf{a}\}$ is a structural connective associated with $[\mathsf{a}]$ and $\langle\mathsf{a}\rangle$. [5]

1.5 Contribution

We mainly focused on the proof theories of PAL and EAK, which became the basis of the flourishing field of epistemic logic, and constructed cut-free labelled sequent calculi for them based on existing Hilbert-style proof systems. Specifically, in PAL, our labelled sequent calculus was closely related with the study of Maffezioli and Negri [51], where a labelled sequent calculus for PAL was constructed, but Balbiani et al. [6] suggested that this system is not semantically complete for Kripke semantics. We specified this problem and revised it to be a complete calculus. We also focused on the soundness of a labelled sequent calculus for PAL where the usual definition for the validity of a sequent was not adequate, which has not been suggested by previous works, and we provided a different and more suitable definition for it. Then, we investigated the completeness theorem for our calculus. In PAL, the Kripke model can be restricted by an announcement, i.e., some possible world(s) where a contradiction occurs that can be deleted, and this causes a difficulty for a direct proof of the completeness theorem. Therefore, for the proof, we also found that a different but suitable Kripke semantics was required, which we called 'link-cutting semantics,' where only the accessibility relation can be restricted by an announcement. Also, our calculus was founded on modal logic K, the most basic modal logic, as a starting point of the construction of our calculus, but the extensions from K to other modal logics were also

[5] In addition to related works mentioned above, a nested sequent calculus for EAK is provided by Dyckhoff and Sadrzadeh [24], but the syntax underlying their system varies from the normal syntax of EAK, so we do not refer to it in detail.

given by providing additional inference rules corresponding to frame properties. The extension to modal logic S5 is particularly significant since S5 is the usual basis of epistemic logic. In EAK, there exists a calculus by [33] is complete for an unusual semantics, the final coalgebra semantics. In this thesis, we, based on the argument of the labelled sequent calculus for PAL, constructed a labelled sequent calculus for EAK that was semantically complete for the *standard* Kripke semantics. Since EAK is a generalization of PAL, a number of methods for the construction of a labelled sequent calculus for PAL could be applicable to the construction of a calculus for EAK; however, we particularly pay attention to how to deal with the composition of accessibility relations and action relations in a syntactic way. The treatment of these relations was the core of our calculus for EAK and they differentiate it from [3]. We also provide extensions from modal logic K to other modal logics, including S5, as in the case of PAL. Moreover, recently, Intuitionistic PAL (IntPAL) was proposed by Ma et al. [49], and we provided a cut-free labelled sequent calculus for it. The construction of the calculus also follows a similar method to that of PAL; however, since IntPAL employs a bi-relational Kripke semantics, which is one of the standard semantics for intuitionistic modal logics, we face a different difficulty from the deletion of world(s). We settle the problem by making use of Simpson's solution [78]. Intuitionistic epistemic logic has a philosophically profound meaning as it can be regarded to provide a strict sense of knowledge, which is justified by evidence for knowledge. We expect that our cut-free calculus is valuable to give concrete evidence. There is an underlying paper on each of three labelled sequent calculi. Studies of three sequent calculi for PAL, EAK and IntPAL are based on the author's paper [61], [60] and [59] respectively.

The outline of this thesis is as follows: Chapter 2 provides technical preliminaries of basic multi-modal logics and labelled sequent calculi for multi-modal logics, semantics and their applications for PAL, and those of EAK; Chapter 3 introduces our first calculus, which is for PAL, and shows the cut-elimination theorem, as well as the soundness and completeness results; and Chapter 4 introduces our second calculus, which is for EAK and shows the same results as above. Chapter 5 provides the language and bi- relational Kripke semantics of IntPAL, and then introduces our third calculus, in which we also show the results of the cut-elimination, soundness and completeness.

Chapter 2

Preliminaries

2.1 Multi-modal logic (ML)

Let us get started with Multi-modal Logic (ML for short), the foundation of epistemic logics and dynamic epistemic logics. ML contains a finite set $\mathsf{Mod} = \{\Box, \Box', \Box'', \ldots\}$ of modal operators (modalities) is added to classical propositional logic. In fact, the ordinary (multi-agent) epistemic logic is no different from (multi-) modal logic **S5** (in Section 2.1.4) but only modalities are interpreted as states of knowledge. In this section, we mainly refer to [11, 17]. Section 2.1.1 gives the language and Kripke semantics of ML, while Section 2.1.2 and Section 2.1.3 introduce two different proof systems and their basic results respectively.

2.1.1 Language $\mathcal{L}_{ML}$ and Kripke semantics

Let $\mathsf{Prop} = \{p, q, r, \ldots\}$ be a countably infinite set of propositional atoms and $\mathsf{Mod} = \{\Box, \Box', \Box'', \ldots\}$ a nonempty finite set of modalities. Then the set $\mathcal{L}_{ML} = \{A, B, C, \ldots\}$ of formulas of ML is inductively defined by BNF as follows ($p \in \mathsf{Prop}$, $\Box \in \mathsf{Mod}$):

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box A.$$

Definition 2.1.1. Let A, B be any formulas in $\mathcal{L}_{ML}$ and p be any propositional atom in Prop. Then ordinary logical connectives are defined as follows:

$$\begin{array}{ll} A \wedge B := \neg(A \to \neg B), & A \vee B := \neg A \to B, \\ \bot := p \wedge \neg p, & \Diamond A := \neg\Box\neg A, \\ A \leftrightarrow B := (A \to B) \wedge (B \to A), & \top := \bot \to \bot, \\ \bigwedge\{A_1, ..., A_n\} := A_1 \wedge \cdots \wedge A_n, & \bigwedge \varnothing := \top. \end{array}$$

Kripke Semantics Let us consider Kripke semantics. A (*Kripke*) *frame* is a pair $\mathfrak{F} = (W, (R_\Box)_{\Box \in \mathsf{Mod}})$ where W is a non-emptyset of elements, called (*possible*) *worlds*, and each $R_\Box \subseteq W \times W$ is a binary relation on W, called an *accessibility relation*. We call a pair $\mathfrak{M} = (\mathfrak{F}, V)$ a (*Kripke*) *model* if $\mathfrak{F}$ is a frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a *valuation function* which assigns a subset of W to a propositional atom. Given a model

$\mathfrak{M}$, $w \in \mathcal{D}(\mathfrak{M})$ and $A \in \mathcal{L}_{ML}$, we define the *satisfaction relation* $\mathfrak{M}, w \Vdash A$ as follows ($\Box \in \mathsf{Mod}$):

$$
\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \Vdash \neg A & \text{iff} & \mathfrak{M}, w \nVdash A, \\
\mathfrak{M}, w \Vdash A \to B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ implies } \mathfrak{M}, w \Vdash B, \\
\mathfrak{M}, w \Vdash \Box A & \text{iff} & \text{for all } v \in W : wR_{\Box}v \text{ implies } \mathfrak{M}, v \Vdash A.
\end{array}
$$

The set W of worlds in a model $\mathfrak{M}$ is also called the *domain* of $\mathfrak{M}$, denoted by $\mathcal{D}(\mathfrak{M})$. We write a class of frames by $\mathbb{F}$ etc. and a class of models by $\mathbb{M}$ etc.

Definition 2.1.2 (Validity)**.** Let A be any formula in $\mathcal{L}_{ML}$.

(i) A formula A is *valid on a model* $\mathfrak{M}$ (notation: $\mathfrak{M} \Vdash A$) if $\mathfrak{M}, w \Vdash A$ holds for any $w \in \mathcal{D}(\mathfrak{M})$.

(ii) A formula A is *valid on a frame* $\mathfrak{F} = (W, (R_{\Box})_{\Box \in \mathsf{Mod}})$ (notation: $\mathfrak{F} \Vdash A$) if $(\mathfrak{F}, V) \Vdash A$ holds for any valuation V.

(ii) A formula A is *valid on* $\mathbb{M}$ (notation: $\mathbb{M} \Vdash A$) if $\mathfrak{M} \Vdash A$ holds for any model $\mathfrak{M}$ in a class of models $\mathbb{M}$.

(iv) A formula A is *valid on* $\mathbb{F}$ (notation: $\mathbb{F} \Vdash A$) if $\mathfrak{F} \Vdash A$ holds for any frame $\mathfrak{F}$ in a class $\mathbb{F}$ of frames.

Frame definability We consider here the correspondence between a formula and a condition of frames. Then we define the basic concept of modal logic underlying in this thesis.

Definition 2.1.3 (Definability)**.** Let A be a formula in $\mathcal{L}_{ML}$ and $\mathbb{F}$ a class of frames. We say that A *defines* $\mathbb{F}$ if $\mathfrak{F} \in \mathbb{F}$ iff $\mathfrak{F} \Vdash A$ for all $\mathfrak{F}$.

Here are five widely known formulas each of which is given a traditional name such as $\mathbf{T}_{\Box}, \mathbf{B}_{\Box}, \mathbf{4}_{\Box}, \mathbf{5}_{\Box}$ and $\mathbf{D}_{\Box}$ ($\Box \in \mathsf{Mod}$) as follows:

$$
\begin{array}{lll}
\mathbf{T}_{\Box} := \Box p \to p, & \mathbf{4}_{\Box} := \Box p \to \Box\Box p, & \mathbf{D}_{\Box} := \Box p \to \Diamond p. \\
\mathbf{B}_{\Box} := p \to \Box\Diamond p, & \mathbf{5}_{\Box} := \Diamond p \to \Box\Diamond p, &
\end{array}
$$

In addition to that, we introduce other five well-known names of conditions on accessibility relations $(R_{\Box})_{\Box \in \mathsf{Mod}}$ in a frame. These conditions are shown in Table 2.1.

Table 2.1: Frame properties

$R_{\Box}$ is *reflexive*	$wR_{\Box}w$ for all $w \in W$
$R_{\Box}$ is *symmetric*	$wR_{\Box}v$ implies $vR_{\Box}w$ for all $w, v \in W$
$R_{\Box}$ is *transitive*	$wR_{\Box}v$ and $vR_{\Box}u$ imply $wR_{\Box}u$ for all $w, v, u \in W$
$R_{\Box}$ is *Euclidean*	$wR_{\Box}v$ and $wR_{\Box}u$ imply $vR_{\Box}u$ for all $w, v, u \in W$
$R_{\Box}$ is *serial*	$wR_{\Box}v$ for all $w \in W$ for some $v \in W$

When we say a list $(R_{\Box})_{\Box \in \mathsf{Mod}}$ of accessibility relations is reflexive, every accessibility relation $R_{\Box}$ in the list is reflexive. Then we can show the following proposition.

Proposition 2.1.1. Let $\mathfrak{F}$ be a frame and $R_\Box$ be any accessibility relation in $\mathfrak{F}$. Then the following hold:

$\mathfrak{F} \Vdash \mathbf{T}_\Box$	iff	$R_\Box$ is *reflexive*,	$\mathfrak{F} \Vdash \mathbf{5}_\Box$	iff	$R_\Box$ is *Euclidean*,
$\mathfrak{F} \Vdash \mathbf{B}_\Box$	iff	$R_\Box$ is *symmetric*,	$\mathfrak{F} \Vdash \mathbf{D}_\Box$	iff	$R_\Box$ is *serial*.
$\mathfrak{F} \Vdash \mathbf{4}_\Box$	iff	$R_\Box$ is *transitive*,			

As a result of Proposition 2.1.1, formulas $\mathbf{T}_\Box, \mathbf{B}_\Box, \mathbf{4}_\Box, \mathbf{5}_\Box$ and $\mathbf{D}_\Box$ ($\Box \in \mathsf{Mod}$) define classes of frames which satisfy the corresponding frame property respectively, e.g., $\mathbf{T}_\Box$ defines the class of $R_\Box$-reflexive frames. In what follows, we use the following set:

$$\mathsf{FrameAxiom} := \{\mathbf{T}_\Box, \mathbf{B}_\Box, \mathbf{4}_\Box, \mathbf{5}_\Box, \mathbf{D}_\Box \mid \Box \in \mathsf{Mod}\}.$$

Definition 2.1.4. Let Σ be a subset of $\mathsf{FrameAxiom}$. Then we write $\mathbb{F}_\Sigma$ to mean the class of frames which is defined by $\bigwedge \Sigma$. Further, let us also define the class $\mathbb{M}_\Sigma$ of models by $\mathbb{M}_\Sigma := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_\Sigma \text{ and } V \text{ is a valuation } V \text{ on } \mathfrak{F}\}$.

2.1.2 Hilbert-system HK of Multi-modal logic

We will introduce two proof systems: Hilbert-system in this section and labelled sequent calculus in the next section. Hilbert-system **HK** is given in Table 2.2, where axiom (K) and inference rule (Nec) are added to the Hilbert-system of classical propositional logic. When we add one or more formulas in $\mathsf{FrameAxiom}$ as additional axiom

Table 2.2: Hilbert-system for ML: **HK**

Modal axiom scheme	($Taut$)	all instantiations of propositional tautologies
	(K)	$\Box(A \to B) \to (\Box A \to \Box B)$ ($\Box \in \mathsf{Mod}$)
Inference rules	(MP)	From A and $A \to B$, infer B
	(Nec)	From A, infer $\Box A$ ($\Box \in \mathsf{Mod}$)

schemes (each of which can define a class of frame) to the set of axiom scheme of **HK**, we obtain Hilbert-systems other than **HK** as follows.

Definition 2.1.5 (Extensions of **HK**)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. When each element of Σ is added to **HK** as an axiom scheme by replacing p with an arbitrary formula A, *the extension of* **HK** is the resulting Hilbert-system $\mathbf{HK}\Sigma$.

For example, if the set $\{\mathbf{T}_\Box, \mathbf{B}_{\Box'}\}$ are added to **HK**, we obtain Hilbert-system $\mathbf{HK}\{\mathbf{T}_\Box\mathbf{B}_{\Box'}\}$. Hilbert-systems with some particular combinations of axiom schemes are given names.

$$\begin{array}{ll}
\mathbf{HT} := \mathbf{HK}\{\mathbf{T}_\Box \mid \Box \in \mathsf{Mod}\}, & \mathbf{HS5} := \mathbf{HK}\{\mathbf{T}_\Box, \mathbf{5}_\Box \mid \Box \in \mathsf{Mod}\}, \\
\mathbf{HB} := \mathbf{HK}\{\mathbf{T}_\Box, \mathbf{B}_\Box \mid \Box \in \mathsf{Mod}\}, & \mathbf{HD} := \mathbf{HK}\{\mathbf{D}_\Box \mid \Box \in \mathsf{Mod}\}. \\
\mathbf{HS4} := \mathbf{HK}\{\mathbf{T}_\Box, \mathbf{4}_\Box \mid \Box \in \mathsf{Mod}\}, &
\end{array}$$

Definition 2.1.6 (Derivation of $\mathbf{HK}\Sigma$)**.** Let $\mathbf{HK}\Sigma$ (where $\Sigma \subseteq \mathsf{FrameAxiom}$) be a Hilbert-system. A *derivation* in $\mathbf{HK}\Sigma$ consists of a sequence of formulas each of which is an instance of an axiom or is the result of applying an inference rule to formula(s) that occur earlier. If A is the last formula in a derivation, then A is *derivable*, and we write $\vdash_{\mathbf{HK}\Sigma} A$. Given a subset $\Gamma \cup \{A\}$ of $\mathcal{L}_{ML}$, A *is derivable from* Γ in $\mathbf{HK}\Sigma$ if there is a finite subset Γ' of Γ such that $\vdash_{\mathbf{HK}\Sigma} \bigwedge \Gamma' \to A$, and we write $\Gamma \vdash_{\mathbf{HK}\Sigma} A$.

Additionally, when A is not derivable in a proof system $\mathbf{HK\Sigma}$ (where $\Sigma \subseteq \mathsf{FrameAxiom}$), we write $\nvdash_{\mathbf{HK\Sigma}} A$. Finally, we define basic MLs. For any $\Sigma \subseteq \mathsf{FrameAxiom}$, modal logic $\mathbf{K\Sigma}$ is the set of all derivable formulas in $\mathbf{HK\Sigma}$. Some modal logics are also given special names in some particular combinations of axiom schemes.

$$\begin{array}{ll} \mathbf{K} := \mathbf{K}\varnothing, & \mathbf{S4} := \mathbf{K}\{\mathbf{T}_\Box, \mathbf{4}_\Box \mid \Box \in \mathsf{Mod}\}, \\ \mathbf{T} := \mathbf{K}\{\mathbf{T}_\Box \mid \Box \in \mathsf{Mod}\}, & \mathbf{S5} := \mathbf{K}\{\mathbf{T}_\Box, \mathbf{5}_\Box \mid \Box \in \mathsf{Mod}\}, \\ \mathbf{B} := \mathbf{K}\{\mathbf{T}_\Box, \mathbf{B}_\Box \mid \Box \in \mathsf{Mod}\}, & \mathbf{D} := \mathbf{K}\{\mathbf{D}_\Box \mid \Box \in \mathsf{Mod}\}. \end{array}$$

Soundness and Completeness The soundness and completeness theorems are integral parts of a proof system, and the completeness of **HK** is especially important through this thesis since the semantic completeness theorems of DELs depend on the completeness of **HK** (we will see it in the next section).

Theorem 2.1.1 (Soundness of $\mathbf{HK\Sigma}$)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. If $\vdash_{\mathbf{HK\Sigma}} A$, then $\mathbb{M}_\Sigma \Vdash A$, for any formula $A \in \mathcal{L}_{ML}$.

We briefly look at a proof of the completeness theorem of $\mathbf{HK\Sigma}$ with a similar argument in [11, Section 4.3].

Definition 2.1.7 (Maximal $\mathbf{K\Sigma}$-consistent set)**.** Let $\Gamma \subseteq \mathcal{L}_{ML}$. Then Γ is $\mathbf{K\Sigma}$-*consistent* if $\bot$ is not derivable from Γ ($\Gamma \nvdash_{\mathbf{HK\Sigma}} \bot$). Γ is *maximal* if $A \in \Gamma$ or $\neg A \in \Gamma$ for any $A \in \mathcal{L}_{ML}$. Γ is a *maximally* $\mathbf{K\Sigma}$-*consistent set* (a $\mathbf{K\Sigma}$-MCS for short) if Γ is maximal and $\mathbf{K\Sigma}$-consistent.

Lemma 2.1.1 (Lindenbaum)**.** Every $\mathbf{K\Sigma}$-consistent set of formulas is a subset of a $\mathbf{K\Sigma}$-MCS.

Lemma 2.1.2. If Γ is a $\mathbf{K\Sigma}$-MCS, then

(i) $\Gamma \vdash_{\mathbf{HK\Sigma}} A$ implies $A \in \Gamma$ (Deductively closed),

(ii) $A \in \Gamma$ iff $\neg A \notin \Gamma$,

(iii) $A \to B \in \Gamma$ iff $A \in \Gamma$ implies $B \in \Gamma$,

(iv) if $\Box A \notin \Gamma$, then $\{\neg A\} \cup \{B \mid \Box B \in \Gamma\}$ is $\mathbf{K\Sigma}$-consistent.

Definition 2.1.8 (Canonical model)**.** *The canonical model* $\mathfrak{M}^{\mathbf{K\Sigma}} = (W^{\mathbf{K\Sigma}}, (R^{\mathbf{K\Sigma}}_\Box)_{\Box \in \mathsf{Mod}}, V^{\mathbf{K\Sigma}})$ is defined as follows:

$$\begin{array}{lll} W^{\mathbf{K\Sigma}} & = & \{\Gamma \mid \Gamma \textit{ is a } \mathbf{K}\Sigma\textit{-MCS}\}, \\ \Gamma R^{\mathbf{K\Sigma}}_\Box \Delta & \textit{iff} & \{A \mid \Box A \in \Gamma\} \subseteq \Delta \quad (\Box \in \mathsf{Mod}), \\ V^{\mathbf{K\Sigma}}(p) & = & \{\Gamma \in W^{\mathbf{K\Sigma}} \mid p \in \Gamma\}. \end{array}$$

Lemma 2.1.3 (Truth lemma)**.** For any formula $A \in \mathcal{L}_{ML}$ and any $\mathbf{K\Sigma}$-MCS Γ, $A \in \Gamma$ iff $\mathfrak{M}^{\mathbf{K\Sigma}}, \Gamma \Vdash A$.

Lemma 2.1.4. Let $\mathfrak{M}^{\mathbf{K\Sigma}} = (W^{\mathbf{K\Sigma}}, (R^{\mathbf{K\Sigma}}_\Box)_{\Box \in \mathsf{Mod}}, V^{\mathbf{K\Sigma}}\}$ be the canonical model for modal logic $\mathbf{K\Sigma}$. Then the following hold:

(i) if $\vdash_{\mathbf{HK\Sigma}} \Box A \to A$ for all formulas A, then $R^{\mathbf{K\Sigma}}_\Box$ is reflexive,

(ii) if $\vdash_{\mathbf{HK\Sigma}} A \to \Box\Diamond A$ for all formulas A, then $R_{\Box}^{\mathbf{K\Sigma}}$ is symmetric,

(iii) if $\vdash_{\mathbf{HK\Sigma}} \Box A \to \Box\Box A$ for all formulas A, then $R_{\Box}^{\mathbf{K\Sigma}}$ is transitive,

(iv) if $\vdash_{\mathbf{HK\Sigma}} \Diamond A \to \Box\Diamond A$ for all formulas A, then $R_{\Box}^{\mathbf{K\Sigma}}$ is Euclidean,

(v) if $\vdash_{\mathbf{HK\Sigma}} \Box A \to \Diamond A$ for all formulas A, then $R_{\Box}^{\mathbf{K\Sigma}}$ is serial.

Theorem 2.1.2 (Completeness of **HKΣ**)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. If $\mathbb{M}_\Sigma \Vdash A$, then $\vdash_{\mathbf{HK\Sigma}} A$, for any formula $A \in \mathcal{L}_{ML}$.

Proof. Fix any $A \in \mathcal{L}_{ML}$. We show the contraposition i.e., if $\nvdash_{\mathbf{HK\Sigma}} A$, then A is not valid. Suppose $\nvdash_{\mathbf{HK\Sigma}} A$ which is equivalent to $\{\neg A\} \nvdash_{\mathbf{HK\Sigma}} \bot$. So, $\{\neg A\}$ is $\mathbf{K\Sigma}$-consistent. By Lemma 2.1.1, there is a $\mathbf{K\Sigma}$-MCS Δ such that $\{\neg A\} \subseteq \Delta$. By Lemma 2.1.2 (Truth lemma), $\mathfrak{M}^{\mathbf{K\Sigma}}, \Gamma \Vdash \neg A$. By Lemma 2.1.3, the canonical model $\mathfrak{M}^{\mathbf{K\Sigma}}$ satisfies the corresponding frame property(ies). Therefore, we obtain $\mathfrak{M}^{\mathbf{K\Sigma}}, \Gamma \nVdash A$ as desired. □

Decidability and Finite model property We add one more basic property of MLs i.e., decidability. To establish the notion of decidability, we need to mention the finite model property.

Definition 2.1.9 (Finite model property)**.** Let $\mathbf{\Sigma}$ be a subset of $\mathsf{FrameAxiom}$. $\mathbf{K\Sigma}$ has the *finite model property* iff each non-theorem of $\mathbf{K\Sigma}$ is false in some finite model in $\mathbb{M}_\Sigma$.

Definition 2.1.10 (Decidability)**.** A system $\mathbf{K\Sigma}$ of modal logic is *decidable* if there is an effective method[1] for deciding of any formula whether or not it is a theorem of the system.

The following is a well-known result of modal logics.

Theorem 2.1.3. Modal logics **K**, **T**, **B**, **S4**, **S5** and **D** have the finite model property.

Its proof is conducted by the standard filteration-method (c.f. [17, Theorem 5.21] and [65, Section 5] for uni-modal logic).[2]

Theorem 2.1.4. A system of modal logic $\mathbf{K\Sigma}$ is decidable if $\mathbf{K\Sigma}$ has the finite model property.

[1] In [19], a method (procedure) M is said to be effective (or mechanical) if (1) M is set out in terms of a finite number of exact instructions (each instruction being expressed by means of a finite number of symbols); (2) M will, if carried out without error, produce the desired result in a finite number of steps; (3) M can (in practice or in principle) be carried out by a human being unaided by any machinery save paper and pencil; (4) M demands no insight or ingenuity on the part of the human being carrying it out.

[2] A different approach for the finite model property of multi-modal logics is by the fusion of modal logics in [12, Chapter 15]. For any modal logics $\mathbf{X}_1$ and $\mathbf{X}_2$ where they have disjoint modal operators $\Box_1, ..., \Box_n$ and $\Box_{n+1}, ..., \Box_{n+m}$ respectively, the *fusion* $\mathbf{X}_1 \otimes \mathbf{X}_2$ of $\mathbf{X}_1$ and $\mathbf{X}_2$ is the smallest multi-modal logic containing $\Box_1, ..., \Box_n, \Box_{n+1}, ..., \Box_{n+m}$. Then the following [12, Theorem 4 in Chapter 15] holds: if both $\mathbf{X}_1$ and $\mathbf{X}_2$ are modal logics having the finite model property, then their fusion $\mathbf{X}_1 \otimes \mathbf{X}_2$ has the finite model property as well. Let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be modal logics containing $\Box$ and $\Box'$ ($\Box, \Box' \in \mathsf{Mod}$ and $\Box \neq \Box'$) respectively. Since they are uni-modal logics, the finite model property holds respectively. Then by the above theorem, the fusion $\mathbf{Y}_1 \otimes \mathbf{Y}_2$ has the finite model property. By doing the same step finite times, we obtain that a multi-modal logic containing a finite number of modalities $\Box_1, \Box_2, ..., \Box_n \in \mathsf{Mod}$ has the property, since an arbitrary multi-modal logic is equal to a fusion $\mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_n$.

Proof (*Outline*). Assume $\mathbf{K}\Sigma$ has the finite model property. Let us consider an effective method for deciding of any formula A whether it is a theorem of the system. We call such a method for deciding A is a theorem *a positive test*, and call such a method for deciding A is a non-theorem *a negative test*. By assumption, $\mathbf{K}\Sigma$ clearly has a finite number of axioms, so there is a positive test.

Next, a negative test is given as follows. If $\mathfrak{M}$ has a finite domain, there is an effective method to check whether an arbitrary formula X is valid at $\mathfrak{M}$. Let us call such a method δ. Besides, we have an effective method to permutate all finite models, and let us call it γ. By the methods of δ and γ, we have a complete enumeration $\mathfrak{M}_1, \mathfrak{M}_2, ... \in \mathbb{M}_\Sigma$ of finite models in each of which $\bigwedge \Sigma$ is valid (the validity is checked by δ). If A is a non-theorem of $\mathbf{K}\Sigma$, then A is false in some $\mathfrak{M}_k \in \mathbb{M}_\Sigma$. To find such a model $\mathfrak{M}_k$, the falsity of A is checked at $\mathfrak{M}_i$ by the method of δ, and if A is false at the model, then it is a countermodel of A, else A is checked by the next model $\mathfrak{M}_{i+1}$. Starting from this procedure from $i = 1$, a countermodel $\mathfrak{M}_k$ of A can be found in a finite step, and therefore, we obtain the way to check whether A is a theorem or not can be checked by alternating these tests. □

Corollary 2.1.1. Multi-modal logics $\mathbf{K}, \mathbf{T}, \mathbf{B}, \mathbf{S4}, \mathbf{S5}$ and $\mathbf{D}$ are decidable.

2.1.3 Labelled sequent calculus G3K

We introduce in this section one of the most uniform approaches for sequent calculus for ML, *labelled sequent calculus* **G3K** by Negri and von Plato [57]. and Negri [56]. **G3K** is a basic G3-style sequent calculus for modal logic **K**, where each formula has a label corresponding to a world of a domain in Kripke semantics. In fact, the labelled sequent calculus can be regarded as a formalized version of this Kripke semantics. We note that a G3-style sequent calculus (or G3-system) is a sequent calculus that does not have any structural rules, and outstanding features of **G3K** are that all inference rules are height-preserving invertible and that the contraction rules are admissible[3]. The specific introduction of the G3-system itself can be found in Troelstra and Schwichtenberg [80] and Negri and von Plato [57], and in this section we follow [57]'s introduction.

Let $\mathsf{Var} = \{x, y, z, \ldots\}$ be a countably infinite set of variables. Then, given any $x, y \in \mathsf{Var}$ and any formula A, we say $x{:}A$ is a *labelled formula*, and say $x\mathsf{R}_\Box y$ is a *relational atom* for any modality $\Box \in \mathsf{Mod}$. Intuitively, the labelled formula $x{:}A$ corresponds to '$\mathfrak{M}, x \Vdash A$.' We also use the term, *labelled expressions* to indicate that they are either labelled formulas or relational atoms, and we denote them by $\mathfrak{A}$, $\mathfrak{B}$, etc. A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of finite multi-sets of labelled expressions. The set of inference rules of **G3K** is given in Table 2.3. Hereinafter, for any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is derivable in **G3K**, we write $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow \Delta$. We call labelled expression $\mathfrak{A}$ in the lowersequent at each inference rule *principal* if $\mathfrak{A}$ is not in either Γ or Δ.

[3] The definitions of the height-preserving invertibility and admissibility will be soon given in this section. We would like to remark a short history of the G3-system. According to von Plato [85], Ketonen [42] obtained the invertibility of inference rules for classical propositional logic (CL), and Curry [20] showed the *height-preserving* invertibility of them. Moreover, the height-preserving admissibility of the contraction rules (G3-system for CL) was given by Dragalin [22]. Subsequently, Troelstra and Schwichtenberg [80] gave a G3-system in the intuitionistic single suceedent calculus.

Table 2.3: Labelled sequent calculus for ML : **G3K**

(Initial Sequent)

$$x{:}p, \Gamma \Rightarrow \Delta, x{:}p \qquad x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta, x\mathsf{R}_\Box y$$

(Rules for propositional connectives)

$$\frac{\Gamma \Rightarrow \Delta, x{:}A}{x{:}\neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \qquad \frac{x{:}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}\neg A}\ (R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}A \quad x{:}B, \Gamma \Rightarrow \Delta}{x{:}A \to B, \Gamma \Rightarrow \Delta}\ (L \to) \qquad \frac{x{:}A, \Gamma \Rightarrow \Delta, x{:}B}{\Gamma \Rightarrow \Delta, x{:}A \to B}\ (R \to)$$

$$\frac{}{x{:}\bot, \Gamma \Rightarrow \Delta}\ (L\bot)$$

(Rules for modal operators)

$$\frac{y{:}A, x{:}\Box A, x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta}{x{:}\Box A, x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta}\ (L\Box) \qquad \frac{x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta, y{:}A}{\Gamma \Rightarrow \Delta, x{:}\Box A}\ (R\Box)\dagger$$

† y does not appear in the lower sequent.

Extensions of G3K Similar to the situation of **HK**, extensions of **G3K** are obtained by adding to **G3K** one or more of the following rules shown in Table 2.4, which also correspond to the frame properties. Let * be a function such that $\mathsf{FrameAxiom} \to \{(ref_\Box), (sym_\Box), (tra_\Box), (euc_\Box), (ser_\Box) \mid \Box \in \mathsf{Agt}\}$, and defined as follows:

$$\mathbf{T}_\Box{}^* := (ref_\Box), \quad \mathbf{B}_\Box{}^* := (sym_\Box), \quad \mathbf{4}_\Box{}^* := (tra_\Box),$$
$$\mathbf{5}_\Box{}^* := (euc_\Box), \quad \mathbf{D}_\Box{}^* := (ser_\Box).$$

Let Σ be a subset of $\mathsf{FrameAxiom}$ then Σ^* is defined to be the set $\{\mathbf{X}^* \mid \mathbf{X} \in \Sigma\}$.

Definition 2.1.11 (Extensions of **G3K**)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. A G3-system $\mathbf{G3K}\Sigma^*$ is an extension of **G3K**, when each element of Σ^* is added to **G3K** as

Table 2.4: Rules of G3-system for frame properties

$$\frac{x\mathsf{R}_\Box x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (ref_\Box) \qquad \frac{x\mathsf{R}_\Box z, x\mathsf{R}_\Box y, y\mathsf{R}_\Box z, \Gamma \Rightarrow \Delta}{x\mathsf{R}_\Box y, y\mathsf{R}_\Box z, \Gamma \Rightarrow \Delta}\ (tra_\Box)$$

$$\frac{y\mathsf{R}_\Box x, x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_\Box y, \Gamma \Rightarrow \Delta}\ (sym_\Box) \qquad \frac{y\mathsf{R}_\Box z, x\mathsf{R}_\Box y, x\mathsf{R}_\Box z, \Gamma \Rightarrow \Delta}{x\mathsf{R}_\Box y, x\mathsf{R}_\Box z, \Gamma \Rightarrow \Delta}\ (euc_\Box)$$

$$\frac{x\mathsf{R}_\Box v, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (ser_\Box)\dagger$$

† v does not appear in the lowersequent.

an inference rule.

G3-systems with some particular combinations of inference rules are given names.

$\mathbf{G3T} := \mathbf{G3K}\{(ref_{\Box}) \mid \Box \in \mathsf{Mod}\}$,
$\mathbf{G3B} := \mathbf{G3K}\{(sym_{\Box}) \mid \Box \in \mathsf{Mod}\}$,
$\mathbf{G3S4} := \mathbf{G3K}\{(ref_{\Box}), (tra_{\Box}) \mid \Box \in \mathsf{Mod}\}$,
$\mathbf{G3S5} := \mathbf{G3K}\{(ref_{\Box}), (euc_{\Box}) \mid \Box \in \mathsf{Mod}\}$,
$\mathbf{G3D} := \mathbf{G3K}\{(ser_{\Box}) \mid \Box \in \mathsf{Mod}\}$.

Features of $\mathbf{G3K\Sigma^*}$ Let $\mathbf{G3K\Sigma^*}$ be an arbitrary extension of **G3K**. We introduce some definitions and outstanding features of $\mathbf{G3K\Sigma^*}$. Each extension has properties of G3-system such as the height-preserving invertibility of inference rules and the admissibility of the contraction rules and cut-admissibility as well as the completeness for the corresponding Kripke semantics.

Definition 2.1.12 (Derivation of $\mathbf{G3K\Sigma^*}$)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. A *derivation* of sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3K\Sigma^*}$ is a tree of sequents satisfying the following conditions:

(1) The uppermost sequent of the tree is an initial sequent or a conclusion of $(L\bot)$.

(2) Every sequent in the tree except the uppermost sequent(s) is a lowersequent of an inference rule of **G3K**.

(3) The lowest sequent is $\Gamma \Rightarrow \Delta$.

Given a sequent $\Gamma \Rightarrow \Delta$, it is *derivable in* $\mathbf{G3K\Sigma^*}$ and we write $\vdash_{\mathbf{G3K\Sigma^*}} \Gamma \Rightarrow \Delta$ if there is a derivation of the sequent. The height of the derivation of $\Gamma \Rightarrow \Delta$ is the maximum length of branches of the derivation, and we write $\vdash^n_{\mathbf{G3K\Sigma^*}} \Gamma \Rightarrow \Delta$ to be explicit on the meaning of $\vdash_{\mathbf{G3K\Sigma^*}} \Gamma \Rightarrow \Delta$ at derivation height n.

Definition 2.1.13 (Admissible)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. A rule is *admissible* if, whenever the premise(s) of the rule is derivable in $\mathbf{G3K\Sigma^*}$, the conclusion of the rule is derivable in $\mathbf{G3K\Sigma^*}$. A rule is height-preserving admissible if, whenever the premise(s) of the rule is derivable in $\mathbf{G3K\Sigma^*}$ with the derivation height at most n, the conclusion of the rule is derivable in $\mathbf{G3K\Sigma^*}$ with the derivation height at most n.

Definition 2.1.14 (Invertible)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. A rule is *height-preserving invertible* in $\mathbf{G3K\Sigma^*}$ if, whenever the conclusion of the rule is derivable with the derivation height at most n, premise(s) of the rule is also derivable with the derivation height at most n.

Proposition 2.1.2. Let Σ be a subset of $\mathsf{FrameAxiom}$. All the inference rules of $\mathbf{G3K\Sigma^*}$ are height-preserving invertible.

Proposition 2.1.3. The following structural rules are height-preserving admissible:

$$\frac{\Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lw), \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rw), \quad \frac{\mathfrak{A}, \mathfrak{A}, \Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lc), \quad \frac{\Gamma \Rightarrow \Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rc).$$

Theorem 2.1.5. Let Σ be a subset of $\mathsf{FrameAxiom}$. The following rule (Cut) is admissible in $\mathbf{G3K}\Sigma^*$,

$$\frac{\Gamma \Rightarrow \Delta, x{:}A \quad x{:}A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ (Cut),$$

where labelled formula $x{:}A$ in (Cut) is called a *cut expression*.

Now, we move onto the part of the soundness theorem of $\mathbf{G3K}\Sigma^*$. For the theorem, we extend Kripke semantics to cover the labelled expressions. Given any model $\mathfrak{M}$, we say that $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ is an *assignment*.

Definition 2.1.15. Let $\mathfrak{M}$ be a model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment:

$$\begin{array}{lll} \mathfrak{M}, f \Vdash x{:}A & \textit{iff} & \mathfrak{M}, f(x) \Vdash A, \\ \mathfrak{M}, f \Vdash x\mathsf{R}_\Box y & \textit{iff} & (f(x), f(y)) \in R_\Box. \end{array}$$

Definition 2.1.16 (Validity for sequents)**.** Let $\Gamma \Rightarrow \Delta$ be any sequent.

- $\Gamma \Rightarrow \Delta$ is *valid on* a model $\mathfrak{M}$ if for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \Vdash \mathfrak{B}$.

- $\Gamma \Rightarrow \Delta$ is *valid on* a class $\mathbb{M}$ of frames if $\Gamma \Rightarrow \Delta$ is valid on $\mathfrak{M}$ for all $\mathfrak{M}$ in a class $\mathbb{M}$ of models.

Let us recall Definition 2.1.4 by which the class $\mathbb{M}_\Sigma$ of models is defined by $\mathbb{M}_\Sigma := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_\Sigma \text{ and } V \text{ is a valuation } V \text{ on } \mathfrak{F}\}$. The following results of soundness and completeness are shown in Negri and von Plato [58, Theorem 11.27, Theorem 11.28].

Theorem 2.1.6 (Soundness and completeness of $\mathbf{G3K}\Sigma^*$)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. $\Gamma \Rightarrow \Delta$ is valid on $\mathbb{M}_\Sigma$ iff $\vdash_{\mathbf{G3K}\Sigma^*} \Gamma \Rightarrow \Delta$.

Combining Theorem 2.1.1 and Theorem 2.1.2 and Theorem 2.1.6, we have the following.

Corollary 2.1.2. Let Σ be a subset of $\mathsf{FrameAxiom}$ and A a formula of $\mathcal{L}_{ML}$. Then the following are equivalent:

(i) $\mathbb{M}_\Sigma \Vdash A$,

(ii) $\vdash_{\mathbf{HK}\Sigma} A$,

(iii) $\vdash_{\mathbf{G3K}\Sigma^*} \Rightarrow x{:}A$.

2.1.4 Multi-agent epistemic logic

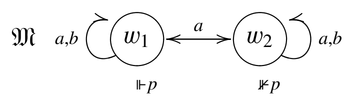
Multi-agent epistemic logic is basically the same as Multi-modal Logic **S5** where a *finite* set Agt of agents $a, b, c, \ldots$ are given as an index set of modal operators, and so the set Mod of modal operators is defined by $\{\Box_a \mid a \in \mathsf{Agt}\}$ and also a list of accessibility relations in a model is given with the index set of agents as $(R_a)_{a \in \mathsf{Agt}}$.[4] It is

[4]In epistemic logics, operator $\Box_a$ is sometimes written as K_a by the initial of "Know."

usually assumed that R_a is an *equivalent relation* in Kripke semantics for the standard epistemic logic. A modal operator $\Box_a$ is called a *knowledge operator* and interpreted as 'agent a knows that'; for example a formula $\Box_a A$ reads 'agent a knows that A.' By using such operators, a sentence like 'agent a knows A and agent b doesn't know B' can be formally expressed as $\Box_a A \wedge \neg\Box_b B$. Moreover, a nesting knowledge of distinct agents is also representable such as $\Box_a \Box_b A$ which reads 'agent a knows that agent b knows that A.'

In terms of epistemic logic, the formulas $\mathbf{4}_{\Box_a}$ $(\Box_a A \to \Box_a \Box_a A)$ and $\mathbf{5}_{\Box_a}$ $(\neg\Box_a A \to \Box_a \neg\Box_a A)$ are called *positive introspection* and *negative introspection* respectively, since $\mathbf{4}_{\Box_a}$ may be interpreted as 'if agent a knows A, he/she knows that their own knowledge state of A,' and similarly $\mathbf{5}_{\Box_a}$ may be interpreted as 'if agent a does not know that A, he/she knows their own knowledge state of A.' Besides, the formula $\mathbf{T}_{\Box_a}$ $(\Box_a A \to A)$ is called *truth-axiom* since this means here that what is known is actually true.[5]

Example 2.1.1. Let us consider $\mathsf{Agt} = \{a, b\}$ and the following model such as: $\mathfrak{M} = (W, R_a, R_b, V) = (\{w_1, w_2\}, W^2, id_W, V)$ where $V(p) = \{w_1\}$. This model can be shown in graphic forms as follows.



Intuitively, for any agent x, a bidirectional arrow of x between two worlds stands for that x cannot distinguish between two, and he/she is ignorant of sentences which distinguish the two worlds. The above model $\mathfrak{M}$ stands for the knowledge state of both a and b where a is ignorant of p but b is not. In other words, agent a does not know whether p holds. This formally means that formula $\neg(\Box_a p \vee \Box_a \neg p)$ is valid in $\mathfrak{M}$ but not valid in the case of agent b,

2.2 Public Announcement Logic (PAL)

In this section, we introduce the first DEL in the thesis, and this becomes the basis of other DELs. Section 2.2.1 introduces the language and Kripke semantics of PAL, Section 2.2.2 gives specific examples (one of which is called the Muddy children puzzle) of how to formalize knowledge-change through public announcements. Section 2.2.3 introduces Hilbert-system **HPAL**.

2.2.1 Language $\mathcal{L}_{PAL}$ and Kripke semantics

First of all, we address the language of PAL. Let $\mathsf{Prop} = \{p, q, r, \ldots\}$ be a countably infinite set of propositional atoms and $\mathsf{Agt} = \{a, b, c, \ldots\}$ a finite set of agents. Then the set $\mathcal{L}_{PAL} = \{A, B, C, \ldots\}$ of formulas of PAL is inductively defined as follows ($p \in \mathsf{Prop}$, $a \in \mathsf{Agt}$):

[5]Formula $\mathbf{T}_{\Box_a}$ implies a quite strong meaning in terms of knowledge. Actually, doxastic logic (logic of belief) usually includes $\mathbf{D}_{\Box_a}$ instead of that.

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box_a A \mid [A]A.$$

Other logical connectives are defined as usual (see Definition 2.1.1), and $\langle A \rangle B$ is defined by $\neg[A]\neg B$. $[A]B$ reads 'after public announcement of A, it holds that B.'

Example 2.2.1. Let us consider a propositional atom p to read 'it will rain tomorrow'. Then a formula $\neg(\Box_a p \lor \Box_a \neg p)$ means that a does not know whether it will rain tomorrow or not, and $[\neg p]\Box_a \neg p$ means that after a public announcement (e.g., a weather report) of $\neg p$, a knows that it will not rain tomorrow.

We should now consider Kripke semantics of PAL in which we mainly follow the semantics introduced in van Ditmarsch et al. [83]. We call $\mathfrak{M} = (W, (R_a)_{a \in \mathsf{Agt}}, V)$ a *model* if W is a nonempty set of worlds, $R_a \subseteq W \times W$, and V is a valuation function which assigns a propositional atom to a subset of W. W is also called the *domain* of $\mathfrak{M}$, denoted by $\mathcal{D}(\mathfrak{M})$. Next, let us define the satisfaction relation.

Definition 2.2.1 (The satisfaction relation)**.** Given a model $\mathfrak{M}$, $w \in \mathcal{D}(\mathfrak{M})$, and $A \in \mathcal{L}_{PAL}$, we define $\mathfrak{M}, w \Vdash A$ as follows:

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \Vdash \neg A & \text{iff} & \mathfrak{M}, w \nVdash A, \\
\mathfrak{M}, w \Vdash A \to B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ implies } \mathfrak{M}, w \Vdash B, \\
\mathfrak{M}, w \Vdash \Box_a A & \text{iff} & \text{for all } v \in W : wR_a v \text{ implies } \mathfrak{M}, v \Vdash A \ (a \in \mathsf{Agt}), \\
\mathfrak{M}, w \Vdash [A]B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ implies } \mathfrak{M}^A, w \Vdash B,
\end{array}$$

where the restriction $\mathfrak{M}^A$, at the definition of the announcement operator, is the restricted model to the truth set of A, defined as $\mathfrak{M}^A = (W^A, (R_a^A)_{a \in \mathsf{Agt}}, V^A)$ with

$$\begin{array}{lll}
W^A & := & \{x \in W \mid \mathfrak{M}, x \Vdash A\}, \\
R_a^A & := & R_a \cap (W^A \times W^A), \\
V^A(p) & := & V(p) \cap W^A \ (p \in \mathsf{Prop}).
\end{array}$$

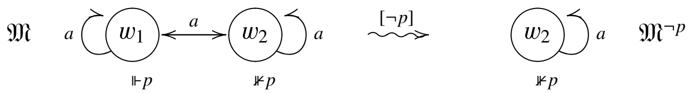
As above, the restriction of a model is based on the restriction of the set of worlds, so that this can be said to be the *world-deleting* semantics of PAL, and this will be distinguished from the *link-cutting* semantics in Section 3.4. In the semantics above, we do not assume any requirement on accessibility relations $(R_a)_{a \in \mathsf{Agt}}$ (not assuming R_a is an equivalent relation), since the previous works [6, 51] also start with a model with an arbitrary accessibility relation, we also follow them in this respect. The validity of formula A is defined similarly as in the case of ML.

Definition 2.2.2. A formula $A \in \mathcal{L}_{PAL}$ is *valid* in a model $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash A$ for all $w \in \mathcal{D}(\mathfrak{M})$.

2.2.2 Examples of knowledge-change in PAL

We now have a complete set of PAL i.e., its language, Kripke semantics and Hilbert-system, but a reader who is not familiar with PAL may not easily see what it is. The following each example formally represents knowledge-change of agents through models and it might help for understanding the heart of PAL.

Example 2.2.2. First of all, we formalize Example 2.2.1 with models as follows. Let us consider $\mathsf{Agt} = \{a\}$ and the following two models such as: $\mathfrak{M} = (\{w_1, w_2\}, W^2, V)$ where $V(p) = \{w_1\}$, and $\mathfrak{M}^{\neg p} = (\{w_2\}, \{(w_2, w_2)\}, V^{\neg p})$ where $V^{\neg p}(p) = \varnothing$. These models can be shown in graphic forms as follows.



In $\mathfrak{M}$, agent a does not know whether p or $\neg p$ (i.e., $\neg(\Box_a p \vee \Box_a \neg p)$ is valid in $\mathfrak{M}$), but after announcement of $\neg p$, agent a comes to know $\neg p$ in the restricted model $\mathfrak{M}$ to $\neg p$.

Example 2.2.3 (Muddy children puzzle)**.** Logical puzzles are sometimes utilized for formal expressions of knowledge-change as concrete examples of DELs. 'the Muddy children puzzle' introduced in [68, Example 2.17] and [83, pp.93-96] is one such puzzle and is probably the most famous in knowledge-representation.

The situation is that there are three children *Ann*, *Bill* and *Cath*, and they may have mud on their foreheads. They can see each others' foreheads but not their own forehead. In other words, they know if children other than oneself are muddy or not. This is the initial situation. Then their father who knows who is muddy said 'at least one of you is muddy' (1st public announcement). Then, the father said 'tell me if you know who is muddy,' and no child said anything (2nd public announcement). After that, the father repeats the request, and Ann and Bill said 'yes, I know' (3rd public announcement).Who is muddy?

Let Agt be $\{a, b, c\}$ each of whose elements represents Ann, Bill and Cath respectively, and let x_knows be a formula $\Box_x xm \vee \Box_x \neg xm$ ($x \in \mathsf{Agt}$) which intuitively means that agent x knows whether x is muddy or not (propositional atom xm reads agent x is muddy). Then we formalize the changes of the children's knowledge.

First, let us formalize the three public announcements.

- The first public announcement 'at least one of you is muddy,' can be expressed as a formula in $\mathcal{L}_{PAL}$ as '$am \vee bm \vee cm$,' and let the formula be *somebody_is_muddy*.
- The second public announcement (in this context public information) is the children's reaction to the father's request 'tell me if you know who is muddy,' and it is no body said anything.' The announcement (information) can also be formalized as '$\neg a\_knows \wedge \neg b\_knows \wedge \neg c\_knows$'
- The third public announcement is also the children's reaction to the father's same request, and Ann and Bill said 'yes, I know,' but Cath said nothing, i.e., Cath is still ignorant of who is exactly muddy. The announcement can also be formalized as '$a\_knows \wedge b\_knows \wedge \neg c\_knows$'

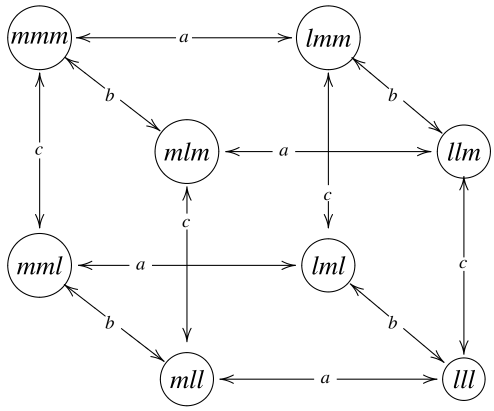
Children's initial knowledge-states in which they do not know if their own forehead is muddy can be formalized through a model as follows. Let Agt be $\{a, b, c\}$ and model

$\mathfrak{M} = (W, R_a, R_b, R_c, V)$ where

$$
\begin{array}{ll}
W = \{xyz \mid x, y, z \in \{m, l\}\}, & V(am) = \{mmm, mlm, mll, mml\}, \\
R_a = \{(xyz, x'yz) \mid x, x', y, z \in \{m, l\}\}, & V(bm) = \{lml, lmm, mml, mmm\}, \\
R_b = \{(xyz, xy'z) \mid x, y, y', z \in \{m, l\}\}, & V(cm) = \{llm, lmm, mlm, mmm\}. \\
R_c = \{(xyz, xyz') \mid x, y, z, z' \in \{m, l\}\}, &
\end{array}
$$

Each world represents the states of children's forehead, e.g., *mll* means *a* is muddy, *b* is clean and *c* is clean. Model $\mathfrak{M}$ representing the initial state of children's knowledge can be shown in graphic forms as follows (omitting reflexivity).

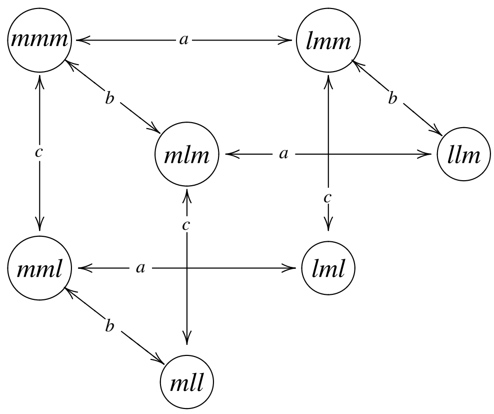
• Initial model $\mathfrak{M}$



As we have mentioned in the previous section, a bidirectional arrow of agent *x* linking two worlds means in epistemic logics that *x* does not distinguish the two; therefore, this model suggests that each children does not distinguish a world where their own forehead is muddy from a world where it is clean.

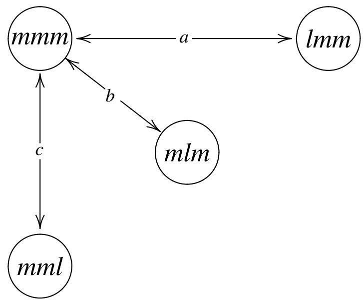
But, after the first announcement *somebody_is_muddy*, the model is modified since this formula does not hold at world *lll* where every child is clean, i.e., '$\mathfrak{M}, lll \nVdash somebody\_is\_muddy$.' This restricted model is written as $\mathfrak{M}^{somebody\_is\_muddy}$ and depicted as follows:

• Second model $\mathfrak{M}^{somebody\_is\_muddy}$



Next, announcement $\neg a\_knows \wedge \neg b\_knows \wedge \neg c\_knows$ modifies model $\mathfrak{M}^{somebody\_is\_muddy}$ and the secondly restricted model is written as $(\mathfrak{M}^{somebody\_is\_muddy})^{\neg a\_knows \wedge \neg b\_knows \wedge \neg c\_knows}$, in which every child knows at least two children are muddy.

• Third model $(\mathfrak{M}^{somebody\_is\_muddy})^{\neg a\_knows \wedge \neg b\_knows \wedge \neg c\_knows}$



The next restricted model after $a\_knows \wedge b\_knows \wedge \neg c\_knows$ is depicted as follows:

• Forth model $((\mathfrak{M}^{somebody\_is\_muddy})^{\neg a\_knows \wedge \neg b\_knows \wedge \neg c\_knows})^{a\_knows \wedge b\_knows \wedge \neg c\_knows}$

Finally, we obtain the model consisting of only one world *mml*. That means the answer is only Cath is clean.

2.2.3 Hilbert-system HPAL of PAL

Table 2.5: Hilbert-system for PAL : **HPAL**

Modal Axioms	$(Taut)$	all instantiations of propositional tautologies
	(K)	$\Box_a(A \to B) \to (\Box_a A \to \Box_a B)$
Recursion Axioms	$(RA1)$	$[A]p \leftrightarrow (A \to p)$
	$(RA2)$	$[A](B \to C) \leftrightarrow ([A]B \to [A]C)$
	$(RA3)$	$[A]\neg B \leftrightarrow (A \to \neg[A]B)$
	$(RA4)$	$[A]\Box_a B \leftrightarrow (A \to \Box_a[A]B)$
	$(RA5)$	$[A][B]C \leftrightarrow [A \wedge [A]B]C$
Inference Rules	(MP)	From A and $A \to B$, infer B
	$(Nec\Box_a)$	From A, infer $\Box_a A$

Hilbert-system **HPAL** is defined in Table 2.5 below where there are some axioms with announcement operators as additional axioms to the Hilbert-system of **HK**. These five additional axioms (from (RA1) to (RA5)) are called *recursion axioms* (or *reduction axioms*). They exist for reducing each of the theorems of **HPAL** into theorems of Hilbert-system **HK**. The proof of the completeness and soundness theorem of **HPAL** is carried out by a translation method whose basic idea was given in the previous work [68, Theorem 2.7]. We define the derivation and derivability of **HPAL** in the same manner as that of **HK** in Definition 2.1.6. Then we briefly look at established results.

Theorem 2.2.1 (Soundness of **HPAL**)**.** For any formula A, A is valid only if A is derivable in **HPAL**.

In the case of the soundness theorem, it suffices to show the validity of **HPAL**'s recursion axioms, which is straightforward.

For the proof of the completeness theorem, we follow the proof in [83, pp.186-7] whose essential idea is that every formula in $\mathcal{L}_{PAL}$ is reducible into a formula in $\mathcal{L}_{ML}$ by the following translation function $t : \mathcal{L}_{PAL} \to \mathcal{L}_{ML}$. In other words, translated formula $t(A) \in \mathcal{L}_{ML}$ is semantically equivalent to the original formula $A \in \mathcal{L}_{PAL}$, which is guaranteed by the validity of the recursion axioms. The translation function $t : \mathcal{L}_{PAL} \to \mathcal{L}_{ML}$ and the complexity function $c : \mathcal{L}_{PAL} \to \mathbb{N}$ are given in the following.

Definition 2.2.3 (Complexity)**.** The complexity function $c : \mathcal{L}_{PAL} \to \mathbb{N}$ is inductively defined as follows:

$$\begin{array}{ll} c(p) = 1, & c(\Box_a A) = 1 + c(A), \\ c(\neg A) = 1 + c(A), & c([A]B) = (4 + c(A)) \cdot c(B). \\ c(A \to B) = 1 + max\{c(A), c(B)\}, & \end{array}$$

Definition 2.2.4 (Translation)**.** The translation function $t : \mathcal{L}_{PAL} \to \mathcal{L}_{ML}$ is inductively defined as follows:

$$\begin{array}{ll} t(p) = p, & t([A]p) = t(A \to p), \\ t(\neg A) = \neg t(A), & t([A]B \to C) = t([A]B \to [A]C), \\ t(A \to B) = t(A) \to t(B), & t([A]\Box_a B) = t(A \to \Box_a [A]B), \\ t(\Box_a A) = \Box_a t(A), & t([A][B]C) = t([A \wedge [A]B]C). \end{array}$$

By these settings, we can easily show the following lemma.

Lemma 2.2.1. For any $A, B, C \in \mathcal{L}_{PAL}$, the following hold:

$$\begin{array}{llll} (1) & c([A]p) > c(A \to p), & (4) & c([A]\Box_a B) > c(A \to \Box_a [A]B), \\ (2) & c([A]\neg B) > c(A \to \neg [A]B), & (5) & c([A][B]C) > c([A \wedge [A]B]C). \\ (3) & c([A]B \to C) > c([A]B \to [A]C), & & \end{array}$$

Proof. We only check (5), and it is trivial by the following equations:

$$\begin{array}{rcl} c([A][B]C) & = & (4 + c(A)) \cdot ((4 + c(B)) \cdot c(C)) \\ & = & (16 + 4c(B) + 4c(A) + c(A)c(B)) \cdot c(C) \\ & & \\ c([A \wedge [A]B]C) & = & (4 + c(\neg(A \to \neg[A]B))) \cdot c(C) \\ & = & (6 + \max\{c(A), 1 + c([A]B)\}) \cdot c(C) \\ & = & (6 + \max\{c(A), 1 + 4c(B) + c(A)c(B)\}) \cdot c(C) \\ & = & (7 + 4c(B) + c(A)c(B)) \cdot c(C) \end{array}$$

Therefore, (5) holds. □

Lemma 2.2.2. For any $A \in \mathcal{L}_{PAL}$, $\vdash_{\mathbf{HPAL}} A \leftrightarrow t(A)$ holds.

Theorem 2.2.2 (Completeness of **HPAL**)**.** For any formula A, A is valid only if A is derivable in **HPAL**.

Proof. Suppose A is valid. So, since $A \leftrightarrow t(A)$ is valid by Lemma 2.2.2 and Theorem 2.2.1 (the soundness of **HPAL**) , we obtain $t(A)$ is valid. Then by Theorem 2.1.2 (the completeness of **HK**), we obtain $\vdash_{\mathbf{HK}} t(A)$ and trivially $\vdash_{\mathbf{HPAL}} t(A)$ holds; therefore, $\vdash_{\mathbf{HPAL}} A$ by Lemma 2.2.2 again. □

Added to them, the decidability of PAL may be shown easily with the help of the recursion axioms and Corollary 2.1.1.

Corollary 2.2.1. PAL is decidable.

Proof. We show that there is an effective method for deciding of any formula $A \in \mathcal{L}_{PAL}$ whether or not it is a theorem of PAL. Fix any $A \in \mathcal{L}_{PAL}$. Note that translation $t : \mathcal{L}_{PAL} \to \mathcal{L}_{EL}$ is inductively and so provides an effective method. Then since modal logic **K** is decidable, $t(A) \in \mathcal{L}_{ML}$ can be decided whether it is a theorem of **K** or not. □

2.3 Logic of Epistemic Actions and Knowledge (EAK)

Section 2.3.1 gives the language and Kripke semantics of EAK, Section 2.3.2 shows examples of EAK and Section 2.3.3 briefly looks at the soundness and completeness results of its Hilbert-system.

2.3.1 Language $\mathcal{L}_{EAK}$ and Kripke semantics

We define the language and Kripke semantics of EAK, we mainly follow the definition of EAK as given in van Ditmarsch et al. [83]. Let $\mathsf{Agt} = \{a, b, c, \ldots\}$ be a finite set of *agents* and $\mathsf{Prop} = \{p, q, r, \ldots\}$ a countably infinite set of *propositional atoms*. An (S5) *action frame* is a pair $(\mathsf{S}, (\sim_a)_{a \in \mathsf{Agt}})$ where S is a non-empty *finite* set of actions and $\sim_a$ is an equivalence relation on S, which represents agent a's uncertainty like PAL. In what follows, we use an element of a countable set $\mathsf{Evt} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{s}, \mathsf{t}, \ldots\}$ as a *meta-variable* to refer to an action.

Definition 2.3.1. We define the set $\mathcal{L}_{EAK} = \{A, B, \ldots\}$ of all formulas of EAK and the set of all (S5) *action models* $\mathsf{M} = (\mathsf{S}, (\sim_a)_{a \in \mathsf{Agt}}, \mathsf{pre})$ by simultaneous induction as follows ($p \in \mathsf{Prop}$, $a \in \mathsf{Agt}$, and $\mathsf{a} \in \mathsf{S}$)[6]:

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box_a A \mid [\mathsf{a}^{\mathsf{M}}]A,$$

where $(\mathsf{S}, (\sim_a)_{a \in \mathsf{Agt}})$ is an action frame, pre is a function which assigns an $\mathcal{L}_{EAK}$-formula $\mathsf{pre}(\mathsf{b})$ to each action $\mathsf{b} \in \mathsf{S}$, and an expression a^{M} is an abbreviation of a *pointed action model* (M, a). We read $[\mathsf{a}^{\mathsf{M}}]A$ as 'after an action a^{M} occurs, A holds.' Other logical connectives are defined as usual (see Definition 2.1.1).

[6]Ditmarsch et al. [83] includes union of actions $\mathsf{a}^{\mathsf{M}} \cup \mathsf{b}^{\mathsf{N}}$ in the language, but we omit $[\mathsf{a}^{\mathsf{M}} \cup \mathsf{b}^{\mathsf{N}}]A$ as $[\mathsf{a}^{\mathsf{M}}]A \wedge [\mathsf{b}^{\mathsf{N}}]A$.

For any action model $\mathsf{M} = (\mathsf{S}, (\sim_a)_{a\in\mathsf{Agt}}, \mathsf{pre})$, we use M as a superscript of S, $\sim_a$ and pre such as S^{M}, $\sim_a^{\mathsf{M}}$ and $\mathsf{pre}^{\mathsf{M}}$ to emphasize that they belong to the action model M. PEvt is used to denote the set $\{\mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}, \ldots\}$ of all pointed action models.

Definition 2.3.2 (Composition of Actions)**.** Given any two action models M and N, the *composition* of the actions $\mathsf{M};\mathsf{N}$ is the action model such that:

$$\begin{array}{lcl} \mathsf{S}^{\mathsf{M};\mathsf{N}} & = & \mathsf{S}^{\mathsf{M}} \times \mathsf{S}^{\mathsf{N}}, \\ (\mathsf{a}, \mathsf{a}')\sim_a^{\mathsf{M};\mathsf{N}}(\mathsf{b}, \mathsf{b}') & \text{iff} & \mathsf{a}\sim_a^{\mathsf{M}}\mathsf{b} \textit{ and } \mathsf{a}'\sim_a^{\mathsf{N}}\mathsf{b}', \\ \mathsf{pre}^{\mathsf{M};\mathsf{N}}((\mathsf{a}, \mathsf{b})) & = & \mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \wedge [\mathsf{a}^{\mathsf{M}}]\mathsf{pre}^{\mathsf{N}}(\mathsf{b}). \end{array}$$

Given any pointed action models a^{M} and b^{N}, the composition of the pointed action models, $\mathsf{a}^{\mathsf{M}};\mathsf{b}^{\mathsf{N}}$, is the pointed action model such that $(\mathsf{a}, \mathsf{b})^{\mathsf{M};\mathsf{N}}$ with $\mathsf{M};\mathsf{N}$.

Note that the above action model $(\mathsf{a}, \mathsf{b})^{\mathsf{M};\mathsf{N}}$ is a pointed action model (but only with a complex name of action (a, b)) by the definition above, and so it is included in PEvt.

Kripke semantics of EAK is, in fact, exactly the same as the that of PAL. A *model* $\mathfrak{M}$ is a triple $(W, (R_a)_{a\in\mathsf{Agt}}, V)$ such that W is a non-empty set of *worlds* (W of $\mathfrak{M}$ is also written as $\mathcal{D}(\mathfrak{M})$), An accessibility relations $(R_a)_{a\in\mathsf{Agt}}$ is an Agt-indexed family of binary relations on W (a ranges over Agt) and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function. Like the case of PAL, we define EAK based on modal logic **K**; therefore, we do not assume any frame property on R_a.

Given a model $\mathfrak{M}$ and a world $w \in \mathcal{D}(\mathfrak{M})$, the satisfaction relation $\mathfrak{M}, w \Vdash A$ for a formula A is inductively defined as follows:

$$\begin{array}{lll} \mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \\ \mathfrak{M}, w \Vdash \neg A & \text{iff} & \mathfrak{M}, w \nVdash A, \\ \mathfrak{M}, w \Vdash A \to B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ implies } \mathfrak{M}, w \Vdash B, \\ \mathfrak{M}, w \Vdash \Box_a A & \text{iff} & \text{for all } v \in W : \ wR_a v \text{ implies } \mathfrak{M}, v \Vdash A, \\ \mathfrak{M}, w \Vdash [\mathsf{a}^{\mathsf{M}}]A & \text{iff} & \mathfrak{M}, w \Vdash \mathsf{pre}(\mathsf{a}) \text{ implies } \mathfrak{M}^{\otimes\mathsf{M}}, (w, \mathsf{a}) \Vdash A, \end{array}$$

where $\mathfrak{M}^{\otimes\mathsf{M}} = (W^{\otimes\mathsf{M}}, (R_a^{\otimes\mathsf{M}})_{a\in\mathsf{Agt}}, V^{\otimes\mathsf{M}})$ is the *updated model* of $\mathfrak{M}$ by an action model M and it is defined as:

$$\begin{array}{lll} W^{\otimes\mathsf{M}} & = & \{(w, \mathsf{a}) \in W \times \mathsf{S}^{\mathsf{M}} \mid \mathfrak{M}, w \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})\}, \\ (w, \mathsf{a})R_a^{\otimes\mathsf{M}}(v, \mathsf{b}) & \text{iff} & wR_a v \text{ and } \mathsf{a}\sim_a^{\mathsf{M}}\mathsf{b}, \\ (w, \mathsf{a}) \in V^{\otimes\mathsf{M}}(p) & \text{iff} & w \in V(p), \end{array}$$

where $a \in \mathsf{Agt}$ and $p \in \mathsf{Prop}$. A formula A is *valid* if $\mathfrak{M}, w \Vdash A$ holds in any model $\mathfrak{M}$ and any world $w \in \mathcal{D}(\mathfrak{M})$. Intuitively, $\mathfrak{M}^{\otimes\mathsf{M}}$ means $\mathfrak{M}$ updated by action M. We briefly give an example which will show a way how EAK expresses a changing knowledge state, by taking an example of an action model Read (the simplest example of 'private announcement') in [83, p.166]. Additionally, *multiple updates* $(\cdots(\mathfrak{M}^{\otimes\mathsf{M}_1})^{\otimes\cdots})^{\otimes\mathsf{M}_n}$ on $\mathfrak{M}$ are also possible, which we write as $\mathfrak{M}^{\otimes\mathsf{M}_1\otimes\cdots\otimes\mathsf{M}_n}$ for simplicity. Each Greek letters $\alpha, \beta, \ldots$ indicates a finite list $\mathsf{a}_1^{\mathsf{M}_1}, \ldots, \mathsf{a}_n^{\mathsf{M}_n}$ of pointed action models, and ϵ is the empty list. Moreover, if α is a list $\mathsf{a}_1^{\mathsf{M}_1}, \ldots, \mathsf{a}_n^{\mathsf{M}_n}$ of pointed action models, then we define $\alpha_{evt} := (\mathsf{a}_1, \ldots, \mathsf{a}_n)$ and $\alpha_{mdl} := (\mathsf{M}_1, \ldots, \mathsf{M}_n)$, and $\alpha_{evt} := \epsilon$ and $\alpha_{mdl} := \epsilon$ if α is ϵ. The symbol $\mathfrak{M}^{\otimes\alpha_{mdl}}$ indicates $\mathfrak{M}^{\otimes\mathsf{M}_1\otimes\mathsf{M}_2\otimes\cdots\otimes\mathsf{M}_n}$ when $\alpha_{mdl} = (\mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_n)$, and $\mathfrak{M}^{\otimes\alpha_{mdl}}$ indicates $\mathfrak{M}$ when $\alpha = \epsilon$.

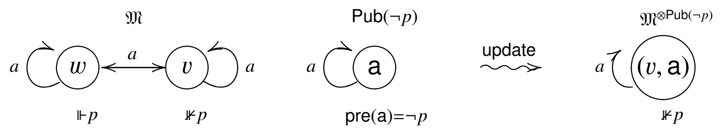
2.3.2 Examples of knowledge-change in EAK

In this section, we look at some examples of change of knowledge states in EAK. At first, it is significant to mention that EAK is a generalized version of PAL and this logic is able to emulate PAL (Excise 6.14 in [83]). Let us consider action model $\mathsf{Pub}(A) = (\{\mathsf{a}\}, (\sim_a)_{a \in \mathsf{Agt}}, \mathsf{pre})$ where each $\sim_a$ is the identity relation and $\mathsf{pre}(\mathsf{a}) = A$ for any formula $A \in \mathcal{L}_{EAK}$.

Proposition 2.3.1 (PAL in EAK)**.** The following equivalence holds, $\mathfrak{M}, w \Vdash [\mathsf{a}^{\mathsf{Pub}(A)}]B$ iff $\mathfrak{M}, w \Vdash [A]B$ for any formulas $A, B \in \mathcal{L}_{ML}$ and any model $\mathfrak{M}$.

Now, let us see a specific example of the emulation of PAL by EAK.

Example 2.3.1. Let $\mathsf{Agt} = \{a\}$ and Kripke mode $\mathfrak{M} = (\{w, v\}, W^2, V)$ where $V(p) = \{w\}$, and action model $\mathsf{Pub}(\neg p) = (\{\mathsf{a}\}, \{(\mathsf{a}, \mathsf{a})\}, \mathsf{pre})$ where $\mathsf{pre}(\mathsf{a}) = \neg p$. These models can be shown in graphic forms as follows.



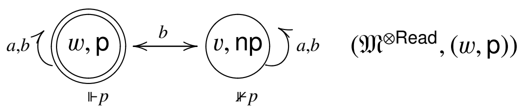
Next, let us look at a peculiar example of knowledge change by EAK. This is called 'private announcement.'

Example 2.3.2. Suppose there are two agents a and b, and neither of them knows whether p. Then only a reads a letter where p is written. As a consequence, a's knowledge changes and she knows p, but b does not. Let $\mathsf{Agt} = \{a, b\}$. Then a model $\mathfrak{M}$ and an action model Read are defined as follows:

$\mathfrak{M} = (W, R_a, R_b, V) = (\{w_1, w_2\}, W^2, W^2, V)$
  where $V(p) = \{w_1\}$,
$\mathsf{Read} = (\mathsf{S}, \sim_a, \sim_b, \mathsf{pre}) = (\{\mathsf{p}, \mathsf{np}\}, id_{\mathsf{S}}, \mathsf{S}^2, \mathsf{pre})$
  where $\mathsf{pre}(\mathsf{p}) = p, \mathsf{pre}(\mathsf{np}) = \neg p$.

This situation of the agent a and b can be semantically formalized by a pointed Kripke model $(\mathfrak{M}, w)$, a pointed action model $(\mathsf{Read}, \mathsf{p})$, and these two models are depicted as follows.

Then the pointed updated model $(\mathfrak{M}^{\otimes \mathsf{Read}}, (w, \mathsf{p}))$ is as follows.



Each double circle indicates the given or resulting point (world or action) which stands for the actual world or actual action. For any agent x, a bidirectional arrow of x between two worlds (or actions) intuitively stands for that agent x cannot distinguish between two, and x is ignorant of the reality (or actually what occurred) if one side of the arrow is the actual world (or action). Pointed Kripke model $(\mathfrak{M}, w)$ stands for the initial knowledge state of both a and b where both are ignorant of p (the actual world w). Pointed action model $(\mathsf{Read}, \mathsf{p})$ stands for an action such that only a reads the letter containing information p, and that is because a does not have her bidirectional arrow between the two worlds. The updated model $(\mathfrak{M}^{\otimes \mathsf{Read}}, (w, \mathsf{p}))$ stands for the knowledge state of both a and b where a knows p but b is still ignorant of p.

2.3.3 Hilbert-system HEAK of EAK

Hilbert-system **HEAK** of EAK was introduced by Baltag et al. [8], and this system is defined in Table 2.6 where the axioms for action operators are added to the Hilbert-system of modal logic **K**. These additional axioms (from $(RA1)$ to $(RA5)$) are often called *recursion axioms*, as they express a way of reducing each formula of **HEAK** equivalently into a formula of K.

Table 2.6: Hilbert-system for EAK : **HEAK**

Modal Axioms	*(Taut)*	all instantiations of propositional tautologies
	(K)	$\Box_a(A \to B) \to (\Box_a A \to \Box_a B)$
Recursion Axioms	$(RA1)$	$[\mathsf{a}^{\mathsf{M}}]p \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to p)$
	$(RA2)$	$[\mathsf{a}^{\mathsf{M}}]\neg A \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to \neg[\mathsf{a}^{\mathsf{M}}]A)$
	$(RA3)$	$[\mathsf{a}^{\mathsf{M}}](A \to B) \leftrightarrow [\mathsf{a}^{\mathsf{M}}]A \to [\mathsf{a}^{\mathsf{M}}]B$
	$(RA4)$	$[\mathsf{a}^{\mathsf{M}}]\Box_a A \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to \bigwedge_{\mathsf{a} \sim_a^{\mathsf{M}} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A)$
	$(RA5)$	$[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A \leftrightarrow [\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A$
Inference Rules	(MP)	From A and $A \to B$, infer B
	$(Nec\Box_a)$	From A, infer $\Box_a A$

The completeness theorem of **HEAK** can be shown by a similar argument in [8, Proposition 4.5].

Theorem 2.3.1 (Soundness of **HEAK**)**.** For any formula A, A is valid only if A is derivable in **HEAK**.

In the case of the soundness theorem, it suffices to show the validity of **HEAK**'s recursion axioms, which is straightforward.

For the proof of the completeness theorem, we follow the proof in [83, pp.186-7] whose essential idea is that every formula in $\mathcal{L}_{EAK}$ is reducible into a formula in $\mathcal{L}_{ML}$ by the following translation function t. The translation function $t : \mathcal{L}_{EAK} \to \mathcal{L}_{ML}$ and the complexity function $c : \mathcal{L}_{EAK} \to \mathbb{N}$ are defined.

Definition 2.3.3 (Complexity)**.** The complexity function $c : \mathcal{L}_{EAK} \to \mathbb{N}$ is inductively defined as follows:

$c(p) = 1$,	$c(\Box_a A) = 1 + c(A)$,
$c(\neg A) = 1 + c(A)$,	$c([\mathsf{a}^{\mathsf{M}}]A) = (4 + c(\mathsf{a}^{\mathsf{M}})) \cdot c(A)$,
$c(A \to B) = 1 + max\{c(A), c(B)\}$,	$c(\mathsf{a}^{\mathsf{M}}) = \max\{c(\mathsf{pre}^{\mathsf{M}}(\mathsf{x})) \mid \mathsf{x} \in \mathsf{S}^{\mathsf{M}}\}$.

Definition 2.3.4 (Translation)**.** The translation function $t : \mathcal{L}_{EAK} \to \mathcal{L}_{ML}$ is inductively defined as follows[7]:

$$
\begin{array}{ll}
t(p) = p, & t([\mathsf{a}^{\mathsf{M}}]p) = t(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p), \\
t(\neg A) = \neg t(A), & t([\mathsf{a}^{\mathsf{M}}]A \to B) = t([\mathsf{a}^{\mathsf{M}}]A \to [\mathsf{a}^{\mathsf{M}}]B), \\
t(A \to B) = t(A) \to t(B), & t([\mathsf{a}^{\mathsf{M}}]\neg A) = t(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \neg[\mathsf{a}^{\mathsf{M}}]A), \\
t(\Box_a A) = \Box_a t(A), & t([\mathsf{a}^{\mathsf{M}}]\Box_a A) = \bigwedge_{\mathsf{a} \sim_a^{\mathsf{M}} \mathsf{x}} t(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \Box_a[\mathsf{x}^{\mathsf{M}}]A), \\
 & t([\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A) = t([\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A).
\end{array}
$$

By these settings, we can easily show the following lemma.

Lemma 2.3.1. For any $A, B, C \in \mathcal{L}_{EAK}$, the following hold:

(1) $c([\mathsf{a}^{\mathsf{M}}]p) > c(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p)$,
(2) $c([\mathsf{a}^{\mathsf{M}}]\neg A) > c(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \neg[\mathsf{pre}^{\mathsf{M}}(\mathsf{a})]B)$,
(3) $c([\mathsf{a}^{\mathsf{M}}]A \to B) > c([\mathsf{pre}^{\mathsf{M}}(\mathsf{a})]A \to [\mathsf{pre}^{\mathsf{M}}(\mathsf{a})]B)$,
(4) $c([\mathsf{a}^{\mathsf{M}}]\Box_a B) > c(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \Box_a[\mathsf{pre}^{\mathsf{M}}(\mathsf{a})]B)$,
(5) $c([\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A) > c([\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A)$.

Lemma 2.3.2. For any $A \in \mathcal{L}_{EAK}$, $\vdash_{\mathbf{HEAK}} A \leftrightarrow t(A)$ holds.

Theorem 2.3.2 (Completeness of **HEAK**)**.** For any formula A, A is valid only if A is derivable in **HEAK**.

Proof. Suppose A is valid. So, since $A \leftrightarrow t(A)$ is valid by Lemma 2.3.2 and Theorem 2.3.1 (the soundness of **HEAK**), we obtain $t(A)$ is valid. Then by Theorem 2.1.2 (the completeness of **HK**), we obtain $\vdash_{\mathbf{HK}} t(A)$ and trivially $\vdash_{\mathbf{HEAK}} t(A)$ holds; therefore, $\vdash_{\mathbf{HPAL}} A$ by Lemma 2.3.2 again. □

Added to them, the decidability of EAK may be shown easily as the case of PAL.

Corollary 2.3.1. EAK is decidable.

Proof. We show that there is an effective method for deciding of any formula $A \in \mathcal{L}_{EAK}$ whether or not it is a theorem of EAK. Fix any $A \in \mathcal{L}_{EAK}$. Note that translation $t : \mathcal{L}_{EAK} \to \mathcal{L}_{EL}$ is inductively and so provides an effective method. Then since modal logic **K** is decidable, $t(A) \in \mathcal{L}_{ML}$ can be decided whether it is a theorem of **K** or not. □

[7] In [83, p.195], $t([\mathsf{a}^{\mathsf{M}}]\Box_a A)$ is defined by $t(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \Box_a[\mathsf{a}^{\mathsf{M}}]A)$, but since it is not sufficient when we look at (RA4) of EAK, we change it. (In (RA4), $[\mathsf{a}^{\mathsf{M}}]\Box_a A$ is equivalent to the conjunctive formula whose conjuncts are the elements of the set $\{\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \Box_a[\mathsf{x}^{\mathsf{M}}]A \mid \mathsf{a} \sim_a^{\mathsf{M}} \mathsf{x}\}$, so $\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \Box_a[\mathsf{a}^{\mathsf{M}}]A$ which only looks at a is not equivalent to $[\mathsf{a}^{\mathsf{M}}]\Box_a A$ and the translation fails.)

Chapter 3

Labelled sequent calculus for PAL

We have introduced in Preliminaries a basic proof system for PAL, Hilbert system **HPAL**; however an easier system to calculate theorems should be desirable, since Hilbert systems are, in general, hard to handle for proving theorems. One possible candidate for such a proof system is a celebrated Gentzen-style sequent calculus [30], where a basic unit of a derivation is the notion of a *sequent*

$$\Gamma \Rightarrow \Delta,$$

which consists of two lists (or multi-sets or sets) of formulas. How can we read $\Gamma \Rightarrow \Delta$ intuitively? There are at least two ways of reading it. First, we may read it as 'if all formulas in Γ hold, then some formula in Δ holds'. Second, we may also read it as 'it is not the case that all formulas in Γ hold and all formulas in Δ fail'. We may wonder if these two readings are equivalent, but in fact the equivalence depends on an underlying logic. For example, two readings are equivalent in the classical propositional logic, provided we understand that 'a formula A holds' by 'A is true in a given truth assignment' and 'A fails' by 'A is false under the assignment' (note that, under these readings, A does not hold if and only if A fails). One of the most uniform approaches for sequent calculus for modal logic is *labelled sequent calculus* (c.f., [57]), where each formula has a label corresponding to an element of a domain in Kripke semantics for modal logic. The proof system we are concerned with in this paper is one of the variants of labelled sequent calculus. An existing labelled sequent calculus for PAL, named **G3PAL**, was devised by Maffezioli and Negri [51]; however, a deficiency of **G3PAL** has been pointed out by Balbiani et al. [6]. They stated that there are some valid formulas such as $[p \wedge p]A \leftrightarrow [p]A$ which may be underivable in **G3PAL**. Here, we also suggest a different defect in it. In brief, because **G3PAL** does not have inference rules relating to accessibility relations, there exists a problem in case of deriving one of axioms of **HPAL**. Therefore, we introduce a revised labelled sequent calculus **GPAL** (with the rule of cut, **GPAL**$^{+}$) to compensate for the deficiency by adding some rules for accessibility relations.

Moreover, we especially focus on the soundness theorem of **GPAL**, since there is a hidden factor behind the definition of the validity of the sequent $\Gamma \Rightarrow \Delta$, of which the researchers of this field (e.g., [6, 51]) seemingly have not made a point. In particular, we notice that the above two readings of a sequent in our setting are not equivalent and that the notion of validity based on the first reading of a sequent is not sufficient to prove the soundness of our calculus for Kripke semantics; however, we employ the notion of validity based on the second reading of a sequent to establish **GPAL**'s soundness. One of the reasons why two notions of validity are not equivalent consists of deleting worlds by a (truthful) public announcement. In fact, we will show the completeness of our calculus for PAL's another semantics, a version of the link-cutting semantics by van Benthem and Liu [82] where only the accessibility relation is restricted in a model and two notions of validity become equivalent.

The outline of Chapter 3 is as follows: Section 3.1 reviews an existing labelled sequent calculus **G3PAL** by [51] and specifies which part of **G3PAL** is problematic. Section 3.2 introduces our calculus **GPAL**, a revised version of **G3PAL**, and we show that all the theorems of **HPAL** are derivable in **GPAL**$^{+}$ (Theorem 3.2.1), and establish the cut elimination theorem of **GPAL**$^{+}$ (Theorem 3.2.2). Section 3.3 focuses on its soundness theorem (Theorem 3.3.1) in terms of two notions of validity based on the above two readings of a sequent. Section 3.4 introduces the link-cutting semantics of PAL to provide a direct proof of the completeness of **GPAL** for the link-cutting semantics (Theorem 3.4.1). Section 3.5 extends the basis of **GPAL** from K to S5.

3.1 Sequent calculus for PAL

A labelled sequent calculus called **G3PAL** has been provided by [51] based on G3-system for modal logic K.

3.1.1 G3PAL

In order to introduce **G3PAL**, as in [51], it is better to explicitly confirm the satisfaction relation with a list of formulas, that restricts a model, since the following inference rules of **G3PAL** are all obtained from those satisfaction relations. We denote finite lists $(A_1, A_2, ..., A_n)$ of formulas by α,β, etc., and do the empty list by ϵ from here and after. As an abbreviation, for any list $\alpha = (A_1, A_2, ..., A_n)$ of formulas , we define $\mathfrak{M}^{\alpha}$ inductively as: $\mathfrak{M}^{\alpha} := \mathfrak{M}$ (if $\alpha = \epsilon$), and $\mathfrak{M}^{\alpha} := (\mathfrak{M}^{\beta})^{A_n} = (W^{\beta,A_n}, (R_a^{\beta,A_n})_{a\in\mathsf{Agt}}, V^{\beta,A_n})$ (if $\alpha = \beta, A_n$). We may also denote $(\mathfrak{M}^{\beta})^{A_n}$ by $\mathfrak{M}^{\beta,A_n}$ for simplicity. The satisfaction relation with restricting formulas is shown as follows ($a \in \mathsf{Agt}$):

$\mathfrak{M}^{\alpha,A}, w \Vdash p$	iff	$\mathfrak{M}^{\alpha}, w \Vdash A$ and $\mathfrak{M}^{\alpha}, w \Vdash p$,
$\mathfrak{M}^{\alpha}, w \Vdash \neg A$	iff	$\mathfrak{M}^{\alpha}, w \nVdash A$,
$\mathfrak{M}^{\alpha}, w \Vdash A \to B$	iff	$\mathfrak{M}^{\alpha}, w \Vdash A$ implies $\mathfrak{M}^{\alpha}, w \Vdash B$,
$\mathfrak{M}^{\alpha}, w \Vdash \Box_a A$	iff	for all $v \in W : wR_a^{\alpha}v$ implies $\mathfrak{M}^{\alpha}, v \Vdash A$,
$\mathfrak{M}^{\alpha}, w \Vdash [A]B$	iff	$\mathfrak{M}^{\alpha}, w \Vdash A$ implies $\mathfrak{M}^{\alpha,A}, w \Vdash B$,

where $p \in \mathsf{Prop}, A, B \in \mathcal{L}_{PAL}, \mathfrak{M}$ is any model, $w \in \mathcal{D}(\mathfrak{M})$, and α is any list of formulas. According to Kripke semantics defined in Section 2.2, $(w, v) \in R_a^{\alpha,A}$ is equivalent to the following conjunction:

$$(w, v) \in R_a^{\alpha,A} \text{ iff } (w, v) \in R_a^{\alpha} \text{ and } \mathfrak{M}^{\alpha}, w \Vdash A \text{ and } \mathfrak{M}^{\alpha}, v \Vdash A.$$

A point to notice here is that from an accessibility relation with restricting formulas, we may obtain three conjuncts.

Now we will introduce **G3PAL**. Let $\mathsf{Var} = \{x, y, z, \ldots\}$ be a countably infinite set of variables. Then, given any $x, y \in \mathsf{Var}$, any list of formulas α and any formula A, we say $x{:}^{\alpha}A$ is a *labelled formula*, and that, for any agent $a \in \mathsf{Agt}$, $x\mathsf{R}_a^{\alpha}y$ is a *relational atom*. Intuitively, the labelled formula $x{:}^{\alpha}A$ corresponds to '$\mathfrak{M}^{\alpha}, x \Vdash A$' and is to read 'after a sequence α of public announcements, x still survives[1] and A holds at x', and the relational atom $x\mathsf{R}_a^{\alpha}y$ is to read 'after a sequence α of public announcements both x and y survive and we can still access from x to y'. We also use the term, *labelled expressions* to indicate that they are either labelled formulas or relational atoms, and we denote them by $\mathfrak{A}, \mathfrak{B}$, etc. A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of finite multi-sets of labelled expressions. The set of inference rules of **G3PAL** is given in Table 3.1. For any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is derivable in **G3PAL**, we write $\vdash_{\mathbf{G3PAL}} \Gamma \Rightarrow \Delta$. The rules of (*Lat*) and (*Rat*) are obtained from the above satisfaction relation, hence if there is an announcement A and a propositional atom p, we get p with the restricting formula A. In the case of (L[.]) and (R[.]), although the satisfaction relation of the announcement operator is the same as that of implication only with the exception of restricting formulas, the rules, (L[.]) and (R[.]), are (probably) modified for G3-system. The last two rules (L_{cmp}) and (R_{cmp}) are for dealing with the proof of (RA5) of **HPAL** (we will discuss them shortly afterwards). Other inference rules result naturally from the semantics. As we have referred to in the previous paragraph, while we could have sound inference rules corresponding to restricted relational atoms, there is, actually, no rule of relational atoms in **G3PAL**, and due to this fact, **G3PAL** may not have an ability to derive one of the reduction axioms, (RA4).

3.1.2 Problems of G3PAL

As we mentioned, Balbiani et al. [6] suggested that there is a valid formula such as $[B \wedge B]A \leftrightarrow [B]A$ which cannot be derivable in **G3PAL**, but their short paper does not contain the argument regarding the underivability of such a formula. So, we, in the following, show that a particular case of one direction of $[B \wedge B]A \leftrightarrow [B]A$, which is $[p \wedge p]\Box_a p \rightarrow [p]\Box_a p$, is not derivable in **G3PAL**.

Proposition 3.1.1. The following holds:

$$\nvdash_{\mathbf{G3PAL}} \Rightarrow x{:}[p \wedge p]\Box_a p \rightarrow [p]\Box_a p,$$

where $p \in \mathsf{Prop}, x \in \mathsf{Var}$.

Proof. Suppose for a contradiction that there is a derivation of **G3PAL** for $\Rightarrow x{:}[p \wedge p]\Box_a p \rightarrow [p]\Box_a p$, and fix such a derivation $\mathcal{D}$. The last applied rule of $\mathcal{D}$ must be $(R \rightarrow)$ which is the only applicable rule **G3PAL**, and so we obtain $\vdash_{\mathbf{G3PAL}}$ $x{:}[p \wedge$

[1]The notion of *survival* will be referred in Section 3.4.2 more specifically.

Table 3.1: Labelled sequent calculus for PAL : **G3PAL**

(Initial Sequent)

$$x{:}^{\epsilon}p, \Gamma \Rightarrow \Delta, x{:}^{\epsilon}p$$

(Rules for propositional connectives)

$$\frac{}{x{:}^{\alpha}\bot, \Gamma \Rightarrow \Delta}\ (L\bot)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}{x{:}^{\alpha}\neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\neg A}\ (R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L \to) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B}\ (R \to)$$

(Rules for knowledge operators)

$$\frac{y{:}^{\alpha}A, x{:}^{\alpha}\Box_a A, x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a A, x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}\ (L\Box_a) \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\Box_a A}\ (R\Box_a)\dagger$$

† y does not appear in the lower sequent.

(Rules for PAL)

$$\frac{x{:}^{\alpha}A, x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat)$$

$$\frac{x{:}^{\alpha,A}B, x{:}^{\alpha}[A]B, x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}\ (L[.]) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha,A,B}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B}C, \Gamma \Rightarrow \Delta}\ (L_{cmp}) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B}C}\ (R_{cmp})$$

$p]\Box_a p \ \Rightarrow\ x{:}[p]\Box_a p$. Then the second last applied rule of $\mathcal{D}$ must be $(R[.])$ which is the only applicable rule, and so on. By doing so, we may find that there is only one possibility to construct derivation $\mathcal{D}$ for the sequent of the statement by the application of $(R \to)$, $(R[.])$, $(R\Box_a)$ and (Rat), each of which is the only applicable rule to each corresponding lowersequent as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{x\mathsf{R}_a^{p}y, x{:}[p \wedge p]\Box_a p, x{:}p \Rightarrow y{:}p \quad x\mathsf{R}_a^{p}y, x{:}[p \wedge p]\Box_a p, x{:}p \Rightarrow y{:}p}{x\mathsf{R}_a^{p}y, x{:}[p \wedge p]\Box_a p, x{:}p \Rightarrow y{:}^{p}p}\ (Rat)}{x{:}[p \wedge p]\Box_a p, x{:}p \Rightarrow x{:}^{p}\Box_a p}\ (R\Box_a)}{x{:}[p \wedge p]\Box_a p \Rightarrow x{:}[p]\Box_a p}\ (R[.])}{\Rightarrow x{:}[p \wedge p]\Box_a p \to [p]\Box_a p}\ (R \to)$$
.

However, the uppermost sequents above are both not initial sequents, and no inference rule in **G3PAL** can be applicable to either of them, so both are not derivable. Therefore, the attempt for deriving $\Rightarrow x{:}[p \wedge p]\Box_a p \to [p]\Box_a p$ in **G3PAL** fails. A contradiction. $\Box$

Next, we also noticed another but the same rooted problem in **G3PAL**. Maffezioli and Negri stated, in Section 5 of [51], that **G3PAL** may derive all inference rules and axioms of **HPAL**, namely if $\vdash_{\mathbf{HPAL}} A$, then $\vdash_{\mathbf{G3PAL}} \Rightarrow x{:}^{\epsilon}A$ (for any A and x). Nevertheless, there are, in fact, some problems in deriving (RA4):

$$[A]\Box_a B \leftrightarrow (A \to \Box_a[A]B).$$

This axiom cannot be derived in **G3PAL**. Let us look at possible but plausible attempts to derive both directions of (RA4). First, a possible attempt of deriving the direction from right to left is given as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{A}\Box_a B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\ ?\\ x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B, x\mathsf{R}^A_a y \Rightarrow y{:}^{A}B\end{array}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B \Rightarrow x{:}^{A}\Box_a B}\,(R\Box_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \Box_a[A]B \Rightarrow x{:}^{A}\Box_a B}\,(L\to)}{x{:}^{\epsilon}A \to \Box_a[A]B \Rightarrow x{:}^{\epsilon}[A]\Box_a B}\,(R[.])}{\Rightarrow x{:}^{\epsilon}(A \to \Box_a[A]B) \to [A]\Box_a B}\,(R\to) \qquad (*)$$

Starting from the bottom sequent, the bottom sequent of $\mathcal{D}_1$ is clearly derivable, but it is difficult to find the way to go step forward from the right uppermost sequent of the derivation. The problem here is that A in $x\mathsf{R}^A_a y$ and ϵ in $x{:}^{\epsilon}\Box_a[A]B$ on the left side of the sequent do not match, and therefore we cannot apply the rule $(L\Box_a)$.

Secondly, the other direction of (RA4) also seemingly cannot be derived by **G3PAL**. A possible attempt to derive it may be as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ ?\\ y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y, x{:}^{A}\Box_a B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_a B \Rightarrow y{:}^{A}B\end{array}}{x\mathsf{R}^{\epsilon}_a y, x{:}^{A}\Box_a B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_a B \Rightarrow y{:}^{\epsilon}[A]B}\,(R[.])}{x{:}^{A}\Box_a B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_a B \Rightarrow x{:}^{\epsilon}\Box_a[A]B}\,(R\Box_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_a B \Rightarrow x{:}^{\epsilon}\Box_a[A]B}\,(L[.])}{x{:}^{\epsilon}[A]\Box_a B \Rightarrow x{:}^{\epsilon}A \to \Box_a[A]B}\,(R\to)}{\Rightarrow x{:}^{\epsilon}[A]\Box_a B \to (A \to \Box_a[A]B)}\,(R\to) \qquad (**)$$

The derivation also comes to a dead end (in fact, the rule $(L[.])$ is applicable infinitely many times, but no new labelled expression is obtained by the application). The problem here is also that ϵ in $x\mathsf{R}^{\epsilon}_a y$ and A in $x{:}^{A}\Box_a B$ on the left side of the left uppermost sequent do not match, and again the rule $(L\Box_a)$ cannot be applied.[2].

[2] In fact, the rule of cut makes $(**)$ derivable, but if we follow $(L\Box_a)$, the rules of (L_{cmp}) and (R_{cmp}) may be indispensable to derive $(RA5)$. Nevertheless, when we reformulate $(L\Box_a)$ in a natural way (as $(L\Box'_a)$ in the next section), (L_{cmp}) and (R_{cmp}) are derivable (Lemma 3.2.1). We selected to reduce the inference rules rather than keeping G3-system.

In brief, for applying the rule $(L\Box_a)$, α in $x\mathsf{R}_a^{\alpha}y$, and β in $x{:}^{\beta}\Box_a B$ must be the same and $(L\Box_a)$ is indispensable for deriving both directions of (RA4); however there seems no way to make them equal in **G3PAL**. To settle the problems, we introduce rules for relational atoms for decomposing $x\mathsf{R}_a^{A}y$ into $x\mathsf{R}_a^{\epsilon}y$ and related labelled formulas.

3.2 Revised calculus GPAL

In this section, we revise **G3PAL** to make it possible to cope with (RA4) of **HPAL**. Let us examine the problem of $(*)$ first. To overcome the dead end of the derivation, we introduce rules of the relational atom with a list of formulas, i.e., $(Lrel_a1)$, $(Lrel_a2)$, $(Lrel_a3)$ and $(Rrel_a)$, and it is not trivial if these rules are derivable in **G3PAL**. Here are our additional rules:

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a1) \quad \frac{y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a2) \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a3)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A}y}\ (Rrel_a)$$

These inference rules are obtained in PAL's Kripke semantics. Namely, as we have already seen in Section 3.1.1, any restricted accessibility relation $wR_a^{\alpha,A}v$ is equivalent to the conjunction of the following such as: $wR_a^{\alpha}v$ and $\mathfrak{M}^{\alpha}, w \Vdash A$ and $\mathfrak{M}^{\alpha}, v \Vdash A$. These three conjuncts correspond to three $(Lrel_ai)$ rules and three uppersequents of $(Rrel_a)$. If we use $(Lrel_a3)$ to the dead end of $(*)$, $x\mathsf{R}_a^{\epsilon}y$ which we desire is obtained and it is obvious that the new emerged sequent is derivable as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots\ \mathcal{D}_2}{y{:}^{A}B, y{:}^{\epsilon}[A]B, x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{A}B}}{y{:}^{\epsilon}[A]B, x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{A}B}\ (L[.])}{x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{A}B}\ (L\Box_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_a[A]B, x\mathsf{R}_a^{A}y \Rightarrow y{:}^{A}B}\ (Lrel_a3)$$
$$\vdots$$
,

where $\mathcal{D}_2$ can be given since $y{:}^{A}B \Rightarrow y{:}^{A}B$ is clearly derivable.

However, in the case of $(**)$, the additional inference rules are not sufficient to make the branch reach initial sequent(s). This is because the new rules could not be applied to $x\mathsf{R}^{\epsilon}y$ and they will not change the situation. To settle the problem, we reformulate the rule of $(L\Box_a)$ in a semantically natural way. Our reformulated rule $(L\Box'_a)$ is then defined as follows.

$$\frac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y \quad y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a A, \Gamma \Rightarrow \Delta}\ (L\Box'_a)$$

It is necessary to note that, by this change of the rule, we need to depart from G3-system. [3] Although a solution with keeping G3-style might be a better solution than ours, we choose the semantically natural way to reformulate the rule ($L\Box_a$) first, and at the same time we reformulate the rule ($L[.]$) in a natural form.

3.2.1 GPAL

Now, we introduce our revised calculus, **GPAL**. The definition of **GPAL** is presented in Table 3.2. Hereinafter, we use the following abbreviation in a derivation for drawing simpler derivations:

$$\frac{\frac{\textit{Initial Seq.}}{}}{\mathfrak{A}, \Gamma \Rightarrow \Delta, \mathfrak{A}}$$

which is obvious by the rules (Lw) and (Rw). Besides, we also use the following derivable rules:

$$\frac{x{:}^{\alpha}A, x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \wedge B}\ (R\wedge)$$

$$\frac{x\mathsf{R}_a v, v{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Diamond_a A, \Gamma \Rightarrow \Delta}\ (L\Diamond_a)\dagger \quad \frac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a y \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\Diamond_a A}\ (R\Diamond_a).$$

where † means that v does not appear in the lower c-sequent.
Let us now show the derivations of (RA4) of **HPAL**.

Proposition 3.2.1. $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}[A]\Box_a B \leftrightarrow (A \to \Box_a[A]B)$

Proof. We may find a derivation of $x{:}^{\epsilon}[A]\Box_a B \to (A \to \Box_a[A]B)$ in **GPAL** as follows:

$$\mathcal{D} = \left\{ \frac{\frac{\frac{\textit{Initial Seq.}}{}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}A} \quad \frac{\frac{\textit{Initial Seq.}}{}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{\epsilon}A} \quad \frac{\frac{\textit{Initial Seq.}}{}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}_a^{\epsilon}y \Rightarrow, x\mathsf{R}_a^{\epsilon}y}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}_a^{\epsilon}y \Rightarrow x\mathsf{R}_a^{A}y}\ (Rrel) \right.$$

[3] Of course, there might still exist a possibility to keep G3-system with the additional rules for relational atoms. As mentioned in this page and the previous footnote, the rule ($L\Box_a$) may contain a difficulty for a labelled G3-system for PAL, since the matching of restricting formulas can be problematic; and so we reformulate the rules as ($L\Box'_a$). But one of examiners of this thesis, Makoto Kanazawa, gave us a comment regarding another formulation of ($L\Box_a$) as follows:

$$\frac{x{:}^{\alpha}\Box_a A, \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y \quad x{:}^{\alpha}\Box_a A, y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a A, \Gamma \Rightarrow \Delta}.$$

This might be an adequate candidate for forming a G3-system for PAL. Since $x{:}^{\alpha}\Box_a A$ can be interpreted as $\forall y(x\mathsf{R}_a^{\alpha}y \to y{:}^{\alpha}A)$, the rule above is naturally obtained by the combination of the rules such as ($L\forall$) and ($L\to$) in the existing G3-systems for predicate logic.

$$
\begin{prooftree}
\AxiomC{$\vdots\ \mathcal{D}$}
\noLine
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{A}_{a}y$}
\RightLabel{(Rw)}
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{A}B, x\mathsf{R}^{A}_{a}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{A}B, x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{A}B$}
\RightLabel{$(L\Box'_{a})$}
\BinaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, x{:}^{A}\Box_{a}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{A}B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Box_{a}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}[A]B$}
\RightLabel{$(R\Box_{a})$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Box_{a}B \Rightarrow x{:}^{\epsilon}\Box_{a}[A]B$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_{a}[A]B$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_{a}B \Rightarrow x{:}^{\epsilon}\Box_{a}[A]B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$x{:}^{\epsilon}[A]\Box_{a}B \Rightarrow x{:}^{\epsilon}A \to \Box_{a}[A]B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A]\Box_{a}B \to (A \to \Box_{a}[A]B)$}
\end{prooftree}
.
$$

We may also find a derivation of $x{:}^{\epsilon}(A \to \Box_{a}[A]B) \to [A]\Box_{a}B$ in **GPAL** as follows:

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{A}\Box_{a}B, x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y$}
\RightLabel{$(Lrel_{a}3)$}
\UnaryInfC{$x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}A \Rightarrow y{:}^{A}B, y{:}^{\epsilon}A$}
\RightLabel{$(Lrel_{a}2)$}
\UnaryInfC{$x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B, y{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{A}B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$y{:}^{\epsilon}[A]B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B$}
\RightLabel{$(L\Box'_{a})$}
\BinaryInfC{$x\mathsf{R}^{A}_{a}y, x{:}^{\epsilon}\Box_{a}[A]B \Rightarrow y{:}^{A}B$}
\RightLabel{$(R\Box_{a})$}
\UnaryInfC{$x{:}^{\epsilon}\Box_{a}[A]B \Rightarrow x{:}^{A}\Box_{a}B$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}^{\epsilon}\Box_{a}[A]B, x{:}^{\epsilon}A \Rightarrow x{:}^{A}\Box_{a}B$}
\RightLabel{$(L\to)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \Box_{a}[A]B \Rightarrow x{:}^{A}\Box_{a}B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \to \Box_{a}[A]B \Rightarrow x{:}^{\epsilon}[A]\Box_{a}B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow (x{:}^{\epsilon}A \to \Box_{a}[A]B) \to [A]\Box_{a}B$}
\end{prooftree}
.
$$

□

As we can see above, the proof of (RA4) in **GPAL** can be done thanks to the rules of relational atoms.

Moreover, **GPAL**$^{+}$ is defined to be **GPAL** with the following rule (*Cut*),

$$
\frac{\Gamma \Rightarrow \Delta, \mathfrak{A} \quad \mathfrak{A}, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Cut).
$$

Labelled expression $\mathfrak{A}$ in (*Cut*) is called a *cut expression*, and we say that $\mathfrak{A}$ is a *principal expression* of an inference rule of **GPAL**$^{+}$ if $\mathfrak{A}$ is newly introduced on the left uppersequent or the right uppersequent by the rule of **GPAL**$^{+}$.

Let us briefly summarize our revised calculus in order. **GPAL** is different from **G3PAL** in respect to the following features:

(i) **GPAL** is based on Gentzen's standard sequent calculus [30] but not in G3-system, and so it contains structural rules.

(ii) **GPAL** includes rules for relational atoms which **G3PAL** lacks.

(iii) $(L[.])$ and $(L\Box_{a})$ are redefined in a semantically natural way, and each of them is denoted by $(L[.]')$ and $(L\Box'_{a})$ in **GPAL**.

(iv) **GPAL** does not contain (L_{cmp}) and (R_{cmp}) of **G3PAL**, but without them it can derive (RA5). These rules are also derivable in **GPAL**$^{+}$ (see Proposition 3.2.2).

Table 3.2: Revised labelled sequent calculus for PAL: **GPAL**

(Initial sequents)

$$x{:}^{\alpha}A \Rightarrow x{:}^{\alpha}A \qquad x\mathsf{R}_a^{\alpha}v \Rightarrow x\mathsf{R}_a^{\alpha}v$$

(Structural Rules)

$$\frac{\Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lw) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rw)$$

$$\frac{\mathfrak{A}, \mathfrak{A}, \Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lc) \qquad \frac{\Gamma \Rightarrow \Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rc)$$

(Rules for propositional connectives)

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}{x{:}^{\alpha}\neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \qquad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\neg A}\ (R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L\to) \qquad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B}\ (R\to)$$

(Rules for knowledge operators)

$$\frac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y \quad y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a A, \Gamma \Rightarrow \Delta}\ (L\Box_a') \qquad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\Box_a A}\ (R\Box_a)\dagger$$

† y does not appear in the lower sequent.

(Rules for PAL)

$$\frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat') \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat')$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, \Gamma \Rightarrow \Delta}\ (L[.]') \qquad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a1) \qquad \frac{y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a2) \qquad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a3)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A}y}\ (Rrel_a)$$

(v) (*Lat*) and (*Rat*) are redefined taking into account of the notion of survival, and each of them is denoted by (*Lat′*) and (*Rat′*) in **GPAL**.

The last two features have not been mentioned so far, and the last feature of **GPAL** will be considered at the beginning of Section 3.4. In this paragraph, we focus on feature (iv). In [51], the following rules

$$\frac{x{:}^{\alpha,A,B}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B}C, \Gamma \Rightarrow \Delta}\ (L_{cmp}) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B}C}\ (R_{cmp})$$

are required to derive (RA5) of **HPAL**:

$$[A][B]C \leftrightarrow [A \wedge [A]B]C.$$

In what follows, however, we reveal that these rules of (L_{cmp}) and (R_{cmp}) are not necessary in the set of inference rules of **GPAL**. Let us see the details. First, let us define the length of a labelled expression $\mathfrak{A}$.

Definition 3.2.1. For any formula A, $\ell(A)$ is equal to the number of the propositional atoms and the logical connectives in A.

$$\ell(\alpha) = \begin{cases} 0 & \text{if } \alpha = \epsilon \\ \ell(\beta) + \ell(A) & \text{if } \alpha = \beta, A \end{cases}$$

$$\ell(\mathfrak{A}) = \begin{cases} \ell(\alpha) + \ell(A) & \text{if } \mathfrak{A} = x{:}^{\alpha}A \\ \ell(\alpha) + 1 & \text{if } \mathfrak{A} = x\mathsf{R}_a^{\alpha}y \end{cases}$$

Then, let us show the following lemma.

Lemma 3.2.1. For any $A, B \in \mathsf{Form}$, $x, y \in \mathsf{Var}$ and for any list α, β of formulas,

(i) $\vdash_{\mathbf{GPAL}} x{:}^{\alpha,A,B,\beta}C \Rightarrow x{:}^{\alpha,A\wedge[A]B,\beta}C$,

(ii) $\vdash_{\mathbf{GPAL}} x{:}^{\alpha,A\wedge[A]B,\beta}C \Rightarrow x{:}^{\alpha,A,B,\beta}C$,

(iii) $\vdash_{\mathbf{GPAL}} x\mathsf{R}_a^{\alpha,A,B,\beta}y \Rightarrow x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y$,

(iv) $\vdash_{\mathbf{GPAL}} x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y \Rightarrow x\mathsf{R}_a^{\alpha,A,B,\beta}y$.

Proof. The proofs of (i), (ii), (iii) and (iv) are done simultaneously by double induction on C and β. We only see the case where C is of the form $\Box_a D$ and the case where C is of the form $[D]E$, because the derivability of the other sequents (ii), (iii) and (iv) can also be shown similarly. First, let us consider the case where C is of the form $\Box_a D$. Let γ be (α, A, B, β) and θ be $(\alpha, A \wedge [A]B, \beta)$.

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{x\mathsf{R}_a^{\theta}y \Rightarrow x\mathsf{R}_a^{\gamma}y}}{x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D, x\mathsf{R}_a^{\gamma}y}\ (Rw) \qquad \dfrac{\dfrac{\vdots\ \mathcal{D}_2}{y{:}^{\gamma}D \Rightarrow y{:}^{\theta}D}}{y{:}^{\gamma}D, x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D}\ (Lw)}{\dfrac{x{:}^{\gamma}\Box_a D, x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D}{x{:}^{\gamma}\Box_a D \Rightarrow x{:}^{\theta}\Box_a D}\ (R\Box_a)}\ (L\Box_a')$$

Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are obtained by induction hypothesis, since the length of the labelled expressions is reduced. We may need to pay attention to the length of the labelled expression at the bottom sequent of $\mathcal{D}_1$, but according to Definition 3.2.1, $\ell(x{:}^{\gamma}\Box_a D) > \ell(x\mathsf{R}_a^{\gamma}y)$ (for any γ).

Second, let us consider the case where C is of the form $[D]E$. Let γ be (α, A, B, β) and θ be $(\alpha, A \wedge [A]B, \beta)$.

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_3}{x{:}^{\theta}D \Rightarrow x{:}^{\gamma}D}}{x{:}^{\theta}D \Rightarrow x{:}^{\gamma}D, x{:}^{\theta,D}E}\,(Rw) \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_4}{x{:}^{\gamma,D}E \Rightarrow x{:}^{\theta,D}E}}{x{:}^{\gamma,D}E, x{:}^{\theta}D \Rightarrow x{:}^{\theta,D}E}\,(Lw)}{\dfrac{x{:}^{\gamma}[D]E, x{:}^{\theta}D \Rightarrow x{:}^{\theta,D}E}{x{:}^{\gamma}[D]E \Rightarrow x{:}^{\theta}[D]E}\,(R[.])}\,(L[.]')$$

The derivations $\mathcal{D}_3$ and $\mathcal{D}_4$ are obtained by induction hypotheses. □

Now with the help of the rule (Cut), we can also show the derivability of more general rules than (L_{cmp}) and (R_{cmp}) of **G3PAL** as follows:

Proposition 3.2.2. The following rules (L'_{cmp}) and (R'_{cmp}) are derivable in $\mathbf{GPAL}^+$.

$$\frac{x{:}^{\alpha,A,B,\beta}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B,\beta}C, \Gamma \Rightarrow \Delta}\ (L'_{cmp}) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B,\beta}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B,\beta}C}\ (R'_{cmp})$$

where $a \in \mathsf{Agt}, A, B, C \in \mathsf{Form}$ and α and β are arbitrary lists of formulas.

Proof. It is shown immediately from Lemma 3.2.1 and (Cut). [4] □

3.2.2 All theorems of HPAL are derivable in GPAL$^+$

We first define the substitution of variables in labelled expressions.

Definition 3.2.2. Let $\mathfrak{A}$ be any labelled expression. Then the substitution of x for y in $\mathfrak{A}$, denoted by $\mathfrak{A}[x/y]$, is defined by

$$\begin{array}{lcl} z[x/y] & := & z \quad (\text{if } y \neq z) \\ z[x/y] & := & x \quad (\text{if } y = z) \\ (z{:}^{\alpha}A)[x/y] & := & (z[x/y]){:}^{\alpha}A \\ (z\mathsf{R}_a^{\alpha}w)[x/y] & := & (z[x/y])\mathsf{R}_a^{\alpha}(w[x/y]) \end{array}$$

Substitution $[x/y]$ to a multi-set Γ of labelled expressions is defined as

$$\Gamma[x/y] := \{\mathfrak{A}[x/y] \mid \mathfrak{A} \in \Gamma\}.$$

Next, for a preparation of Theorem 3.2.1, we show the next lemma.

Lemma 3.2.2 (Substitution lemma)**.**
(i) $\vdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{GPAL}} \Gamma[x/y] \Rightarrow \Delta[x/y]$ for any $x, y \in \mathsf{Var}$.
(ii) $\vdash_{\mathbf{GPAL}^+} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{GPAL}^+} \Gamma[x/y] \Rightarrow \Delta[x/y]$ for any $x, y \in \mathsf{Var}$.

[4] The following rules are also derivable in $\mathbf{GPAL}^+$.

$$\frac{x\mathsf{R}_a^{\alpha,A,B,\beta}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y, \Gamma \Rightarrow \Delta}\ (L_{cmpr}) \qquad \frac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A,B,\beta}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y}\ (R_{cmpr})$$

Proof. By induction on the height of the derivation, we go through almost the same procedure in the proof in Negri and von Plato [58, p.194]. □

Finally, let us show the following theorem:

Theorem 3.2.1. For any formula A, if $\vdash_{\mathbf{HPAL}} A$, then $\vdash_{\mathbf{GPAL}^+} \Rightarrow x{:}^{\epsilon}A$ (for any x).

Proof. The proof is carried out by the height of the derivation in **HPAL**. Since the case of reduction axiom (RA5) has been shown in Proposition 3.2.1, let us prove the other base cases (the derivation height of **HPAL** is equal to 0).

• Case of (RA1)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}p, x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}p, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}p$}
\RightLabel{(Lat')}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}p \Rightarrow x{:}^{\epsilon}p$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]p \Rightarrow x{:}^{\epsilon}p$}
\RightLabel{$(R\to)$}
\UnaryInfC{$x{:}^{\epsilon}[A]p \Rightarrow x{:}^{\epsilon}A \to p$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A]p \to (A \to p)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{A}p$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}p, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}p$}
\RightLabel{(Rat')}
\UnaryInfC{$x{:}^{\epsilon}p, x{:}^{\epsilon}A \Rightarrow x{:}^{A}p$}
\RightLabel{$(L\to)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}A \to p \Rightarrow x{:}^{A}p$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \to p \Rightarrow x{:}^{\epsilon}[A]p$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(A \to p) \to [A]p$}
\RightLabel{$(R\wedge)$}
\BinaryInfC{$\Rightarrow x{:}^{\epsilon}[A]p \leftrightarrow (A \to p)$}
\end{prooftree}
$$

• Case of (RA2): left to right

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A](B \to C) \Rightarrow x{:}^{\epsilon}A, x{:}^{A}C$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A, x{:}^{A}C$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B, x{:}^{A}C$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L\to)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}B \to C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A](B \to C), x{:}^{A}B \Rightarrow x{:}^{A}C$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A](B \to C) \Rightarrow x{:}^{A}C$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A](B \to C) \Rightarrow x{:}^{\epsilon}[A]C$}
\RightLabel{$(R\to)$}
\UnaryInfC{$x{:}^{\epsilon}[A](B \to C) \Rightarrow x{:}^{\epsilon}[A]B \to [A]C$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A](B \to C) \to ([A]B \to [A]C)$}
\end{prooftree}
$$

• Case of (RA2): right to left

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B, x{:}^{A}C$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}[A]B, x{:}^{A}C$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A, x{:}^{A}C$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]C, x{:}^{A}B \Rightarrow x{:}^{A}C$}
\RightLabel{$(L\to)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B \to [A]C, x{:}^{A}B \Rightarrow x{:}^{A}C$}
\RightLabel{$(R\to)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B \to [A]C \Rightarrow x{:}^{A}B \to C$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}[A]B \to [A]C \Rightarrow x{:}^{\epsilon}[A](B \to C)$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}([A]B \to [A]C) \to [A](B \to C)$}
\end{prooftree}
$$

• Case of (RA3)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{\epsilon}\neg[A]B$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{A}B$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B \Rightarrow x{:}^{A}B$}
\RightLabel{$(R\neg)$}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}\neg[A]B, x{:}^{A}B$}
\RightLabel{$(L\neg)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\neg B \Rightarrow x{:}^{\epsilon}\neg[A]B$}
\RightLabel{$(L[.]')$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\neg B \Rightarrow x{:}^{\epsilon}\neg[A]B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$x{:}^{\epsilon}[A]\neg B \Rightarrow x{:}^{\epsilon}A \to \neg[A]B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A]\neg B \to (A \to \neg[A]B)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{A}\neg B$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(R\neg)$}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{A}B, x{:}^{A}\neg B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}[A]B, x{:}^{A}\neg B$}
\RightLabel{$(L\neg)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\neg[A]B \Rightarrow x{:}^{A}\neg B$}
\RightLabel{$(L\to)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \neg[A]B \Rightarrow x{:}^{A}\neg B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \to \neg[A]B \Rightarrow x{:}^{\epsilon}[A]\neg B$}
\RightLabel{$(R\to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(A \to \neg[A]B) \to [A]\neg B$}
\RightLabel{$(R\wedge)$}
\BinaryInfC{$\Rightarrow x{:}^{\epsilon}[A]\neg B \leftrightarrow (A \to \neg[A]B)$}
\end{prooftree}
$$

• Case of (RA5): left to right where Lemma 3.2.1 is required.

$$
\dfrac{
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A][B] \Rightarrow \mathfrak{A}, x{:}^{\epsilon}A}
\quad
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{A}B, x{:}^{\epsilon}A \Rightarrow \mathfrak{A}, x{:}^{\epsilon}A}
\quad
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{A}B, x{:}^{\epsilon}A \Rightarrow \mathfrak{A}, x{:}^{A}B}
\quad
\dfrac{\dfrac{\text{Lemma 3.2.1}}{x{:}^{A,B}C, \Rightarrow \mathfrak{A}}}{x{:}^{A,B}C, x{:}^{A}B, x{:}^{\epsilon}A \Rightarrow \mathfrak{A}}\;(Lw)}{x{:}^{A}[B]C, x{:}^{A}B, x{:}^{\epsilon}A \Rightarrow \mathfrak{A}}\;(L[.]')}{x{:}^{A}B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A][B]C \Rightarrow \mathfrak{A}}\;(L[.]')}
{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A][B]C \Rightarrow \mathfrak{A}}\;(R\rightarrow)}
{\dfrac{x{:}^{\epsilon}A \wedge [A]B, x{:}^{\epsilon}[A][B]C \Rightarrow \mathfrak{A}}{\dfrac{x{:}^{\epsilon}[A][B]C \Rightarrow x{:}^{\epsilon}[A \wedge [A]B]C}{\Rightarrow x{:}^{\epsilon}[A][B]C \rightarrow [A \wedge [A]B]C}\;(R\rightarrow)}\;(R[.])}\;(L\wedge)
$$

where $\mathfrak{A} = x{:}^{A \wedge [A]B}C$.

• Case of (RA5): right to left where Lemma 3.2.1 is required.

$$
\dfrac{
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A, x{:}^{A,B}C}
\quad
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B, x{:}^{A,B}C}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}[A]B, x{:}^{A,B}C}\;(R[.])}
{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A \wedge [A]B, x{:}^{A,B}C}\;(R\wedge)
\quad
\dfrac{\dfrac{\text{Lemma 3.2.1}}{x{:}^{A \wedge [A]B}C \Rightarrow x{:}^{A,B}C}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A \wedge [A]B}C \Rightarrow x{:}^{A,B}C}\;(Lw)}
{\dfrac{x{:}^{\epsilon}A, x{:}^{\epsilon}[A \wedge [A]B]C, x{:}^{A}B \Rightarrow x{:}^{A,B}C}{\dfrac{x{:}^{\epsilon}A, x{:}^{\epsilon}[A \wedge [A]B]C \Rightarrow x{:}^{A}[B]C}{\dfrac{x{:}^{\epsilon}[A \wedge [A]B]C \Rightarrow x{:}^{\epsilon}[A][B]C}{\Rightarrow x{:}^{\epsilon}[A \wedge [A]B]C \rightarrow [A][B]C}\;(R\rightarrow)}\;(R[.])}\;(R[.])}\;(L[.]')
$$

In the inductive step, we show the inference rules, (MP) and (Nec), by **GPAL**.

Case of (MP)**:** It is shown with (Cut).

$$
\dfrac{\dfrac{\textit{Assumption}}{\Rightarrow x{:}^{\epsilon}A}
\quad
\dfrac{\dfrac{\textit{Assumption}}{\Rightarrow x{:}^{\epsilon}A \rightarrow B}
\quad
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B, x{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}B, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}}{x{:}^{\epsilon}A \rightarrow B, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}\;(L\rightarrow)}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}\;(Cut)}
{\Rightarrow x{:}^{\epsilon}B}\;(Cut)
$$

Case of $(Nec\Box_a)$**:** In the case, we show the admissibility of the following rule:

$$
\dfrac{\Rightarrow x{:}^{\epsilon}A}{\Rightarrow x{:}^{\epsilon}\Box_a A}\;(Nec\Box_a).
$$

Suppose $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}A$. By Lemma 3.2.2, we obtain $\vdash_{\mathbf{GPAL}} \Rightarrow y{:}^{\epsilon}A$ where variable y does not appear in the derivation of $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}A$. Therefore, we obtain $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}\Box_a A$ by the application of (Lw) and $(R\Box_a)$.

□

3.2.3 Cut Elimination of $\mathbf{GPAL}^+$

Here we prove the (syntactic) cut elimination theorem of $\mathbf{GPAL}^+$.

Theorem 3.2.2 (Cut elimination theorem of $\mathbf{GPAL}^+$)**.** For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{GPAL}^+} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$.

Proof. The proof is carried out using Ono and Komori's method [67] introduced in the reference [41] by Kashima where we employ the following rule (*Ecut*) instead of the usual method of 'mix cut'. We denote the n-copies of the same labelled expression $\mathfrak{A}$ by $\mathfrak{A}^n$ (when $n = 0$, $\mathfrak{A}^n = \varnothing$), and (*Ecut*) is defined as follows:

$$\frac{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \quad \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)$$

where $n, m \geq 0$. The theorem is proven by double induction on the height of the derivation and the length of cut expression $\mathfrak{A}$ of (*Ecut*). If $n = 0$ or $m = 0$, the lowersequent of (*Ecut*) is derivable by using (*Lw*) and (*Rw*) mutiple times instead of (*Ecut*), so we in the following consider the case where $n \neq 0$ and $m \neq 0$. The proof is divided into four cases. In brief,

(1) at least one of the uppersequents of (*Ecut*) is an initial sequent;

(2) the last inference rule of either uppersequents of (*Ecut*) is a structural rule;

(3) the last inference rule of either uppersequents of (*Ecut*) is a non-structural rule[5], and the principal expression introduced by the rule is not the cut expression; and

(4) the last inference rules of two uppersequents of (*Ecut*) are both non-structural rules, and the principal expressions introduced by the rules used on the uppersequents of (*Ecut*) are both cut expressions.

Case of (1) where the right uppersequents of (*Ecut*) is initial sequent $x{:}A \Rightarrow x{:}A$. In this case, we obtain the following part of derivation:

$$\frac{\dfrac{\vdots\ \mathcal{D}}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n} \quad \dfrac{\mathit{Initial\ Seq.}}{x{:}^{\alpha}A \Rightarrow x{:}^{\alpha}A}}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}\ (Ecut)$$

This is transformed into the derivation:

$$\frac{\dfrac{\vdots\ \mathcal{D}}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n}}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}\ (Rc)$$

where (*Rc*) is applied $n - 1$ times. Similarly to the above, we can show the case where the left uppersequent is an initial sequent.

[5]Non-structural rules indicate the all inference rules except (*Lc*), (*Rc*), (*Lw*) and (*Rw*).

Case of (2) where the right uppersequent of (*Ecut*) is structural rule (*Lc*) which contracts the same expression as the cut expressin.

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m+1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{(x{:}^{\alpha}A)^m, \Gamma' \Rightarrow \Delta'}\,(Lc)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m+1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut).$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Lc*).

Case of (2) where the right uppersequent of (*Ecut*) is structural rule (*Lc*) which contracts a different expression from the cut expressin.

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m}, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta'\end{matrix}}{(x{:}^{\alpha}A)^m, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Lc)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m}, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta'\end{matrix}}{\Gamma, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lc).$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Lc*).

Case of (2) where one of the uppersequents of (*Ecut*) is structural rule (*Lw*) which reduces the same formula as the cut formula.

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m-1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{(x{:}^{\alpha}A)^m, \Gamma' \Rightarrow \Delta'}\,(Lw)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m-1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut).$$

Note that ($Ecut$) is applicable, even if $m - 1 = 0$. Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule (Lw).

Case of (2) where one of the uppersequents of ($Ecut$) is structural rule (Lw) which reduces a different formula from the cut formula.

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^m, \Gamma' \Rightarrow \Delta'\end{array}}{(x{:}^{\alpha}A)^m, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Lw)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, (x{:}^{\alpha}A)^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^{m-1}, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lw) \quad .$$

Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule (Lw).

Case of (3) where one of the uppersequents of ($Ecut$) is inference rule ($R\neg$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}\neg A}\,(R\neg)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}\neg A}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, x{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}\neg A}\,(R\neg)$$

Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule ($R\neg$).

Case of (3) where one of the uppersequents of ($Ecut$) is inference rule ($L\neg$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A\end{array}}{\mathfrak{A}^m, x{:}^{\alpha}\neg A, \Gamma' \Rightarrow \Delta'}\,(L\neg)}{x{:}^{\alpha}\neg A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\ (Ecut)}{x{:}^{\alpha}\neg A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (L\neg)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($L\neg$).

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule ($R \rightarrow$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}B\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A \rightarrow B}\ (R\rightarrow)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A \rightarrow B}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, x{:}^{\alpha}A, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}B\end{array}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}B}\ (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A \rightarrow B}\ (R\rightarrow)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($R \rightarrow$).

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule ($L \rightarrow$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ x{:}^{\alpha}B, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{\mathfrak{A}^m, x{:}^{\alpha}A \rightarrow B, \Gamma' \Rightarrow \Delta'}\ (L\rightarrow)}{x{:}^{\alpha}A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\ (Ecut) \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m, x{:}^{\alpha}B, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha}B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)}{x{:}^{\alpha}A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (L\rightarrow)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($L \rightarrow$).

Case of (3) where one of the uppersequents of $(Ecut)$ is inference rule $(R\Box_a)$.

$$
\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, x\mathsf{R}^{\alpha}_a y,\Gamma'\Rightarrow\Delta', y{:}^{\alpha}A\end{matrix}}{\mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x{:}^{\alpha}\Box_a A}\,(R\Box_a)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x{:}^{\alpha}\Box_a A}\,(Ecut)
$$

If y does not appear in $\Gamma\Rightarrow\Delta,\mathfrak{A}^n$, it does not matter and leave y as it is. We consider the case where y appears in the sequent. In this case, label y is, by Lemma 3.2.2, replaced with z which does not appear in both $\Gamma\Rightarrow\Delta,\mathfrak{A}^n$ and $\mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x{:}^{\alpha}\Box_a A$, and let the derivation of $\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z,\Gamma'\Rightarrow\Delta', z{:}^{\alpha}A$ be $\mathcal{D}'_2$. Then the derivation is transformed into the following:

$$
\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\qquad \begin{matrix}\vdots\ \mathcal{D}'_2\\ \mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z,\Gamma'\Rightarrow\Delta', z{:}^{\alpha}A\end{matrix}}{x\mathsf{R}^{\alpha}_a z,\Gamma,\Gamma'\Rightarrow\Delta,\Delta', z{:}^{\alpha}A}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x{:}^{\alpha}\Box_a A}\,(R\Box_a)
$$

Case of (3) where one of the uppersequents of $(Ecut)$ is inference rule $(L\Box'_a)$.

$$
\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x\mathsf{R}_a y\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_3\\ y{:}^{\alpha}A,\mathfrak{A}^m,\Gamma'\Rightarrow\Delta'\end{matrix}}{\mathfrak{A}^m, x{:}^{\alpha}\Box_a A,\Gamma'\Rightarrow\Delta'}\,(L\Box'_a)}{x{:}^{\alpha}\Box_a A,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)
$$

This is transformed into the derivation:

$$
\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x\mathsf{R}_a y\end{matrix}}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x\mathsf{R}_a y}\,(Ecut)\quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m, y{:}^{\alpha}A,\Gamma'\Rightarrow\Delta'\end{matrix}}{y{:}^{\alpha}A,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)}{x{:}^{\alpha}\Box_a A,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(L\Box'_a)
$$

Similarly to this, we can show the case where the left uppersequent of $(Ecut)$ is structural rule $(L\Box'_a)$.

Case of (3) where one of the uppersequents of $(Ecut)$ is inference rule (Rat').

$$
\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x{:}^{\alpha}p\end{matrix}}{\mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x{:}^{\alpha,A}p}\,(Rat')}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x{:}^{\alpha,A}p}\,(Ecut)
$$

This is transformed into the derivation:

$$
\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\Delta,\mathfrak{A}^n\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow\Delta', x{:}^{\alpha}p\end{matrix}}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x{:}^{\alpha}p}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta', x{:}^{\alpha,A}p}\,(Rat')
$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Rat′*).

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule (*Lat′*).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha} p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha,A} p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}\,(Lat')}{x{:}^{\alpha,A} p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha} p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha} p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{x{:}^{\alpha,A} p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lat')$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Lat′*).

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule (*R*[.]).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha} A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha,A} B\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}[A]B}\,(R[.])}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}[A]B}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, x{:}^{\alpha} A, \Gamma' \Rightarrow \Delta', x{:}^{\alpha,A} B\end{array}}{x{:}^{\alpha} A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha,A} B}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}[A]B}\,(R[.])$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*R*[.]).

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule (*L*[.]′).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha} A\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ x{:}^{\alpha,A} B, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{\mathfrak{A}^m, x{:}^{\alpha}[A]B, \Gamma' \Rightarrow \Delta'}\,(L[.]')}{x{:}^{\alpha}[A]B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\,(Ecut) \quad \dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_3}{\mathfrak{A}^m, x{:}^{\alpha,A}B, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha,A}B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{x{:}^{\alpha}[A]B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(L[.]')$$

Similarly to this, we can show the case where the left uppersequent of $(Ecut)$ is structural rule $(L[.]')$.

Case of (3) where one of the uppersequents of $(Ecut)$ is inference rule $(Rrel_a)$.

$$\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', y{:}^{\alpha}A} \quad \overset{\vdots\,\mathcal{D}_3}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A} \quad \overset{\vdots\,\mathcal{D}_4}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x\mathsf{R}_a^{\alpha}y}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x\mathsf{R}_a^{\alpha,A}y}\,(Rrel_a)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x\mathsf{R}_a^{\alpha,A}y}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\,(Ecut) \quad \dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_3}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', y{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', y{:}^{\alpha}A}\,(Ecut) \quad \dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_4}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', x\mathsf{R}_a^{\alpha}y}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x\mathsf{R}_a^{\alpha}y}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x\mathsf{R}_a^{\alpha,A}y}\,(Rrel_a)$$

Similarly to this, we can show the case where the left uppersequent of $(Ecut)$ is structural rule $(Rrel_a)$.

Case of (3) where one of the uppersequents of $(Ecut)$ is inference rule $(Lrel_a)$.

$$\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\,\mathcal{D}_2}{x{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}}{\mathfrak{A}^m, x\mathsf{R}_a^{\alpha,A}y, \Gamma' \Rightarrow \Delta'}\,(Lrel_a 1)}{x\mathsf{R}_a^{\alpha,A}y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, x{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{x\mathsf{R}_a^{\alpha,A}y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lrel_a)$$

Similarly to this, we can show the case where the left uppersequent of $(Ecut)$ is structural rule $(Lrel_a 1)$. Moreover, the case of $(Lrel_a 2)$ and $(Lrel_a 3)$ can also be shown similarly.

Case of (4) where both sides of $\mathfrak{A}$ are $x{:}^{\alpha}\neg A$ and principal, when we obtain the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow\Delta,(x{:}^{\alpha}\neg A)^{n-1}\end{array}}{\Gamma\Rightarrow\Delta,(x{:}^{\alpha}\neg A)^{n}}\,(R\neg)\qquad\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}\neg A)^{m-1},\Gamma'\Rightarrow\Delta',x{:}^{\alpha}A\end{array}}{(x{:}^{\alpha}\neg A)^{m},\Gamma'\Rightarrow\Delta'}\,(L\neg)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1^{+}\\ \Gamma\Rightarrow\Delta,(x{:}^{\alpha}\neg A)^{n}\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}\neg A)^{m-1},\Gamma'\Rightarrow\Delta',x{:}^{\alpha}A\end{array}}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta',x{:}^{\alpha}A}\,(Ecut)\quad\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow\Delta,(x{:}^{\alpha}\neg A)^{n-1}\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_2^{+}\\ (x{:}^{\alpha}\neg A)^{m},\Gamma'\Rightarrow\Delta'\end{array}}{x{:}^{\alpha}A,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)}{\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ are $x{:}^{\alpha}A\rightarrow B$ and principal, when we obtain the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow\Delta,x{:}^{\alpha}B,(x{:}^{\alpha}A\rightarrow B)^{n-1}\end{array}}{\Gamma\Rightarrow\Delta,(x{:}^{\alpha}A\rightarrow B)^{n}}\,(R\rightarrow)\quad\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow\Delta',x{:}^{\alpha}A\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_3\\ x{:}^{\alpha}B,(x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow\Delta'\end{array}}{(x{:}^{\alpha}A\rightarrow B)^{m},\Gamma'\Rightarrow\Delta'}\,(L\rightarrow)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\mathcal{A}=\left\{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1^{+}\\ \Gamma\Rightarrow\Delta,(x{:}^{\alpha}A\rightarrow B)^{n}\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow\Delta',x{:}^{\alpha}A\end{array}}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta',x{:}^{\alpha}A}\,(Ecut)\right.$$

$$\mathcal{A}'=\left\{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1^{+}\\ \Gamma\Rightarrow\Delta,(x{:}^{\alpha}A\rightarrow B)^{n}\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_3\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},x{:}^{\alpha}B,\Gamma'\Rightarrow\Delta'\end{array}}{x{:}^{\alpha}B,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)\right.$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{A}\\ \Gamma,\Gamma'\Rightarrow\Delta,\Delta',x{:}^{\alpha}A\end{array}\quad\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow\Delta,x{:}^{\alpha}B,(x{:}^{\alpha}A\rightarrow B)^{n-1}\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_{2,3}^{+}\\ (x{:}^{\alpha}A\rightarrow B)^{m},\Gamma'\Rightarrow\Delta'\end{array}}{x{:}^{\alpha}A,\Gamma\Rightarrow\Delta,x{:}^{\alpha}B}\,(Ecut)\quad\begin{array}{c}\vdots\ \mathcal{A}'\\ x{:}^{\alpha}B,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'\end{array}}{x{:}^{\alpha}A,\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\,(Ecut)}{\Gamma,\Gamma,\Gamma,\Gamma',\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta,\Delta',\Delta',\Delta'}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ are $x{:}^{\alpha}\Box_a A$ and principal, when we obtain the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x\mathsf{R}_a^{\alpha}y,\Gamma\Rightarrow\Delta,y{:}^{\alpha}A,(x{:}^{\alpha}\Box_a A)^{n-1}\end{array}}{\Gamma\Rightarrow\Delta,(x{:}^{\alpha}\Box_a A)^{n}}\,(R\rightarrow)\quad\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}\Box_a A)^{m-1},\Gamma'\Rightarrow\Delta',x\mathsf{R}_a^{\alpha}v\end{array}\quad\begin{array}{c}\vdots\ \mathcal{D}_3\\ v{:}^{\alpha}A,(x{:}^{\alpha}\Box_a A)^{m-1},\Gamma'\Rightarrow\Delta'\end{array}}{(x{:}^{\alpha}\Box_a A)^{m},\Gamma'\Rightarrow\Delta'}\,(L\rightarrow)}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\mathcal{A} = \left\{ \dfrac{\overset{\vdots\ \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}\Box_a A)^n} \quad \overset{\vdots\ \mathcal{D}_2}{(x{:}^{\alpha}\Box_a A)^{m-1}, \Gamma' \Rightarrow \Delta', x\mathsf{R}^{\alpha}_a v}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x\mathsf{R}^{\alpha}_a v}\,(Ecut) \right.$$

$$\mathcal{A}' = \left\{ \dfrac{\overset{\vdots\ \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}\Box_a A)^n} \quad \overset{\vdots\ \mathcal{D}_3}{(x{:}^{\alpha}\Box_a A)^{m-1}, v{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'}}{v{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut) \right.$$

Additionally, by using Lemma 3.2.2 to the bottom sequent of $\mathfrak{D}_1$, we obtain the derivation $\mathcal{D}'_1$ whose bottom sequent is $x\mathsf{R}^{\alpha}_a v, \Gamma \Rightarrow \Delta, v{:}^{\alpha}A, (x{:}^{\alpha}\Box_a A)^{n-1}$.

$$\cfrac{\cfrac{\overset{\vdots\ \mathcal{A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x\mathsf{R}^{\alpha}_a v} \quad \cfrac{\cfrac{\overset{\vdots\ \mathcal{D}'_1}{x\mathsf{R}^{\alpha}_a v, \Gamma \Rightarrow \Delta, v{:}^{\alpha}A, (x{:}^{\alpha}\Box_a A)^{n-1}} \quad \overset{\vdots\ \mathcal{D}^+_{2,3}}{(x{:}^{\alpha}\Box_a A)^m, \Gamma' \Rightarrow \Delta'}}{x\mathsf{R}^{\alpha}_a v, \Gamma \Rightarrow \Delta, v{:}^{\alpha}A}\,(Ecut) \quad \overset{\vdots\ \mathcal{A}'}{v{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{x\mathsf{R}^{\alpha}_a v, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ are $x{:}^{\alpha}[A]B$ and principal, when we obtain the following derivation:

$$\cfrac{\cfrac{\overset{\vdots\ \mathcal{D}_1}{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B, (x{:}^{\alpha}[A]B)^{n-1}}}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}[A]B)^n}\,(R[.]) \quad \cfrac{\overset{\vdots\ \mathcal{D}_2}{(x{:}^{\alpha}[A]B)^{m-1}, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A} \quad \overset{\vdots\ \mathcal{D}_3}{x{:}^{\alpha,A}B, (x{:}^{\alpha}[A]B)^{m-1}, \Gamma' \Rightarrow \Delta'}}{(x{:}^{\alpha}[A]B)^m, \Gamma' \Rightarrow \Delta'}\,(L[.]')}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\mathcal{A} = \left\{ \dfrac{\overset{\vdots\ \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}[A]B)^n} \quad \overset{\vdots\ \mathcal{D}_2}{(x{:}^{\alpha}[A]B)^{m-1}, \Gamma' \Rightarrow \Delta', x{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\,(Ecut) \right.$$

$$\mathcal{A}' = \left\{ \dfrac{\overset{\vdots\ \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, (x{:}^{\alpha}[A]B)^n} \quad \overset{\vdots\ \mathcal{D}_3}{(x{:}^{\alpha}[A]B)^{m-1}, x{:}^{\alpha,A}B, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha,A}B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut) \right.$$

$$\cfrac{\cfrac{\overset{\vdots\ \mathcal{A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A} \quad \cfrac{\cfrac{\overset{\vdots\ \mathcal{D}_1}{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B, (x{:}^{\alpha}[A]B)^{n-1}} \quad \overset{\vdots\ \mathcal{D}^+_{2,3}}{(x{:}^{\alpha}[A]B)^m, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}\,(Ecut) \quad \overset{\vdots\ \mathcal{A}'}{x{:}^{\alpha,A}B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{x{:}^{\alpha}A, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ are $x\mathsf{R}_a^{\alpha,A}y$ and principal. When we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x{:}^{\alpha}A} \quad \dfrac{\vdots\ \mathcal{D}_2}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, y{:}^{\alpha}A} \quad \dfrac{\vdots\ \mathcal{D}_3}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x\mathsf{R}_a^{\alpha}y}}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n}}\ (Rrel_a) \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_4}{x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta'}}{(x\mathsf{R}_a^{\alpha,A}y)^{m}, \Gamma' \Rightarrow \Delta'}\ (Lrel_a3)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)\ ,$$

it is transformed into the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x{:}^{\alpha}A} \quad \dfrac{\vdots\ \mathcal{D}_4'}{(x\mathsf{R}_a^{\alpha,A}y)^{m}, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\ (Ecut) \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_{123}'}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n}} \quad \dfrac{\vdots\ \mathcal{D}_4}{x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\ (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Rc)/(Lc)\ ,$$

where ($Ecut$) to the two uppersequents is applicable by induction hypothesis, since the derivation height of ($Ecut$) is reduced by comparison with the original derivation. Additionally, the application of ($Ecut$) to the lowersequents is also allowed by induction hypothesis, since the length of the cut expression is reduced, namely $\ell(x{:}^{\alpha}A) < \ell(x\mathsf{R}_a^{\alpha,A}y)$. □

As a corollary of Theorem 3.2.2, the consistency of **GPAL**$^+$ is shown.

Corollary 3.2.1 (Consistency of **GPAL**). The empty sequent $\Rightarrow$ cannot be derived in **GPAL**$^+$.

Proof. Suppose for contradiction that $\Rightarrow$ is derivable in **GPAL**$^+$. By Theorem 3.2.2, $\Rightarrow$ is derivable in **GPAL**; however, there is no inference rule in **GPAL** which can derive the empty sequent. This is a contradiction. □

3.3 Soundness of GPAL

Now, we switch the subject to the soundness theorem of **GPAL**. For the theorem, we extend Kripke semantics of PAL to cover the labelled expressions. Given any model $\mathfrak{M}$, we say that $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ is an *assignment*.

Definition 3.3.1. Let $\mathfrak{M}$ be a model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment.

$$\begin{array}{lll} \mathfrak{M}, f \Vdash x{:}^{\alpha}A & \text{iff} & \mathfrak{M}^{\alpha}, f(x) \Vdash A \text{ and } f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}) \\ \mathfrak{M}, f \Vdash x\mathsf{R}_a^{\epsilon}y & \text{iff} & (f(x), f(y)) \in R_a \\ \mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha,A}y & \text{iff} & \mathfrak{M}, f \models x\mathsf{R}_a^{\alpha}y \text{ and } \mathfrak{M}^{\alpha}, f(x) \Vdash A \text{ and } \mathfrak{M}^{\alpha}, f(y) \Vdash A \end{array}$$

Here we have to be careful of the fact that $f(x)$ and $f(y)$ above must be defined in $\mathcal{D}(\mathfrak{M}^{\alpha})$. In the clause $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$, for example, $f(x)$ should survive (well-defined) in the restricted model $\mathfrak{M}^{\alpha}$. Taking into account this fact, it is essential that we pay attention to the negation of $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$.

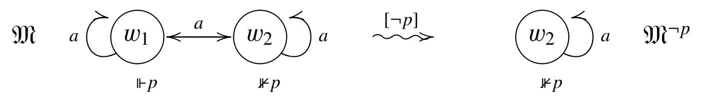
Proposition 3.3.1. $\mathfrak{M}, f \nVdash x{:}^{\alpha}A$ iff $f(x) \notin \mathcal{D}(\mathfrak{M}^{\alpha})$ or ($f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and $\mathfrak{M}^{\alpha}, f(x) \nVdash A$).

As far as, we know, this point has not been suggested in previous works (e.g., [6, 51]). Then, the reader may wonder if the following 'natural' definition of the validity for sequents (which we call *s-valid*) also works. The following notion can be regarded as an implementation of the reading of a sequent $\Gamma \Rightarrow \Delta$ as 'if all of the antecedent Γ hold, then some of the consequents Δ hold'.

Definition 3.3.2 (s-validity)**.** $\Gamma \Rightarrow \Delta$ is *s-valid* in $\mathfrak{M}$ if, for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \Vdash \mathfrak{B}$.

However, following this natural definition of validity of sequents, we come to a deadlock on the way to prove the soundness theorem, especially in the case of rules for logical negation, as we can see the following proposition with Example 2.2.2.

Recall (Example 2.2.2)**.** First of all, we formalize Example 2.2.1 with models as follows. Let us consider $\mathsf{Agt} = \{a\}$ and the following two models such as: $\mathfrak{M} = (\{w_1, w_2\}, W^2, V)$ where $V(p) = \{w_1\}$, and $\mathfrak{M}^{\neg p} = (\{w_2\}, \{(w_2, w_2)\}, V^{\neg p})$ where $V^{\neg p}(p) = \varnothing$. These models can be shown in graphic forms as follows.



In $\mathfrak{M}$, agent a does not know whether p or $\neg p$ (i.e., $\neg(\Box_a p \vee \Box_a \neg p)$ is valid in $\mathfrak{M}$), but after announcement of $\neg p$, agent a comes to know $\neg p$ in the restricted model $\mathfrak{M}$ to $\neg p$.

Proposition 3.3.2. There is a model $\mathfrak{M}$ such that $(R\neg)$ of **GPAL** does not preserve s-validity in $\mathfrak{M}$.[6]

Proof. We use the same model as in Example 2.2.2, and consider the particular instance of the application of $(R\neg)$ is as follows:

$$\frac{x{:}^{\neg p} p \Rightarrow}{\Rightarrow x{:}^{\neg p} \neg p}\ (R\neg)$$

We show that the uppersequent is s-valid in $\mathfrak{M}$ but the lowersequent is not s-valid in $\mathfrak{M}$, and so $(R\neg)$ does not preserve s-validity in this case. Note that w_1 does not survive after $\neg p$, i.e., $w_1 \notin \mathcal{D}(\mathfrak{M}^{\neg p}) = \{w_2\}$.

First, we show that $x{:}^{\neg p} p \Rightarrow$ is s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ for any assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. So, we fix any $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. We divide our argument into: $f(x) = w_1$ or $f(x) = w_2$. If $f(x) = w_1$, $f(x)$ does not survive after $\neg p$, and so $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ by Proposition 3.3.1. If $f(x) = w_2$, $f(x)$ survives after $\neg p$ but $f(x) \notin \varnothing = V(p) \cap \mathcal{D}(\mathfrak{M}^{\neg p})$, which implies $\mathfrak{M}^{\neg p}, f(x) \nVdash p$ hence $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ by Proposition 3.3.1.

Second, we show that $\Rightarrow x{:}^{\neg p} \neg p$ is not s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \nVdash x{:}^{\neg p} \neg p$ for some assignment $f : \mathsf{Var} \to W$. We fix some $f : \mathsf{Var} \to W$ such that $f(x) = w_1$. Since $f(x) \notin \mathcal{D}(\mathfrak{M}^{\neg p})$ ($f(x)$ does not survive after $\neg p$), $\mathfrak{M}, f \nVdash x{:}^{\neg p} \neg p$ by Proposition 3.3.1, as desired. □

[6]This proposition and the following definition of t-validity are suggested by Katsuhiko Sano.

Therefore, Proposition 3.3.2 forces us to abandon the notion of s-validity and have an alternative notion of validity. Here we recall the second intuitive reading (in the introduction) of sequent $\Gamma \Rightarrow \Delta$ as 'it is not the case that all of the antecedents Γ hold and all of the consequents fail.' In order to realize the idea of 'failure', we first introduce the syntactic notion of the negated form $\overline{\mathfrak{A}}$ of a labelled expression $\mathfrak{A}$ and then provide the semantics $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$ with such negated forms, where we may read $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$ as '$\mathfrak{A}$ fails in $\mathfrak{M}$ under f.' Moreover, with this definition, our second notion of validity of a sequent, which we call *t-valid*,[7] is defined.

Definition 3.3.3 (t-validity)**.** Let $\mathfrak{M}$ be a model and $f : \mathsf{Var} \to D(\mathfrak{M})$ an assignment. Then,

$$\begin{array}{lll} \mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A} & \text{iff} & \mathfrak{M}^{\alpha}, f(x) \Vdash \neg A \text{ and } f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}), \\ \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\epsilon}y} & \text{iff} & (f(x), f(y)) \notin R_a, \\ \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha,A}y} & \text{iff} & \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y} \text{ or } \mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A} \text{ or } \mathfrak{M}, f \Vdash \overline{y{:}^{\alpha}A}. \end{array}$$

We say that $\Gamma \Rightarrow \Delta$ is *t-valid* in $\mathfrak{M}$ if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$.

In this definition, we explicitly gave a condition of survival that $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$, e.g., in $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$. Therefore, '$x{:}^{\alpha}A$ fails in $\mathfrak{M}$ under f' means that $f(x)$ survives after α but A is false at $f(x)$ in $\mathfrak{M}^{\alpha}$. The following proposition shows that the clauses for relational atoms and their negated forms characterize what they intend to capture.

Proposition 3.3.3. For any model $\mathfrak{M}$, assignment f, $a \in \mathsf{Agt}$ and $x, y \in \mathsf{Var}$,

(i) $\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha}y$ iff $(f(x), f(y)) \in R_a^{\alpha}$,
(ii) $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$ iff $(f(x), f(y)) \notin R_a^{\alpha}$.

Proof. Both are easily shown by induction of α. Let us consider the case of $\alpha = \alpha', A$ in the proof of (ii).

We show $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha',A}y}$ iff $(f(x), f(y)) \in R_a^{\alpha',A}$. $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha',A}y}$ is, by Definition 3.3.3 and the induction hypothesis, equivalent to $(f(x), f(y)) \in R_a^{\alpha'}$ and $\mathfrak{M}^{\alpha'}, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha'}, f(y) \Vdash A$. That is also equivalent to $(f(x), f(y)) \in R_a^{\alpha',A}$. □

Following this, we may prove the soundness of **GPAL** properly. Let Γ is a finite set of labelled expressions. Then in what follows, we write $\mathfrak{M}, f \Vdash \Gamma$ to mean $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Gamma}$ to mean $\mathfrak{M}, f \Vdash \overline{\mathfrak{A}}$ for all $\mathfrak{A} \in \Gamma$.

Theorem 3.3.1 (Soundness of **GPAL**)**.** Given any sequent $\Gamma \Rightarrow \Delta$ in **GPAL**, if $\vdash_{\mathbf{GPAL}}$ $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is t-valid in every model $\mathfrak{M}$.

Proof. The proof is carried out by induction of the height of the derivation of $\Gamma \Rightarrow \Delta$ in **GPAL**.

Base case: we show that $x\mathsf{R}_a^{\alpha}v \Rightarrow x\mathsf{R}_a^{\alpha}v$ is t-valid. Suppose for contradiction that $\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha}v$ and $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}v}$. By Proposition 3.3.3, this is impossible.

[7] We note that t-validity is close to the validity in the tableaux method of PAL [7].

Case where the last applied rule is of the form $(L\neg)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash x{:}^{\alpha}\neg A$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$. Then, $\mathfrak{M}, f \Vdash x{:}^{\alpha}\neg A$ iff $\mathfrak{M}^{\alpha}, f(x) \Vdash \neg A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$. By Definition 3.3.3, we obtain $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$.

Case where the last applied rule is of the form $(R\neg)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\neg A}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$. Then, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\neg A}$ iff $\mathfrak{M}^{\alpha}, f(x) \nVdash \neg A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$, which is equivalent to: $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$. By Definition 3.3.1, $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$. So, the contraposition has been shown.

Case where the last applied rule is of the form $(L \to)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash x{:}^{\alpha}A \to B$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$ or $\mathfrak{M}, f \Vdash x{:}^{\alpha}B$. Then, $\mathfrak{M}, f \Vdash x{:}^{\alpha}A \to B$ iff $(\mathfrak{M}^{\alpha}, f(x) \Vdash \neg A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}))$ or $(\mathfrak{M}^{\alpha}, f(x) \Vdash B$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}))$. By Definition 3.3.1, we obtain the goal as desired.

Case where the last applied rule is of the form $(R \to)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A \to B}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}B}$. Then, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A \to B}$ iff $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha}, f(x) \nVdash B$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$. By Definitions 3.3.1 and 3.3.3, we obtain the goal as desired.

Case where the last applied rule is of the form $(L\Box_a')$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash x^{\alpha}{:}\Box_a A$ and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$ or $\mathfrak{M}, f \Vdash y{:}^{\alpha}A$. Then, from $\mathfrak{M}, f \Vdash x{:}^{\alpha}\Box_a A$, we obtain $(f(x), f(y)) \notin R_a^{\alpha}$ or $\mathfrak{M}^{\alpha}, f(y) \Vdash A$. Suppose the former disjunct, i.e., $(f(x), f(y)) \notin R_a^{\alpha}$, which is, by Proposition 3.3.3, $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$. Then, suppose the latter disjunct $\mathfrak{M}^{\alpha}, f(y) \Vdash A$. By definition, this is equivalent to $\mathfrak{M}, f \Vdash y :^{\alpha} A$. Then, the contraposition has been shown.

Case where the last applied rule is of the form $(R\Box_a)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\Box_a A}$. Fix such f. Then, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\Box_a A}$ iff $f(x)R_a^{\alpha}v$ and $\mathfrak{M}^{\alpha}, v \nVdash A$ for some $v \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$. Fix such $v \in \mathcal{D}(\mathfrak{M}^{\alpha})$. It suffices to show that there is some $f' : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f' \Vdash x\mathsf{R}_a^{\alpha}y$ and $\mathfrak{M}, f' \Vdash \overline{x{:}^{\alpha}A}$ where y is not x and does not appear in Γ and Δ. Define f' such that $f'(x) = v$ if $x = y$ and otherwise $f'(x) = f(x)$. Therefore, by the definition of f', we obtain $f'(x)R_a^{\alpha}f'(y)$ and $\mathfrak{M}^{\alpha}, f'(y) \nVdash A$ and $f'(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ By Definitions 3.3.1 and 3.3.3, we obtain the goal as desired.

Case where the last applied rule is of the form (Lat')**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash x{:}^{\alpha,A}p$, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^{\alpha}p$. Then, $\mathfrak{M}, f \Vdash x{:}^{\alpha,A}p$

implies $f(x) \in V^\alpha(p)$, which is equivalent to $\mathfrak{M}^\alpha, f(x) \Vdash p$. By Definition 3.3.1, we obtain the goal as desired.

Case where the last applied rule is of the form (Rat')**:** Similar to the above, we show the contraposition. Suppose there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}p}$. By Definition 3.3.3, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$ is equivalent to $\mathfrak{M}^{\alpha,A}, f(x) \Vdash \neg p$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$. By $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$, we obtain $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and $\mathfrak{M}^\alpha, f(x) \Vdash A$. It follows from $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha,A}, f(x) \Vdash \neg p$ that $f(x) \notin V^\alpha(p)$. This is equivalent to $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}p}$. Then, the contraposition has been shown.

Case where the last applied rule is of the form $(L[.])$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash x{:}^\alpha[A]B$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha A}$ or $\mathfrak{M}, f \Vdash x{:}^{\alpha,A}B$. Then, $\mathfrak{M}, f \Vdash x{:}^\alpha[A]B$ iff $(\mathfrak{M}^\alpha, f(x) \Vdash \neg A$ or $\mathfrak{M}^{\alpha,A}, f(x) \Vdash B)$ and $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$. By Definition 3.3.1 and 3.3.3, we obtain the goal as desired.

Case where the last applied rule is of the form $(R[.])$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha[A]B}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^\alpha A$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}B}$. Then, $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha[A]B}$ iff $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha,A}, f(x) \nVdash B$ and $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$. From $\mathfrak{M}^\alpha, f(x) \Vdash A$, we obtain $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$. Then, by Definition 3.3.1 and 3.3.3, we obtain the goal as desired.

Case where the last applied rule is of the form $(Lrel1)$**:** We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha,A}y$, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^\alpha A$. Then, $\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha,A}y$ is equivalent to $\mathfrak{M}, f \models x\mathsf{R}_a^\alpha y$ and $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $\mathfrak{M}^\alpha, f(y) \Vdash A$. By $\mathfrak{M}^\alpha, f(x) \Vdash A$ and Definition 3.3.1, we obtain the goal as desired.

Case where the last applied rule is of the form $(Lrel2)$ **and** $(Lrel3)$**:** Similar to the above.

Case where the last applied rule is of the form $(Rrel)$**:** As before, we show the contraposition. Suppose there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha,A}y}$. Fix such f. By Definition 3.3.3, $\overline{x\mathsf{R}_a^{\alpha,A}y}$ is equivalent to $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$ or $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha A}$ or $\mathfrak{M}, f \Vdash \overline{y{:}^\alpha A}$. This is what we want to show.

□

For the following corollary, we prepare the next proposition.

Proposition 3.3.4. If $\Rightarrow x{:}^\epsilon A$ is t-valid in a model $\mathfrak{M}$, then A is valid in $\mathfrak{M}$.

Proof. Suppose that $\Rightarrow x{:}^\epsilon A$ is t-valid in $\mathfrak{M}$. So, it is not the case that there exists some assignment f such that $\mathfrak{M}, f \Vdash \overline{x{:}^\epsilon A}$. Equivalently, for all assignments f, $\mathfrak{M}, f \nVdash \overline{x{:}^\epsilon A}$. For any assignment f, $\mathfrak{M}, f \nVdash \overline{x{:}^\epsilon A}$ is equivalent to $\mathfrak{M}, f(x) \Vdash A$ because $f(x) \in \mathcal{D}(\mathfrak{M})$.

So, it follows that $\mathfrak{M}, f(x) \Vdash A$ for all assignments f. Then, it is immediate to see that A is valid in $\mathfrak{M}$, as required.

$\square$

Then an indirect proof of completeness of **GPAL** can be provided as follows:

Corollary 3.3.1. Given any formula A and label $x \in \mathsf{Var}$, the following are equivalent.

(i) A is valid on all models.
(ii) $\vdash_{\mathbf{HPAL}} A$
(iii) $\vdash_{\mathbf{GPAL^+}} \Rightarrow x{:}^{\epsilon}A$
(iv) $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}A$

Proof. The direction from (i) to (ii) is established by Fact 1 and the direction from (ii) to (iii) is shown by Theorem 3.2.1. Then, the direction from (iii) to (iv) is established by the admissibility of (*Cut*) (Theorem 3.2.2). Finally, the direction from (iv) to (i) is shown by Theorem 3.3.1 and Proposition 3.3.4. $\square$

3.4 Completeness of GPAL for Link-cutting semantics

Let us denote by **GPALw** as the resulting sequent calculus of replacing (*Lat′*) and (*Rat′*) of **GPAL** with the following modified version of (*Lat*) and (*Rat*) in **G3PAL**:

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat1) \quad \frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat2) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat).$$

We checked that all results needed to show Corollary 3.3.1 hold also for **GPALw**, and so we can establish the similar result to Corollary 3.3.1 also for **GPALw**. While (*Rat*) *does* preserve t-validity in a model $\mathfrak{M}$ by the similar argument to the proof of Theorem 3.3.1, we remark that one premise $\Gamma \Rightarrow \Delta, x{:}^{\alpha}A$ of (*Rat*) becomes redundant when we prove that (*Rat*) preserves t-validity in a model. This is because, for any assignment f, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$ already implies that A holds at $f(x)$ after α, i.e., $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$. We realize that this difference between **GPALw** and **GPAL** comes from the difference between the standard Kripke semantics (this is also called the *world-deleting semantics* in Section 2.2.1) and *the link-cutting semantics*[8].

As we have seen in Corollary 3.3.1 **GPAL** is complete for a formula with respect to the world-deleting semantics of PAL; however, it is not complete for an arbitrary sequent.[9] Namely, there is a sequent which is t-valid in the world-deleting semantics but it is not derivable in **GPAL**. Let us consider sequent $\Rightarrow x{:}^{\perp}p$.

[8]As far as we know, van Benthem and Liu [82, p.166] first provide an idea of link-cutting semantics of PAL. Their underlying idea is: cutting the links (pairs in an accessibility relation) between A-zone and $\neg A$-zone. Then, they state that all valid formulas in the resulting semantics are also the same as those in the world-deleting semantics [82, Fact 1]. Their semantics is similar but different to our semantics above. Hansen [34, p.145] touches on the same link-cutting semantics as ours in the public announcement extension of hybrid logic (an extended modal logic), but he does not investigate the semantics in detail there. A variant of our link-cutting semantics is also explained for logic of belief in [83], though the notion of public announcement there is not truthful and this is why the announcement there is called the ‘introspective announcement.’

[9]The argument in this paragraph is not included in [61].

Proposition 3.4.1. $\Rightarrow x{:}^{\perp}p$ is t-valid in the world-deleting semantics.

Proof. We show t-validity of the sequent. Suppose for a contradiction that there is an assignment $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \not\Vdash \overline{x{:}^{\perp}p}$. Fix such f. Then we have $\mathfrak{M}, f \Vdash \overline{x{:}^{\perp}p}$ which is equivalent to $\mathfrak{M}^{\perp}, f(x) \nVdash p$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\perp})$. However, since $\mathcal{D}(\mathfrak{M}^{\perp}) = \emptyset$, we obtain a contradiction. □

Proposition 3.4.2. $\nvdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\perp}p$.

Proof. It suffices to show that sequent $\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m$ is not derivable in **GPAL** for all $n, m \in \mathbb{N}$ by induction of the height of the derivation.

Case of height$= 0$**.** Since $\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m$ is not an initial sequent, it is not derivable.

Case of height $= k$**.** We obtain the following possibilities:

$$\frac{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^{m-1}}{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m}\ (Rw), \qquad \frac{\Rightarrow (x{:}p)^{n-1}, (x{:}^{\perp}p)^{m}}{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m}\ (Rw),$$

$$\frac{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^{m+1}}{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m}\ (Rc), \qquad \frac{\Rightarrow (x{:}p)^{n+1}, (x{:}^{\perp}p)^{m}}{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m}\ (Rc),$$

$$\frac{\Rightarrow (x{:}p)^{n+1}, (x{:}^{\perp}p)^{m-1}}{\Rightarrow (x{:}p)^n, (x{:}^{\perp}p)^m}\ (Rat).$$

Applying the induction hypothesis to each of the uppersequent (height $k-1$) and subsequently applying the same rule, we obtain that the sequent is not derivable in **GPAL**.

□

As a result of Propositions 3.4.1 and 3.4.2, we conclude the following corollary.

Corollary 3.4.1. The following does not hold: for any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is t-valid in the world-deleting semantics, then $\vdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$.

In what follows, we introduce our version of the link-cutting semantics of PAL and provide a direct proof of completeness of **GPAL** for link-cutting semantics. [10] The specific definition of the link-cutting version of PAL's semantics is given as follows, where we keep the symbol $\Vdash$ for the previous world-deleting semantics of PAL and use the new symbol '$\models$' for the satisfaction relation for the link-cutting semantics.

Definition 3.4.1 (Link-cutting semantics of PAL)**.** Given a model $\mathfrak{M}$, $w \in \mathcal{D}(\mathfrak{M})$ and a formula A, $\mathfrak{M}, w \models A$ is defined by

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$,
$\mathfrak{M}, w \models \neg A$	iff	$\mathfrak{M}, w \not\models A$,
$\mathfrak{M}, w \models A \to B$	iff	$\mathfrak{M}, w \models A$ implies $\mathfrak{M}, w \models B$,
$\mathfrak{M}, w \models \Box_a A$	iff	for all $v \in W : wR_a v$ implies $\mathfrak{M}, v \models A$, and
$\mathfrak{M}, w \models [A]B$	iff	$\mathfrak{M}, w \models A$ implies $\mathfrak{M}^{A!}, w \models B$,

[10] Thanks to a comment from Makoto Kanazawa in the annual meeting of MLG2014 in Japan, we noticed that the link-cutting semantics may be suitable for our labelled sequent calculus of PAL.

where the restriction $\mathfrak{M}^{A!}$ is defined by triple $(W, (R_a^{A!})_{a \in \mathsf{Agt}}, V)$ with

$$R_a^{A!} := R_a \cap ([\![A]\!]_{\mathfrak{M}} \times [\![A]\!]_{\mathfrak{M}}), \quad \text{where } [\![A]\!]_{\mathfrak{M}} := \{x \in W \mid \mathfrak{M}, x \models A\}.$$

According to this definition, only the accessibility relation is restricted to A in $\mathfrak{M}^{A!}$, and the set of worlds and valuation stay as they were. Similar to the world-deleting semantics, we can also define the notion of validity in a model. The following soundness of **HPAL** for the link-cutting semantics is straightforward.

Proposition 3.4.3. If A is a theorem of **HPAL**, A is valid in every model $\mathfrak{M}$ for the link-cutting semantics.

As before, for any list $\alpha = (A_1, A_2, ..., A_n)$ of formulas , we define $\mathfrak{M}^{\alpha!}$ inductively as: $\mathfrak{M}^{\alpha!} := \mathfrak{M}$ (if $\alpha = \epsilon$), and $\mathfrak{M}^{\alpha!} := (\mathfrak{M}^{\beta!})^{A_n!} = (W, (R_a^{\beta!, A_n!})_{a \in \mathsf{Agt}}, V)$ (if $\alpha = \beta, A_n$). Now we can show that the corresponding notions to s- and t-validity become equivalent under our link-cutting semantics.

Definition 3.4.2. Let $\mathfrak{M}$ be a model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment.

$$\begin{array}{lll}
\mathfrak{M}, f \models x{:}^{\alpha} A & \text{iff} & \mathfrak{M}^{\alpha!}, f(x) \models A \\
\mathfrak{M}, f \models x \mathsf{R}_a^{\epsilon} y & \text{iff} & (f(x), f(y)) \in R_a \\
\mathfrak{M}, f \models x \mathsf{R}_a^{\alpha, A} y & \text{iff} & \mathfrak{M}, f \models x \mathsf{R}_a^{\alpha} y \text{ and } \mathfrak{M}^{\alpha!}, f(x) \models A \text{ and } \mathfrak{M}^{\alpha!}, f(y) \models A
\end{array}$$

By this definition, the next proposition immediately follows.

Proposition 3.4.4. For any model $\mathfrak{M}$, assignment f, $a \in \mathsf{Agt}$ and $x, y \in \mathsf{Var}$,

$$\mathfrak{M}, f \models x \mathsf{R}_a^{\alpha} y \quad \text{iff} \quad (f(x), f(y)) \in R_a^{\alpha!}$$

The semantics of the negated form of a labelled expression $\overline{\mathfrak{A}}$ is also defined as before.

Definition 3.4.3. Let $\mathfrak{M}$ be a model and $f : \mathsf{Var} \to D(\mathfrak{M})$ an assignment. Then,

$$\begin{array}{lll}
\mathfrak{M}, f \models \overline{x{:}^{\alpha} A} & \text{iff} & \mathfrak{M}^{\alpha!}, f(x) \not\models A, \\
\mathfrak{M}, f \models \overline{x \mathsf{R}_a^{\epsilon} y} & \text{iff} & (f(x), f(y)) \notin R_a, \\
\mathfrak{M}, f \models \overline{x \mathsf{R}_a^{\alpha, A} y} & \text{iff} & \mathfrak{M}, f \Vdash \overline{x \mathsf{R}_a^{\alpha} y} \text{ or } \mathfrak{M}, f \not\models x{:}^{\alpha} A \text{ or } \mathfrak{M}, f \not\models y{:}^{\alpha} A
\end{array}$$

Now we may confirm that, based on the semantics, t-validity and s-validity are equivalent since $\mathfrak{M}, f \not\models \overline{\mathfrak{B}}$ is equivalent to $\mathfrak{M}, f \models \mathfrak{B}$ in this semantics.

Proposition 3.4.5. Under the link-cutting semantics, a sequent $\Gamma \Rightarrow \Delta$ is s-valid in a model $\mathfrak{M}$ iff it is t-valid in $\mathfrak{M}$.

Proof. Suppose $\Gamma \Rightarrow \Delta$ is t-valid in $\mathfrak{M}$. In other words, if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \models \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \models \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Equivalently, for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$, $\mathfrak{M}, f \models \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \models \mathfrak{B}$. $\square$

Because the notion of survival is expelled, the definition of the satisfaction of labelled expressions becomes wholly natural. Thus, we do not need to worry about the notion of survival of worlds in this link-cutting semantics.

Hereafter in this section we consider possibly infinite multi-sets of labelled expressions. That is, we call $\Gamma \Rightarrow \Delta$ an infinite sequent if Γ or Δ are infinite multi-sets. We use the notation $\vdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$ to mean that there are finite multi-sets Γ' and Δ' of labelled expressions such that $\vdash_{\mathbf{GPAL}} \Gamma' \Rightarrow \Delta'$ in the ordinary sense and $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. To establish the completeness result of **GPAL** for the link-cutting semantics, we first introduce the notion of saturation as follows.

Definition 3.4.4. A possibly infinite sequent $\Gamma \Rightarrow \Delta$ is *saturated* if it satisfies the following:

(*unprov*) $\Gamma \Rightarrow \Delta$ is not derivable in **GPAL**,

$(\to l)$ if $x{:}^{\alpha}A \to B \in \Gamma$, then $x{:}^{\alpha}A \in \Delta$ or $x{:}^{\alpha}B \in \Gamma$,

$(\to r)$ if $x{:}^{\alpha}A \to B \in \Delta$, then $x{:}^{\alpha}A \in \Gamma$ and $x{:}^{\alpha}B \in \Delta$,

$(\neg l)$ if $x{:}^{\alpha}\neg A \in \Gamma$, then $x{:}^{\alpha}A \in \Delta$,

$(\neg r)$ if $x{:}^{\alpha}\neg A \in \Delta$, then $x{:}^{\alpha}A \in \Gamma$,

$(\Box_a l)$ if $x{:}^{\alpha}\Box_a A \in \Gamma$, then $x\mathsf{R}_a^{\alpha}y \in \Delta$ or $y{:}^{\alpha}A \in \Gamma$ for any label y,

$(\Box_a r)$ if $x{:}^{\alpha}\Box_a A \in \Delta$, then $x\mathsf{R}_a^{\alpha}y \in \Gamma$ and $y{:}^{\alpha}A \in \Delta$ for some label y,

$([.]l)$ if $x{:}^{\alpha}[A]B \in \Gamma$, then $x{:}^{\alpha}A \in \Delta$ or $x{:}^{\alpha,A}B \in \Gamma$,

$([.]r)$ if $x{:}^{\alpha}[A]B \in \Delta$, then $x{:}^{\alpha}A \in \Gamma$ and $x{:}^{\alpha,A}B \in \Delta$,

(*atl*) if $x{:}^{\alpha,A}p \in \Gamma$, then $x{:}^{\alpha}p \in \Gamma$,

(*atr*) if $x{:}^{\alpha,A}p \in \Delta$, then $x{:}^{\alpha}p \in \Delta$,

(*rell*) if $x\mathsf{R}_a^{\alpha,A}y \in \Gamma$, then $x{:}^{\alpha}A \in \Gamma$ and $y{:}^{\alpha}A \in \Gamma$, and $x\mathsf{R}_a^{\alpha}y \in \Gamma$, and

(*relr*) if $x\mathsf{R}_a^{\alpha,A}y \in \Delta$, then $x{:}^{\alpha}A \in \Delta$ or $y{:}^{\alpha}A \in \Delta$, or $x\mathsf{R}_a^{\alpha}y \in \Delta$.

We show the next lemma which states that any underivable sequent in **GPAL** can be extended to a (possibility infinite) saturated sequent.

Lemma 3.4.1. Let $\Gamma \Rightarrow \Delta$ be a finite sequent. If $\nvdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$, then there exists a possibility infinite saturated sequent $\Gamma^+ \Rightarrow \Delta^+$ where $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$.

Proof. Suppose that there is a finite sequent $\Gamma \Rightarrow \Delta$ such that $\nvdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$. Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be an enumeration of all labelled expressions such that each labelled expression appears infinitely many times. We inductively construct an infinite sequence $(\Gamma_i \Rightarrow \Delta_i)_{i \in \mathbb{N}}$ of finite sequents such that $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$ at each $i \in \mathbb{N}$ as follows and define $\Gamma^+ \Rightarrow \Delta^+$ as the 'limit' of such sequence.

Let $\Gamma_0 \Rightarrow \Delta_0$ be $\Gamma \Rightarrow \Delta$ as the basis of $\Gamma_i \Rightarrow \Delta_i$, and by the supposition $\nvdash_{\mathbf{GPAL}} \Gamma_0 \Rightarrow \Delta_0$. The $i+1$-th step consists of the procedures to define an underivable $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ from $\Gamma_i \Rightarrow \Delta_i$ depending on the shape of the labelled expression $\mathfrak{A}_i$. In the $i+1$-th step, one of the following operations is executed.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}A \to B$ and $\mathfrak{A}_i \in \Gamma_i$: Because $\Gamma_i \Rightarrow \Delta_i$ is underivable, either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $x{:}^{\alpha}B, \Gamma_i \Rightarrow \Delta_i$ is also underivable by $(L \to)$. Then we choose one underivable sequent as $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}A \to B$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}B$. By $(R \to)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}\neg A$ and $\mathfrak{A}_i \in \Gamma_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$. Because of $(L\neg)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}\neg A$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i$. Because of $(R\neg)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x :^{\alpha} [A]B$ and $\mathfrak{A}_i \in \Gamma_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ as either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $x{:}^{\alpha,A}B, \Gamma_i \Rightarrow \Delta_i$. Because of $(L[.])$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x :^{\alpha} [A]B$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha,A}B$. Because of $(R[.])$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha,A}p$ and $\mathfrak{A}_i \in \Gamma_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}p, \Gamma_i \Rightarrow \Delta_i$. Because of (Lat') and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha,A}p$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}p$. Because of (Rat') and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}\Box_a A$ and $\mathfrak{A}_i \in \Gamma_i$: Let $\{y_1, ..., y_n\}$ be the set of all labels appearing in $\Gamma_i \Rightarrow \Delta_i$. Suppose we have constructed $(\Gamma_i^{(k)} \Rightarrow \Delta_i^{(k)})_{1 \leq k \leq \ell}$ such that $(\Gamma_i^{(k)} \Rightarrow \Delta_i^{(k)})$ is underivable, $\Gamma_i^{(k)} \subseteq \Gamma_i^{(k+1)}$, and $\Delta_i^{(k)} \subseteq \Delta_i^{(k+1)}$. Because of $(L\Box_a)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i^{(l)} \Rightarrow \Delta_i^{(l)}$, either $\Gamma_i^{(l)} \Rightarrow \Delta_i^{(l)}, x\mathsf{R}_a^{\alpha}y_{\ell+1}$ or $y_{l+1}{:}A, \Gamma_i^{(l)} \Rightarrow \Delta_i^{(l)}$ is underivable, and we choose one underivable sequent as $\Gamma_i^{(l+1)} \Rightarrow \Delta_i^{(l+1)}$. Then we define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i^{(n)} \Rightarrow \Delta_i^{(n)}$, and $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is underivable by construction.

Case where $\mathfrak{A}_i$ is of the form $x{:}^{\alpha}\Box_a A$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x\mathsf{R}_a^{\alpha}y, \Gamma_i \Rightarrow \Delta_i, y{:}^{\alpha}A$, where y is a fresh variable that does not appear in $\Gamma_i \Rightarrow \Delta_i$. Because of $(R\Box_a)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x\mathsf{R}_a^{\alpha,A}y$ and $\mathfrak{A}_i \in \Gamma_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, y{:}^{\alpha}A, x\mathsf{R}_a^{\alpha}y, \Gamma_i \Rightarrow \Delta_i$. Because of $(Lrel)$ and $\nvdash_{\mathbf{GPAL}} \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Case where $\mathfrak{A}_i$ is of the form $x\mathsf{R}_a^{\alpha,A}y$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ as either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $\Gamma_i \Rightarrow \Delta_i, y{:}^{\alpha}A$ or $\Gamma_i \Rightarrow \Delta_i, x\mathsf{R}_a^{\alpha}y$. Because of $(Rrel)$ and $\nvdash_{\mathbf{GPAL}}$ $\Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also underivable.

Otherwise: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i$.

Finally, let $\Gamma^+ \Rightarrow \Delta^+$ be the union $\bigcup_{i\in\mathbb{N}} \Gamma_i \Rightarrow \bigcup_{i\in\mathbb{N}} \Delta_i$. Then, it is routine to check that $\Gamma^+ \Rightarrow \Delta^+$ is saturated and $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$. □

We now prove the completeness of **GPAL** for the link-cutting semantics.

Theorem 3.4.1. If a sequent $\Gamma \Rightarrow \Delta$ is s-valid in every model $\mathfrak{M}$ for the link-cutting semantics, then $\vdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$.

Proof. We show its contraposition, and so suppose $\nvdash_{\mathbf{GPAL}} \Gamma \Rightarrow \Delta$. By Lemma 3.4.1, there exists a saturated sequent $\Gamma^+ \Rightarrow \Delta^+$ such that $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$. Using the saturated sequent, we construct the derived model $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ from the saturated sequent $\Gamma^+ \Rightarrow \Delta^+$.

- W is a set of all labels appearing in $\Gamma^+ \Rightarrow \Delta^+$,
- $xR_a^{\epsilon}y$ iff $x\mathsf{R}_a^{\epsilon}y \in \Gamma^+$,
- $x \in V(p)$ iff $x{:}p \in \Gamma^+$.

In addition to this, let $f : \mathsf{Var} \to W$ be an arbitrary assignment such that $f(x) = x$ (if x is in W). Then , we can establish the following two items:

(i) $\mathfrak{A} \in \Gamma^+$ implies $\mathfrak{M}, f \models \mathfrak{A}$,

(ii) $\mathfrak{A} \in \Delta^+$ implies $\mathfrak{M}, f \not\models \mathfrak{A}$.

The second item implies that $\mathfrak{M}, f(x) \not\models A$ hence A is not valid in the derived model $\mathfrak{M}$. The proof for these two items is conducted by simultaneous induction on the length of $\mathfrak{A}$. Here we only look at the cases where $\mathfrak{A}$ is $x{:}^{\alpha,A}p$ or $x{:}^{\alpha}\Box_a A$.

Case where $\mathfrak{A}$ is $x{:}^{\epsilon}p$: (i) and (ii) are trivial by the definition of $\mathfrak{M}$.

Case where $\mathfrak{A}$ is $x{:}^{\alpha,A}p$: (i) If $x{:}^{\alpha,A}p \in \Gamma^+$, then by saturatedness, we have $x{:}^{\alpha}p \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}p$ is obtained. This is equivalent to $\mathfrak{M}^{\alpha}, f(x) \models p$, i.e., $f(x) \in V(p)$. Hence $\mathfrak{M}, f \models x{:}^{\alpha,A}p$.
(ii) If $x{:}^{\alpha,A}p \in \Delta^+$, then by the saturatedness, we have $x{:}^{\alpha}p \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}p$ is obtained. This is equivalent to $f(x) \notin V(p)$, and so $\mathfrak{M}, f \not\models x{:}^{\alpha,A}p$.

Case where $\mathfrak{A}$ is $x{:}^{\alpha}\neg A$: (i) If $x{:}^{\alpha}\neg A \in \Gamma^+$. By Definition 3.4.4, $x{:}^{\alpha}A \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}A$ is obtained. This is equivalent to $\mathfrak{M}, f \models x{:}^{\alpha}\neg A$.
(ii) If $x{:}^{\alpha}\neg A \in \Delta^+$, then by the saturatedness, we have $x{:}^{\alpha}A \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}A$ is obtained. This is equivalent to $\mathfrak{M}, f \not\models x{:}^{\alpha}\neg A$.

Case where $\mathfrak{A}$ is $x{:}^{\alpha}A \to B$: (i) If $x{:}^{\alpha}A \to B \in \Gamma^+$. By Definition 3.4.4, $x{:}^{\alpha}A \in \Delta^+$ or $x{:}^{\alpha}B \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}A$ or $\mathfrak{M}, f \models x{:}^{\alpha}B$ is obtained. This is equivalent to $\mathfrak{M}, f \models x{:}^{\alpha}A \to B$.
(ii) If $x{:}^{\alpha}A \to B \in \Delta^+$, then by the saturatedness, we have $x{:}^{\alpha}A \in \Gamma^+$ and $x{:}^{\alpha}B \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}A$ and $\mathfrak{M}, f \not\models x{:}^{\alpha}B$ are obtained. This is equivalent to $\mathfrak{M}, f \not\models x{:}^{\alpha}A \to B$.

Case where $\mathfrak{A}$ is $x{:}^{\alpha}\Box_a A$: (i) Suppose $x{:}^{\alpha}\Box_a A \in \Gamma^+$. What we show is $\mathfrak{M}, f \models x{:}^{\alpha}\Box_a A$, i.e., for all $y \in \mathcal{D}(\mathfrak{M})$, $xR_a^{\alpha!}y$ implies $\mathfrak{M}^{\alpha!}, y \models A$. So, fix any $y \in \mathcal{D}(\mathfrak{M})$ such that $xR_a^{\alpha!}y$. Now it suffices to show $\mathfrak{M}^{\alpha!}, y \models A$. By Proposition 3.4.4, we have $\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha}y$. Suppose for contradiction that $x\mathsf{R}_a^{\alpha}y \in \Delta^+$. By induction hypothesis, $\mathfrak{M}, f \not\models x\mathsf{R}_a^{\alpha}y$. A contradiction. Therefore, $x\mathsf{R}_a^{\alpha}y \notin \Delta^+$. Since $\Gamma^+ \Rightarrow \Delta^+$ is saturated and $x{:}^{\alpha}\Box_a A \in \Gamma^+$, we have $x\mathsf{R}_a^{\alpha}y \in \Delta^+$ or $y{:}^{\alpha}A \in \Gamma^+$. It follows that $y{:}^{\alpha}A \in \Gamma^+$, hence $\mathfrak{M}^{\alpha!}, y \models A$ by induction hypothesis. (ii) Suppose $x{:}^{\alpha}\Box_a A \in \Delta^+$. By Definition 3.4.4, $x\mathsf{R}_a^{\alpha}y \in \Gamma^+$ and $y{:}^{\alpha}A \in \Delta^+$, for some y. By induction hypothesis, $\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha}y$ and $\mathfrak{M}, f \not\models y{:}^{\alpha}A$, for some y. By Proposition 3.4.4, the definition of f and Definition 3.3.1, $(x, f(y)) \in R_a^{\alpha!}$ and $\mathfrak{M}^{\alpha!}, f(y) \not\models A$, for some y. Then, we get the goal: $\mathfrak{M}, f \not\models x{:}^{\alpha}\Box_a A$.

Case where $\mathfrak{A}$ is $x{:}^{\alpha}[A]B$: (i) If $x{:}^{\alpha}[A]B \in \Gamma^+$. By Definition 3.4.4, $x{:}^{\alpha}A \in \Delta^+$ or $x{:}^{\alpha,A}B \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}A$ or $\mathfrak{M}, f \models x{:}^{\alpha}B$ is obtained. This is equivalent to $\mathfrak{M}, f \models x{:}^{\alpha}A \to B$.
(ii) If $x{:}^{\alpha}A \to B \in \Delta^+$, then by the saturatedness, we have $x{:}^{\alpha}A \in \Gamma^+$ and $x{:}^{\alpha}B \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}A$ and $\mathfrak{M}, f \not\models x{:}^{\alpha}B$ are obtained. This is equivalent to $\mathfrak{M}, f \not\models x{:}^{\alpha}A \to B$.

Case where $\mathfrak{A}$ is $x\mathsf{R}_a^{\epsilon}y$: (i) and (ii) are trivial by the definition of $\mathfrak{M}$.

Case where $\mathfrak{A}$ is $x\mathsf{R}_a^{\alpha,A}y$: (i) If $x\mathsf{R}_a^{\alpha,A}y \in \Gamma^+$. By Definition 3.4.4, $x{:}^{\alpha}A \in \Gamma^+$ and $y{:}^{\alpha}A \in \Gamma^+$ and $x\mathsf{R}_a^{\alpha}y \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}A$ and $\mathfrak{M}, f \models y{:}^{\alpha}A$ and $\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha}y$ are obtained. This is equivalent to $\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha,A}y$.
(ii) If $x\mathsf{R}_a^{\alpha,A}y$, then by the saturatedness, we have $x{:}^{\alpha}A \in \Delta^+$ or $y{:}^{\alpha}A \in \Delta^+$ or $x\mathsf{R}_a^{\alpha}y \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}A$ or $\mathfrak{M}, f \not\models y{:}^{\alpha}A$ or $\mathfrak{M}, f \not\models x\mathsf{R}_a^{\alpha}y$ is obtained. This is equivalent to $\mathfrak{M}, f \not\models x\mathsf{R}_a^{\alpha,A}y$.

□

Since in the link-cutting semantics **GPAL** is complete for any sequent by Theorem 3.4.1, there is a counter-model for the underivable sequent $\Rightarrow x{:}^{\perp}p$ in **GPAL** (this sequent is t-valid but not derivable as we have seen in Propositions 3.4.1 and 3.4.2).

Proposition 3.4.6. There is a model where sequent $\Rightarrow x{:}^{\perp}p$ does not hold in the link-cutting semantics.

Proof. Let $\mathsf{Agt} = a$ for simplicity. Consider model $\mathfrak{M} = (\{w\}, \varnothing, V)$ where $V(p) = \varnothing$, and $f : \mathsf{Var} \to \{w\}$ such that $f(x) = w$. We show that in $\mathfrak{M}, f \not\models\Rightarrow x{:}^{\perp}p$. This is, by Definition 3.4.2, to $\mathfrak{M}^{\perp!}, w \not\models p$, and then by Definition 3.4.1, we obtain $w \notin V(p) = \varnothing$. It trivially holds. □

Corollary 3.4.2. Given any formula A and label $x \in \mathsf{Var}$, the following are equivalent.

(i) A is valid on all models for the world-deleting semantics.

(ii) $\vdash_{\mathbf{HPAL}} A$

(iii) $\vdash_{\mathbf{GPAL}^+} \Rightarrow x{:}^{\epsilon}A$

(iv) $\vdash_{\mathbf{GPAL}} \Rightarrow x{:}^{\epsilon}A$

(v) A is valid on all models for the link-cutting semantics.

Proof. The direction from (v) to (iv) is established by Theorem 3.4.1 and the direction from (ii) to (v) is shown by Propostion 3.4.3. Then, Corollary 3.3.1 implies the equivalence between five items. □

3.5 Extensions of PAL from K to S5

The labelled sequent calculus **GPAL** we have seen so far is based on **K**. On the other hand, epistemic logics including PAL are based on **S5** in general, so we now extend **GPAL** to **GPAL** with other modal systems including **S5**. We follow the definitions for extended Kripke semantics by frame properties in Chapter 2, but the index set Mod of modalities is here substituted for Agt.[11]

Recall (Definition 2.1.4)**.** Let Σ be a subset of $\mathsf{FrameAxiom}$. Then we write $\mathbb{F}_\Sigma$ to mean the class of frames which is defined by $\bigwedge \Sigma$. Further, let us also define the class $\mathbb{M}_\Sigma$ of models by $\mathbb{M}_\Sigma := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_\Sigma$ and V is a valuation V on $\mathfrak{F}\}$.

Proposition 3.5.1. Let $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. For all $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V) \in \mathbb{M}_\Sigma$ and all non-empty subsets X of $\mathcal{D}(\mathfrak{M})$, $\mathfrak{M}^X$ is also a member of $\mathbb{M}_\Sigma$, where $\mathfrak{M}^X = (X, (R^X_a)_{a\in\mathsf{Agt}}, V^X)$ with $R^X_a := R_a \cap X \times X$, $V^X(p) := V(p) \cap X$ ($p \in \mathsf{Prop}$).

Note that $\mathfrak{M}^{W^A} = \mathfrak{M}^A$

Proof. Fix any $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ in class $\mathbb{M}_\Sigma$ and fix any non-empty $X \subseteq W$. By the assumption $\mathfrak{M} \in \mathbb{M}_\Sigma$ i.e., $(W, (R_a)_{a\in\mathsf{Agt}}) \in \mathbb{F}_\Sigma$ and V is a valuation on $(W, (R_a)_{a\in\mathsf{Agt}})$. We show $(X, (R^X_a)_{a\in\mathsf{Agt}}) \in \mathbb{F}_\Sigma$, and V^X is a valuation on $(X, (R^X_a)_{a\in\mathsf{Agt}})$. Since $X \subseteq W$, the latter is trivial.

So in what follows, we show the former $(X, (R^X_a)_{a\in\mathsf{Agt}}) \in \mathbb{F}_\Sigma$ (class $\mathbb{F}_\Sigma$ of frames is defined by $\bigwedge \Sigma$) which is by Definition 2.1.3 equivalent to $(X, (R^X_a)_{a\in\mathsf{Agt}}) \Vdash \bigwedge \Sigma$. For any $A \in \Sigma$, R_a has a frame property defined by A. In the following, it suffices to show that R^S_a has also the property.

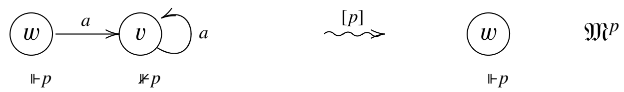
Case where $\mathbf{T}_a \in \Sigma$. In this case, we have $(W, (R_a)_{a\in\mathsf{Agt}}) \Vdash \mathbf{T}_a$.So by Definition 2.1 R_a is reflexive. Fix any world $x \in X$, and show $x R^X_a x$. Since R_a is reflexive, and so we get the goal R^S_a is also reflexive by $(x, x) \in X \times X$.

[11] The argument in this section is not included in [61].

Case where $\mathbf{5}_a \in \Sigma$. In this case, we have $(W, (R_a)_{a \in \mathsf{Agt}}) \Vdash \mathbf{5}_a$.So by Definition 2.1 R_a is Euclidean. Fix any $x, y, z \in X$. Suppose $xR_a^X y$ and $xR_a^X z$, and show $yR_a^X z$. We have $xR_a y$ and $xR_a z$ and $y, z \in X$, by the assumption $xR_a^X y$ and $xR_a^X z$. Since R_a is Euclidean i.e., $xR_a y$ and $xR_a z$ jointly imply $yR_a z$, we get $yR_a z$ and the goal $yR_a^X z$.

Other cases regarding $\mathbf{B}_a$ and $\mathbf{4}_a$ can be shown similarly. $\square$

Note that Proposition 3.5.1 does not hold, if $\mathbf{D}_a$ is included, since consider model $\mathfrak{M} = (W, R_a, V) = (\{w, v\}, \{(w, v), (v, v)\}, V) \in \mathbb{M}_{\{\mathbf{D}_a\}}$ where $V(p) = \{w\}$ and the restricted model $\mathfrak{M}^p$ by announcement of p.



The restricted model $\mathfrak{M}^p$ does not satisfy seriarity i.e., $\mathfrak{M}^p \notin \mathbb{M}_{\{\mathbf{D}_a\}}$.

As the case of extensions of **HK**, when we add one or more formulas in $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ as additional axiom schemes to the set of axiom scheme of **HPAL**, we obtain Hilbert-systems other than **HPAL** as follows.

Definition 3.5.1 (Extensions of **HPAL**)**.** Let Σ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. When all elements of Σ is added to **HPAL** as an axiom scheme by replacing p with an arbitrary formula A, *the extension of* **HPAL** by Σ is the resulting Hilbert-system denoted by $\mathbf{HPAL}\Sigma$.

We give names to Hilbert-systems with some particular combinations of axiom schemes.

$$\begin{array}{ll} \mathbf{HPAL_T} := \mathbf{HPAL}\{\mathbf{T}_a \mid a \in \mathsf{Agt}\}, & \mathbf{HPAL_{S4}} := \mathbf{HPAL}\{\mathbf{T}_a, \mathbf{4}_a \mid a \in \mathsf{Agt}\}, \\ \mathbf{HPAL_B} := \mathbf{HPAL}\{\mathbf{T}_a, \mathbf{B}_a \mid a \in \mathsf{Agt}\}, & \mathbf{HPAL_{S5}} := \mathbf{HPAL}\{\mathbf{T}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}. \end{array}$$

For any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, public announcement logic $\mathbf{PAL}\Sigma$ is the set of all derivable formulas in $\mathbf{HPAL}\Sigma$. We name some $\mathbf{PAL}\Sigma$.

$$\begin{array}{ll} \mathbf{PAL_T} := \mathbf{PAL}\{\mathbf{T}_a \mid a \in \mathsf{Agt}\}, & \mathbf{PAL_{S4}} := \mathbf{PAL}\{\mathbf{T}_a, \mathbf{4}_a \mid a \in \mathsf{Agt}\}, \\ \mathbf{PAL_B} := \mathbf{PAL}\{\mathbf{T}_a, \mathbf{B}_a \mid a \in \mathsf{Agt}\}, & \mathbf{PAL_{S5}} := \mathbf{PAL}\{\mathbf{T}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}. \end{array}$$

Corollary 3.5.1. Let $\mathbf{X}$ be an element of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a\}$. $\mathbf{PAL_X}$ is decidable, i.e., there is an effective method for deciding whether or not any formula is a theorem of PAL.

Proof. Fix any $A \in \mathcal{L}_{PAL}$. Then since by Corollary 2.1.1 modal logic $\mathbf{X} \in \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a\}$ is decidable. Besides, Note that translation $t : \mathcal{L}_{PAL} \to \mathcal{L}_{EL}$ is inductive and so provides an effective method. $t(A) \in \mathcal{L}_{ML}$ can be decided whether it is a theorem of $\mathbf{PAL_X}$. $\square$

Theorem 3.5.1 (Soundness and completeness of $\mathbf{HPAL}\Sigma$)**.** Let Σ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ and $A \in \mathcal{L}_{PAL}$. Then the following holds:

$$\mathbb{M}_\Sigma \Vdash A \text{ iff } \vdash_{\mathbf{HPAL}\Sigma} A.$$

Proof. The proof is carried out by the same step as in 2.2.2. $\square$

Extensions of GPAL Let us define the extensions of **GPAL**. We add to **GPAL** one or more of the additional rules in Table 3.3 which correspond to the frame properties respectively.

Table 3.3: Rules for frame properties

$$\frac{x\mathsf{R}^{\epsilon}_{a}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (ref_a) \qquad \frac{\Gamma \Rightarrow \Delta, x\mathsf{R}^{\epsilon}_{a}y \quad \Gamma \Rightarrow \Delta, y\mathsf{R}^{\epsilon}_{a}z \quad x\mathsf{R}^{\epsilon}_{a}z, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (tra_a)$$

$$\frac{\Gamma \Rightarrow \Delta, x\mathsf{R}^{\epsilon}_{a}y \quad y\mathsf{R}^{\epsilon}_{a}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (sym_a) \qquad \frac{\Gamma \Rightarrow \Delta, x\mathsf{R}^{\epsilon}_{a}y \quad \Gamma \Rightarrow \Delta, x\mathsf{R}^{\epsilon}_{a}z \quad y\mathsf{R}^{\epsilon}_{a}z, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (euc_a)$$

Let * be a function from $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ to $\{(ref_a), (sym_a), (tra_a), (euc_a), (ser_a) \mid a \in \mathsf{Agt}\}$, defined by:

$$\begin{array}{ll} \mathbf{T}_a{}^* := (ref_a), & \mathbf{4}_a{}^* := (tra_a), \\ \mathbf{B}_a{}^* := (sym_a), & \mathbf{5}_a{}^* := (euc_a). \end{array}$$

Let Σ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ then Σ^* is defined to be the set $\{\mathbf{X}^* \mid \mathbf{X} \in \Sigma\}$.

Definition 3.5.2 (Extensions of **GPAL**)**.** Let Σ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. A labelled sequent calculus $\mathbf{GPAL\Sigma}^*$ is an extension of **GPAL**, when each element of Σ^* is added to **GPAL** as an inference rule.

Some particular combinations of inference rules are given names.

$\mathbf{GPAL_T} := \mathbf{GPAL}\{(ref_a) \mid a \in \mathsf{Agt}\}$,
$\mathbf{GPAL_B} := \mathbf{GPAL}\{(sym_a) \mid a \in \mathsf{Agt}\}$,
$\mathbf{GPAL_{S4}} := \mathbf{GPAL}\{(ref_a), (tra_a) \mid a \in \mathsf{Agt}\}$,
$\mathbf{GPAL_{S5}} := \mathbf{GPAL}\{(ref_a), (euc_a) \mid a \in \mathsf{Agt}\}$,

We denote each $\mathbf{GPAL\Sigma}^*$ with (Cut) by $\mathbf{GPAL\Sigma}^{*+}$.

Theorem 3.5.2. For any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, if $\vdash_{\mathbf{HPAL\Sigma}} A$, then $\vdash_{\mathbf{GPAL\Sigma}^*} \Rightarrow x{:}^{\epsilon}A$ (for any x) for any formula $A \in \mathcal{L}_{PAL}$.

Proof. Fix any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. The proof is carried out by the height of the derivation in **HPALΣ**, and it suffices to show the derivability of the additional cases in $\mathbf{GPAL\Sigma}^*$ to the proof of Theorem 3.2.1 i.e., the cases of $\mathbf{T}_a$, $\mathbf{B}_a$, $\mathbf{4}_a$ and $\mathbf{5}_a$, (where $a \in \mathsf{Agt}$).

Case where $\mathbf{T}_a \in \Sigma$. In this case, we show $\vdash_{\mathbf{GPAL\Sigma}^*} \Rightarrow x{:}\mathbf{T}_a$ where $(ref_a) \in \Sigma^*$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}_a x \Rightarrow x{:}A, x\mathsf{R}_a x}}{\Rightarrow x{:}A, x\mathsf{R}_a x}\ (ref_a) \quad \dfrac{\textit{Initial Seq.}}{x{:}A \Rightarrow x{:}A}}{x{:}\Box_a A \Rightarrow x{:}A}\ (L\Box_a)}{\Rightarrow x{:}\Box_a A \to A}\ (R \to)$$

Case where $\mathbf{B}_a \in \Sigma$. In this case, we show $\vdash_{\mathbf{GPAL\Sigma^*}} \Rightarrow x{:}\mathbf{B}_a$ where $(sym_a) \in \Sigma^*$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y \Rightarrow y\mathsf{R}_a x, x\mathsf{R}_a y} \quad \dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, y\mathsf{R}_a x \Rightarrow y\mathsf{R}_a x}}{x\mathsf{R}_a y \Rightarrow y\mathsf{R}_a x}\,(sym_a)}{x{:}A, x\mathsf{R}_a y \Rightarrow y\mathsf{R}_a x}\,(Lw) \quad \dfrac{\mathit{Initial\ Seq.}}{x{:}A, x\mathsf{R}_a y \Rightarrow x{:}A}}{\dfrac{x{:}A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A}{\dfrac{x{:}A \Rightarrow x{:}\Box_a\Diamond_a A}{\Rightarrow x{:}A \to \Box_a\Diamond_a A}\,(R\to)}\,(R\Box_a)}\,(R\Diamond_a)$$

Case where $\mathbf{4}_a \in \Sigma$. In this case, we show $\vdash_{\mathbf{GPAL\Sigma^*}} \Rightarrow x{:}\mathbf{4}_a$ where $(tra) \in \Sigma^*$.

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow x\mathsf{R}_a z, x\mathsf{R}_a y} \quad \dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow x\mathsf{R}_a z, y\mathsf{R}_a z} \quad \dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a z, x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow x\mathsf{R}_a z}}{x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow x\mathsf{R}_a z}\,(tra_a) \right.$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow z{:}A, x\mathsf{R}_a z\end{array} \quad \dfrac{\mathit{Initial\ Seq.}}{z{:}A, x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow z{:}A}}{x{:}\Box_a A, x\mathsf{R}_a y, y\mathsf{R}_a z \Rightarrow z{:}A}\,(L\Box_a)}{\dfrac{x{:}\Box_a A, x\mathsf{R}_a y \Rightarrow y{:}\Box_a A}{\dfrac{x{:}\Box_a A \Rightarrow x{:}\Box_a\Box_a A}{\Rightarrow x{:}\Box_a A \to \Box_a\Box_a A}\,(R\to)}\,(R\Box_a)}\,(R\Box_a)$$

Case where $\mathbf{5}_a \in \Sigma$. In this case, we show $\vdash_{\mathbf{GPAL\Sigma^*}} \Rightarrow x{:}\mathbf{5}_a$ where $(euc_a) \in \Sigma^*$.

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, x\mathsf{R}_a z \Rightarrow y\mathsf{R}_a z, x\mathsf{R}_a y} \quad \dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, x\mathsf{R}_a z \Rightarrow y\mathsf{R}_a z, x\mathsf{R}_a z} \quad \dfrac{\mathit{Initial\ Seq.}}{y\mathsf{R}_a z, x\mathsf{R}_a y, x\mathsf{R}_a z \Rightarrow y\mathsf{R}_a z}}{x\mathsf{R}_a y, x\mathsf{R}_a z \Rightarrow y\mathsf{R}_a z}\,(euc_a) \right.$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ x\mathsf{R}_a y, x\mathsf{R}_a z \Rightarrow y\mathsf{R}_a z\end{array}}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y\mathsf{R}_a z}\,(Rw) \quad \dfrac{\mathit{Initial\ Seq.}}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow z{:}A}}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond_a A}\,(R\Diamond_a)}{\dfrac{x{:}\Diamond_a A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A}{\dfrac{x{:}\Diamond_a A \Rightarrow x{:}\Box_a\Diamond_a A}{\Rightarrow x{:}\Diamond_a A \to \Box_a\Diamond_a A}\,(R\to)}\,(R\Box_a)}\,(L\Diamond_a)$$

□

Theorem 3.5.3 (Soundness of **GPALΣ**). For any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, given any sequent $\Gamma \Rightarrow \Delta$ in **GPALΣ**, if $\vdash_{\mathbf{GPAL\Sigma}} \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is t-valid in every model $\mathfrak{M} \in \mathbb{M}_\Sigma$.

Proof. Fix any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. We show the additional cases to the proof of Theorem 3.3.1, and so it suffices to show that any additional rule keeps t-validity in any corresponding model to the rule.

Case of (ref_a)**:** Fix any R_a-reflexive model $\mathfrak{M}$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $(f(x), f(x)) \in R_a$. This is trivial by the R_a-reflexive model $\mathfrak{M}$.

Case of (sym_a)**:** Fix any R_a-symmetric model $\mathfrak{M}$. We show the contraposition. Suppose that there is some $f' : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$, $\mathfrak{M}, f' \Vdash \Gamma$, and $\mathfrak{M}, f' \Vdash \overline{\Delta}$. Fix such f'. We show that $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ is not t-valid or $y\mathsf{R}_a x, \Gamma \Rightarrow \Delta$ is not t-valid. So, suppose $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ is t-valid i.e., for all f', $\mathfrak{M}, f' \Vdash \Gamma$ implies $\mathfrak{M}, f' \nVdash \overline{\Delta}$ and $\mathfrak{M}, f' \nVdash \overline{x\mathsf{R}_a y}$. Then we show $y\mathsf{R}_a x, \Gamma \Rightarrow \Delta$ is not t-valid i.e., there exists f', $\mathfrak{M}, f' \Vdash y\mathsf{R}_a x$ and $\mathfrak{M}, f' \Vdash \Gamma$ and $\mathfrak{M}, f' \Vdash \overline{\Delta}$. Take f' as f. Now, it suffices to show that $\mathfrak{M}, f \Vdash y\mathsf{R}_a x$ (i.e., $f(y)R_a f(x)$). From the suppositions, we obtain $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a y}$ (i.e., $f(x)R_a f(y)$). So, we trivially obtain $f(y)R_a f(x)$ from the R_a-symmetric model $\mathfrak{M}$.

Case of (tra_a)**:** Fix any R_a-transitive model $\mathfrak{M}$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. We show that $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ is not t-valid or $\Gamma \Rightarrow \Delta, y\mathsf{R}_a z$ is not t-valid or $x\mathsf{R}_a z, \Gamma \Rightarrow \Delta$ is not t-valid. So, suppose $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ and $\Gamma \Rightarrow \Delta, y\mathsf{R}_a z$ are t-valid i.e., for all f', $\mathfrak{M}, f' \Vdash \Gamma$ implies $\mathfrak{M}, f' \nVdash \overline{\Delta}$ and $\mathfrak{M}, f' \nVdash \overline{x\mathsf{R}_a y}$, and for all f', $\mathfrak{M}, f' \Vdash \Gamma$ implies $\mathfrak{M}, f' \nVdash \overline{\Delta}$ and $\mathfrak{M}, f' \nVdash \overline{y\mathsf{R}_a z}$. Then we show $\Gamma \Rightarrow \Delta, x\mathsf{R}_a z$ is not t-valid i.e., there exists f', $\mathfrak{M}, f' \Vdash x\mathsf{R}_a z$ and $\mathfrak{M}, f' \Vdash \Gamma$ and $\mathfrak{M}, f' \Vdash \overline{\Delta}$. Take such f' as f. Now, it suffices to show that $\mathfrak{M}, f \Vdash x\mathsf{R}_a z$ (i.e., $f(x)R_a f(z)$). From the suppositions, we obtain $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a y}$ (i.e., $f(x)R_a f(y)$) and $\mathfrak{M}, f \nVdash \overline{y\mathsf{R}_a z}$ (i.e., $f(y)R_a f(z)$). So, we trivially obtain $f(x)R_a f(z)$ from the R_a-transitive model $\mathfrak{M}$.

Case of (euc_a)**:** Fix any R_a-Euclidean model $\mathfrak{M}$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. We show that $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ is not t-valid or $\Gamma \Rightarrow \Delta, x\mathsf{R}_a z$ is not t-valid or $y\mathsf{R}_a z, \Gamma \Rightarrow \Delta$ is not t-valid. So, suppose $\Gamma \Rightarrow \Delta, x\mathsf{R}_a y$ and $\Gamma \Rightarrow \Delta, x\mathsf{R}_a z$ are t-valid i.e., for all f', $\mathfrak{M}, f' \Vdash \Gamma$ implies $\mathfrak{M}, f' \nVdash \overline{\Delta}$ and $\mathfrak{M}, f' \nVdash \overline{x\mathsf{R}_a y}$, and for all f', $\mathfrak{M}, f' \Vdash \Gamma$ implies $\mathfrak{M}, f' \nVdash \overline{\Delta}$ and $\mathfrak{M}, f' \nVdash \overline{x\mathsf{R}_a z}$. Then we show $\Gamma \Rightarrow \Delta, y\mathsf{R}_a z$ is not t-valid i.e., there exists f', $\mathfrak{M}, f' \Vdash y\mathsf{R}_a z$ and $\mathfrak{M}, f' \Vdash \Gamma$ and $\mathfrak{M}, f' \Vdash \overline{\Delta}$. Take such f' as f. Now, it suffices to show that $\mathfrak{M}, f \Vdash y\mathsf{R}_a z$ (i.e., $f(y)R_a f(z)$). From the suppositions, we obtain $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a y}$ (i.e., $f(x)R_a f(y)$) and $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a z}$ (i.e., $f(y)R_a f(z)$). So, we trivially obtain $f(y)R_a f(z)$ from the R_a-Euclidean model $\mathfrak{M}$.

□

Each extension enjoys the cut elimination theorem.

Theorem 3.5.4 (Cut elimination theorem of $\mathbf{GPAL\Sigma}^{*+}$)**.** For any $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, and any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{GPAL\Sigma}^{*+}} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{GPAL\Sigma}^{*}} \Gamma \Rightarrow \Delta$.

Proof. The proof goes through the same procedure as in the proof of Theorem 3.2.2 with the rule of (*Ecut*), and the proof is divided into four cases. In brief,

(1) at least one of the uppersequents of (*Ecut*) is an initial sequent;

(2) the last inference rule of either uppersequents of (*Ecut*) is a structural rule;

(3) the last inference rule of either uppersequents of (*Ecut*) is a non-structural rule[12], and the principal expression introduced by the rule is not the cut expression; and

[12]Inference rules for frame properties are categorized as non-structural rules.

(4) the last inference rules of two uppersequents of (*Ecut*) are both non-structural rules, and the principal expressions introduced by the rules used on the uppersequents of (*Ecut*) are both cut expressions.

It suffices to show additional cases for $(ref_a), (sym_a), (tra_a)$ and (euc_a) in addition to the proof of Theorem 3.2.2. Since there is no principal expression(s) introduced by the uppersequent(s), we do not have the case (4) where cut expression $\mathfrak{A}$s on both sides of uppersequents are principal expressions. The other cases where only one of the cut expressions is introduced by the right uppersequent or the left uppersequent are straightforward. We look at one of such cases.

Case of (3) where one of the uppersequents of (*Ecut*) is inference rule (ref_a).

$$\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_2}{x\mathsf{R}_a y, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}\,(ref_a)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n} \quad \dfrac{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, x\mathsf{R}_a y, \Gamma' \Rightarrow \Delta'}}{x\mathsf{R}_a y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(ref_a)$$

Every other case can be shown similarly to this.

□

Then the corollary below holds.

Corollary 3.5.2. Given a formula A, $x \in \mathsf{Var}$, $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, the following statements are all equivalent.

(i) $\mathbb{M}_\Sigma \Vdash A$,

(ii) $\vdash_{\mathbf{HPAL\Sigma}} A$,

(iii) $\vdash_{\mathbf{GPAL\Sigma^{*+}}} \Rightarrow x{:}^{\epsilon}A$,

(iv) $\vdash_{\mathbf{GPAL\Sigma^{*}}} \Rightarrow x{:}^{\epsilon}A$.

Proof. The direction from (i) to (ii) is shown by Theorem 3.5.1 and the direction from (ii) to (iii) is established by Theorem 3.5.2. Then, the direction from (iii) to (iv) is established by the admissibility of (*Cut*) (Theorem 3.5.4). Finally, the direction from (iv) to (i) is shown by Theorem 3.5.3. □

Remark The extension of **GPAL** is done by the above argument, and we come to a happy conclusion that **GPAL** can be extended to be based on modal logic **S5** which is the standard basis of epistemic logics. On the other hand, there is a different candidate of a rule corresponding reflexivity instead of (ref_a) as follows:

$$\frac{x\mathsf{R}_a^{\alpha}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (ref'_a)$$

where a relational atom on the upper sequent contains a list of formulas. However, if we introduce the rule of (ref'_a) without any condition in a naive manner, then the soundness theorem for the world-deleting semantics does not hold.

Proposition 3.5.2. Let $\Sigma \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ such that $\mathbf{T}_a \in \Sigma$. Then there is a model $\mathfrak{M} \in \mathbb{M}_\Sigma$ such that (ref'_a) of $\mathbf{GPAL}\Sigma^*$ does not preserve t-validity in $\mathfrak{M}$.

Proof. Consider a model $\mathfrak{M}' = (\{w\}, (w, w), V)$ where $V(p) = \varnothing$. and consider the particular instance of the application of (ref'_a) is as follows:

$$\frac{x\mathsf{R}^{\perp}x \Rightarrow}{\Rightarrow}\ (ref'_a)$$

We show that the uppersequent is t-valid in $\mathfrak{M}$ but the lowersequent is not t-valid in $\mathfrak{M}$. Note that no world does not survive after an announcement of $\perp$, so $\mathfrak{M}, f \Vdash x\mathsf{R}^{\perp}x$ is false and the uppersequent is t-valid. On the other hand, the lowersequent, the empty-sequent, is not t-valid. □

Chapter 4

Labelled sequent calculus for EAK

For the last decade, several studies of semantical developments of EAK have emerged for the sake of capturing characteristics regarding knowledge; there are also some proof-theoretic studies of EAK such as a tableaux calculus for EAK by Aucher et al. [3, 4], a display calculus for it by Frittella et al. [28] and a nested calculus for it by Dyckhoff et al. [24]. We, in this paper, construct a labelled calculus of EAK based on the study of the previous chapter.

The outline of Chapter 4 is as follows. In Section 4.1, we give our labelled sequent calculus for EAK (**GEAK**) based on the study of Chapter 3. In Section 4.2, we establish admissibility of the *cut* rule in **GEAK**. In Section 4.3, we prove the soundness theorem and then give a proof of the completeness theorem of **GEAK** as a corollary. In Section 4.5, we extend the basis of **GPAL** from K to S5.

Table 4.1: Hilbert-system for EAK : **HEAK**

Modal Axioms	all instantiations of propositional tautologies	
	(K)	$\Box_a(A \to B) \to (\Box_a A \to \Box_a B)$
Recursion Axioms	$(RA1)$	$[\mathsf{a}^{\mathsf{M}}]p \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to p)$
	$(RA2)$	$[\mathsf{a}^{\mathsf{M}}]\neg A \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to \neg[\mathsf{a}^{\mathsf{M}}]A)$
	$(RA3)$	$[\mathsf{a}^{\mathsf{M}}](A \to B) \leftrightarrow ([\mathsf{a}^{\mathsf{M}}]A \to [\mathsf{a}^{\mathsf{M}}]B)$
	$(RA4)$	$[\mathsf{a}^{\mathsf{M}}]\Box_a A \leftrightarrow (\mathsf{pre}(\mathsf{a}) \to \bigwedge_{\mathsf{a} \sim_a^{\mathsf{M}} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A)$
	$(RA5)$	$[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A \leftrightarrow [\mathsf{a}^{\mathsf{M}};\mathsf{b}^{\mathsf{N}}]A$
Inference Rules	(MP)	From A and $A \to B$, infer B
	$(Nec\Box_a)$	From A, infer $\Box_a A$

4.1 Labelled sequent calculus GEAK

We define our labelled calculus **GEAK** for EAK. Let $\mathsf{Var} = \{x, y, z, ...\}$ be a countable set of *action variable* For any $x \in \mathsf{Var}$ and any finite list α of pointed action models, a pair $\langle x, \alpha\rangle$ is called a *label*, and we *abuse* x for an abbreviation of $\langle x, \epsilon\rangle$, when it does not cause any confusion.

For any formula A and label $\langle x, \alpha\rangle$, we call $\langle x, \alpha\rangle{:}A$ a *labelled formula*. Similarly, for any agent $a \in \mathsf{Agt}$ and any lists α,β of actions, an expression $\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$ is defined to be a *relational atom* if

(1) $\alpha_{mdl} = \beta_{mdl}$ and

(2) $\mathsf{a}_i {\sim}_a^{\mathsf{M}_i} \mathsf{b}_i$ holds for any $1 \leq i \leq n$, where $\alpha = (\mathsf{a}_1^{\mathsf{M}_1}, \ldots, \mathsf{a}_n^{\mathsf{M}_n})$ and $\beta = (\mathsf{b}_1^{\mathsf{M}_1}, \ldots, \mathsf{b}_n^{\mathsf{M}_n})$.

A *labelled expression* $\mathfrak{A}$ is either a labelled formula or a relational atom.

Note that every labelled expression of **GPAL** is expressible in a labelled expression of **GEAK**, i.e., **GPAL** is a special case of **GEAK**. Let us recall action model $\mathsf{Pub}(A)$ in Example 2.3.1 by which PAL can be emulated by EAK, and by the action model any labelled expression of **GPAL** is also can be emulated by **GEAK**. Let $\mathsf{Pub}(A) = (\{\mathsf{a}\}, (\sim_a)_{a \in \mathsf{Agt}}, \mathsf{pre})$ be an action model where each $\sim_a$ is the identity relation and $\mathsf{pre}(\mathsf{a}) = A$ for any formula $A \in \mathcal{L}_{EAK}$. Then consider labelled relation $x\mathsf{R}_a^A y$ of **GPAL**, and it is expressed by labelled relation $\langle x, \mathsf{a}^{\mathsf{Pub}(A)}\rangle \mathsf{R}_a \langle y, \mathsf{a}^{\mathsf{Pub}(A)}\rangle$ of **GEAK**.

Definition 4.1.1. The length of a labelled expression $\ell(\mathfrak{A})$, a formula $\ell(A)$ and an action model $\ell(\mathsf{M})$ is defined as follows:

$$\begin{array}{ll}
\ell(\langle x, \alpha\rangle{:}A) := \ell(\alpha) + \ell(A), & \ell(\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle) := \ell(\alpha), \\
\ell(p) := 1, & \ell(A \to B) := \ell(A) + \ell(B) + 1, \\
\ell(*A) := \ell(A) + 1 \text{ where } * \in \{\neg, \Box_a\}, & \ell([\mathsf{a}^{\mathsf{M}}]A) := \ell(\mathsf{M}) + \ell(A) + 1, \\
\ell(\mathsf{M}) := \max\{\ell(\mathsf{pre}^{\mathsf{M}}(\mathsf{x})) \mid \mathsf{x} \in \mathsf{S}^{\mathsf{M}}\}, & \ell(\mathsf{a}_1^{\mathsf{M}_1}, \ldots, \mathsf{a}_n^{\mathsf{M}_n}) = \ell(\mathsf{M}_1) + \cdots + \ell(\mathsf{M}_n).
\end{array}$$

A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of multi-sets Γ, Δ of labelled expressions. The existence of the pointed action model a^{M} in our syntax of EAK forces us to handle many *branches* in a naturally constructed sequent calculus. For example, we may consider a set

$$\{x{:}\mathsf{pre}(\mathsf{x}) \Rightarrow y{:}\mathsf{pre}(\mathsf{y}) \mid \mathsf{p} \sim_b \mathsf{x} \text{ and } \mathsf{x} \sim_a \mathsf{y}\}$$

of sequents in the setting of Example 2.3.2. In order to handle such several branches simultaneously in a sequent calculus, we introduce the notation

$$x{:}\mathsf{pre}(\mathsf{x}) \Rightarrow y{:}\mathsf{pre}(\mathsf{y}) \parallel \mathsf{p} \sim_b \mathsf{x}, \mathsf{x} \sim_a \mathsf{y}$$

for representing the above set. In general, we keep a countable proper subset $\mathsf{CVar} = \{\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots\}$ of Act for comprehension variables and define that a *collective sequent* (simply a *c-sequent*) is an expression:

$$\Gamma \Rightarrow \Delta \parallel \Sigma,$$

where $\Gamma \Rightarrow \Delta$ is a sequent, and Σ is a finite set $\{\mathsf{s}_1 {\sim}_{a_1} \mathsf{t}_1, ..., \mathsf{s}_n {\sim}_{a_n} \mathsf{t}_n\}$ of actions relations, and s_i or t_i from Act is assumed to be an element of CVar and all the variables from

CVar occurring in $\Gamma \Rightarrow \Delta$ are bounded in Σ, i.e., they are a subset of all the variables from CVar occurring in Σ. Throughout the paper we use Greek letter Σ for a finite set $\{\mathsf{s}_1{\sim_{a_1}}\mathsf{t}_1, ..., \mathsf{s}_n{\sim_{a_n}}\mathsf{t}_n\}$ of actions relations. For any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, we simply write $\Gamma \Rightarrow \Delta$, when $\Sigma = \varnothing$.

We now introduce the set of rules of **GEAK** which is presented in Table 4.2. We call labelled expression $\mathfrak{A}$ in the lower c-sequent at each inference rule *principal* if $\mathfrak{A}$ is not in either Γ or Δ. In this table, all rules are given for a c-sequent for the reason of drawing smaller and simpler derivations in the following, but the inference rules can be defined for an ordinary sequent.

Definition 4.1.2 (Derivable)**.** A *derivation* of c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ in **GEAK** is a finite tree of c-sequents satisfying the following conditions:

(i) The uppermost c-sequent of the tree is an initial sequent of **GEAK**.

(ii) Every c-sequent in the tree except the uppermost c-sequent(s) is a lower c-sequent of an inference rule of **GEAK**.

(iii) The lowest c-sequent is $\Gamma \Rightarrow \Delta \parallel \Sigma$.

Given a c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, it is *derivable in* **GEAK** and we write $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$ if there is a derivation of the c-sequent; and especially if there exists a derivation of the c-sequent which is restricted to action models $\mathsf{M}_1, ..., \mathsf{M}_n$ which appear in the derivation, we say, for emphasizing the fact, it is *derivable in* **GEAK** *under action models* $\mathsf{M}_1, ..., \mathsf{M}_n$ and write $\mathsf{M}_1, ..., \mathsf{M}_n \vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$.[1] For any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, when $\Gamma \Rightarrow \Delta \parallel \Sigma$ is derivable in **GEAK**, we write $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$.

Hereinafter, we use the following abbreviations in a derivation for drawing simpler derivations:

$$\dfrac{\textit{Initial Seq.}}{\mathfrak{A}, \Gamma \Rightarrow \Delta, \mathfrak{A} \parallel \Sigma}$$

This abbreviation is obvious by the rules (Lw) and (Rw). Besides, the following usual inference rules for the defined logical connectives are all derivable in **GEAK**:

$$\dfrac{\Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}A, \langle x, \alpha\rangle{:}B \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}A \vee B, \parallel \Sigma}\ (R\vee) \qquad \dfrac{\langle x, \alpha\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma \quad \langle x, \alpha\rangle{:}B, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x, \alpha\rangle{:}A \vee B, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\vee)$$

$$\dfrac{\langle x, \alpha\rangle{:}A, \langle x, \alpha\rangle{:}B, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x, \alpha\rangle{:}A \wedge B, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\wedge) \qquad \dfrac{\Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}A \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}B \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}A \wedge B \parallel \Sigma}\ (R\wedge)$$

$$\dfrac{x\mathsf{R}_a v, \langle v, \alpha\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x, \alpha\rangle{:}\Diamond_a A, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\Diamond_a)\dagger \qquad \dfrac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a y \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle y, \alpha\rangle{:}A \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x, \alpha\rangle{:}\Diamond_a A \parallel \Sigma}\ (R\Diamond_a)$$

where † means that v does not appear in the lower c-sequent.

Let us look at specific derivations of **GEAK**. Let $\mathsf{M} = (\mathsf{S}, \sim_a, \mathsf{pre}) = (\{\mathsf{a}, \mathsf{b}\}, \mathsf{S}^2, \mathsf{pre}\})$ where $\mathsf{pre}(\mathsf{a}) = \mathsf{pre}(\mathsf{b}) = p$. Sequent $\Rightarrow x{:}\mathsf{pre}(\mathsf{x}) \to \mathsf{pre}(\mathsf{b}) \parallel \mathsf{a} \sim_a \mathsf{x}$ is derivable in

[1] In the case that c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ and a derivation of it do not include any action model, we write $\varepsilon \vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$ to emphasize the case. We remark that this should be distinguished from $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$ defined above.

Table 4.2: Labelled sequent calculus for EAK : **GEAK**

(Initial c-sequents)

$$\langle x,\alpha\rangle{:}A \Rightarrow \langle x,\alpha\rangle{:}A \parallel \Sigma \quad \langle x,\alpha\rangle\mathsf{R}_a\langle y,\beta\rangle \Rightarrow \langle x,\alpha\rangle\mathsf{R}_a\langle y,\beta\rangle \parallel \Sigma$$

(Structural Rules)

$$\frac{\Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta, \mathfrak{A} \parallel \Sigma}\ (Rw) \quad \frac{\Gamma \Rightarrow \Delta \parallel \Sigma}{\mathfrak{A}, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lw) \quad \frac{\Gamma \Rightarrow \Delta, \mathfrak{A}, \mathfrak{A} \parallel \Sigma}{\Gamma \Rightarrow \Delta, \mathfrak{A} \parallel \Sigma}\ (Rc) \quad \frac{\mathfrak{A}, \mathfrak{A}, \Gamma \Rightarrow \Delta \parallel \Sigma}{\mathfrak{A}, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lc)$$

(Rules for the propositional connectives)

$$\frac{\langle x,\alpha\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}\neg A \parallel \Sigma}\ (R\neg) \quad \frac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A \parallel \Sigma}{\langle x,\alpha\rangle{:}\neg A, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\neg)$$

$$\frac{\langle x,\alpha\rangle{:}A, \Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}B \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A \to B \parallel \Sigma}\ (R\to) \quad \frac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A \parallel \Sigma \quad \langle x,\alpha\rangle{:}B, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha\rangle{:}A \to B, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\to)$$

(Rules for the knowledge operator)

$$\frac{\langle x,\epsilon\rangle\mathsf{R}_a\langle y,\epsilon\rangle, \Gamma \Rightarrow \Delta, \langle y,\epsilon\rangle{:}A \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\epsilon\rangle{:}\Box_a A \parallel \Sigma}\ (R\Box_a 1)\ *_1 \quad \frac{\Gamma \Rightarrow \Delta, \langle x,\epsilon\rangle\mathsf{R}_a\langle y,\epsilon\rangle \parallel \Sigma \quad \langle y,\epsilon\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\epsilon\rangle{:}\Box_a A, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\Box_a 1)$$

$$\frac{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle\mathsf{R}_a\langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle, \Gamma \Rightarrow \Delta, \langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle{:}A \parallel \Sigma, \mathsf{a}_1 \sim_a^{\mathsf{M}_1} \mathsf{x}_1, ..., \mathsf{a}_k \sim_a^{\mathsf{M}_k} \mathsf{x}_k}{\Gamma \Rightarrow \Delta, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A \parallel \Sigma}\ (R\Box_a 2)\ *_1$$

$$\frac{\Gamma \Rightarrow \Delta, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle\mathsf{R}_a\langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k}\rangle \parallel \Sigma \quad \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k}\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L\Box_a 2)\ *_2$$

(Rules for the action operators)

$$\frac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}p \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle{:}p \parallel \Sigma}\ (Rat) \quad \frac{\langle x,\alpha\rangle{:}p, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle{:}p, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lat)$$

$$\frac{\langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta, \langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle{:}A \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]A \parallel \Sigma}\ (R[.]) \quad \frac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma \quad \langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle{:}A, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]A, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (L[.])$$

(Rules for the relational atoms)

$$\frac{\langle x,\alpha\rangle\mathsf{R}_a\langle y,\beta\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle\mathsf{R}_a\langle y,\beta,\mathsf{b}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lrel1)$$

$$\frac{\langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle\mathsf{R}_a\langle y,\beta,\mathsf{b}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lrel2) \quad \frac{\langle y,\beta\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{b}), \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle\mathsf{R}_a\langle y,\beta,\mathsf{b}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Lrel3)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle\mathsf{R}_a\langle y,\beta\rangle \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle y,\beta\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{b}) \parallel \Sigma}{\Gamma \Rightarrow \Delta, \langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle\mathsf{R}_a\langle y,\beta,\mathsf{b}^{\mathsf{M}}\rangle \parallel \Sigma}\ (Rrel)$$

$*_1$ $v \in \mathsf{Var}$ and each $\mathsf{x}_i \in \mathsf{CVar}$ do not appear in the lower c-sequent.

$*_2$ Each b_i is an action such that $(\mathsf{a}_i, \mathsf{b}_i) \in \sim_a^{\mathsf{M}_i}$ or $\mathsf{a}_i \sim_a^{\mathsf{M}_i} \mathsf{b}_i \in \Sigma$.

GEAK under M:

$$\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}\mathsf{pre(b)} \Rightarrow x{:}\mathsf{pre(b)}} \qquad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}p \Rightarrow x{:}p}}{x{:}\mathsf{pre(a)} \Rightarrow x{:}\mathsf{pre(b)}}\ \text{(Def. of M)}}{\dfrac{x{:}\mathsf{pre(x)} \Rightarrow x{:}\mathsf{pre(b)} \parallel \mathsf{a} \sim_a \mathsf{x}}{\Rightarrow x{:}\mathsf{pre(x)} \to \mathsf{pre(b)} \parallel \mathsf{a} \sim_a \mathsf{x}}\ (R\to)}\ \text{(Def. of M)}$$

where (Def. of M) means to use the definition of action model M i.e., $\mathsf{M} \vdash_{\mathbf{GEAK}} \Rightarrow x{:}\mathsf{pre(x)} \to \mathsf{pre(b)} \parallel \mathsf{a}{\sim_a}\mathsf{x}$. Next, let us take a look a bit more complicated example. Using the action model Read in Example 2.3.1, and we show one of the exercises in [83, p.166].

Example 4.1.1. Let us recall action model Read in the setting of Example 2.3.2 where $\mathsf{Read} = (\mathsf{S}, \sim_a, \sim_b, \mathsf{pre}) = (\{\mathsf{p}, \mathsf{np}\}, \{(\mathsf{p}, \mathsf{p}), (\mathsf{np}, \mathsf{np})\}, \mathsf{S}^2, \mathsf{pre}\})$ with $\mathsf{pre(p)} = p$ and $\mathsf{pre(np)} = \neg p$. In the setting, we can show that $[\mathsf{p}^{\mathsf{Read}}]\Box_b(\Box_a p \vee \Box_a \neg p)$ is derivable under Read in **GEAK** as follows (intuitively, this formula means that after the agent a reads a letter containing p, the agent b knows that a knows whether p).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{z{:}p, w{:}p \Rightarrow z{:}p}}{z{:}\mathsf{pre(y)}, w{:}\mathsf{pre(z)}, w{:}p \Rightarrow z{:}p \parallel \mathsf{p}{\sim_a}\mathsf{y}, \mathsf{p}{\sim_a}\mathsf{z}}\ \text{(Def. of M)} \quad \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{z{:}\neg p, w{:}p \Rightarrow z{:}p, w{:}p}}{z{:}\neg p, w{:}\neg p, w{:}p \Rightarrow z{:}p}\ (L\neg)}{z{:}\mathsf{pre(y)}, w{:}\mathsf{pre(z)}, w{:}p \Rightarrow z{:}p \parallel \mathsf{np}{\sim_a}\mathsf{y}, \mathsf{np}{\sim_a}\mathsf{z}}\ \text{(Def. of M)}}{z{:}\mathsf{pre(y)}, w{:}\mathsf{pre(z)}, w{:}p \Rightarrow z{:}p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}, \mathsf{x}{\sim_a}\mathsf{z}}\ \text{(Def. of M)}}{z{:}\mathsf{pre(y)}, w{:}\mathsf{pre(z)}, \langle w, \mathsf{z}^{\mathsf{Read}}\rangle{:}p \Rightarrow z{:}p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}, \mathsf{x}{\sim_a}\mathsf{z}}\ (Lat)}{\langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle z, \mathsf{y}^{\mathsf{Read}}\rangle, \langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle w, \mathsf{z}^{\mathsf{Read}}\rangle, \langle w, \mathsf{z}^{\mathsf{Read}}\rangle{:}p \Rightarrow z{:}p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}, \mathsf{x}{\sim_a}\mathsf{z}}\ (Lrel3)}{\langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle z, \mathsf{y}^{\mathsf{Read}}\rangle, \langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle w, \mathsf{z}^{\mathsf{Read}}\rangle \Rightarrow z{:}p, \langle w, \mathsf{z}^{\mathsf{Read}}\rangle{:}\neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}, \mathsf{x}{\sim_a}\mathsf{z}}\ (R\neg)}{\langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle z, \mathsf{y}^{\mathsf{Read}}\rangle \Rightarrow z{:}p, \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a \neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}}\ (R\Box_a 2)}{\langle y, \mathsf{x}^{\mathsf{Read}}\rangle \mathsf{R}_a \langle z, \mathsf{y}^{\mathsf{Read}}\rangle \Rightarrow \langle z, \mathsf{y}^{\mathsf{Read}}\rangle{:}p, \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a \neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}, \mathsf{x}{\sim_a}\mathsf{y}}\ (Rat)}{\Rightarrow \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a p, \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a \neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}}\ (R\Box_a)}{\Rightarrow \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a p \vee \Box_a \neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}}\ (R\vee)}{x{:}\mathsf{pre(p)}, \langle x, \mathsf{p}^{\mathsf{Read}}\rangle \mathsf{R}_b \langle y, \mathsf{x}^{\mathsf{Read}}\rangle \Rightarrow \langle y, \mathsf{x}^{\mathsf{Read}}\rangle{:}\Box_a p \vee \Box_a \neg p \parallel \mathsf{p}{\sim_b}\mathsf{x}}\ (Lw)}{x{:}\mathsf{pre(p)} \Rightarrow \langle x, \mathsf{p}^{\mathsf{Read}}\rangle{:}\Box_b(\Box_a p \vee \Box_a \neg p)}\ (R\Box_b)}{\Rightarrow x{:}[\mathsf{p}^{\mathsf{Read}}]\Box_b(\Box_a p \vee \Box_a \neg p)}\ (R[.])$$

The derivation above is restricted to the action model Read, and so $\mathsf{Read} \vdash_{\mathbf{GEAK}} \Rightarrow x{:}[\mathsf{p}^{\mathsf{Read}}]\Box_b(\Box_a p \vee \Box_a \neg p)$.

4.2 Cut elimination of GEAK$^+$

In this section, we provide a proof of the cut elimination theorem of **GEAK**. For preparations for the proof of the theorem, we show the substitution lemma. The result of substitution $\mathfrak{A}[y/x]$ (y is substituted by x in $\mathfrak{A}$) is defined as follows:

Definition 4.2.1. Let u, v be any elements in Var.

$$
\begin{array}{lcl}
x[v/u] & := & x \ (\textit{if } u \neq x), \\
x[v/u] & := & v \ (\textit{if } u = x), \\
(\langle x, \alpha\rangle{:}A)[v/u] & := & \langle x[v/u], \alpha\rangle{:}A, \\
(\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle)[v/u] & := & \langle x[v/u], \alpha\rangle \mathsf{R}_a \langle y[v/u], \beta\rangle.
\end{array}
$$

For a multi-set Γ of labelled expressions, $\Gamma[y/x]$ denotes the set $\{\mathfrak{A}[y/x] \mid \mathfrak{A} \in \Gamma\}$. For a preparation of the cut-admissibility theorem, we show the following lemma.

Lemma 4.2.1 (Substitution lemma)**.** If $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\vdash_{\mathbf{GEAK}} \Gamma[v/u] \Rightarrow \Delta[v/u] \parallel \Sigma$ with the same derivation height, for any $u, v \in \mathsf{Var}$.

Proof. This proof is done in a similar manner to the proof in Negri and von Plato [58, p.194]. Suppose $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$. We show $\vdash_{\mathbf{GEAK}} \Gamma[v/u] \Rightarrow \Delta[v/u] \parallel \Sigma$ by induction on the height of its derivation. Fix any $u, v \in \mathsf{Var}$. The base case where $\vdash_{\mathbf{GEAK}} \Gamma[v/u] \Rightarrow \Delta[v/u] \parallel \Sigma$ is an initial c-sequent is trivial; and, we see one of cases in its induction step where the last applied rule is $(R\Box_a 2)$. In this case, we have a part of the derivation as follows:

$$
\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle \mathsf{R}_a \langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_n^{\mathsf{M}_n}\rangle, \Gamma \Rightarrow \Delta, \langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_n^{\mathsf{M}_n}\rangle{:}A \parallel \Sigma, \mathsf{a}_1 {\sim}_a^{\mathsf{M}_1} \mathsf{x}_1, ..., \mathsf{a}_n {\sim}_a^{\mathsf{M}_n} \mathsf{x}_n\end{array}}{\Gamma \Rightarrow \Delta, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle{:}\Box_a A \parallel \Sigma} \ (R\Box_a 2)
$$

If $u = v$, variable v should be replaced with variable o which does not appear in $\Gamma \Rightarrow \Delta, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle{:}\Box_a A \parallel \Sigma$ by Lemma 4.2.1. After that, we apply the induction hypothesis and $(R\Box_a 2)$ to the resulting c-sequent. Observe that $o[v/u] = o$. Then whate obtain what we desired at exactly the same height as that of the assumption. □

Theorem 4.2.1 (Cut elimination $\mathbf{GEAK}^+$)**.** For any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, if $\vdash_{\mathbf{GEAK}^+} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$. In particular, if $\mathsf{M}_1, ..., \mathsf{M}_k \vdash_{\mathbf{GEAK}^+} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\mathsf{M}_1, ..., \mathsf{M}_k \vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$.

Proof. The proof is carried out with Ono and Komori's method [67] of (*Ecut*). (*Ecut*) is given as follows:

$$
\dfrac{\Gamma \Rightarrow \Delta, \mathfrak{A}^m \parallel \Sigma \quad \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} \ (Ecut)
$$

where $x, y \geq 0$ and $\mathfrak{A}$ is called a *cut expression*. The theorem is proven by double induction on the height of the derivation and the length of the cut expression $\ell(\mathfrak{A})$ of (*Ecut*). The proof is divided into four cases:

(1) At least one of the upper c-sequents of (*Ecut*) is an initial c-sequent,

(2) The last inference rule of either upper c-sequent of (*Ecut*) is a structural rule,

(3) The last inference rule of either upper c-sequent of (*Ecut*) is a non-structural rule, and the principal expression introduced by the rule is not a cut expression, and

(4) The last inference rules of two upper c-sequents of (*Ecut*) are both non-structural rules, and the principal expressions introduced by the rules used on the upper c-sequents of (*Ecut*) are both cut expressions.

Case of (1) where one of upper c-sequents of (*Ecut*) is an initial c-sequent.
In this case, we obtain the following derivation:

$$\dfrac{\dfrac{\textit{Initial Seq.}}{\mathfrak{A} \Rightarrow \mathfrak{A} \parallel \Sigma} \quad \begin{matrix} \vdots\ \mathcal{D} \\ \mathfrak{A}^n, \Gamma \Rightarrow \Delta \parallel \Sigma \end{matrix}}{\mathfrak{A}, \Gamma \Rightarrow \Delta \parallel \Sigma}\ (Ecut)$$

This can be transformed into a derivation without (*Cut*) by using (*Rc*) multiple times to the lower c-sequent of $\mathcal{D}$.

Case of (2) where the right upper c-sequent of (*Ecut*) is structural rule (*Rc*) which contracts the same expression as the cut expressin. In this case, we obtain the following derivation:

$$\dfrac{\dfrac{\begin{matrix} \vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, \mathfrak{A}^{m+1} \parallel \Sigma \end{matrix}}{\Gamma \Rightarrow \Delta, \mathfrak{A}^m \parallel \Sigma}\ (Rc) \quad \begin{matrix} \vdots\ \mathcal{D}_2 \\ \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma \end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)$$

This is transformed into the following derivation.

$$\dfrac{\begin{matrix} \vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, \mathfrak{A}^{m+1} \parallel \Sigma \end{matrix} \quad \begin{matrix} \vdots\ \mathcal{D}_2 \\ \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma \end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule (*Rc*).

Case of (2) where the right upper c-sequent of (*Ecut*) is structural rule (*Lc*) which contracts a different expression from the cut expressin.

$$\dfrac{\begin{matrix} \vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, (\langle x, \alpha \rangle{:}A)^n \parallel \Sigma \end{matrix} \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}_2 \\ (\langle x, \alpha \rangle{:}A)^m, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta' \parallel \Sigma \end{matrix}}{(\langle x, \alpha \rangle{:}A)^m, \mathfrak{B}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\ (Lc)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{matrix} \vdots\ \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, (\langle x, \alpha \rangle{:}A)^n \parallel \Sigma \end{matrix} \quad \begin{matrix} \vdots\ \mathcal{D}_2 \\ (\langle x, \alpha \rangle{:}A)^m, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta' \parallel \Sigma \end{matrix}}{\Gamma, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Lc) \quad .$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule (*Lc*).

Case of (2) where one of upper c-sequents of (*Ecut*) is structural rule (*Rw*) which reduces the same formula as the cut formula.

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (\langle x,\alpha\rangle{:}A)^{n-1} \parallel \Sigma}}{\Gamma \Rightarrow \Delta, (\langle x,\alpha\rangle{:}A)^{n} \parallel \Sigma}\ (Rw) \qquad \overset{\vdots\ \mathcal{D}_2}{(\langle x,\alpha\rangle{:}A)^{m}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (\langle x,\alpha\rangle{:}A)^{n-1} \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{(\langle x,\alpha\rangle{:}A)^{m}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut).$$

Note that (*Ecut*) is applicable, even if $n - 1 = 0$. Similarly to this, we can show the case where the right upper c-sequent of (*Ecut*) is structural rule (*Rw*).

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule ($L\neg$).

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A, \mathfrak{A}^{n} \parallel \Sigma}}{\langle x,\alpha\rangle{:}\neg A, \Gamma \Rightarrow \Delta, \mathfrak{A}^{n} \parallel \Sigma}\ (L\neg) \qquad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^{m}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle x,\alpha\rangle{:}\neg A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A, \mathfrak{A}^{n} \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^{m}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x,\alpha\rangle{:}A \parallel \Sigma}\ (Ecut)}{\langle x,\alpha\rangle{:}\neg A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (L\neg)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule ($L\neg$).

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule ($R \rightarrow$).

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^{n} \parallel \Sigma} \qquad \dfrac{\overset{\vdots\ \mathcal{D}_2}{\langle x,\alpha\rangle{:}A, \mathfrak{A}^{m}, \Gamma' \Rightarrow \Delta', \langle x,\alpha\rangle{:}B \parallel \Sigma}}{\mathfrak{A}^{m}, \Gamma' \Rightarrow \Delta', \langle x,\alpha\rangle{:}A \rightarrow B \parallel \Sigma}\ (R \rightarrow)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x,\alpha\rangle{:}A \rightarrow B \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^{n} \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^{m}, \langle x,\alpha\rangle{:}A, \Gamma' \Rightarrow \Delta', \langle x,\alpha\rangle{:}B \parallel \Sigma}}{\langle x,\alpha\rangle{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x,\alpha\rangle{:}B \parallel \Sigma}\ (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x,\alpha\rangle{:}A \rightarrow B \parallel \Sigma}\ (R \rightarrow)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule ($R \rightarrow$).

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule ($L \rightarrow$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \langle x, \alpha\rangle{:}B, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\mathfrak{A}^m, \langle x, \alpha\rangle{:}A \rightarrow B, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(L \rightarrow)}{\langle x, \alpha\rangle{:}A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma}\,(Ecut) \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m, \langle x, \alpha\rangle{:}B, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\langle x, \alpha\rangle{:}B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)}{\langle x, \alpha\rangle{:}A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(L \rightarrow)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule ($L \rightarrow$).

Case of (3) where the last inference rule of left upper c-sequents of (*Ecut*) is ($R\Box_a 2$) which is not the cut expression. Let $\Sigma' = \{\mathsf{a}_1 \sim_a^{\mathsf{M}_1} \mathsf{x}_1, ..., \mathsf{a}_k \sim_a^{\mathsf{M}_k} \mathsf{x}_k\}$. In this case, we obtain the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle \mathsf{R}_a \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle, \Gamma \Rightarrow \Delta, \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle{:}A, \mathfrak{A}^m \parallel \Sigma \cup \Sigma'\end{array}}{\Gamma \Rightarrow \Delta, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A, \mathfrak{A}^m \parallel \Sigma}\,(R\Box_a 2) \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

Since each $\mathsf{x}_i \in \mathsf{CVar}$ does not appear in the lower c-sequents, it does not also appear in $\mathfrak{A}, \Gamma'$ and Δ'. Therefore, even if Σ' are added to Σ, its provability does not obviously change with the same height of the derivation, and we obtain $\vdash_{\mathbf{GEAK}} \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma \cup \Sigma'$. Then we may transform the derivation into the following:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle \mathsf{R}_a \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle, \Gamma \Rightarrow \Delta, \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle{:}A, \mathfrak{A}^m \parallel \Sigma \cup \Sigma'\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2'\\ \mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma \cup \Sigma'\end{array}}{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle \mathsf{R}_a \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle y, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_k^{\mathsf{M}_k}\rangle{:}A \parallel \Sigma \cup \Sigma'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(R\Box_a 2)$$

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule ($L\Box_a 2$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle \mathsf{R}_a \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k}\rangle \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k}\rangle{:}A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\mathfrak{A}^m, \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(L\Box_a 2)}{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k} \rangle \mathsf{R}_a \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k} \rangle \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k} \rangle \mathsf{R}_a \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k} \rangle \parallel \Sigma}\ (Ecut) \quad \dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_3}{\mathfrak{A}^m, \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k} \rangle{:}A, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_k^{\mathsf{M}_k} \rangle{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (Ecut)}{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k} \rangle{:}\Box_a A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\ (L\Box_a 2)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule ($L\Box_a 2$).

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule (*Rat*).

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \dfrac{\overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha \rangle{:}p \parallel \Sigma}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}p \parallel \Sigma}\ (Rat)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}p \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha \rangle{:}p \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha \rangle{:}p \parallel \Sigma}\ (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}p \parallel \Sigma}\ (Rat)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule (*Rat*), and the case of (*Lat*) is also similar to this.

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule (*R*[.]).

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \dfrac{\overset{\vdots\ \mathcal{D}_2}{\langle x, \alpha \rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}B \parallel \Sigma}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta', \langle x, \alpha \rangle{:}[\mathsf{a}^{\mathsf{M}}]B \parallel \Sigma}\ (R[.])}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha \rangle{:}[\mathsf{a}^{\mathsf{M}}]B \parallel \Sigma}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, \langle x, \alpha \rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma' \Rightarrow \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}B \parallel \Sigma}}{\langle x, \alpha \rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle{:}B \parallel \Sigma}\ (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha \rangle{:}[\mathsf{a}^{\mathsf{M}}]B \parallel \Sigma}\ (R[.])$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule (*R*[.]). The case of (*R*[.]) is also similar to this.

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule (*Lrel*$_a$1).

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{matrix}}{\mathfrak{A}^m, \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(Lrel_a 1)}{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{matrix}}{\langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)}{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Lrel_a 1)$$

Similarly to this, we can show the case where the left upper c-sequent of (*Ecut*) is structural rule (*Lrel*$_a$1). Moreover, the case of (*Lrel*$_a$2) and (*Lrel*$_a$3) can also be shown similarly.

Case of (4) where both sides of $\mathfrak{A}$ is $\langle x, \alpha\rangle{:}\neg A$ and principal, when we obtain the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta, (\langle x, \alpha\rangle{:}\neg A)^{n-1} \parallel \Sigma\end{matrix}}{\Gamma \Rightarrow \Delta, (\langle x, \alpha\rangle{:}\neg A)^n \parallel \Sigma}\,(R\neg) \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ (\langle x, \alpha\rangle{:}\neg A)^{m-1}, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma\end{matrix}}{(\langle x, \alpha\rangle{:}\neg A)^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(L\neg)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1^+\\ \Gamma \Rightarrow \Delta, (\langle x, \alpha\rangle{:}\neg A)^n \parallel \Sigma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ (\langle x, \alpha\rangle{:}\neg A)^{m-1}, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha\rangle{:}A \parallel \Sigma}\,(Ecut) \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta, (\langle x, \alpha\rangle{:}\neg A)^{n-1} \parallel \Sigma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2^+\\ (\langle x, \alpha\rangle{:}\neg A)^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{matrix}}{\langle x, \alpha\rangle{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta' \parallel \Sigma}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ in (*Ecut*) are $\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_k^{\mathsf{M}_k}\rangle{:}\Box_a A$ and principal expressions. Let us consider the case where $k = 1$ for simplicity, and $\mathfrak{A} = \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A$. In this case, we obtain the following derivation.

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle v, \mathsf{x}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta, \langle v, \mathsf{x}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{m-1} \parallel \Sigma, \mathsf{a}{\sim}_a^{\mathsf{M}}\mathsf{x}\end{matrix}}{\Gamma \Rightarrow \Delta, \mathfrak{A}^m \parallel \Sigma}\,(R\Box_a 2) \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^{n-1}, \Gamma' \Rightarrow \Delta', \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle \parallel \Sigma\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_3\\ \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{n-1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{matrix}}{\mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(L\Box_a 2)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

First, replace v with y in the left upper c-sequent by Lemma 4.2.1. Next, since we know that $\mathsf{a} \sim_a^{\mathsf{M}} \mathsf{b}$ by $\langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle$ in the middle upper c-sequent and the condition of an action relation, we have the following.

$$\vdash_{\mathbf{GEAK}} \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta, \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{m-1} \parallel \Sigma$$

Then the derivation above can be transformed into the following:

$$\mathcal{A} = \left\{ \dfrac{\dfrac{\overset{\vdots\, \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, \mathfrak{A}^y \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_2}{p\mathfrak{A}^{n-1}, \Gamma' \Rightarrow \Delta', \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle \parallel \Sigma} (Ecut)_2 \quad \dfrac{\overset{\vdots\, \mathcal{D}_1 \textit{ and Lemma } 4.2.1}{\langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma \Rightarrow \Delta, \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{m-1} \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_{23}^+}{\mathfrak{A}^n, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{b}^{\mathsf{M}}\rangle, \Gamma', \Gamma \Rightarrow \Delta, \Delta', \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} (Ecut)_1}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} (Ecut)_4 \right.$$

$$\cfrac{\cfrac{\overset{\vdots\, \mathcal{A}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} \quad \cfrac{\overset{\vdots\, \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, \mathfrak{A}^y \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_3}{\langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{n-1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle y, \mathsf{b}^{\mathsf{M}}\rangle{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} (Ecut)_3}{\Gamma, \Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta', \Delta' \parallel \Sigma} (Ecut)_5}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} (Lc)/(Rc)$$

where $(Ecut)_{1,2,3}$ are applicable by induction hypothesis, since the derivation height of $(Ecut)$ is reduced by comparison with the original derivation. Besides, the application of $(Ecut)_{4,5}$ is also allowed by induction hypothesis, where $\ell(\mathfrak{A})$ is reduced as follows: $\ell(\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A) > \ell(\langle x, \mathsf{b}^{\mathsf{M}}\rangle{:}A)$ and $\ell(\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A) > \ell(\langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle x, \mathsf{b}^{\mathsf{M}}\rangle)$.

Case of (4) where both sides of $\mathfrak{A}$ in $(Ecut)$ are $\langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]A$ and principal expressions. Let $\mathfrak{A}$ be $\langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]A$.

$$\dfrac{\dfrac{\overset{\vdots\, \mathcal{D}_1}{\langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta, \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{n-1} \parallel \Sigma}}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} (R[.]) \quad \dfrac{\overset{\vdots\, \mathcal{D}_2}{\mathfrak{A}^{m-1}, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_3}{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{m-1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma} (L[.])}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} (Ecut)$$

This is transformed into the derivation:

$$\mathcal{A} = \left\{ \dfrac{\dfrac{\overset{\vdots\, \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_2}{\mathfrak{A}^{m-1}, \Gamma' \Rightarrow \Delta', \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma} (Ecut) \quad \dfrac{\overset{\vdots\, \mathcal{D}_1}{\langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma \Rightarrow \Delta, \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{n-1} \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_{2,3}^+}{\mathfrak{A}^{n-1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} (Ecut)}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} (Ecut) \right.$$

$$\cfrac{\cfrac{\overset{\vdots\, \mathcal{A}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A \parallel \Sigma} \quad \cfrac{\overset{\vdots\, \mathcal{D}_1^+}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_3}{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A, \mathfrak{A}^{m-1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}A, , \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} (Ecut)}{\Gamma, \Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta', \Delta' \parallel \Sigma} (Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma} (Lc)/(Rc)$$

Case of (4) where both sides of $\mathfrak{A}$ are $\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle$ and principal. When we obtain the following derivation:

$$\mathcal{A} = \left\{ \dfrac{\overset{\vdots\, \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^{n\text{-}1}, \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_2}{\Gamma \Rightarrow \Delta, \mathfrak{A}^{n\text{-}1}, \langle y, \beta\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{b}) \parallel \Sigma} \quad \overset{\vdots\, \mathcal{D}_3}{\Gamma \Rightarrow \Delta, \mathfrak{A}^{n\text{-}1}, \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle \parallel \Sigma}}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} (Rrel_a) \right.$$

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{A}\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_4\\ \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \mathfrak{A}^{m\text{-}1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(Lrel_a3)}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta' \parallel \Sigma}\,(Ecut) \quad .$$

It is transformed into the following derivation:

$$\mathcal{A} = \left\{ \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^{n\text{-}1}, \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_4^{+}\\ \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma}\,(Ecut) \right.$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{A}\\ \Gamma,\Gamma' \Rightarrow \Delta,\Delta', \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \parallel \Sigma\end{array} \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_{123}^{+}\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_4\\ \langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \mathfrak{A}^{m\text{-}1}, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \Gamma,\Gamma' \Rightarrow \Delta,\Delta' \parallel \Sigma}\,(Ecut)}{\Gamma,\Gamma,\Gamma',\Gamma' \Rightarrow \Delta,\Delta,\Delta',\Delta' \parallel \Sigma}\,(Ecut)}{\overline{\Gamma,\Gamma' \Rightarrow \Delta,\Delta' \parallel \Sigma}}\,(Rc)/(Lc) \quad ,$$

where $(Ecut)$ to the two upper c-sequents is applicable by induction hypothesis, since the derivation height of $(Ecut)$ is reduced by comparison with the original derivation. Additionally, the application of $(Ecut)$ to the lower c-sequents is also allowed by induction hypothesis, since the length of the cut expression is reduced, namely $\ell(\langle x,\alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})) < \ell(\langle x,\alpha,\mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y,\beta,\mathsf{b}^{\mathsf{M}}\rangle)$. □

4.3 All derivable formulas in HEAK are derivable in GEAK$^+$

Our task in this sections is to establish that the derivability of our sequent system **GEAK** is equal to that of **HEAK**. In other words, we show the theorem that every derivable formula in **HEAK** is also derivable in **GEAK**. To show this requires the following trivial derivable rules in **GEAK** (for the case of (RA4)) and one lemma (for (RA5)).

$$\dfrac{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}A(\mathsf{v}) \parallel \Sigma, \mathsf{a}{\sim}_a^{\mathsf{M}}\mathsf{v}}{\Gamma \Rightarrow \Delta, \langle x,\alpha\rangle{:}\bigwedge_{\mathsf{a}\sim_a^{\mathsf{M}}\mathsf{x}} A \parallel \Sigma}\,(R{\wedge}{\wedge})\dagger \qquad \dfrac{\langle x,\alpha\rangle{:}A(\mathsf{b}), \Gamma \Rightarrow \Delta \parallel \Sigma}{\langle x,\alpha\rangle{:}\bigwedge_{\mathsf{a}\sim_a^{\mathsf{M}}\mathsf{x}} A, \Gamma \Rightarrow \Delta \parallel \Sigma}\,(L{\wedge}{\wedge})\ddagger$$

† $\mathsf{v} \in \mathsf{CVar}$ does not appear in the lower c-sequent.

‡ b is in $\{\mathsf{x} \mid \mathsf{a}{\sim}_a^{\mathsf{M}}\mathsf{x}\}$.

where $A(\mathsf{b})$ (or $A(\mathsf{y})$) means that b (or y) possibly appears in formula A. These are only generalizations of $(L\wedge)$ and $(R\wedge)$.

Lemma 4.3.1. For any finite lists α,β of actions, any formula A, any finite set Σ of relational atoms, the following hold:

(i) $\vdash_{\mathbf{GEAK}} \langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\beta\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\beta'\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\beta\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{b}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}},\beta\rangle \parallel \Sigma$

(ii) $\vdash_{\mathbf{GEAK}} \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\beta\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{b}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}},\beta\rangle \Rightarrow \langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\beta\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\beta'\rangle \parallel \Sigma$

(iii) $\vdash_{\mathbf{GEAK}} \langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\beta\rangle{:}A \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\beta\rangle{:}A \parallel \Sigma$

(iv) $\vdash_{\mathbf{GEAK}} \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\beta\rangle{:}A \Rightarrow \langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\beta\rangle{:}A \parallel \Sigma$

Proof. The proofs of (i), (ii), (iii) and (iv) are simultaneously conducted by double induction on $\ell(\mathfrak{A})$ and the length of β $(= \beta')$. We only look at the proof of (i) (other cases can be shown similarly).

Base case: (i) where $\beta = \epsilon$.
We show the following, and it is straightforward to construct a derivation with $(Rrel)/(Lrel)$ and $(R\wedge)/(L\wedge)$. Note that $(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}}$ is included in PAct by Definition 2.3.2.

$$\vdash_{\mathbf{GEAK}} \langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}}\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}}\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}}\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{b}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}}\rangle \parallel \Sigma$$

Induction step of (i) where $\beta = (\gamma,\mathsf{c}^{\mathsf{O}})$ and $\beta' = (\gamma',\mathsf{c}'^{\mathsf{O}})$

$$\dfrac{\dfrac{\dfrac{\textit{Induction hypothesis of }(\mathrm{i})}{\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\gamma'\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\gamma\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{u}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}},\gamma'\rangle \parallel \Sigma}}{\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma,\mathsf{c}^{\mathsf{O}}\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\gamma',\mathsf{c}'^{\mathsf{O}}\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\gamma\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{b}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}},\gamma'\rangle \parallel \Sigma}\,(Lrel1) \quad \vdots\,\mathcal{D}_1 \quad \vdots\,\mathcal{D}_2}{\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma,\mathsf{c}^{\mathsf{O}}\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\gamma',\mathsf{c}'^{\mathsf{O}}\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\gamma,\mathsf{c}^{\mathsf{O}}\rangle\mathsf{R}_a\langle y,\alpha',\mathsf{b}^{\mathsf{M}},\mathsf{b}'^{\mathsf{N}},\gamma',\mathsf{c}'^{\mathsf{O}}\rangle \parallel \Sigma}\,(Rrel)$$

$\mathcal{D}_1$ (and similarly $\mathcal{D}_2$) is immediately given by $(Lrel2)$ and the induction hypothesis of (iii) as follows:

$$\mathcal{D}_1 = \left\{ \dfrac{\dfrac{\textit{Induction hypothesis of }(\mathrm{iii})}{\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma\rangle{:}\mathsf{pre}^{\mathsf{O}}(\mathsf{c}) \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\gamma\rangle{:}\mathsf{pre}^{\mathsf{O}}(\mathsf{c}) \parallel \Sigma}}{\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma,\mathsf{c}^{\mathsf{O}}\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\gamma',\mathsf{c}'^{\mathsf{O}}\rangle \Rightarrow \langle x,\alpha,\mathsf{a}^{\mathsf{M}},\mathsf{a}'^{\mathsf{N}},\gamma\rangle{:}\mathsf{pre}^{\mathsf{O}}(\mathsf{c}) \parallel \Sigma}\,(Lrel2) \right.$$

where note that $\ell(\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma,\mathsf{c}^{\mathsf{O}}\rangle\mathsf{R}_a\langle y,\alpha',(\mathsf{b},\mathsf{b}')^{\mathsf{M;N}},\gamma',\mathsf{c}'^{\mathsf{O}}\rangle) > \ell(\langle x,\alpha,(\mathsf{a},\mathsf{a}')^{\mathsf{M;N}},\gamma\rangle{:}\mathsf{pre}^{\mathsf{O}}(\mathsf{c}))$.

□

Theorem 4.3.1. For any formula A, if $\vdash_{\mathbf{HEAK}} A$, then $\vdash_{\mathbf{GEAK}^+} \Rightarrow \langle x,\epsilon\rangle{:}A$ for any $x \in \mathsf{Var}$.

Proof. Suppose $\vdash_{\mathbf{HEAK}} A$, and fix any $x \in \mathsf{Var}$. The proof is conducted by induction on the height of derivation of **HEAK**. We pick up some significant base cases (the derivation height of **HEAK** is equal to 0).

(RA1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}p, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}p, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}p}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x,\mathsf{a}^{\mathsf{M}}\rangle{:}p \Rightarrow x{:}p}\,(Lat)}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}]p \Rightarrow x{:}p}\,(L[.])}{x{:}[\mathsf{a}^{\mathsf{M}}]p \Rightarrow x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p}\,(R\to)}{\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]p \to (\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p)}\,(R\to) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}p, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})} \quad \dfrac{\textit{Initial Seq.}}{x{:}p, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}p}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p \Rightarrow x{:}p}\,(L\to)}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p \Rightarrow \langle x,\mathsf{a}^{\mathsf{M}}\rangle{:}p}\,(Rat)}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]p}\,(R[.])}{\Rightarrow x{:}(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p) \to [\mathsf{a}^{\mathsf{M}}]p}\,(R\to)}{\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]p \leftrightarrow (\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to p)}\,(R\wedge)$$

(RA2: Left to Right)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}) \Rightarrow x{:}\neg[\mathsf{a}^{\mathsf{M}}]A, x{:}\mathsf{pre}(\mathsf{a})$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A \Rightarrow x{:}\mathsf{pre}(\mathsf{a})$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A$}
\RightLabel{$(L\neg)$}
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A, \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A \Rightarrow$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), x{:}[(\mathsf{M}, \mathsf{s})]A, \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A \Rightarrow$}
\RightLabel{$(R\neg)$}
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A \Rightarrow x{:}\neg[\mathsf{a}^{\mathsf{M}}]A$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), [\mathsf{a}^{\mathsf{M}}]\neg A \Rightarrow x{:}\neg[\mathsf{a}^{\mathsf{M}}]A$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]\neg A \Rightarrow x{:}\mathsf{pre}(\mathsf{a}) \rightarrow \neg[\mathsf{a}^{\mathsf{M}}]A$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]\neg A \rightarrow (\mathsf{pre}(\mathsf{a}) \rightarrow \neg[\mathsf{a}^{\mathsf{M}}]A)$}
\end{prooftree}
$$

(RA2: Right to Left)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A$}
\RightLabel{$(R[.])$}
\UnaryInfC{$\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]A,$}
\RightLabel{$(L\neg)/(R\neg)$}
\UnaryInfC{$x{:}\neg[\mathsf{a}^{\mathsf{M}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}\neg[\mathsf{a}^{\mathsf{M}}]A, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A$}
\RightLabel{$(L \rightarrow)$}
\BinaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \rightarrow \neg[\mathsf{a}^{\mathsf{M}}]A, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\neg A$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \rightarrow \neg[\mathsf{a}^{\mathsf{M}}]A \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]\neg A$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \rightarrow \neg[\mathsf{a}^{\mathsf{M}}]A) \rightarrow [\mathsf{a}^{\mathsf{M}}]\neg A$}
\end{prooftree}
$$

(RA3: Left to Right)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B, \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A$}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B, x{:}[\mathsf{a}^{\mathsf{M}}]A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B$}
\RightLabel{$(L \rightarrow)$}
\BinaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]A \rightarrow [\mathsf{a}^{\mathsf{M}}]B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}B$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]A \rightarrow [\mathsf{a}^{\mathsf{M}}]B, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}A \rightarrow B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]A \rightarrow [\mathsf{a}^{\mathsf{M}}]B \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}](A \rightarrow B)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}([\mathsf{a}^{\mathsf{M}}]A \rightarrow [\mathsf{a}^{\mathsf{M}}]B) \rightarrow [\mathsf{a}^{\mathsf{M}}](A \rightarrow B)$}
\end{prooftree}
$$

(RA4: Left to Right)

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}) \Rightarrow x{:} \bigwedge_{\mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A, x{:}\mathsf{pre}(\mathsf{a})$}
\AxiomC{$\vdots\ \mathcal{D}$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$\langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A, \Gamma \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{y}$}
\RightLabel{$(L\Box_a 2)$}
\BinaryInfC{$\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A, \Gamma \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{y}$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A, x\mathsf{R}_a y \Rightarrow y{:}[\mathsf{y}^{\mathsf{M}}]A \parallel \mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{y}$}
\RightLabel{$(R\Box_a 1)$}
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A \Rightarrow x{:}\Box_a[\mathsf{y}^{\mathsf{M}}]A \parallel \mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{y}$}
\RightLabel{$(R\wedge\wedge)$}
\UnaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A \Rightarrow x{:} \bigwedge_{\mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}\mathsf{pre}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}]\Box_a A \Rightarrow x{:} \bigwedge_{\mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$x{:}[\mathsf{a}^{\mathsf{M}}]\Box_a A \Rightarrow x{:}\mathsf{pre}(\mathsf{a}) \rightarrow \bigwedge_{\mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]\Box_a A \rightarrow (\mathsf{pre}(\mathsf{a}) \rightarrow \bigwedge_{\mathsf{a} \sim^{\mathsf{M}}_{a} \mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A)$}
\end{prooftree}
$$

where $\Gamma = \{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), x\mathsf{R}_a y\}$ and $\mathcal{D}$ is the following derivation:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{\Gamma \Rightarrow x\mathsf{R}_a y \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}} \quad \dfrac{\textit{Initial Seq.}}{\Gamma \Rightarrow x{:}\mathsf{pre}(\mathsf{a}) \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}} \quad \dfrac{\textit{Initial Seq.}}{\Gamma \Rightarrow y{:}\mathsf{pre}(\mathsf{y}) \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}}{\Gamma \Rightarrow \langle x, \mathsf{a}\rangle \mathsf{R}_a \langle y, \mathsf{y}\rangle \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(Rrel)}{\Gamma \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A, \langle x, \mathsf{a}\rangle \mathsf{R}_a \langle y, \mathsf{y}\rangle \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(Rw) \right. .$$

(RA4: Right to Left)

$$\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), x\mathsf{R}_a y \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A, x\mathsf{R}_a y \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}} \quad \vdots\,\mathcal{D}}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), x\mathsf{R}_a y, x{:}\Box_a[\mathsf{y}^{\mathsf{M}}]A \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(L\Box_a 1)}{x{:}\Box_a[\mathsf{y}^{\mathsf{M}}]A, \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{y}^{\mathsf{M}}\rangle \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(Lc)/(Lrel1)/(Lrel3)}{x{:}\bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A, \langle x, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \mathsf{y}^{\mathsf{M}}\rangle \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(L\wedge\wedge)}{x{:}\bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A}\,(R\Box_a 2)}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A}\,(Lw)}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\Box_a A}\,(L\to)}{\dfrac{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]\Box_a A}{\Rightarrow x{:}(\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \to \bigwedge_{\mathsf{a}\sim^{\mathsf{M}}_a\mathsf{x}} \Box_a[\mathsf{x}^{\mathsf{M}}]A) \to [\mathsf{a}^{\mathsf{M}}]\Box_a A}\,(R\to)}\,(R[.])$$

where $\mathcal{D}$ is the following derivation:

$$\mathcal{D} = \left\{ \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}) \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A, y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}) \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}} \quad \dfrac{\textit{Initial Seq.}}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), y{:}[\mathsf{y}^{\mathsf{M}}]A \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(L[.])}{y{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{y}), x\mathsf{R}_a y, y{:}[\mathsf{y}^{\mathsf{M}}]A \Rightarrow \langle y, \mathsf{y}^{\mathsf{M}}\rangle{:}A \parallel \mathsf{a}\sim^{\mathsf{M}}_a\mathsf{y}}\,(Lw) \right. .$$

(RA5: Left to Right)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}]\mathsf{pre}^{\mathsf{N}}(\mathsf{b}) \Rightarrow x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}}{x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}\,(L\wedge)}{x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}\,(Lw) \quad \begin{matrix}\vdots\,\mathcal{D}\\ \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}[\mathsf{b}^{\mathsf{N}}]A, x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A\end{matrix}}{x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})), x{:}[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A}\,(L[.])}{\dfrac{x{:}[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A}{\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A \to [\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A}\,(R\to)}\,(R[.])$$

A derivation $\mathcal{D}$ is given as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})} \quad \dfrac{\textit{Initial Seq.}}{\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b})}}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}]\mathsf{pre}^{\mathsf{N}}(\mathsf{b}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b})}\,(L\to)}{x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b})}\,(L\wedge)}{x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A, \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b})}\,(Lw) \quad \dfrac{\dfrac{\textit{Lemma } 4.3.1}{\langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A}}{\langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A, x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A}\,(Lw)}{\langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}[\mathsf{b}^{\mathsf{N}}]A, x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a}, \mathsf{b})) \Rightarrow \langle x, (\mathsf{a}, \mathsf{b})^{\mathsf{M;N}}\rangle{:}A}\,(L[.])$$

(RA5: Right to Left)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \mathfrak{A}}}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}]\mathsf{pre}^{\mathsf{N}}(\mathsf{b})}\,(R[.])}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a},\mathsf{b}))}\,(R\wedge)}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A, x{:}\mathsf{pre}^{\mathsf{M;N}}((\mathsf{a},\mathsf{a}'))}\,(Lw) \quad \dfrac{\dfrac{\textit{Lemma } 4.3.1}{\langle x, (\mathsf{a},\mathsf{b})^{\mathsf{M;N}}\rangle{:}A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A}}{\langle x, (\mathsf{a},\mathsf{b})^{\mathsf{M;N}}\rangle{:}A, \mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A}\,(Lw)}{\mathfrak{A}, x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}, \mathsf{b}^{\mathsf{N}}\rangle{:}A}\,(L[.])}{x{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a}), x{:}[\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A \Rightarrow \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}[\mathsf{b}^{\mathsf{N}}]A}\,(R[.])}{x{:}[\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A \Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A}\,(R[.])}{\Rightarrow x{:}[\mathsf{a}^{\mathsf{M}}; \mathsf{b}^{\mathsf{N}}]A \rightarrow [\mathsf{a}^{\mathsf{M}}][\mathsf{b}^{\mathsf{N}}]A}\,(R \rightarrow)
$$

where $\mathfrak{A} = \langle x, \mathsf{a}^{\mathsf{M}}\rangle{:}\mathsf{pre}^{\mathsf{N}}(\mathsf{b})$.

In induction step, we show the admissibility of the inference rules **HEAK**, such as (MP) and $(Nec\Box_a)$, and their proofs are similar to proofs in Theorem 3.2.1 (as we have seen in it, to show the admissibility of (MP) requires (Cut)). $\square$

4.4 Soundness of GEAK

Let us move on to a proof of the soundness theorem of **GEAK**. For the soundness theorem, we expand the definition of the satisfaction relation to the labelled expression and the c-sequent. Hereinafter we denote $(w, (\mathsf{a}_1, \mathsf{a}_2, ..., \mathsf{a}_n))$ for $(\cdots((w, \mathsf{a}_1), \mathsf{a}_2), ..., \mathsf{a}_n)$.

Definition 4.4.1. Let $\mathfrak{M}$ be a model and f be an assignment function $f : \mathsf{Var} \rightarrow \mathcal{D}(\mathfrak{M})$, α be any finite list of actions. $\mathfrak{M}, f \Vdash \mathfrak{A}$ is defined as follows:

$$
\begin{array}{lll}
\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A & \textit{iff} & \mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash A, \\
 & & \quad \textit{and } (f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}}), \\
\mathfrak{M}, f \Vdash \langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle & \textit{iff} & (f(x), f(y)) \in R_a, \\
\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle & \textit{iff} & \mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle \\
 & & \quad \textit{and } \mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \\
 & & \quad \textit{and } \mathfrak{M}^{\otimes\beta_{mdl}}, (f(y), \alpha'_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{b}).
\end{array}
$$

In Section 3.3 of the previous chapter, we gave light on the notion of *survival* of a world in the definition of satisfaction of the labelled expressions in PAL. The notion should be also considered in EAK, otherwise the soundness does not hold like in the case of PAL shown in Section 3.3. Specifically, note that at the satisfaction of the labelled formula $\langle x, \alpha\rangle{:}A$, not only the labelled formula is true by the valuation, but also a corresponding world $(f(x), \alpha_{evt})$ must exist or survive in the updated domain $\mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}})$. Otherwise $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash A$ is ill-defined. Following the idea of Section 3.3, it is sufficient to pay attention to the negated form of the labelled expression $\overline{\mathfrak{A}}$ taking into the condition of survival of a world which must also survive in the updated domain. With the notion of survival, $\mathfrak{M}, f \Vdash \overline{\mathfrak{A}}$ is defined as follows:

Definition 4.4.2. Let f be an assignment function $f : \mathsf{Var} \rightarrow \mathcal{D}(\mathfrak{M})$ (for any $\mathfrak{M}$), α be any finite list of actions. $\mathfrak{M}, f \Vdash \overline{\mathfrak{A}}$ is defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, f \Vdash \overline{\langle x, \alpha \rangle{:}A} & \textit{iff} & \mathfrak{M}^{\otimes \alpha_{mdl}}, (f(x), \alpha_{evt}) \nVdash A \\
& & \quad \textit{and } (f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}}), \\
\mathfrak{M}, f \Vdash \overline{\langle x, \epsilon \rangle \mathsf{R}_a \langle y, \epsilon \rangle} & \textit{iff} & (f(x), f(y)) \notin R_a, \\
\mathfrak{M}, f \Vdash \overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}} \rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}} \rangle} & \textit{iff} & \mathfrak{M}, f \Vdash \overline{\langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle} \\
& & \quad \textit{or } \mathfrak{M}^{\otimes \alpha_{mdl}}, (f(x), \alpha_{evt}) \nVdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a}) \\
& & \quad \textit{or } \mathfrak{M}^{\otimes \beta_{mdl}}, (f(y), \beta_{evt}) \nVdash \mathsf{pre}^{\mathsf{M}}(\mathsf{b}).
\end{array}$$

Additionally, it should be clarified that these semantic definitions for relational atoms are connected with an accessibility relation.

Proposition 4.4.1. The following equivalent relations hold.

(1) $\mathfrak{M}, f \Vdash \langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle$ *iff* $((f(x), \alpha_{act}), (f(y), \beta_{act})) \in R_a^{\otimes \alpha_{mdl}}$

(2) $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle}$ *iff* $((f(x), \alpha_{act}), (f(y), \beta_{act})) \notin R_a^{\otimes \alpha_{mdl}}$

Proof. Both can be straightforwardly shown by induction on the length of α $(= \beta)$. We show the contraposition of (2) i.e., $\mathfrak{M}, f \nVdash \overline{\langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle}$ *iff* $((f(x), \alpha_{act}), (f(y), \beta_{act})) \in R_a^{\otimes \alpha_{mdl}}$. The base where $\alpha = \beta = \epsilon$ is trivial by Definition 4.4.1. Next, we look at the case where $\alpha = (\alpha', \mathsf{a}^{\mathsf{M}})$ and $\beta = (\beta', \mathsf{b}^{\mathsf{M}})$. $\mathfrak{M}, f \nVdash \overline{\langle x, \alpha', \mathsf{a}^{\mathsf{M}} \rangle \mathsf{R}_a \langle y, \beta', \mathsf{b}^{\mathsf{M}} \rangle}$. By Definition 4.4.1, that is equivalent to $\mathfrak{M}, f \nVdash \overline{\langle x, \alpha' \rangle \mathsf{R}_a \langle y, \beta' \rangle}$ and $\mathfrak{M}, f \Vdash \langle x, \alpha' \rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ and $\mathfrak{M}, f \Vdash \langle y, \beta' \rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{b})$. By the induction hypothesis and the semantic reasoning, this is also equivalent to $((f(x), \alpha'_{act}, \mathsf{a}), (f(y), \beta'_{act}, \mathsf{b})) \in R_a^{\otimes \alpha'_{mdl} \otimes \mathsf{M}}$. □

Following the definition of the validity for a sequent (Definition 3.3.3) in Chapter 3, the validity of c-sequents is defined as follows.

Definition 4.4.3 (Validity of a c-sequent)**.** We say that sequent $\Gamma \Rightarrow \Delta$ is t-valid in $\mathfrak{M}$ if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Furthermore, c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ is t-valid if every sequent in $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$ is valid.

Proposition 4.4.2. For any model $\mathfrak{M}$, assignment f, $a \in \mathsf{Agt}$ and $x, y \in \mathsf{Var}$,

(i) $\mathfrak{M}, f \Vdash \langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle$ iff $((f(x), \alpha_{evt}), (f(y), \beta_{evt})) \in R_a^{\otimes \alpha_{mdl}}$,

(ii) $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha \rangle \mathsf{R}_a \langle y, \beta \rangle}$ iff $((f(x), \alpha_{evt}), (f(y), \beta_{evt})) \notin R_a^{\otimes \alpha_{mdl}}$.

Proof. Both are easily shown by induction on the number of α $(= \beta)$. Let us consider the case of $\alpha = (\alpha', \mathsf{a}^{\mathsf{M}})$ (and $\beta = (\beta', \mathsf{b}^{\mathsf{M}})$) in the proof of (ii).

We show $\mathfrak{M}, f \nVdash \overline{\langle x, \alpha', \mathsf{a}^{\mathsf{M}} \rangle \mathsf{R}_a \langle y, \beta', \mathsf{b}^{\mathsf{M}} \rangle}$ iff $((f(x), \alpha_{evt}, \mathsf{a}), (f(y), \beta'_{evt}, \mathsf{b})) \in R_a^{\alpha_{mdl} \otimes \mathsf{M}}$. $\mathfrak{M}, f \nVdash \overline{\langle x, \alpha', \mathsf{a}^{\mathsf{M}} \rangle \mathsf{R}_a \langle y, \beta', \mathsf{b}^{\mathsf{M}} \rangle}$ is, by Definition 4.4.3 and the induction hypothesis, equivalent to $((f(x), \alpha'_{evt}), (f(y), \beta'_{evt})) \in R_a^{\alpha'_{mdl}}$ and $\mathfrak{M}^{\otimes \alpha_{mdl}}, (f(x), \alpha'_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ and $\mathfrak{M}^{\otimes \alpha'_{mdl}}, (f(y), \beta'_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{b})$. That is also equivalent to $((f(x), \alpha_{evt}, \mathsf{a}), (f(y), \beta'_{evt}, \mathsf{b})) \in R_a^{\alpha_{mdl} \otimes \mathsf{M}}$. □

Hereinafter, we use the next notation. Let Γ is a finite set of labelled expressions. Then in what follows, we write $\mathfrak{M}, f \Vdash \Gamma$ to mean $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Gamma}$ to mean $\mathfrak{M}, f \Vdash \overline{\mathfrak{A}}$ for all $\mathfrak{A} \in \Gamma$.

Theorem 4.4.1 (Soundness of **GEAK**)**.** For any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, if $\vdash_{\mathbf{GEAK}} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\Gamma \Rightarrow \Delta \parallel \Sigma$ is t-valid in every model $\mathfrak{M}$.

Proof. We show each inference rule preserve t-validity.

Base case: Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show that $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle \Rightarrow \mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$ is t-valid. Suppose for contradiction that $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle}$. By Proposition 4.4.2, this is impossible.

The case where the last applied rule is of the form $(L\neg)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}\neg A$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}A}$. Then, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}\neg A$ iff $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \Vdash \neg A$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}})$. By Definition 4.4.3, we obtain $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}A}$.

The case where the last applied rule is of the form $(R\neg)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$, and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}\neg A}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A$. Then, $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}\neg A}$ iff $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \nVdash \neg A$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}})$, which is equivalent to: $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \Vdash A$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}})$. By Definition 4.4.1, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A$. So, the contraposition has been shown.

The case where the last applied rule is of the form $(L \to)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A \to B$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}A}$ or $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}B$. Then, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A \to B$ iff $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \Vdash \neg A$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}}))$ or $(\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \Vdash B$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}}))$. By Definition 4.4.1, we obtain the goal as desired.

The case where the last applied rule is of the form $(R \to)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}A \to B}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}A$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}B}$. Then, $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}A \to B}$ iff $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \Vdash A$ and $\mathfrak{M}^{\otimes \alpha_{mdl}}, f(x) \nVdash B$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes \alpha_{mdl}})$. By Definitions 4.4.1 and 4.4.2, we obtain the goal as desired.

The case where the last applied rule is of the form $(L\Box_a 1)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ For all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \langle x, \epsilon\rangle{:}\Box_a A$ and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle}$ or $\mathfrak{M}, f \Vdash \langle y, \epsilon\rangle{:}A$. Then, from $\mathfrak{M}, f \Vdash \langle x, \epsilon\rangle{:}\Box_a A$, we obtain $((f(x), \epsilon), (f(y), \epsilon)) \notin R_a$ or $\mathfrak{M}, f(y) \Vdash A$. Suppose the former disjunct which is, by Proposition 4.4.2, $\mathfrak{M}, f \Vdash \overline{\langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle}$. Then, suppose the latter disjunct $\mathfrak{M}, f(y) \Vdash A$. By definition, this is equivalent to $\mathfrak{M}, f \Vdash \langle y, \epsilon\rangle{:}A$. Then, the contraposition has been shown.

The case where the last applied rule is of the form $(R\Box_a 1)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \epsilon\rangle{:}\Box_a A}$. Fix such f. Then, $\mathfrak{M}, f \Vdash \overline{\langle x, \epsilon\rangle{:}\Box_a A}$ iff $(f(x), \epsilon) R_a (v, \epsilon)$ and $\mathfrak{M}, v \nVdash A$ for some $v \in \mathcal{D}(\mathfrak{M})$ and $f(x) \in \mathcal{D}(\mathfrak{M})$. Fix such $v \in \mathcal{D}(\mathfrak{M})$. It suffices to show that there is some $f' : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f' \Vdash \langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle$ and $\mathfrak{M}, f' \Vdash \overline{\langle y, \epsilon\rangle{:}A}$ where y is not x and does not appear in Γ and Δ. Define f' such that $f'(x) = v$ if $x = y$ and otherwise $f'(x) = f(x)$. Therefore, by the definition of f', we obtain $(f'(x), \epsilon) R_a (f'(y), \epsilon)$ and $\mathfrak{M}, f'(y) \nVdash A$ and $f'(x) \in \mathcal{D}(\mathfrak{M})$. By Definitions 4.4.1 and 4.4.2, we obtain the goal as desired.

Case where the last rule is $(L\Box_a 2)$**.** We show the contraposition such that if the lower sequent of the rule $(L\Box_a 2)$ is not t-valid, then some upper sequents of it are not t-valid. Suppose that the lower sequent of $(L\Box_a 2)$ is not t-valid, and by Definition 4.4.3, there is some $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \Vdash \Gamma'$ and $\mathfrak{M}, f \Vdash \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle{:}\Box_a A$ and $\mathfrak{M}, f \Vdash \overline{\Delta'}$. Fix such f. Then it suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle \mathsf{R}_a \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_n^{\mathsf{M}_n}\rangle}$ or $\mathfrak{M}, f \Vdash \langle y, \mathsf{b}_1^{\mathsf{M}_1}, ..., \mathsf{b}_n^{\mathsf{M}_n}\rangle{:}A$ for some $y \in \mathsf{Var}$ and some $\mathsf{b}_1, ..., \mathsf{b}_n$ such that $\mathsf{a}_1 \sim_a^{\mathsf{M}_1} \mathsf{b}_1, ..., \mathsf{a}_n \sim_a^{\mathsf{M}_n} \mathsf{b}_n$. From the supposition, i.e., $\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}, (f(x), \mathsf{a}_1, ..., \mathsf{a}_n) \Vdash \Box_a A$ and $(f(x), \mathsf{a}_1, ..., \mathsf{a}_n) \in \mathcal{D}(\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n})$, we obtain for all $v \in \mathcal{D}(\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n})$, $((f(x), \mathsf{a}_1, ..., \mathsf{a}_n), v) \notin R_a^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}$ or $\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}, v \Vdash A$. Take v as $(f(y), \mathsf{b}_1, ..., \mathsf{b}_n)$ where each b_i satisfies $\mathsf{a}_i \sim_a \mathsf{b}_i$. Then by Proposition 4.4.1-(2) and Definition 4.4.1, we obtain what we desired.

Case where the last rule is $(R\Box_a 2)$**.** We show the contraposition such that if the lower sequent of the rule $(R\Box_a 2)$ is not t-valid, then some upper sequents of it are not t-valid. Suppose that the lower sequent of $(R\Box_a 2)$ is not t-valid, and by Definition 4.4.3, there is some $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \Vdash \Gamma'$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle{:}\Box_a A}$ and $\mathfrak{M}, f \Vdash \overline{\Delta'}$. Fix such f. It suffices to show for some $\mathsf{x}_1, ..., \mathsf{x}_n$ such that $\mathsf{a}_1 \sim_a^{\mathsf{M}_1} \mathsf{x}_1, ..., \mathsf{a}_n \sim_a^{\mathsf{M}_n} \mathsf{x}_n$ there exists f' such that $\mathfrak{M}, f' \Vdash \langle x, \mathsf{a}_1^{\mathsf{M}_1}, ..., \mathsf{a}_n^{\mathsf{M}_n}\rangle \mathsf{R}_a \langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_n^{\mathsf{M}_n}\rangle$ and $\mathfrak{M}, f' \Vdash \overline{\langle v, \mathsf{x}_1^{\mathsf{M}_1}, ..., \mathsf{x}_n^{\mathsf{M}_n}\rangle{:}A}$ where v does not appear in $\Gamma' \cup \Delta'$. From the supposition, i.e., $\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}, (f(x), \mathsf{a}_1, ..., \mathsf{a}_n) \nVdash \Box_a A$ and $(f(x), \mathsf{a}_1, ..., \mathsf{a}_n) \in \mathcal{D}(\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n})$, we obtain for some $(w, \mathsf{y}_1, ..., \mathsf{y}_n) \in \mathcal{D}(\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n})$ such that $((f(x), \mathsf{a}_1, ..., \mathsf{a}_n), (w, \mathsf{y}_1, ..., \mathsf{y}_n)) \in R_a^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}$ and $\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}, (w, \mathsf{y}_1, ..., \mathsf{y}_n) \nVdash A$. Fix such $(w, \mathsf{y}_1, ..., \mathsf{y}_n)$. Define f' such that $f'(x) = w$ if $x = v$ and otherwise $f'(x) = f(x)$. By the definition of f', we obtain $((f'(x), \mathsf{a}_1, ..., \mathsf{a}_n), (f'(v), \mathsf{y}_1, ..., \mathsf{y}_n)) \in R_a^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}$ and $\mathfrak{M}^{\otimes \mathsf{M}_1 \otimes \cdots \otimes \mathsf{M}_n}, (f'(v), \mathsf{y}_1, ..., \mathsf{y}_n) \nVdash A$. Then by Proposition 4.4.1-(1) and Definition 4.4.2, we obtain what we desired.

The case where the last applied rule is of the form (Lat)**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ $(= \{\Gamma \Rightarrow \Delta \mid \Sigma\})$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}p$, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}p$. Then, $\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}p$ implies $(f(x), \alpha_{evt}, \mathsf{a}) \in V^{\otimes \alpha_{mdl} \otimes \mathsf{M}}(p)$, which is equivalent to $\mathfrak{M}^{\otimes \alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash p$. By Definition 4.4.1, we obtain the goal as desired.

The case where the last applied rule is of the form (Rat)**:** Fix any sequent in c-sequent

$\Gamma \Rightarrow \Delta \parallel \Sigma$ (= $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$). We show the contraposition. Suppose there is some f : $\mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$, and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}p}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}p}$. By Definition 4.4.2, $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}p}$ is equivalent to $\mathfrak{M}^{\otimes\alpha_{mdl}\otimes\mathsf{M}}, (f(x), \alpha_{evt}, \mathsf{a}) \nVdash p$ and $(f(x), \alpha_{evt}, \mathsf{a}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}\otimes\mathsf{M}})$. From them, we obtain $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}})$ and $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \nVdash p$.

This is equivalent to $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}p}$ by Proposition 4.4.2. Then, the contraposition has been shown.

The case where the last applied rule is of the form $(L[.])$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ (= $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$). We show the contraposition. Suppose that there is some f : $\mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]B$ and $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}$ or $\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}B$. Then, $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]B$ iff $(\mathfrak{M}^{\alpha,A}, (f(x), \alpha_{evt}) \nVdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ or $\mathfrak{M}^{\otimes\alpha_{mdl}\otimes\mathsf{M}}, (f(x), \alpha_{evt}, \mathsf{a}) \Vdash B)$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}})$. By Definitions 4.4.1 and 4.4.2, we obtain the goal as desired.

The case where the last applied rule is of the form $(R[.])$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ (= $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$). We show the contraposition. Suppose that there is some f : $\mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]B}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle{:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle{:}B}$. Then, $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle{:}[\mathsf{a}^{\mathsf{M}}]B}$ iff $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ and $\mathfrak{M}^{\otimes\alpha_{mdl}\otimes\mathsf{M}}, (f(x), \alpha_{evt}, \mathsf{a}) \nVdash B$ and $(f(x), \alpha_{evt}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}})$. From $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(x), \alpha_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})$, we obtain $(f(x), \alpha_{evt}, \mathsf{a}) \in \mathcal{D}(\mathfrak{M}^{\otimes\alpha_{mdl}\otimes\mathsf{M}})$. Then, by Definitions 4.4.1 and 4.4.2, we obtain the goal as desired.

The case where the last applied rule is of the form $(Lrel1)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ (= $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$). We show the contraposition. Suppose that there is some f : $\mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle$, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$. Then, $\mathfrak{M}, f \Vdash \langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle$ is equivalent to $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$ and $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(y), \alpha_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{a})$ and $\mathfrak{M}^{\otimes\alpha_{mdl}}, (f(y), \beta_{evt}) \Vdash \mathsf{pre}^{\mathsf{M}}(\mathsf{b})$. By $\mathfrak{M}, f \Vdash \langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle$ and Definition 4.4.1, we obtain the goal as desired.

The case where the last applied rule is of the form $(Lrel2)$ **and** $(Lrel3)$**:** Similar to the above.

The case where the last applied rule is of the form $(Rrel)$**:** Fix any sequent in c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ (= $\{\Gamma \Rightarrow \Delta \mid \Sigma\}$). As before, we show the contraposition. Suppose there is some f : $\mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\Delta}$, and $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle}$. Fix such f. By Definition 4.4.2, $\overline{\langle x, \alpha, \mathsf{a}^{\mathsf{M}}\rangle \mathsf{R}_a \langle y, \beta, \mathsf{b}^{\mathsf{M}}\rangle}$ is equivalent to $\mathfrak{M}, f \Vdash \overline{\langle x, \alpha\rangle \mathsf{R}_a \langle y, \beta\rangle}$ or $\mathfrak{M}, f \Vdash \overline{(x, \alpha){:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}$ or $\mathfrak{M}, f \Vdash \overline{(y, \beta){:}\mathsf{pre}^{\mathsf{M}}(\mathsf{a})}$. This is what we want to show.

□

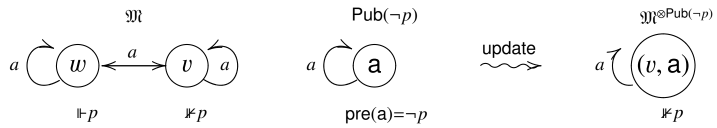
As an additional remark, we here do not empty the following natural and usual definition of the validity which we call *s-validity* for a (c-)sequent.

Definition 4.4.4 (s-validity)**.** $\Gamma \Rightarrow \Delta$ *is s–valid in* $\mathfrak{M}$ *if for all assignment* $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ *such that* $\mathfrak{M}, f \Vdash \mathfrak{A}$ *for all* $\mathfrak{A} \in \Gamma$ *implies* $\mathfrak{M}, f \Vdash \mathfrak{B}$ *for some* $\mathfrak{B} \in \Delta$.

The reason for emptying t-validity instead of s-validity is that a similar argument regarding the failure of s-validity to the argument in Proposition 3.3.2 of **GPAL** also holds in **GDEL**.

Proposition 4.4.3. There is a model $\mathfrak{M}$ such that $(R\neg)$ of **GEAK** does not preserve s-validity in $\mathfrak{M}$.

Proof. We follow the updated model in Example 2.3.1. Let $\mathsf{Agt} = \{a\}$, $\mathfrak{M}^{\mathsf{Pub}(\neg p)} = (\{\mathsf{a}\}, \{(\mathsf{a}, \mathsf{a})\}, \mathsf{pre})$ where $\mathsf{pre}(\mathsf{a}) = \neg p$.



A particular instance of the application of $(R\neg)$ is as follows:

$$\frac{\langle x, \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}p \Rightarrow \| \varnothing}{\Rightarrow \langle x, \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}\neg p \| \varnothing}\ (R \rightarrow)$$

We show that the upper c-sequent is s-valid in $\mathfrak{M}^{\otimes \mathsf{Pub}(\neg p)}$ but the lower c-sequent is not. First, we show that $\langle x, \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}p \Rightarrow$ is s-valid in $\mathfrak{M}^{\mathsf{Pub}(\neg p)}$, i.e., $\mathfrak{M}, f \nVdash \langle x, \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}p$ for any assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ where $\mathcal{D}(\mathfrak{M}) = \{w, v\}$. So, fix any f. We divide our argument into: $f(x) = w$ or $f(x) = v$. If $f(x) = w$, $(f(x), \mathsf{a})$ does not survive after the updated model $\mathfrak{M}^{\otimes \mathsf{Pub}(\neg p)}$ (i.e., $(w, \mathsf{a}) \notin \mathcal{D}(\mathfrak{M}^{\otimes \mathsf{Pub}(\neg p)})$), and so $\mathfrak{M}^{\mathsf{Pub}(\neg p)}, (f(x), \mathsf{a}) \nVdash p$. If $f(x) = v$, world (v, a) survives after $\mathsf{Pub}(\neg p)$ combined, but $(v, \mathsf{a}) \notin V^{\otimes \mathsf{Pub}(\neg p)}(p)$, which also implies $\mathfrak{M}^{\otimes \mathsf{Pub}(\neg p)}, (f(x), \mathsf{a}) \nVdash p$. Thus $\mathfrak{M}, f \nVdash \langle f(x), \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}p$ and the upper c-sequent is s-valid.

Second, we show that $\Rightarrow \langle f(x), \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}\neg p$ is not s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \nVdash \langle f(x), \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}\neg p$ for some assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. Fix f such that $f(x) = w$. Since $(w, \mathsf{a}) \notin \mathcal{D}(\mathfrak{M}^{\mathsf{Pub}(\neg p)})$, we conclude $\mathfrak{M}, f \nVdash \langle f(x), \mathsf{a}^{\mathsf{Pub}(\neg p)}\rangle{:}\neg p$ and the upper c-sequent is s-valid. □

Combining Theorem 4.3.1 (the equality of the derivability between **HEAK** and **GEAK**$^+$), Theorem 4.2.1 (cut elimination of **GEAK**), and 4.4.1 (soundness of **GEAK**) with Theorem 2.3.2 (completeness of **HEAK**), we have the following.

Corollary 4.4.1 (Completeness of **GEAK**)**.** Given any formula A, the following are equivalent:

(i) A is valid on all models,

(ii) $\vdash_{\mathbf{HEAK}} A$,

(iii) $\vdash_{\mathbf{GEAK}^+} \Rightarrow \langle x, \epsilon\rangle{:}A$,

(iv) $\vdash_{\mathbf{GEAK}} \Rightarrow \langle x, \epsilon\rangle{:}A$.

4.5 Extensions of EAK from K to S5

As in the case of extensions of PAL in Section 3.5, we expand the basis of **GEAK** from **K** to other modal logics including **S5** which is the standard basis of epistemic logics.

Proposition 4.5.1. Let $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. For all $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V) \in \mathbb{M}_\Theta$ and all action models of $\mathsf{M} = (\mathsf{S}, (\sim_\mathsf{a})_{\mathsf{a}\in\mathsf{Agt}}, \mathsf{pre})$, $\mathfrak{M}^{\otimes\mathsf{M}}$ is also a member of $\mathbb{M}_\Theta$

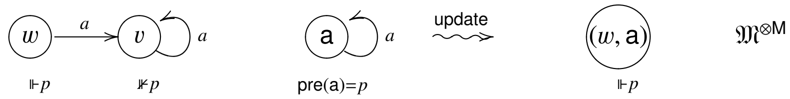
Proof. Fix any $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ in class $\mathbb{M}_\Theta$ and fix an action model M. By the assumption $\mathfrak{M} \in \mathbb{M}_\Theta$ i.e., $(W, (R_a)_{a\in\mathsf{Agt}}) \in \mathbb{F}_\Theta$. We show $(W^{\otimes\mathsf{M}}, (R_a^{\otimes\mathsf{M}})_{a\in\mathsf{Agt}}) \in \mathbb{F}_\Theta$, and so so in what follows, we show for any $A \in \Theta$, R_a has a frame property defined by A. It suffices to show that $R_a^{\otimes\mathsf{M}}$ has also the property.

Case where $\mathbf{T}_a \in \Theta$. Fix any world $(x, \mathsf{s}) \in W^{\otimes\mathsf{M}}$, and show $(x, \mathsf{s})R_a^{\otimes\mathsf{M}}(x, \mathsf{s})$. Since R_a is reflexive, and obtain xR_ax, and since action model M is an S5 model, we obtain $\mathsf{s} \sim_a \mathsf{s}$ as well. By the definition, we obtain the goal $R_a^{\otimes\mathsf{M}}$.

Case where $\mathbf{5}_a \in \Theta$. Fix any $(x, \mathsf{s}), (y, \mathsf{t}), (z, \mathsf{u}) \in W^{\otimes\mathsf{M}}$. Suppose $(x, \mathsf{s})R_a^{\otimes\mathsf{M}}(y, \mathsf{t})$ and $(x, \mathsf{s})R_a^{\otimes\mathsf{M}}(z, \mathsf{u})$, and show $(y, \mathsf{t})R_a^{\otimes\mathsf{M}}(z, \mathsf{u})$. Since R_a is Euclidean i.e., xR_ay and xR_az jointly imply yR_az. By the assumption, we have yR_az, and since action model M is an S5 model, we obtain $\mathsf{s}{\sim_a}\mathsf{t}$ and $\mathsf{s}{\sim_a}\mathsf{u}$ jointly imply $\mathsf{t}{\sim_a}\mathsf{u}$ as well. By the suppositions, we obtain $(x, \mathsf{s})R_a^{\otimes\mathsf{M}}(y, \mathsf{t})$ (i.e., xR_ay and $\mathsf{s}{\sim_a}\mathsf{t}$) and $(x, \mathsf{s})R_a^{\otimes\mathsf{M}}(z, \mathsf{u})$ (i.e., xR_az and $\mathsf{s}{\sim_a}\mathsf{u}$), we obtain yR_az and $\mathsf{t}{\sim_a}\mathsf{u}$. Therefore, we get the goal $(y, \mathsf{t})R_a^{\otimes\mathsf{M}}(z, \mathsf{u})$ as desired.

Other cases regarding $\mathbf{B}_a$ and $\mathbf{4}_a$ can be shown similarly. □

Note that Proposition 4.5.1 does not also hold as in the case of the extensions of EAK like the case in PAL (Section 3.5), if $\mathbf{D}_a$ is included, since consider Kripke model $\mathfrak{M} = (W, R_a, V) = (\{w, v\}, \{(w, v), (v, v)\}, V) \in \mathbb{M}_{\{\mathbf{D}_a\}}$ where $V(p) = \{w\}$ and action model $\mathsf{M} = (\{\mathsf{a}\}, \{(\mathsf{a}, \mathsf{a})\}, \mathsf{pre})$ where $\mathsf{pre}(\mathsf{a}) = p$.



The updated model $\mathfrak{M}^{\otimes\mathsf{M}}$ does not satisfy seriarity i.e., $\mathfrak{M}^{\otimes\mathsf{M}} \notin \mathbb{M}_{\{\mathbf{D}_a\}}$.

As the case of extensions of **HK**, when we add one or more formulas in $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ as additional axiom schemes to the set of axiom scheme of **HEAK**, we obtain Hilbert-systems other than **HEAK** as follows.

Definition 4.5.1 (Extensions of **HEAK**). Let Θ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. When each element of Θ is added to **HEAK** as an axiom scheme by replacing p with an arbitrary formula A, *the extension of* **HEAK** is the resulting Hilbert-system **HEAK**Θ.

We give names to Hilbert-systems with some particular combinations of axiom schemes.

$\mathbf{HEAK_T} := \mathbf{HEAK}\{\mathbf{T}_a \mid a \in \mathsf{Agt}\}$, $\mathbf{HEAK_{S4}} := \mathbf{HEAK}\{\mathbf{T}_a, \mathbf{4}_a \mid a \in \mathsf{Agt}\}$,
$\mathbf{HEAK_B} := \mathbf{HEAK}\{\mathbf{T}_a, \mathbf{B}_a \mid a \in \mathsf{Agt}\}$, $\mathbf{HEAK_{S5}} := \mathbf{HEAK}\{\mathbf{T}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$.

For any $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, Logic of Epistemic Actions and Knowledge $\mathbf{EAK}\Theta$ is the set of all derivable formulas in $\mathbf{HEAK}\Theta$. We name some $\mathbf{EAK}\Theta$.

$$\begin{array}{ll}\mathbf{EAK_T} := \mathbf{EAK}\{\mathbf{T}_a \mid a \in \mathsf{Agt}\}, & \mathbf{EAK_{S4}} := \mathbf{EAK}\{\mathbf{T}_a, \mathbf{4}_a \mid a \in \mathsf{Agt}\}, \\ \mathbf{EAK_B} := \mathbf{EAK}\{\mathbf{T}_a, \mathbf{B}_a \mid a \in \mathsf{Agt}\}, & \mathbf{EAK_{S5}} := \mathbf{EAK}\{\mathbf{T}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}.\end{array}$$

Theorem 4.5.1 (Soundness and completeness of $\mathbf{HEAK}\Theta$)**.** Let Θ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ and $A \in \mathcal{L}_{EAK}$. Then the following holds:

$$\mathbb{M}_\Theta \Vdash A \text{ iff } \vdash_{\mathbf{HEAK}\Theta} A.$$

Proof. The proof is carried out by the same step as in Theorem 2.3.2. □

Corollary 4.5.1. $\mathbf{EAK_X}$ is decidable, where $\mathbf{X}$ be an element of $\{\mathbf{T}, \mathbf{B}, \mathbf{S4}, \mathbf{S5}, \mathbf{D}\}$.

Proof. We show that there is an effective method for deciding of any formula $A \in \mathcal{L}_{EAK}$ whether or not it is a theorem of $\mathbf{EAK_X}$. Fix any $A \in \mathcal{L}_{EAK}$. Note that translation $t : \mathcal{L}_{EAK} \to \mathcal{L}_{EL}$ is inductively defined and so it provides an effective method which is a composition of the two effective methods. Then since modal logic $\mathbf{X}$ is decidable by Corollary 2.1.1, $t(A) \in \mathcal{L}_{ML}$ can be decided whether it is a theorem of $\mathbf{X}$. □

Extensions of GEAK Let us define the extensions of **GEAK**. We add to **GEAK** one or more of the additional rules which correspond to the frame properties.

Table 4.3: Rules for frame properties

$$\frac{\langle x, \epsilon\rangle \mathsf{R}_a \langle x, \epsilon\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta \parallel \Sigma}\ (\mathit{ref}_a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle \parallel \Sigma \quad \langle y, \epsilon\rangle \mathsf{R}_a \langle x, \epsilon\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta \parallel \Sigma}\ (\mathit{sym}_a)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle y, \epsilon\rangle \mathsf{R}_a \langle z, \epsilon\rangle \parallel \Sigma \quad \langle x, \epsilon\rangle \mathsf{R}_a \langle z, \epsilon\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta \parallel \Sigma}\ (\mathit{tra}_a)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle x, \epsilon\rangle \mathsf{R}_a \langle y, \epsilon\rangle \parallel \Sigma \quad \Gamma \Rightarrow \Delta, \langle x, \epsilon\rangle \mathsf{R}_a \langle z, \epsilon\rangle \parallel \Sigma \quad \langle y, \epsilon\rangle \mathsf{R}_a \langle z, \epsilon\rangle, \Gamma \Rightarrow \Delta \parallel \Sigma}{\Gamma \Rightarrow \Delta \parallel \Sigma}\ (\mathit{euc}_a)$$

Let * be a function from $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$ to $\{(\mathit{ref}_a), (\mathit{sym}_a), (\mathit{tra}_a), (\mathit{euc}_a) \mid a \in \mathsf{Agt}\}$ defined as follows:

$$\mathbf{T}_a{}^* := (\mathit{ref}_a), \quad \mathbf{4}_a{}^* := (\mathit{tra}_a), \quad \mathbf{B}_a{}^* := (\mathit{sym}_a), \quad \mathbf{5}_a{}^* := (\mathit{euc}_a).$$

Let Θ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. Then Θ^* is defined to be the set $\{\mathbf{X}^* \mid \mathbf{X} \in \Theta\}$.

Definition 4.5.2 (Extensions of **GEAK**)**.** Let Θ be a subset of $\{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$. A labelled sequent calculus $\mathbf{GEAK}\Theta^*$ is an extension of **GEAK**, when each element of Θ^* is added to **GEAK** as inference rules.

Some particular combinations of inference rules are given names.

> $\mathbf{GEAK_T} := \mathbf{GEAK}\{(\mathit{ref}_a) \mid a \in \mathsf{Agt}\}$,
> $\mathbf{GEAK_B} := \mathbf{GEAK}\{(\mathit{sym}_a) \mid a \in \mathsf{Agt}\}$,
> $\mathbf{GEAK_{S4}} := \mathbf{GEAK}\{(\mathit{ref}_a), (\mathit{tra}_a) \mid a \in \mathsf{Agt}\}$,
> $\mathbf{GEAK_{S5}} := \mathbf{GEAK}\{(\mathit{ref}_a), (\mathit{euc}_a) \mid a \in \mathsf{Agt}\}$,

We call each $\mathbf{GEAK}\Theta^*$ with (Cut) $\mathbf{GEAK}\Theta^{*+}$.

Theorem 4.5.2. For any $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, if $\vdash_{\mathbf{HEAK\Theta}} A$ then $\vdash_{\mathbf{GEAK\Theta^*}} \Rightarrow x{:}^{\epsilon}A$ (for any x), for any formula $A \in \mathcal{L}_{EAK}$.

Proof. Proof is almost the same as Theorem 3.5.2. We look at the following additional cases.

Case of $\mathbf{B}_a$ with (sym_a). In this case, we show $\vdash_{\mathbf{GDEL\Theta^*}} \Rightarrow x{:}\mathbf{B}_a$ where $(\mathit{sym}_a) \in \Theta^*$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A, x\mathsf{R}_a y} \quad \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y\mathsf{R}_a x, x{:}A \Rightarrow y\mathsf{R}_a x} \quad \dfrac{\textit{Initial Seq.}}{y\mathsf{R}_a x, x{:}A \Rightarrow x{:}A}}{y\mathsf{R}_a x, x{:}A \Rightarrow y{:}\Diamond_a A}\,(R\Diamond_a)}{y\mathsf{R}_a x, x{:}A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A}\,(Lw)}{x{:}A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A}\,(\mathit{sym}_a)}{x{:}A \Rightarrow x{:}\Box_a\Diamond_a A}\,(R\Box_a)}{\Rightarrow x{:}A \to \Box_a\Diamond_a A}\,(R \to)$$

Case of $\mathbf{T}_a$ with (ref_a). In this case, we show $\vdash_{\mathbf{GDEL\Theta^*}} \Rightarrow x{:}\mathbf{T}_a$ where $(\mathit{ref}_a) \in \Theta^*$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}_a x \Rightarrow x{:}A, x\mathsf{R}_a x} \quad \dfrac{\textit{Initial Seq.}}{x\mathsf{R}_a x, x{:}A \Rightarrow x{:}A}}{x\mathsf{R}_a x, x{:}\Box_a A \Rightarrow x{:}A}\,(R\Box_a)}{x{:}\Box_a A \Rightarrow x{:}A}\,(ref_a)}{\Rightarrow x{:}\Box_a A \to A}\,(R \to)$$

Case of $\mathbf{5}_a$ with (euc_a). In this case, we show $\vdash_{\mathbf{GDEL\Theta^*}} \Rightarrow x{:}\mathbf{5}_a$ where $(\mathit{euc}_a) \in \Theta^*$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond A, x\mathsf{R}_a y} \quad \dfrac{\textit{Initial Seq.}}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond A, x\mathsf{R}_a z} \quad \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y\mathsf{R}_a z, z{:}A \Rightarrow y\mathsf{R}_a z} \quad \dfrac{\textit{Initial Seq.}}{y\mathsf{R}_a z, z{:}A \Rightarrow z{:}A}}{y\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond A}\,(R\Diamond_a)}{y\mathsf{R}_a z, x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond A}\,(Lw)}{x\mathsf{R}_a y, x\mathsf{R}_a z, z{:}A \Rightarrow y{:}\Diamond A}\,(\mathit{euc}_a)}{x{:}\Diamond_a A, x\mathsf{R}_a y \Rightarrow y{:}\Diamond_a A}\,(L\Diamond_a)}{x{:}\Diamond_a A \Rightarrow x{:}\Box_a\Diamond_a A}\,(R\Box_a)}{\Rightarrow x{:}\Diamond_a A \to \Box_a\Diamond_a A}\,(R \to)$$

$\square$

Theorem 4.5.3 (Soundness of $\mathbf{GEAK\Theta}$)**.** For any $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, given any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$ in $\mathbf{GEAK\Theta}$, if $\vdash_{\mathbf{GEAK\Theta}} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\Gamma \Rightarrow \Delta \parallel \Sigma$ is t-valid in every model $\mathfrak{M} \in \mathbb{M}_{\Theta}$.

Proof. Fix any $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, and suppose $\vdash_{\mathbf{GEAK\Theta}} \Gamma \Rightarrow \Delta \parallel \Sigma$. Fix any model $\mathfrak{M} \in \mathbb{M}_{\Theta}$. Then we show $\Gamma \Rightarrow \Delta \parallel \Sigma$ is t-valid. Fix any sequent $\Gamma' \Rightarrow \Delta'$ in the c-sequent. We show the additional cases to the proof of Theorem 3.3.1, and so it suffices to show that any additional rule keeps t-validity in any corresponding model to the rule.

Case of (ref_a)**:** Fix any R_a-reflexive model $\mathfrak{M}$. We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$, $\mathfrak{M}, f \Vdash \Gamma'$, and $\mathfrak{M}, f \Vdash \overline{\Delta'}$. Fix such f. It suffices to show $(f(x), f(x)) \in R_a$ which is equivalent to $\langle f(x), \epsilon\rangle \mathsf{R}_a \langle f(x), \epsilon\rangle$. This is trivially obtained from the R_a-reflexive model $\mathfrak{M}$.

Other cases can be shown by almost the same way as Proposition 3.5.3. Since $\Gamma' \Rightarrow \Delta'$ is sound and is an arbitrary sequent in $\Gamma \Rightarrow \Delta \parallel \Sigma$, this c-sequent is sound as well. □

Theorem 4.5.4 (Cut elimination theorem of $\mathbf{GEAK\Theta}^{*+}$)**.** For any $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, and any c-sequent $\Gamma \Rightarrow \Delta \parallel \Sigma$, if $\vdash_{\mathbf{GEAK\Theta}^{*+}} \Gamma \Rightarrow \Delta \parallel \Sigma$, then $\vdash_{\mathbf{GEAK\Theta}^{*}} \Gamma \Rightarrow \Delta \parallel \Sigma$.

Proof. It suffices to show additional cases for $(ref_a), (sym_a), (tra_a)$ and (euc_a) in addition to Theorem 4.2. Since there is no principal expression(s) introduced by the upper c-sequent(s), we do not have the case where cut expression $\mathfrak{A}$s on both sides of upper c-sequents are principal expressions. The other cases like only one of cut expressions is introduced by the right upper c-sequent or the left upper c-sequent are straightforward.

The proof goes through the same procedure as in the proof of Theorem 3.2.2 with the rule of (*Ecut*), and the proof is divided into four cases. In brief,

(1) at least one of upper c-sequents of (*Ecut*) is an initial c-sequent;

(2) the last inference rule of either upper c-sequents of (*Ecut*) is a structural rule;

(3) the last inference rule of either upper c-sequents of (*Ecut*) is a non-structural rule, and the principal expression introduced by the rule is not the cut expression; and

(4) the last inference rules of two upper c-sequents of (*Ecut*) are both non-structural rules, and the principal expressions introduced by the rules used on the upper c-sequents of (*Ecut*) are both cut expressions.

It suffices to show additional cases for $(ref_a), (sym_a), (tra_a)$ and (euc_a) in addition to the proof of Theorem 3.2.2. Since there is no principal expression(s) introduced by the upper c-sequent(s), we do not have the case (4) where cut expression $\mathfrak{A}$s on both sides of upper c-sequents are principal expressions. The other cases like only one of cut expressions is introduced by the right upper c-sequent or the left upper c-sequent are straightforward. We look at one of such cases.

Case of (3) where one of upper c-sequents of (*Ecut*) is inference rule (sym_a).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x\mathsf{R}_a y, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ y\mathsf{R}_a x, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow \Delta' \parallel \Sigma}\,(sym_a)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)$$

This is transformed into the derivation:

$$
\dfrac{
\dfrac{
\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, x\mathsf{R}_a y, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{x\mathsf{R}_a y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)
\quad
\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \parallel \Sigma} \quad \overset{\vdots\ \mathcal{D}_3}{\mathfrak{A}^m, y\mathsf{R}_a x, \Gamma' \Rightarrow \Delta' \parallel \Sigma}}{y\mathsf{R}_a x, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Ecut)
}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta' \parallel \Sigma}\,(sym_a)
}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \parallel \Sigma}\,(Lc)/(Rc)
$$

Every other case can be shown similar to this. □

Then the corollary below holds.

Corollary 4.5.2. Given a formula A, $x \in \mathsf{Var}$, $\Theta \subseteq \{\mathbf{T}_a, \mathbf{B}_a, \mathbf{4}_a, \mathbf{5}_a \mid a \in \mathsf{Agt}\}$, the following statements are all equivalent.

(i) $\mathbb{M}_\Theta \Vdash A$,

(ii) $\vdash_{\mathbf{HEAK\Theta}} A$,

(iii) $\vdash_{\mathbf{GEAK\Theta}^{*+}} \Rightarrow x{:}^{\epsilon}A$,

(iv) $\vdash_{\mathbf{GEAK\Theta}^{*}} \Rightarrow x{:}^{\epsilon}A$.

Proof. The direction from (i) to (ii) is shown by Theorem 4.5.1 and the direction from (ii) to (iii) is established by Theorem 4.5.2. Then, the direction from (iii) to (iv) is established by the admissibility of (*Cut*) (Theorem 4.5.4). Finally, the direction from (iv) to (i) is shown by Theorem 4.5.3. □

Chapter 5

Intuitionistic Public Announcement Logic (IntPAL)

Epistemic logics including two major DELs such as PAL and EAK usually employ classical modal logic as their underlying logic; however, we may easily imagine that they can be constructed on a different foundation, intuitionistic modal logic. Intuitionistic PAL (IntPAL), which is a combination of the most basic DEL i.e., PAL and an intuitionistic modal logic, can become the touchstone of intuitionistic DELs.

In the context of epistemic logic, knowledge defined in an intuitionistic system can be regarded as knowledge with verification or evidence (cf. [2, 87]).[1] The area of intuitionistic modal logics, since Fitch [26] proposed, has been developed historically by efforts of several logicians (e.g., [15, 25, 64, 70, 78]). On the foundation of the past studies, Ma et al. [49] recently gave a Hilbert-system of IntPAL (we called it **HIntPAL**), which is based on intuitionistic modal logic IK (or IntK) by Fischer Servi [25] and Simpson [78], is shown to be semantically complete for algebraic semantics. It is our expectation that a sequent calculus brings abundant benefits especially for constructive knowledge requiring verification, since a sequent calculus is usually feasible in computation, compared with Hilbert-system; the calculus can easily be translated into an algorithmic procedure. Thus it can supply a proof for any valid formula i.e., verification for any knowledge that is based on intuitionistic system.

The outline of Chapter 5 is as follows. Section 5.1 provides the birelational Kripke semantics and the Hilbert-system **HIntPAL** for IntPAL. Section 5.3 introduces our calculus **GIntPAL** (with the rule of cut) and shows that all theorems of **HIntPAL** are derivable in **GIntPAL**$^+$ (Theorem 5.5.1). Section 5.6 establishes the cut elimination theorem of **GIntPAL**$^+$ (Theorem 5.6.1) and, as a corollary of the theorem, shows that **GIntPAL**$^+$ is consistent. Section 5.7 tackles the soundness theorem of **GIntPAL**$^+$ (Theorem 5.7.1), and it should be noted that its soundness is not straightforward at all by the following two reasons. First, it depends on a non-trivial choice of the notions of validity of a sequent as suggested in [61]. Second, there is another difficulty, pointed

[1] Regarding recent trends in knowledge-representation with intuitionistic logic, several constructive description logics [13, 21, 53] are proposed to investigate possibly incomplete knowledge.

out in [78], which is peculiar to intuitionist modal logic. Then the semantic completeness of **GIntPAL** (Corollary 5.7.1) is shown through the proven theorems. The last section concludes the paper.

5.1 Language $\mathcal{L}_{IntPAL}$ and birelational Kripke Semantics

First of all, we address the syntax of IntPAL. Let $\mathsf{Prop} = \{p, q, r, \ldots\}$ be a countably infinite set of propositional atoms and $\mathsf{Agt} = \{a, b, c, \ldots\}$ a nonempty finite set of agents. Then the set $\mathcal{L}_{IntPAL} = \{A, B, C, \ldots\}$ of formulas of IntPAL is inductively defined as follows:

$$A ::= p \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \Box_a A \mid \Diamond_a A \mid [A]A \mid \langle A \rangle A,$$

where $p \in \mathsf{Prop}$, $a \in \mathsf{Agt}$. We define $\neg A := A \to \bot$. Also, $\top$ and $A \leftrightarrow B$ are defined as usual. Similar to PAL, $\Box_a A$ reads 'agent a knows that A', and $[A]B$ reads 'after public announcement of A, it holds that B'.

Example 5.1.1. Let us consider a propositional atom p to read 'it will rain tomorrow'. Then a formula $\neg(\Box_a p \vee \Box_a \neg p)$ means that a does not know whether it will rain tomorrow or not, and $[\neg p]\Box_a \neg p$ means that after a public announcement (e.g., a weather report) of $\neg p$, a knows that it will not rain tomorrow.

Let us go on to the semantics of IntPAL. We mainly follow the birelational Kripke semantics introduced in Ma et al. [49], which is based on intuitionistic version of modal logic K. We call $\mathfrak{F} = (W, \leqslant, (R_a)_{a \in \mathsf{Agt}})$ an *IntK-frame* if $(W, \leqslant)$ is a nonempty poset (W is also denoted by $\mathcal{D}(\mathfrak{M})$), $(R_a)_{a \in \mathsf{Agt}}$ is a Agt-indexed family of binary relations on W such that the following two conditions $(F1)$ and $(F2)$ in Simpson [78, p.50] are satisfied:

$$(F1) : (\geqslant \circ R_a) \subseteq (R_a \circ \geqslant),$$
$$(F2) : (R_a \circ \leqslant) \subseteq (\leqslant \circ R_a).$$

We note that $(F1)$ and $(F2)$ are essential features to express a combination of the two different relations such as $\leqslant$ and R_a.

Moreover, a pair $\mathfrak{M} = (\mathfrak{F}, V)$ is an *IntK-model* if $\mathfrak{F}$ is an IntK-frame and $V{:}\mathsf{Prop} \to \mathcal{P}^{\uparrow}(W)$ is a valuation function where

$$\mathcal{P}^{\uparrow}(W) := \{X \in \mathcal{P}(W) \mid x \in X \text{ and } x \leqslant y \text{ jointly imply } y \in X \text{ for all } x, y \in W\},$$

that is, $\mathcal{P}^{\uparrow}(W)$ is the set of all upward closed sets. Next, let us define the satisfaction relation $\mathfrak{M}, w \Vdash A$. Given an IntK-model $\mathfrak{M}$, a world $w \in \mathcal{D}(\mathfrak{M})$, and a formula $A \in \mathcal{L}_{IntPAL}$, we define $\mathfrak{M}, w \Vdash A$ as follows:

$$
\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \Vdash \bot & & Never, \\
\mathfrak{M}, w \Vdash A \wedge B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ and } \mathfrak{M}, w \Vdash B, \\
\mathfrak{M}, w \Vdash A \vee B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ or } \mathfrak{M}, w \Vdash B, \\
\mathfrak{M}, w \Vdash A \to B & \text{iff} & \text{for all } v \in W : w \leqslant v \text{ and } \mathfrak{M}, v \Vdash A \text{ jointly imply } \mathfrak{M}, v \Vdash B, \\
\mathfrak{M}, w \Vdash \Box_a A & \text{iff} & \text{for all } v \in W : w(\leqslant \circ R_a)v \text{ implies } \mathfrak{M}, v \Vdash A, \\
\mathfrak{M}, w \Vdash \Diamond_a A & \text{iff} & \text{for some } v \in W : wR_a v \text{ and } \mathfrak{M}, v \Vdash A, \\
\mathfrak{M}, w \Vdash [A]B & \text{iff} & \text{for all } v \in W : w \leqslant v \text{ and } \mathfrak{M}, v \Vdash A \text{ jointly imply } \mathfrak{M}^A, v \Vdash B, \\
\mathfrak{M}, w \Vdash \langle A \rangle B & \text{iff} & \mathfrak{M}, w \Vdash A \text{ and } \mathfrak{M}^A, w \Vdash B,
\end{array}
$$

where the restriction $\mathfrak{M}^A$, in the definition of the announcement operators, is the restricted IntK-model to the truth set of A, defined as $\mathfrak{M}^A = (\llbracket A \rrbracket_{\mathfrak{M}}, \leqslant^A, (R_a^A)_{a \in \mathsf{Agt}}, V^A)$ with

$$
\begin{array}{lll}
\llbracket A \rrbracket_{\mathfrak{M}} & := & \{w \in W \mid \mathfrak{M}, w \Vdash A\} \\
\leqslant^A & := & \leqslant \cap (\llbracket A \rrbracket_{\mathfrak{M}} \times \llbracket A \rrbracket_{\mathfrak{M}}) \\
R_a^A & := & R_a \cap (\llbracket A \rrbracket_{\mathfrak{M}} \times \llbracket A \rrbracket_{\mathfrak{M}}) \\
V^A(p) & := & V(p) \cap \llbracket A \rrbracket_{\mathfrak{M}} \qquad (p \in \mathsf{Prop}).
\end{array}
$$

We note that the conditions $(F1)$ and $(F2)$ are still satisfied in $\mathfrak{M}^A$. Added to these, the restriction of the composition $(\leqslant \circ R_a)^A$ is defined by $(\leqslant \circ R_a) \cap (\llbracket A \rrbracket_{\mathfrak{M}} \times \llbracket A \rrbracket_{\mathfrak{M}})$.

Definition 5.1.1. A formula A is *valid* in an IntK-model $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash A$ for all $w \in \mathcal{D}(\mathfrak{M})$.

By the above semantics, the important semantic feature, heredity, is preserved as follows.[2]

Proposition 5.1.1 (Hereditary)**.** For all IntK-models $\mathfrak{M}$, for all $w, v \in \mathcal{D}(\mathfrak{M})$, if $\mathfrak{M}, w \Vdash A$ and $w \leqslant v$, then $\mathfrak{M}, v \Vdash A$, for any formula A.

Besides, the following proposition is also significant.

Proposition 5.1.2. $(\leqslant \circ R_a)^A = (\leqslant^A \circ R_a^A)$

Proof. We briefly look at the direction of $\subseteq$. Fix any $v, u \in \mathcal{D}(\mathfrak{M})$ such that $v(\leqslant \circ R_a)^A u$. We show $x(\leqslant^A \circ R_a^A)u$. By the above definition, we have $v(\leqslant \circ R_a)u$ and $(v, u) \in \llbracket A \rrbracket_{\mathfrak{M}} \times \llbracket A \rrbracket_{\mathfrak{M}}$, and then there exists some t, such that $v \leqslant t$ and $tR_a u$. Take such t, and by Proposition 5.1.1, we get $t \in \llbracket A \rrbracket_{\mathfrak{M}}$. Therefore, we conclude $x(\leqslant^A \circ R_a^A)u$. □

We denote finite lists $(A_1, ..., A_n)$ of formulas by α, β, etc., and do the empty list by ϵ. As an abbreviation, for any list $\alpha = (A_1, A_2, ..., A_n)$ of formulas, we naturally define $\mathfrak{M}^\alpha$ inductively as: $\mathfrak{M}^\alpha := \mathfrak{M}$ (if $\alpha = \epsilon$), and $\mathfrak{M}^\alpha := (\mathfrak{M}^\beta)^{A_n} = (W^{\beta, A_n}, (R_a^{\beta, A_n})_{a \in \mathsf{Agt}}, V^{\beta, A_n})$ (if $\alpha = \beta, A_n$). We may also denote $(\mathfrak{M}^\beta)^{A_n}$ by $\mathfrak{M}^{\beta, A_n}$ for simplicity. From Proposition 5.1.2, the next corollary may be easily shown by induction on the number of α.

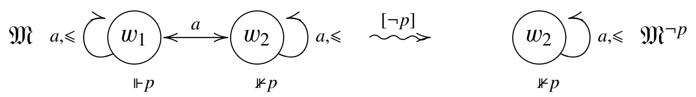
Corollary 5.1.1. $(\leqslant \circ R_a)^\alpha = (\leqslant^\alpha \circ R_a^\alpha)$

[2] Two conditions, $(F1)$ and $(F2)$, are required to show heredity (and validity of axioms) in IntK on which **GIntPAL** is based. In fact, one more condition is added to the two in [49] for some specific purpose in their paper. That is $R_a = (\leqslant \circ R_a) \cap (R_a \circ \geqslant)$.

5.2 Examples of knowledge-change in IntPAL

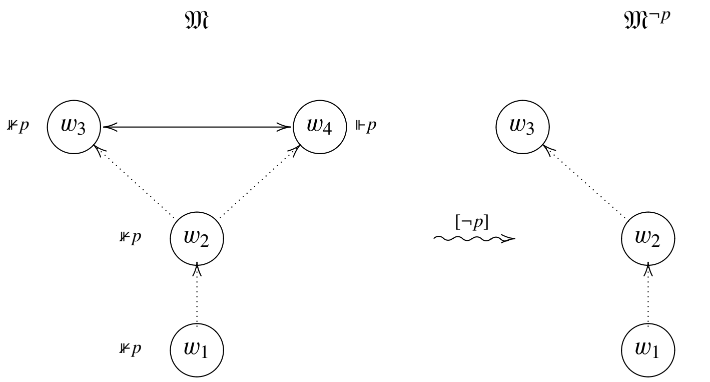
Let us give the following examples to help for understanding the heart of IntPAL.

Example 5.2.1. Let us trace Example 5.1.1 of PAL where $\mathsf{Agt} = \{a\}$; and bidirectional Kripke model $\mathfrak{M} = (W, \leqslant, R_a, V) = (\{w_1, w_2\}, id_W, W^2, V)$ where id_W is the identity relation and $V(p) = \{w_1\}$, [3] and, $\mathfrak{M}^{\neg p} = (\{w_2\}, \{(w_2, w_2)\}, \{(w_2, w_2)\}, V^{\neg p})$ where $V^{\neg p}(p) = \varnothing$. These models are shown in graphic forms as follows.



In $\mathfrak{M}$, agent a does not know whether p or $\neg p$ (i.e., $\neg(\Box_a p \vee \Box_a \neg p)$ is valid in $\mathfrak{M}$). But after announcement of $\neg p$, agent a comes to know $\neg p$ in the restricted model $\mathfrak{M}$ to $\neg p$.

Example 5.2.2. Let us consider another example which is peculiar to IntPAL. Let Agt be $\{a\}$ and the following two models $\mathfrak{M} := (W, \leqslant, R_a, V)$ where $W := \{w_1, w_2, w_3, w_4\}$, $\leqslant := id_W \cup \{(w_1, w_2), (w_2, w_3), (w_2, w_4)\}$, $R_a := id_W \cup \{w_3, w_4\}^2$ and $V(p) = \{w_4\}$, and, $\mathfrak{M}^{\neg p} = (W^{\neg p}, \leqslant, R_a^{\neg p}, V^{\neg p})$ where $W := \{w_1, w_2, w_3\}$, $\leqslant := id_W \cup \{(w_1, w_2), (w_2, w_3)\}$, $R_a := id_W$ and $V^{\neg p}(p) = \varnothing$. These models are shown in graphic forms as follows.



In $\mathfrak{M}$, agent a does not know whether p or $\neg p$ (i.e., $\neg(\Box_a p \vee \Box_a \neg p)$ is valid in $\mathfrak{M}$). But after announcement of $\neg p$, agent a comes to know $\neg p$ in the restricted model $\mathfrak{M}$ to $\neg p$.

5.3 Hilbert-system HIntPAL of IntHPAL

We first introduce a Hilbert-system for IntPAL (**HIntPAL**). It is defined in Table 5.1 , where the axiom (from ($RA1$) to ($RA14$)), called *recursion axioms*, and one inference rule (Nec[.]) are added to the Hilbert-system of IntK. Through the axioms and rules, each theorem of **HIntPAL** may be reduced into a theorem of the Hilbert-system of IntK. And the previous work [49] has shown the completeness theorem of **HIntPAL**.

[3] Note that the above IntK frame satisfies the conditions since $(R_a \circ \leqslant) = (\leqslant \circ R_a) = (\geqslant \circ R_a) = (R_a \circ \leqslant) = \{w_1, w_1\}^2$.

Theorem 5.3.1 (Completeness of HIntPAL)**.** For any formula A, A is valid in all IntK-models iff A is a theorem of **HIntPAL**.

Table 5.1: Hilbert-system for IntPAL : **HIntPAL**

Modal Axioms	$(Taut)$	all instantiations of theorems of intuitionistic propositional logic
	$(IK1)$	$\Box_a(p \to q) \to (\Box_a p \to \Box_a q)$
	$(IK2)$	$\Diamond_a(p \vee q) \to (\Diamond_a p \vee \Diamond_a q))$
	$(IK3)$	$\neg\Diamond_a\bot$
	$(FS1)$	$\Diamond_a(p \to q) \to (\Box_a p \to \Diamond_a q)$
	$(FS2)$	$(\Diamond_a p \to \Box_a q) \to \Box_a(p \to q)$
Recursion Axioms	$(RA1)$	$[A]\bot \leftrightarrow \neg A$
	$(RA2)$	$\langle A\rangle\bot \leftrightarrow \bot$
	$(RA3)$	$[A]p \leftrightarrow (A \to p)$
	$(RA4)$	$\langle A\rangle p \leftrightarrow (A \wedge p)$
	$(RA5)$	$[A](B \vee C) \leftrightarrow A \to \langle A\rangle B \vee \langle A\rangle C$
	$(RA6)$	$\langle A\rangle(B \vee C) \leftrightarrow (\langle A\rangle B \vee \langle A\rangle C)$
	$(RA7)$	$[A](B \wedge C) \leftrightarrow [A]B \wedge [A]C$
	$(RA8)$	$\langle A\rangle(B \wedge C) \leftrightarrow \langle A\rangle B \wedge \langle A\rangle C$
	$(RA9)$	$[A](B \to C) \leftrightarrow \langle A\rangle B \to \langle A\rangle C$
	$(RA10)$	$\langle A\rangle(B \to C) \leftrightarrow A \wedge (\langle A\rangle B \to \langle A\rangle C)$
	$(RA11)$	$[A]\Box_a B \leftrightarrow (A \to \Box_a[A]B)$
	$(RA12)$	$\langle A\rangle\Box_a B \leftrightarrow (A \wedge \Box_a[A]B)$
	$(RA13)$	$[A]\Diamond_a B \leftrightarrow (A \to \Diamond_a[A]B)$
	$(RA14)$	$\langle A\rangle\Diamond_a B \leftrightarrow (A \wedge \Diamond_a\langle A\rangle B)$
Inference Rules	(MP)	From A and $A \to B$, infer B
	$(Nec\Box_a)$	From A, infer $\Box_a A$
	$(Nec[.])$	From A, infer $[B]A$, for any B

It is well-known, since Gentzen [30], that the sequent calculus **LJ** for intuitionistic logic is obtained from the sequent calculus **LK** of classical logic, by restricting the right-hand side of a sequent to at most one formula. It is quite natural to ask if we can obtain an intuitionistic version of **GPAL** by using the same restriction. Therefore, our target of this paper is to construct a labelled sequent calculus (we call it **GIntPAL** and **GIntPAL**$^+$ if it has the cut rule) for **HIntPAL**.

5.4 Labelled sequent calculus GIntPAL

Let $\mathsf{Var} = \{x, y, z, ...\}$ be a countably infinite set of variables. Then, given any $x, y \in \mathsf{Var}$, any list α of formulas and any formula A, we say $x{:}^{\alpha}A$ is a *labelled formula*, and that, for any agent $a \in \mathsf{Agt}$, $x\mathsf{R}_a^{\alpha}y$ is a *relational atom*. Intuitively, the labelled formula $x{:}^{\alpha}A$ corresponds to '$\mathfrak{M}^{\alpha}, x \Vdash A$' and is to read 'after a sequence α of public announcements, x still exists (*survives*) in the restricted domain and A holds at x', and the relational atom $x\mathsf{R}_a^{\alpha}y$ is to read 'after a sequence α of public announcements both x and y exist (survive) and there is a accessibility relation of a from x to y'. We also use the term, *labelled expressions* to indicate that they are either labelled formulas or relational atoms and we denote by $\mathfrak{A}, \mathfrak{B}$, etc. labelled expressions. A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of finite multi-sets of labelled expressions, where at most one labelled expression can appear in Δ. The set of inference rules of **GIntPAL** is shown in Table 5.2. Additionally, for any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is derivable in **GIntPAL**, we write $\vdash_{\mathbf{GIntPAL}} \Gamma \Rightarrow \Delta$. Hereinafter, we use the following abbreviations in a derivation for drawing simpler derivations:

$$\dfrac{\dfrac{\textit{Initial Seq.}}{}}{\mathfrak{A}, \Gamma \Rightarrow \mathfrak{A}} \qquad \dfrac{\dfrac{\textit{Initial Seq.}}{}}{x{:}^{\alpha}\bot, \Gamma \Rightarrow \Delta}$$

both of which are obvious by the rule (Lw). Besides, we also use the following derivable rules:

$$\dfrac{x{:}^{\alpha}A, x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge) \quad \dfrac{\Gamma \Rightarrow x{:}^{\alpha}A \quad \Gamma \Rightarrow x{:}^{\alpha}B}{\Gamma \Rightarrow x{:}^{\alpha}A \wedge B}\ (R\wedge)$$

$$\dfrac{x{:}^{\alpha}A, x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\langle A\rangle B, \Gamma \Rightarrow \Delta}\ (L\langle.\rangle)$$

Moreover, **GIntPAL**$^{+}$ is **GIntPAL** with the following rule (Cut):

$$\dfrac{\Gamma \Rightarrow \mathfrak{A} \quad \mathfrak{A}, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\ (Cut),$$

where $\mathfrak{A}$ in (Cut) is called a *cut expression*. And, we use the term *principal expression* of an inference rule of **GIntPAL**$^{+}$ if a labelled expression is newly introduced on the left uppersequent or the right uppersequent by the rule of **GIntPAL**$^{+}$.

5.5 All theorems of HIntPAL are derivable in GIntPAL^{+}

In this section, we show the set of derivable formulas in **HIntPAL** is equal to the set derivable formulas in **GIntPAL**$^{+}$. Let us define the length of a labelled expression $\mathfrak{A}$ in advance.

Definition 5.5.1. For any formula A, $\ell(A)$ is defined to be the number of the propositional atoms and the logical connectives in A.

$$\ell(\alpha) = \begin{cases} 0 & \text{if } \alpha = \epsilon \\ \ell(\beta) + \ell(A) & \text{if } \alpha = \beta, A \end{cases}$$

$$\ell(\mathfrak{A}) = \begin{cases} \ell(\alpha) + \ell(A) & \text{if } \mathfrak{A} = x{:}^{\alpha}A \\ \ell(\alpha) + 1 & \text{if } \mathfrak{A} = x\mathsf{R}_a^{\alpha}y \end{cases}$$

The following lemma is helpful to make our presentation of derivations shorter.

Lemma 5.5.1. For any labelled expression $\mathfrak{A}$ and any finite multi-set of labelled expressions Γ, $\vdash_{\mathbf{GIntPAL}} \mathfrak{A}, \Gamma \Rightarrow \mathfrak{A}$.

Next, we define the notion of substitution of variables in labelled expressions.

Table 5.2: Labelled sequent calculus for IntPAL : **GIntPAL**
In what follows in this table, Δ contains at most one labelled expression.

(Initial sequents)

$$x{:}^{\alpha}A \Rightarrow x{:}^{\alpha}A \quad x\mathsf{R}_a^{\alpha}v \Rightarrow x\mathsf{R}_a^{\alpha}v$$

$$x{:}^{\alpha}\bot \Rightarrow$$

(Structural Rules)

$$\frac{\Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lw) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathfrak{A}}\ (Rw) \quad \frac{\mathfrak{A}, \mathfrak{A}, \Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lc)$$

(Rules for propositional connectives)

$$\frac{\Gamma \Rightarrow x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L\to) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow x{:}^{\alpha}B}{\Gamma \Rightarrow x{:}^{\alpha}A \to B}\ (R\to)$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge 1) \quad \frac{x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge 2) \quad \frac{\Gamma \Rightarrow x{:}^{\alpha}A \quad \Gamma \Rightarrow x{:}^{\alpha}B}{\Gamma \Rightarrow x{:}^{\alpha}A \wedge B}\ (R\wedge)$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \vee B, \Gamma \Rightarrow \Delta}\ (L\vee) \quad \frac{\Gamma \Rightarrow x{:}^{\alpha}A}{\Gamma \Rightarrow x{:}^{\alpha}A \vee B}\ (R\vee 1) \quad \frac{\Gamma \Rightarrow x{:}^{\alpha}B}{\Gamma \Rightarrow x{:}^{\alpha}A \vee B}\ (R\vee 2)$$

(Rules for knowledge operators)

$$\frac{\Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y \quad y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a A, \Gamma \Rightarrow \Delta}\ (L\Box_a) \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow y{:}^{\alpha}A}{\Gamma \Rightarrow x{:}^{\alpha}\Box_a A}\ (R\Box_a)\dagger$$

$$\frac{x\mathsf{R}_a^{\alpha}y, y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Diamond_a A, \Gamma \Rightarrow \Delta}\ (L\Diamond_a)\dagger \quad \frac{\Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y \quad \Gamma \Rightarrow y{:}^{\alpha}A}{\Gamma \Rightarrow x{:}^{\alpha}\Diamond_a A}\ (R\Diamond_a)$$

† y does not appear in the lowersequent.

(Rules for IntPAL)

$$\frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat) \quad \frac{\Gamma \Rightarrow x{:}^{\alpha}p}{\Gamma \Rightarrow x{:}^{\alpha,A}p}\ (Rat)$$

$$\frac{\Gamma \Rightarrow x{:}^{\alpha}A \quad x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, \Gamma \Rightarrow \Delta}\ (L[.]) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow x{:}^{\alpha,A}B}{\Gamma \Rightarrow x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\langle A\rangle B, \Gamma \Rightarrow \Delta}\ (L\langle.\rangle 1) \quad \frac{x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\langle A\rangle B, \Gamma \Rightarrow \Delta}\ (L\langle.\rangle 2) \quad \frac{\Gamma \Rightarrow x{:}^{\alpha}A \quad \Gamma \Rightarrow x{:}^{\alpha,A}B}{\Gamma \Rightarrow x{:}^{\alpha}\langle A\rangle B}\ (R\langle.\rangle)$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a 1) \quad \frac{y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a 2) \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a 3)$$

$$\frac{\Gamma \Rightarrow x{:}^{\alpha}A \quad \Gamma \Rightarrow y{:}^{\alpha}A \quad \Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y}{\Gamma \Rightarrow x\mathsf{R}_a^{\alpha,A}y}\ (Rrel_a)$$

Definition 5.5.2. Let $\mathfrak{A}$ be any labelled expression. Then the substitution of x for y in $\mathfrak{A}$, denoted by $\mathfrak{A}[x/y]$, is defined by

$$\begin{array}{lcl} z[x/y] & := & z \quad (\text{if } y \neq z) \\ z[x/y] & := & x \quad (\text{if } y = z) \\ (z{:}^{\alpha}A)[x/y] & := & (z[x/y]){:}^{\alpha}A \\ (z\mathsf{R}_a^{\alpha}w)[x/y] & := & (z[x/y])\mathsf{R}_a^{\alpha}(w[x/y]) \end{array}$$

Substitution $[x/y]$ to a multi-set Γ of labelled expressions is defined as

$$\Gamma[x/y] := \{\mathfrak{A}[x/y] \mid \mathfrak{A} \in \Gamma\}.$$

For a preparation of Theorem 5.5.1, we show the next lemma.

Lemma 5.5.2.

(i) $\vdash_{\mathbf{GIntPAL}} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{GIntPAL}} \Gamma[x/y] \Rightarrow \Delta[x/y]$ for any $x, y \in \mathsf{Var}$.

(ii) $\vdash_{\mathbf{GIntPAL^+}} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathbf{GIntPAL^+}} \Gamma[x/y] \Rightarrow \Delta[x/y]$ for any $x, y \in \mathsf{Var}$.

Proof. By induction on the height of the derivation. We go through almost the same procedure in the proof as in Negri et al. [58, p.194]. $\square$

Theorem 5.5.1. For any formula A, if $\vdash_{\mathbf{HIntPAL}} A$, then $\vdash_{\mathbf{GIntPAL^+}} \Rightarrow x{:}^{\epsilon}A$ (for any $x \in \mathsf{Var}$).

Proof. The proof is carried out by the height of the derivation in **HIntPAL**. Let us base cases (the derivation height of **HIntPAL** is equal to 0) except the cases of $(RA3)$ and $(RA11)$ whose derivations are similar to derivations in the proof of Theorem 3.2.1.

The case of $(IK1)$

$$\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}_a^{\epsilon}y, x{:}^{\epsilon}\Box_a p \Rightarrow x\mathsf{R}_a^{\epsilon}y$}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x\mathsf{R}_a^{\epsilon}y \Rightarrow x\mathsf{R}_a^{\epsilon}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}p, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{\epsilon}p$}
\RightLabel{$(L{\to})$}
\BinaryInfC{$x\mathsf{R}_a^{\epsilon}y, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}p$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}q, x\mathsf{R}_a^{\epsilon}y, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}q$}
\RightLabel{$(L{\to})$}
\BinaryInfC{$x\mathsf{R}_a^{\epsilon}y, y{:}^{\epsilon}p \to q, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}q$}
\RightLabel{$(L\Box)$}
\BinaryInfC{$x\mathsf{R}_a^{\epsilon}y, x{:}^{\epsilon}\Box_a(p \to q), x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}q$}
\RightLabel{$(R\Box)$}
\UnaryInfC{$x{:}^{\epsilon}\Box_a(p \to q), x{:}^{\epsilon}\Box_a p \Rightarrow x{:}^{\epsilon}\Box_a q$}
\RightLabel{$(R{\to})$}
\UnaryInfC{$x{:}^{\epsilon}\Box_a(p \to q) \Rightarrow x{:}^{\epsilon}\Box_a p \to \Box_a q$}
\RightLabel{$(R{\to})$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}\Box_a(p \to q) \to (\Box_a p \to \Box_a q)$}
\end{prooftree}$$

The case of $(IK2)$

$$\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}p, x\mathsf{R}_a^{\epsilon}y \Rightarrow x\mathsf{R}_a^{\epsilon}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}p, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{\epsilon}p$}
\RightLabel{$(R\Diamond_a)$}
\BinaryInfC{$y{:}^{\epsilon}p, x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}\Diamond_a p$}
\RightLabel{$(R\vee 1)$}
\UnaryInfC{$y{:}^{\epsilon}p, x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}\Diamond_a p \vee \Diamond_a q$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}q, x\mathsf{R}_a^{\epsilon}y \Rightarrow x\mathsf{R}_a^{\epsilon}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}q, x\mathsf{R}_a^{\epsilon}y \Rightarrow y{:}^{\epsilon}q$}
\RightLabel{$(R\Diamond_a)$}
\BinaryInfC{$y{:}^{\epsilon}q, x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}\Diamond_a q$}
\RightLabel{$(R\vee 2)$}
\UnaryInfC{$y{:}^{\epsilon}q, x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}\Diamond_a p \vee \Diamond_a q$}
\RightLabel{$(L\vee)$}
\BinaryInfC{$y{:}^{\epsilon}(p \vee q), x\mathsf{R}_a^{\epsilon}y \Rightarrow x{:}^{\epsilon}\Diamond_a p \vee \Diamond_a q$}
\RightLabel{$(L\Diamond)$}
\UnaryInfC{$x{:}^{\epsilon}\Diamond_a(p \vee q) \Rightarrow x{:}^{\epsilon}\Diamond_a p \vee \Diamond_a q$}
\RightLabel{$(R \to)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}\Diamond_a(p \vee q) \to (\Diamond_a p \vee \Diamond_a q)$}
\end{prooftree}$$

The case of $(IK3)$

$$\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}\bot, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\bot}}{x{:}^{\epsilon}\Diamond_a\bot \Rightarrow x{:}^{\epsilon}\bot}\,(L\Diamond)}{\Rightarrow x{:}^{\epsilon}\neg\Diamond_a\bot}\,(R\rightarrow)$$

The case of $(FS1)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}p\rightarrow q, x{:}^{\epsilon}\Box_a p \Rightarrow x\mathsf{R}^{\epsilon}_{a}y} \quad \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y} \quad \dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}p}}{x\mathsf{R}^{\epsilon}_{a}y, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}p} \quad \dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}q, x\mathsf{R}^{\epsilon}_{a}y, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}q}}{x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}p\rightarrow q, x{:}^{\epsilon}\Box_a p \Rightarrow y{:}^{\epsilon}q}}{x\mathsf{R}^{\epsilon}_{a}1, y{:}^{\epsilon}p\rightarrow q, x{:}^{\epsilon}\Box_a p \Rightarrow x{:}^{\epsilon}\Diamond_a q}\,(R\Diamond)}{x{:}^{\epsilon}\Diamond_a(p\rightarrow q), x{:}^{\epsilon}\Box_a p \Rightarrow x{:}^{\epsilon}\Diamond_a q}\,(L\Diamond)}{x{:}^{\epsilon}\Diamond_a(p\rightarrow q) \Rightarrow x{:}^{\epsilon}\Box_a p \rightarrow \Diamond_a q}\,(R{\rightarrow})}{\Rightarrow x{:}^{\epsilon}\Diamond_a(p\rightarrow q)\rightarrow(\Box_a p\rightarrow\Diamond_a q)}\,(R{\rightarrow})$$

The case of $(FS2)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y} \quad \dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}p}}{y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_a p}\,(R\Diamond_a) \quad \dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y} \quad \dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}q, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}q}}{x{:}^{\epsilon}\Box_a q, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}q}\,(L\Box_a)}{x{:}^{\epsilon}\Box_a q, y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}q}\,(Lw)}{x{:}^{\epsilon}\Diamond_a p\rightarrow\Box_a q, y{:}^{\epsilon}p, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}q}\,(L\rightarrow)}{x{:}^{\epsilon}\Diamond_a p\rightarrow\Box_a q, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}p\rightarrow q}\,(R\rightarrow)}{x{:}^{\epsilon}\Diamond_a p\rightarrow\Box_a q \Rightarrow x{:}^{\epsilon}\Box_a(p\rightarrow q)}\,(R\Box_a)}{\Rightarrow x{:}^{\epsilon}(\Diamond_a p\rightarrow\Box_a q)\rightarrow\Box_a(p\rightarrow q)}\,(R{\rightarrow})$$

The case of $(RA1)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}\bot \Rightarrow x{:}^{\epsilon}\bot}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\bot \Rightarrow x{:}^{\epsilon}\bot}\,(L[.])}{x{:}^{\epsilon}[A]\bot \Rightarrow x{:}^{\epsilon}\neg A}\,(R\rightarrow)}{\Rightarrow x{:}^{\epsilon}[A]\bot\rightarrow\neg A}\,(R\rightarrow) \quad \dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\bot \Rightarrow x{:}^{A}\bot}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\neg A \Rightarrow x{:}^{A}\bot}\,(L\rightarrow)}{x{:}^{\epsilon}\neg A \Rightarrow x{:}^{\epsilon}[A]\bot}\,(R[.])}{\Rightarrow x{:}^{\epsilon}\neg A\rightarrow[A]\bot}\,(R\rightarrow)}{\Rightarrow x{:}^{\epsilon}[A]\bot\leftrightarrow\neg A}\,(R\wedge)$$

The case of $(RA2)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{A}\bot \Rightarrow x{:}^{\epsilon}\bot}}{x{:}^{\epsilon}\langle A\rangle\bot \Rightarrow x{:}^{\epsilon}\bot}\,(R\langle.\rangle 2)}{\Rightarrow x{:}^{\epsilon}\langle A\rangle\bot\rightarrow\bot}\,(R\rightarrow) \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}\bot \Rightarrow x{:}^{\epsilon}\langle A\rangle\bot}}{\Rightarrow x{:}^{\epsilon}\bot\rightarrow\langle A\rangle\bot}\,(R\rightarrow)}{\Rightarrow x{:}^{\epsilon}\langle A\rangle\bot\leftrightarrow\bot}\,(R\wedge)$$

The case of $(RA4)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}q \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}q, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}q}}{x{:}^{\epsilon}A, x{:}^{A}q \Rightarrow x{:}^{\epsilon}q}\,(Lat)}{x{:}^{\epsilon}A, x{:}^{A}q \Rightarrow x{:}^{\epsilon}(A\wedge q)}\,(R\wedge)}{x{:}^{\epsilon}\langle A\rangle q \Rightarrow x{:}^{\epsilon}(A\wedge q)}\,(R\langle.\rangle) \quad \dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}q, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}q, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}q}}{x{:}^{\epsilon}q, x{:}^{\epsilon}A \Rightarrow x{:}^{A}q}\,(Rat)}{x{:}^{\epsilon}q, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}\langle A\rangle q}\,(L\langle.\rangle)}{x{:}^{\epsilon}(A\wedge q) \Rightarrow x{:}^{\epsilon}\langle A\rangle q}\,(L\wedge)}{\Rightarrow x{:}^{\epsilon}(A\wedge q)\rightarrow\langle A\rangle q}\,(R\rightarrow)}{\Rightarrow x{:}^{\epsilon}\langle A\rangle q\leftrightarrow(A\wedge q)}\,(R\wedge)$$

The case of ($RA5$): left to right

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle C$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(L\vee)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}(B \vee C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A](B \vee C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$x{:}^{\epsilon}[A](B \vee C) \Rightarrow x{:}^{\epsilon}A \rightarrow (\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A](B \vee C) \rightarrow (A \rightarrow (\langle A\rangle B \vee \langle A\rangle C))$}
\end{prooftree}
$$

The case of ($RA5$): right to left

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B \Rightarrow x{:}^{A}(B \vee C)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle C \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(L\vee)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C) \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(L \rightarrow)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}A \rightarrow (\langle A\rangle B \vee \langle A\rangle C) \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \rightarrow (\langle A\rangle B \vee \langle A\rangle C) \Rightarrow x{:}^{\epsilon}[A](B \vee C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(A \rightarrow (\langle A\rangle B \vee \langle A\rangle C)) \rightarrow [A](B \vee C)$}
\end{prooftree}
$$

The case of ($RA6$): left to right

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle C$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(L\vee)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}(B \vee C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}\langle A\rangle(B \vee C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C) \rightarrow (\langle A\rangle B \vee \langle A\rangle C)$}
\end{prooftree}
$$

The case of ($RA6$): right to left

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}\langle A\rangle B \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C)$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(R\vee)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{A}(B \vee C)$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}\langle A\rangle C \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C)$}
\RightLabel{$(L\vee)$}
\BinaryInfC{$x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C) \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \vee C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(\langle A\rangle B \vee \langle A\rangle C) \rightarrow \langle A\rangle(B \vee C)$}
\end{prooftree}
$$

The case of ($RA7$): left to right

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}B$}
\RightLabel{$(L\wedge)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}(B \wedge C) \Rightarrow x{:}^{A}B$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A](B \wedge C) \Rightarrow x{:}^{A}B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}[A](B \wedge C) \Rightarrow x{:}^{\epsilon}[A]B$}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C$}
\RightLabel{$(L\wedge)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}(B \wedge C) \Rightarrow x{:}^{A}C$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A](B \wedge C) \Rightarrow x{:}^{A}C$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}[A](B \wedge C) \Rightarrow x{:}^{\epsilon}[A]C$}
\RightLabel{$(R\wedge)$}
\BinaryInfC{$x{:}^{\epsilon}[A](B \wedge C) \Rightarrow x{:}^{\epsilon}([A]B \wedge [A]C)$}
\RightLabel{$(R \rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A](B \wedge C) \rightarrow ([A]B \wedge [A]C)$}
\end{prooftree}
$$

The case of $(RA7)$: right to left

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}B}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{A}C \Rightarrow x{:}^{A}B}\,(L[.])}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A]C \Rightarrow x{:}^{A}B}\,(L[.]) \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{A}C \Rightarrow x{:}^{A}C}}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A]C \Rightarrow x{:}^{A}C}\,(L[.])}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A]C \Rightarrow x{:}^{A}(B \wedge C)}\,(R\wedge)}{x{:}^{\epsilon}[A]B, x{:}^{\epsilon}[A]C \Rightarrow x{:}^{\epsilon}[A](B \wedge C)}\,(R[.])}{x{:}^{\epsilon}([A]B \wedge [A]C) \Rightarrow x{:}^{\epsilon}[A](B \wedge C)}\,(L\wedge)}{\Rightarrow x{:}^{\epsilon}([A]B \wedge [A]C) \rightarrow [A](B \wedge C)}\,(R \rightarrow)
$$

The case of $(RA8)$: left to right

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}B}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle B}\,(L\langle.\rangle) \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle C}\,(L\langle.\rangle)}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \wedge \langle A\rangle C)}\,(R\wedge)}{x{:}^{\epsilon}A, x{:}^{A}(B \wedge C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \wedge \langle A\rangle C)}\,(L\wedge)}{x{:}^{\epsilon}\langle A\rangle(B \wedge C) \Rightarrow x{:}^{\epsilon}(\langle A\rangle B \wedge \langle A\rangle C)}\,(R\langle.\rangle)}{\Rightarrow x{:}^{\epsilon}\langle A\rangle(B \wedge C) \rightarrow (\langle A\rangle B \wedge \langle A\rangle C)}\,(R \rightarrow)
$$

The case of $(RA8)$: right to left

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}B} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}(B \wedge C)}\,(R\wedge)}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \wedge C)}\,(L\langle.\rangle)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle C, x{:}^{A}B \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \wedge C)}\,(R\langle.\rangle)}{x{:}^{\epsilon}\langle A\rangle B, x{:}^{\epsilon}\langle A\rangle C \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \wedge C)}\,(R\langle.\rangle)}{x{:}^{\epsilon}(\langle A\rangle B \wedge \langle A\rangle C) \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \wedge C)}\,(L\wedge)}{\Rightarrow x{:}^{\epsilon}(\langle A\rangle B \wedge \langle A\rangle C) \rightarrow \langle A\rangle(B \wedge C)}\,(R \rightarrow)
$$

The case of $(RA10)$: left to right

$$
\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{A}C}\,(L \rightarrow)}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}\langle A\rangle C}\,(L\langle.\rangle)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}\langle A\rangle C}\,(R\langle.\rangle)}{x{:}^{\epsilon}A, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}\langle A\rangle B \rightarrow \langle A\rangle C}\,(R \rightarrow)}{x{:}^{\epsilon}A, x{:}^{A}B \rightarrow C \Rightarrow x{:}^{\epsilon}(A \wedge (\langle A\rangle B \rightarrow \langle A\rangle C))}\,(R\wedge)}{x{:}^{\epsilon}\langle A\rangle(B \rightarrow C) \Rightarrow x{:}^{\epsilon}(A \wedge (\langle A\rangle B \rightarrow \langle A\rangle C))}\,(R\langle.\rangle)}{\Rightarrow x{:}^{\epsilon}\langle A\rangle(B \rightarrow C) \rightarrow (A \wedge (\langle A\rangle B \rightarrow \langle A\rangle C))}\,(R \rightarrow)
$$

The case of $(RA10)$: right to left

$$
\dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B \rightarrow \langle A\rangle C \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}\langle A\rangle B}\,(L\langle.\rangle) \quad \dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A}C \Rightarrow x{:}^{A}C}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle C, x{:}^{A}B \Rightarrow x{:}^{A}C}\,(R\langle.\rangle)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B \rightarrow \langle A\rangle C, x{:}^{A}B \Rightarrow x{:}^{A}C}\,(L \rightarrow)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B \rightarrow \langle A\rangle C \Rightarrow x{:}^{A}B \rightarrow C}\,(R \rightarrow)}{x{:}^{\epsilon}A, x{:}^{\epsilon}\langle A\rangle B \rightarrow \langle A\rangle C \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \rightarrow C)}\,(L\langle.\rangle)}{x{:}^{\epsilon}(A \wedge (\langle A\rangle B \rightarrow \langle A\rangle C)) \Rightarrow x{:}^{\epsilon}\langle A\rangle(B \rightarrow C)}\,(L\wedge)}{\Rightarrow x{:}^{\epsilon}(A \wedge (\langle A\rangle B \rightarrow \langle A\rangle C)) \rightarrow \langle A\rangle(B \rightarrow C)}\,(R \rightarrow)
$$

The case of ($RA11$): left to right

$$
\dfrac{
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A}
\qquad
\dfrac{\dfrac{\dfrac{
\dfrac{
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y}
}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{A}_{a}y}\,(Rel)
\quad
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow 1{:}^{A}B}
}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x{:}^{A}\Box_{a}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{A}B}\,(L\Box)
}{x{:}^{\epsilon}A, x{:}^{A}\Box_{a}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}[A]B}\,(R[.])
}{x{:}^{\epsilon}A, x{:}^{A}\Box_{a}B \Rightarrow x{:}^{\epsilon}\Box_{a}[A]B}\,(R\Box)
}{
\dfrac{
\dfrac{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Box_{a}B \Rightarrow x{:}^{\epsilon}\Box_{a}[A]B}{x{:}^{\epsilon}[A]\Box_{a}B \Rightarrow x{:}^{\epsilon}A \to \Box_{a}[A]B}\,(R\to)
}{\Rightarrow x{:}^{\epsilon}[A]\Box_{a}B \to (A \to \Box_{a}[A]B)}\,(R\to)
}\,(L[.])
$$

The case of ($RA11$): right to left

$$
\dfrac{\dfrac{\dfrac{
\dfrac{
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x\mathsf{R}^{A}_{a}y \Rightarrow x{:}^{\epsilon}A}
\quad
\dfrac{
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y}}{x{:}^{\epsilon}A, x\mathsf{R}^{A}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y}\,(Lrel3)
\quad
\dfrac{
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B}
}{x{:}^{\epsilon}A, y{:}^{\epsilon}[A]B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B}\,(L[.])
}{x{:}^{\epsilon}A, x{:}^{\epsilon}\Box_{a}[A]B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B}\,(L\Box)
}{x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \Box_{a}[A]B, x\mathsf{R}^{A}_{a}y \Rightarrow y{:}^{A}B}\,(L\to)
}{x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \Box_{a}[A]B \Rightarrow x{:}^{A}\Box_{a}B}\,(R\Box)
}{x{:}^{\epsilon}A \to \Box_{a}[A]B \Rightarrow x{:}^{\epsilon}[A]\Box_{a}B}\,(R[.])
}{\Rightarrow x{:}^{\epsilon}(A \to \Box_{a}[A]B) \to [A]\Box_{a}B}\,(R\to)
$$

The case of ($RA12$): left to right

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A}}{x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A}\,(L\langle.\rangle 1)
\quad
\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\textit{Initial Seq.}}{y{:}^{A}B, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y}
\quad
\dfrac{\dfrac{
\dfrac{\textit{Initial Seq.}}{y{:}^{A}B, y{:}^{\epsilon}A \Rightarrow y{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{y{:}^{A}B, y{:}^{\epsilon}A \Rightarrow y{:}^{A}B}
}{y{:}^{A}B, y{:}^{\epsilon}A \Rightarrow y{:}^{\epsilon}\langle A\rangle B}\,(L\langle.\rangle)
}{y{:}^{A}B, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}\langle A\rangle B}\,(Lw)
}{y{:}^{A}B, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(R\Diamond_{a})
}{y{:}^{A}B, x\mathsf{R}^{A}_{a}y, x\mathsf{R}^{A}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(Lrel_{a})
}{x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(L\Diamond_{a})
}{x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(L\langle.\rangle 2)
}{x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A \wedge \Diamond_{a}\langle A\rangle B}\,(R\wedge)
}{\Rightarrow x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \to A \wedge \Diamond_{a}\langle A\rangle B}\,(R\to)
$$

The case of ($RA12$): right to left

$$
\dfrac{\dfrac{\dfrac{
\dfrac{\textit{Initial Seq.}}{x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y}
\quad
\dfrac{\dfrac{
\dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}A}
\quad
\dfrac{\textit{Initial Seq.}}{y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{A}B}
}{y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}\langle A\rangle B}\,(L\langle.\rangle)
}{y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}\langle A\rangle B}\,(Lw)
}{y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(R\Diamond_{a})
}{y{:}^{A}B, x\mathsf{R}^{A}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(Lrel)
}{x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(L\Diamond_{a})
}{x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B}\,(Lw)
}{x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A \wedge \Diamond_{a}\langle A\rangle B}\,(R\wedge)
}{x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A \wedge \Diamond_{a}\langle A\rangle B}\,(R\langle.\rangle)
}{\Rightarrow x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \to A \wedge \Diamond_{a}\langle A\rangle B}\,(R\to)
$$

The case of ($RA13$): right to left

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{A}B$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{$(R\Diamond)$}
\BinaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(Lrel)$}
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{A}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(L\Diamond)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(L[.])$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(R\rightarrow)$}
\UnaryInfC{$x{:}^{\epsilon}[A]\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A \rightarrow \Diamond_{a}\langle A\rangle B$}
\RightLabel{$(R\rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}[A]\Diamond_{a}B \rightarrow (A \rightarrow \Diamond_{a}\langle A\rangle B)$}
\end{prooftree}
$$

The case of ($RA13$): right to left2

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\UnaryInfC{$x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}A, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}A, x{:}^{\epsilon}A \Rightarrow y{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}A, x{:}^{\epsilon}A \Rightarrow x\mathsf{R}^{\epsilon}_{a}y$}
\TrinaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}A, x{:}^{\epsilon}A \Rightarrow x\mathsf{R}^{A}_{a}y$}
\RightLabel{$(L\langle.\rangle 1)$}
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}\langle A\rangle B, x{:}^{\epsilon}A \Rightarrow x\mathsf{R}^{A}_{a}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{A}B, x{:}^{\epsilon}A \Rightarrow y{:}^{A}B$}
\RightLabel{$(L\langle.\rangle 2)$}
\UnaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}\langle A\rangle B, x{:}^{\epsilon}A \Rightarrow y{:}^{A}B$}
\RightLabel{$(R\Diamond)$}
\BinaryInfC{$x\mathsf{R}^{\epsilon}_{a}y, y{:}^{\epsilon}\langle A\rangle B, x{:}^{\epsilon}A \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(L\Diamond)$}
\UnaryInfC{$x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B, x{:}^{\epsilon}A \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(L\rightarrow)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}A \rightarrow \Diamond_{a}\langle A\rangle B \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(R[.])$}
\UnaryInfC{$x{:}^{\epsilon}A \rightarrow \Diamond_{a}\langle A\rangle B \Rightarrow x{:}^{\epsilon}[A]\Diamond_{a}B$}
\RightLabel{$(R\rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(A \rightarrow \Diamond_{a}\langle A\rangle B) \rightarrow [A]\Diamond_{a}B$}
\end{prooftree}
$$

The case of ($RA14$): right to left

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x\mathsf{R}^{\epsilon}_{a}y$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{A}B$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B \Rightarrow y{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow y{:}^{\epsilon}\langle A\rangle B$}
\RightLabel{$(R\Diamond)$}
\BinaryInfC{$x{:}^{\epsilon}A, y{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(Lrel)$}
\UnaryInfC{$x{:}^{\epsilon}A, y{:}^{A}B, x\mathsf{R}^{A}_{a}y \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(L\Diamond)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B$}
\RightLabel{$(R\wedge)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{A}\Diamond_{a}B \Rightarrow x{:}^{\epsilon}(A \wedge \Diamond_{a}\langle A\rangle B)$}
\RightLabel{$(R\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \Rightarrow x{:}^{\epsilon}(A \wedge \Diamond_{a}\langle A\rangle B)$}
\RightLabel{$(R\rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B \rightarrow (A \wedge \Diamond_{a}\langle A\rangle B)$}
\end{prooftree}
$$

The case of ($RA14$): right to left2

$$
\begin{prooftree}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow 1{:}^{\epsilon}A$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x\mathsf{R}^{\epsilon}_{a}1$}
\RightLabel{$(Rrel)$}
\TrinaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x\mathsf{R}^{A}_{a}1$}
\RightLabel{(Lw)}
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, 1{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x\mathsf{R}^{A}_{a}1$}
\AxiomC{\textit{Initial Seq.}}
\doubleLine
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow 1{:}^{A}B$}
\RightLabel{$(R\Diamond)$}
\BinaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}A, 1{:}^{A}B, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(L\langle.\rangle)$}
\UnaryInfC{$x{:}^{\epsilon}A, 1{:}^{\epsilon}\langle A\rangle B, x\mathsf{R}^{\epsilon}_{a}1 \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(L\Diamond)$}
\UnaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B \Rightarrow x{:}^{A}\Diamond_{a}B$}
\RightLabel{$(L\langle.\rangle)$}
\BinaryInfC{$x{:}^{\epsilon}A, x{:}^{\epsilon}\Diamond_{a}\langle A\rangle B \Rightarrow x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B$}
\RightLabel{$(L\wedge)$}
\UnaryInfC{$x{:}^{\epsilon}(A \wedge \Diamond_{a}\langle A\rangle B) \Rightarrow x{:}^{\epsilon}\langle A\rangle\Diamond_{a}B$}
\RightLabel{$(R\rightarrow)$}
\UnaryInfC{$\Rightarrow x{:}^{\epsilon}(A \wedge \Diamond_{a}\langle A\rangle B) \rightarrow \langle A\rangle\Diamond_{a}B$}
\end{prooftree}
$$

In the inductive step, we show the admissibility of **HIntPAL**'s inference rules, (MP), ($Nec\Box_a$) and (Nec[.]), by **GIntPAL**$^+$. In induction step, we show the admissibility of the inference rules **HIntPAL**, such as (MP), ($Nec\Box_a$) and (Nec[.]). Proofs of the first two cases are similar to proofs in Theorem 3.2.1. Therefore, we show the inference rule of the last rule.

The case of (Nec[.])**:** In the case, we show the admissibility of the following rule:

$$\frac{\Rightarrow x{:}^{\epsilon}A}{\Rightarrow x{:}^{\epsilon}[B]A}\ (Nec[.]).$$

Suppose $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{\epsilon}A$. It is obvious that $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{\epsilon}A$ implies $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{B}A$ since if there is a derivation of $\Rightarrow x{:}^{\epsilon}A$, there can also be a derivation of $\Rightarrow x{:}^{B}A$ where B is added to the most left side of restricting formulas of each labelled expression appeared in the derivation. Therefore, we obtain $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{B}A$, and by the application of (Lw) and (R[.]), we conclude $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{\epsilon}[B]A$. □

5.6 Cut Elimination of GIntPAL$^+$

Now, we show the rule (Cut) of **GIntPAL**$^+$ is admissible. For a preparation of the cut elimination theorem, we show the following lemma.

Lemma 5.6.1. If a sequent $\Gamma \Rightarrow x{:}^{\alpha}\bot$ can be derivable without using (Cut), then $\Gamma \Rightarrow$ can also be derivable without using (Cut).

Proof. By induction on the height of the derivation. And every case in the inductive step, in which the last applied rule is either (Rw) or one of left rules, can be shown straightforwardly with inductive hypothesis and the same rule as the last rule applied. We only look at the base case.

In the base case, since $\Gamma \Rightarrow x{:}^{\alpha}\bot$ is the initial sequent, Γ should be the singleton $\{x{:}^{\alpha}\bot\}$. Then $x{:}^{\alpha}\bot \Rightarrow$ is also the initial sequent and so derivable. □

Here we prove one of contributions of the paper, the syntactic cut elimination theorem of **GIntPAL**$^+$.

Theorem 5.6.1 (Cut elimination of **GIntPAL**$^+$)**.** For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{GIntPAL}^+}$ $\Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{GIntPAL}}$ $\Gamma \Rightarrow \Delta$.

Proof. The proof is carried out in Ono and Komori's method [67] introduced in the reference [41] by Kashima where we employ the following rule ($Ecut$). We denote the n-copies of the same labelled expression $\mathfrak{A}$ by $\mathfrak{A}^n$, and ($Ecut$) is defined as follows:

$$\frac{\Gamma \Rightarrow \mathfrak{A}^n \quad \mathfrak{A}^m, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\ (Ecut)$$

where $n \leq 1$ and $m \geq 0$. The theorem is shown by double induction on the height of the derivation and the length of the cut expression $\mathfrak{A}$ of ($Ecut$). The proof is divided into four cases:

(1) at least one of uppersequents of (*Ecut*) is an initial sequent;

(2) the last inference rule of either uppersequents of (*Ecut*) is a structural rule;

(3) the last inference rule of either uppersequents of (*Ecut*) is a non-structural rule, and the principal expression introduced by the rule is not a cut expression;

(4) the last inference rules of two uppersequents of (*Ecut*) are both non-structural rules, and the principal expressions introduced by the rules used on the uppersequents of (*Ecut*) are both cut expressions.

We look at one of base cases and one of significant subcases of (4) in which principal expressions introduced by non-structural rules are both cut expressions. We illustrate some cases in the following.

Case of (1) where the right uppersequent is an initial sequent $x{:}^{\alpha}\bot \Rightarrow$ **:** In this case, the form of the derivation is like following:

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow x{:}^{\alpha}\bot} \qquad \dfrac{\textit{Initial Seq.}}{x{:}^{\alpha}\bot \Rightarrow}}{\Gamma \Rightarrow}\ (\textit{Ecut})$$

From the left uppersequent $\Gamma \Rightarrow x{:}^{\alpha}\bot$, we get, without (*Ecut*), the lowersequent $\Gamma \Rightarrow$ by Lemma 5.6.1.

Case of (1) where the left uppersequent is an initial sequent $x{:}^{\alpha}\bot \Rightarrow$ **:** In this case, the form of the derivation is like following:

$$\dfrac{\dfrac{\textit{Initial Seq.}}{x{:}^{\alpha}\bot \Rightarrow} \qquad \dfrac{\vdots}{\mathfrak{A}^n, \Gamma \Rightarrow \Delta}}{x{:}^{\alpha}\bot, \Gamma \Rightarrow \Delta}\ (\textit{Ecut})$$

We obtain the lowersequent from the left uppersequent $x{:}^{\alpha}\bot \Rightarrow$ with applying (*Lw*) and (*Rw*) possibliy multiple times.

Case of (2) where the right uppersequent of (*Ecut*) is structural rule (*Lc*) which contracts the same expression as the cut expressin.

$$\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow (x{:}^{\alpha}A)^n} \qquad \dfrac{\dfrac{\vdots\ \mathcal{D}_2}{(x{:}^{\alpha}A)^{m+1}, \Gamma' \Rightarrow \Delta'}}{(x{:}^{\alpha}A)^{m}, \Gamma' \Rightarrow \Delta'}\ (\textit{Lc})}{\Gamma, \Gamma' \Rightarrow \Delta'}\ (\textit{Ecut})$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow (x{:}^{\alpha}A)^n} \qquad \dfrac{\vdots\ \mathcal{D}_2}{(x{:}^{\alpha}A)^{m+1}, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta'}\ (\textit{Ecut}).$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Lc*).

Case of (2) where the right uppersequent of (*Ecut*) is structural rule (*Lc*) which contracts a different expression from the cut expressin.

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow (x{:}^{\alpha}A)^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^m, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta'\end{array}}{(x{:}^{\alpha}A)^m, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Lc)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow (x{:}^{\alpha}A)^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^m, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \mathfrak{B}, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{\Gamma, \mathfrak{B}, \Gamma' \Rightarrow \Delta'}\,(Lc) \quad .$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule (*Lc*).

Case of (2) where one of uppersequents of (*Ecut*) is structural rule (*Rw*).

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow x{:}^{\alpha}A}\,(Rw) \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^m, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A)^m, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut) .$$

Case of (3) where one of uppersequents of (*Ecut*) is inference rule (*R*¬).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow\end{array}}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}\neg A}\,(R\neg)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}\neg A}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, x{:}^{\alpha}A, \Gamma' \Rightarrow\end{array}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}\neg A}\,(R\neg)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($R\neg$).

Case of (3) where one of uppersequents of (*Ecut*) is inference rule ($L\neg$).

$$\dfrac{\dfrac{}{\Gamma \Rightarrow \mathfrak{A}^n}\vdots\mathcal{D}_1 \quad \dfrac{\dfrac{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A}}{\mathfrak{A}^m, x{:}^{\alpha}\neg A, \Gamma' \Rightarrow}(L\neg)}{x{:}^{\alpha}\neg A, \Gamma, \Gamma' \Rightarrow}(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}A}(Ecut)}{x{:}^{\alpha}\neg A, \Gamma, \Gamma' \Rightarrow}(L\neg)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($L\neg$).

Case of (3) where one of uppersequents of (*Ecut*) is inference rule ($R \rightarrow$).

$$\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\,\mathcal{D}_2}{x{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}B}}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A \rightarrow B}(R \rightarrow)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}A \rightarrow B}(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, x{:}^{\alpha}A, \Gamma' \Rightarrow x{:}^{\alpha}B}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow x{:}^{\alpha}B}(Ecut)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}A \rightarrow B}(R \rightarrow)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($R \rightarrow$).

Case of (3) where one of uppersequents of (*Ecut*) is inference rule ($L \rightarrow$).

$$\dfrac{\overset{\vdots\,\mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\,\mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A} \quad \overset{\vdots\,\mathcal{D}_3}{x{:}^{\alpha}B, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}}{\mathfrak{A}^m, x{:}^{\alpha}A \rightarrow B, \Gamma' \Rightarrow \Delta'}(L \rightarrow)}{x{:}^{\alpha}A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta'}(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A}}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}A}\,(Ecut) \quad \dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\ \mathcal{D}_3}{\mathfrak{A}^m, x{:}^{\alpha}B, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha}B, \Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{x{:}^{\alpha}A \to B, \Gamma, \Gamma' \Rightarrow \Delta'}\,(L\to)$$

Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule ($L \to$).

Case of (3) where one of uppersequents of ($Ecut$) is inference rule ($L\Diamond_a$).

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a y, y{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'}}{\mathfrak{A}^m, x{:}^{\alpha}\Diamond_a A, \Gamma' \Rightarrow \Delta'}\,(L\Diamond_a)}{x{:}^{\alpha}\Diamond_a A, \Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)$$

If y does not appear in $\Gamma \Rightarrow \mathfrak{A}^n$, it does not matter and leave y as it is. We consider the case where y appears in the sequent. In this case, label y is, by Lemma 3.2.2, replaced with z which does not appear in both $\Gamma \Rightarrow \mathfrak{A}^n$ and $\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a y, y{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'$, and let the derivation of $\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z, \Gamma' \Rightarrow \Delta', z{:}^{\alpha}A$ be $\mathcal{D}'_2$. Then the derivation is transformed into the following:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\ \mathcal{D}'_2}{\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z, z{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'}}{x\mathsf{R}^{\alpha}_a z, z{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{x{:}^{\alpha}\Diamond_a A, \Gamma, \Gamma' \Rightarrow \Delta'}\,(L\Diamond_a)$$

Case of (3) where one of uppersequents of ($Ecut$) is inference rule ($R\Box_a$).

$$\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \dfrac{\overset{\vdots\ \mathcal{D}_2}{\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a y, \Gamma' \Rightarrow y{:}^{\alpha}A}}{\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}\Box_a A}\,(R\Box_a)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}\Box_a A}\,(Ecut)$$

If y does not appear in $\Gamma \Rightarrow \mathfrak{A}^n$, it does not matter and leave y as it is. We consider the case where y appears in the sequent. In this case, label y is, by Lemma 5.5.2, replaced with z which does not appear in both $\Gamma \Rightarrow \mathfrak{A}^n$ and $\mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}\Box_a A$, and let the derivation of $\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z, \Gamma' \Rightarrow z{:}^{\alpha}A$ be $\mathcal{D}'_2$. Then the derivation is transformed into the following:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \mathfrak{A}^n} \quad \overset{\vdots\ \mathcal{D}'_2}{\mathfrak{A}^m, x\mathsf{R}^{\alpha}_a z, \Gamma' \Rightarrow z{:}^{\alpha}A}}{x\mathsf{R}^{\alpha}_a z, \Gamma, \Gamma' \Rightarrow z{:}^{\alpha}A}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}\Box_a A}\,(R\Box_a)$$

Case of (3) where one of uppersequents of ($Ecut$) is inference rule ($L\Box_a$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow x\mathsf{R}_a y\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ y{:}^{\alpha}A, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{\mathfrak{A}^m, x{:}^{\alpha}\Box_a A, \Gamma' \Rightarrow \Delta'}\ (L\Box_a)}{x{:}^{\alpha}\Box_a A, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow x\mathsf{R}_a y\end{array}}{\Gamma, \Gamma' \Rightarrow x\mathsf{R}_a y}\ (Ecut) \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m, y{:}^{\alpha}A, \Gamma' \Rightarrow \Delta'\end{array}}{y{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)}{x{:}^{\alpha}\Box_a A, \Gamma, \Gamma' \Rightarrow \Delta'}\ (L\Box_a)$$

Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule ($L\Box_a$).

Case of (3) where one of uppersequents of ($Ecut$) is inference rule (Lat).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha}p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha,A}p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}\ (Lat)}{x{:}^{\alpha,A}p, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ x{:}^{\alpha}p, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha}p, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)}{x{:}^{\alpha,A}p, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Lat)$$

Similarly to this, we can show the case where the left uppersequent of ($Ecut$) is structural rule (Lat).

Case of (3) where one of uppersequents of ($Ecut$) is inference rule ($L[.]$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ x{:}^{\alpha,A}B, \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'\end{array}}{\mathfrak{A}^m, x{:}^{\alpha}[A]B, \Gamma' \Rightarrow \Delta'}\ (L[.])}{x{:}^{\alpha}[A]B, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m, \Gamma' \Rightarrow x{:}^{\alpha}A\end{array}}{\Gamma, \Gamma' \Rightarrow x{:}^{\alpha}A}\ (Ecut) \quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow \mathfrak{A}^n\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m, x{:}^{\alpha,A}B, \Gamma' \Rightarrow \Delta'\end{array}}{x{:}^{\alpha,A}B, \Gamma, \Gamma' \Rightarrow \Delta'}\ (Ecut)}{x{:}^{\alpha}[A]B, \Gamma, \Gamma' \Rightarrow \Delta'}\ (L[.])$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($L[.]$).

Case of (3) where one of uppersequents of (*Ecut*) is inference rule ($Rrel_a$).

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\mathfrak{A}^n\end{array}\quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow y{:}^{\alpha}A\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m,\Gamma'\Rightarrow x{:}^{\alpha}A\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_4\\ \mathfrak{A}^m,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha}y\end{array}}{\mathfrak{A}^m,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha,A}y}\,(Rrel_a)}{\Gamma,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha,A}y}\,(Ecut)$$

This is transformed into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\mathfrak{A}^n\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \mathfrak{A}^m,\Gamma'\Rightarrow x{:}^{\alpha}A\end{array}}{\Gamma,\Gamma'\Rightarrow x{:}^{\alpha}A}\,(Ecut)\quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\mathfrak{A}^n\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \mathfrak{A}^m,\Gamma'\Rightarrow y{:}^{\alpha}A\end{array}}{\Gamma,\Gamma'\Rightarrow y{:}^{\alpha}A}\,(Ecut)\quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \Gamma\Rightarrow\mathfrak{A}^n\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_4\\ \mathfrak{A}^m,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha}y\end{array}}{\Gamma,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha}y}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow x\mathsf{R}_a^{\alpha,A}y}\,(Rrel_a)$$

Similarly to this, we can show the case where the left uppersequent of (*Ecut*) is structural rule ($Rrel_a$).

Case of (4) where both sides of $\mathfrak{A}$ are $x{:}^{\alpha}A\rightarrow B$ and principal, when we obtain the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow x{:}^{\alpha}B\end{array}}{\Gamma\Rightarrow x{:}^{\alpha}A\rightarrow B}\,(R\rightarrow)\quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow x{:}^{\alpha}A\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ x{:}^{\alpha}B,(x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow\Delta'\end{array}}{(x{:}^{\alpha}A\rightarrow B)^{m},\Gamma'\Rightarrow\Delta'}\,(L\rightarrow)}{\Gamma,\Gamma'\Rightarrow\Delta'}\,(Ecut)$$

This is transformed into the derivation:

$$\mathcal{A}=\left\{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1^{+}\\ \Gamma\Rightarrow x{:}^{\alpha}A\rightarrow B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_2\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},\Gamma'\Rightarrow x{:}^{\alpha}A\end{array}}{\Gamma,\Gamma'\Rightarrow x{:}^{\alpha}A}\,(Ecut)\right.$$

$$\mathcal{A}'=\left\{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1^{+}\\ \Gamma\Rightarrow x{:}^{\alpha}A\rightarrow B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{D}_3\\ (x{:}^{\alpha}A\rightarrow B)^{m-1},x{:}^{\alpha}B,\Gamma'\Rightarrow\Delta'\end{array}}{x{:}^{\alpha}B,\Gamma,\Gamma'\Rightarrow\Delta'}\,(Ecut)\right.$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{A}\\ \Gamma,\Gamma'\Rightarrow\Delta',x{:}^{\alpha}A\end{array}\quad \dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\alpha}A,\Gamma\Rightarrow x{:}^{\alpha}B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{A}'\\ x{:}^{\alpha}B,\Gamma,\Gamma'\Rightarrow\Delta'\end{array}}{x{:}^{\alpha}A,\Gamma,\Gamma,\Gamma'\Rightarrow\Delta'}\,(Ecut)}{\Gamma,\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta'}\,(Ecut)}{\Gamma,\Gamma'\Rightarrow\Delta'}\,(Lc)$$

Case of (4): principal expressions are $x\mathsf{R}_a^{\alpha,A}y$: Let us consider the case where both sides of $\mathfrak{A}$ are $x\mathsf{R}_a^{\alpha,A}y$ and principal expressions. When we obtain the derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow x{:}^{\alpha}A\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \Gamma \Rightarrow y{:}^{\alpha}A\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_3\\ \Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y\end{matrix}}{\Gamma \Rightarrow x\mathsf{R}_a^{\alpha,A}y}\,(Lrel_a) \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_4\\ x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta\end{matrix}}{(x\mathsf{R}_a^{\alpha,A}y)^{m}, \Gamma' \Rightarrow \Delta}\,(Lrel_a1)}{\Gamma, \Gamma' \Rightarrow \Delta}\,(Ecut)\quad ,$$

it is transformed into the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow x{:}^{\alpha}A\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}'_{123}\\ \Gamma \Rightarrow x\mathsf{R}_a^{\alpha,A}y\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_4\\ x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta\end{matrix}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta}\,(Ecut)}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta}\,(Lc)$$

where the upper application of $(Ecut)$ is possible by the induction hypothesis, since the derivation height of $(Ecut)$ is reduced by comparison with the original derivation. Besides, the lower application of $(Ecut)$ is also allowed by induction hypothesis, since the length of the cut expression is reduced, namely $\ell(x{:}^{\alpha}A) < \ell(x\mathsf{R}_a^{\alpha,A}y)$.

Case of (4): principal expressions are $x{:}^{\alpha}\Diamond_a A$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}_2\\ \Gamma \Rightarrow y{:}^{\alpha}A\end{matrix}}{\Gamma \Rightarrow x{:}^{\alpha}\Diamond_a A}\,(R\Diamond_a) \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_3\\ x\mathsf{R}_a^{\alpha}z, z{:}^{\alpha}A, (x{:}^{\alpha}\Diamond_a A)^{n-1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{(x{:}^{\alpha}\Diamond_a A)^{n}, \Gamma' \Rightarrow \Delta'}\,(L\Diamond_a)}{\Gamma, \Gamma' \Rightarrow \Delta'}$$

Applying Lemma 5.5.2 to the bottom sequent of $\mathcal{D}_3$, and we obtain $\vdash_{\mathbf{GIntPAL}} x\mathsf{R}_a^{\alpha}y, y{:}^{\alpha}A, (x{:}^{\alpha}\Diamond_a A)^{n}, \Gamma' \Rightarrow \Delta'$. Let the derivation be $\mathcal{D}'_3$. Then we obtain the following derivation.

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_2\\ \Gamma \Rightarrow y{:}^{\alpha}A\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Gamma \Rightarrow x\mathsf{R}_a^{\alpha}y\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \mathcal{D}_{12}\\ \Gamma \Rightarrow x{:}^{\alpha}\Diamond_a A\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{D}'_3\\ y{:}^{\alpha}A, x\mathsf{R}_a^{\alpha}y, (x{:}^{\alpha}\Diamond_a A)^{n-1}, \Gamma' \Rightarrow \Delta'\end{matrix}}{y{:}^{\alpha}A, x\mathsf{R}_a^{\alpha}y, \Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{y{:}^{\alpha}A, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{\Gamma, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta'}\,(Lc)$$

where the upper application of $(Ecut)$ is possible by the induction hypothesis, since the derivation height of $(Ecut)$ is reduced. Besides, the lower two applications of $(Ecut)$ are also possible by induction hypothesis, since the length of the cut expression is reduced. $\square$

As a corollary of Theorem 5.6.1, the consistency of $\mathbf{GIntPAL}^{+}$ is shown.

Corollary 5.6.1. The empty sequent $\Rightarrow$ cannot be derived in $\mathbf{GIntPAL}^+$.

Proof. Suppose for contradiction that $\mathbf{GIntPAL}^+ \vdash\ \Rightarrow$. By Theorem 5.6.1, $\mathbf{GIntPAL} \vdash\ \Rightarrow$ is obtained. However, there is no inference rule in **GIntPAL** which can derive the empty sequent. A contradiction. □

5.7 Soundness of GIntPAL

Now, we switch the subject to the soundness theorem of **GIntPAL**. At first, we define the notion of the satisfaction relation for the labelled expressions, i.e., lift the satisfaction relation for the non-labelled formulas to that of the labelled expressions. Let us say that $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ is an *assignment*, where we recall that Var is the set of all labels.

Definition 5.7.1. Let $\mathfrak{M}$ be an *IntK*-model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment.

$$
\begin{array}{lll}
\mathfrak{M}, f \Vdash x{:}^{\alpha}A & \text{iff} & \mathfrak{M}^{\alpha}, f(x) \Vdash A \text{ and } f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}) \\
\mathfrak{M}, f \Vdash x\mathsf{R}^{\epsilon}_{a}y & \text{iff} & (f(x), f(y)) \in R_a \\
\mathfrak{M}, f \Vdash x\mathsf{R}^{\alpha,A}_{a}y & \text{iff} & (f(x), f(y)) \in R^{\alpha}_{a} \text{ and } \mathfrak{M}^{\alpha}, f(x) \Vdash A \text{ and } \mathfrak{M}^{\alpha}, f(y) \Vdash A
\end{array}
$$

In this definition, we have to be careful of the notion of *survival* as suggested in [61]. In brief, $f(x)$ and $f(y)$ above must be defined in $\mathcal{D}(\mathfrak{M}^{\alpha})$ which may be smaller than $\mathcal{D}(\mathfrak{M})$. In the clause $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$, for example, $f(x)$ should survive in the restricted *IntK*-model $\mathfrak{M}^{\alpha}$. Taking into account of the fact, it is essential that we pay attention to the negation of $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$.

Proposition 5.7.1. $\mathfrak{M}, f \nVdash x{:}^{\alpha}A$ iff $f(x) \notin \mathcal{D}(\mathfrak{M}^{\alpha})$ or ($f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and $\mathfrak{M}^{\alpha}, f(x) \nVdash A$).

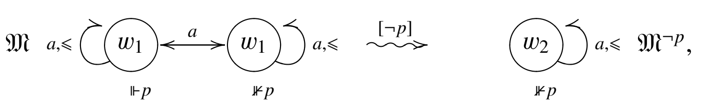
We introduce the first notion of validity for sequents which is defined in a natural and usual way, called *s-validity*, but similar to the cases of **GPAL** and **GEAK** (Proposition 3.3.2 and Proposition 4.4.3) the definition will soon turn out to be inappropriate for showing the soundness theorem.

Definition 5.7.2 (s-validity)**.** $\Gamma \Rightarrow \Delta$ is *s-valid* in $\mathfrak{M}$ if for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \Vdash \mathfrak{B}$.

If we follow s-validity, then we come to a deadlock on the way to prove the soundness theorem, as we can see the following proposition.

Proposition 5.7.2. There is an *IntK*-model $\mathfrak{M}$ such that ($R{\to}$) of **GIntPAL** does not preserve s-validity in $\mathfrak{M}$.

Proof. We use the same models as in Example 5.2.1:



where we note that $V^{\neg p}(p) = \emptyset$. Then we consider a particular instance of $(R{\to})$:[4]

$$\frac{x{:}^{\neg p} p \Rightarrow x{:}^{\neg p}\bot}{\Rightarrow x{:}^{\neg p}\neg p}\ (R{\to})$$

We show that the uppersequent is s-valid in $\mathfrak{M}$ but the lowersequent is not s-valid in $\mathfrak{M}$, and so $(R{\to})$ does not preserve s-validity in this case. Note that w_0 does not survive after $\neg p$, i.e., $w_0 \notin \mathcal{D}(\mathfrak{M}^{\neg p}) = \{w_2\}$. We also note that the semantic clause for $\to$ at a state w becomes classical when w is a single reflexive point.

First, we show that $x{:}^{\neg p} p \Rightarrow$ is s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \not\Vdash x{:}^{\neg p} p$ for any assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. So, we fix any $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. We divide our argument into: $f(x) = w_1$ or $f(x) = w_2$. If $f(x) = w_1$, $f(x)$ does not survive after $\neg p$, and so $\mathfrak{M}, f \not\Vdash x{:}^{\neg p} p$ by Proposition 5.7.1. If $f(x) = w_2$, $f(x)$ survives after $\neg p$ but $f(x) \notin V^{\neg p}(p)(= \emptyset)$, which implies $\mathfrak{M}^{\neg p}, f(x) \not\Vdash p$ hence $\mathfrak{M}, f \not\Vdash x{:}^{\neg p} p$ by Proposition 5.7.1. Therefore, in either case, the uppersequent is valid.

Second, we show that $\Rightarrow x{:}^{\neg p}\neg p$ is not s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \not\Vdash x{:}^{\neg p}\neg p$ for some assignment $f : \mathsf{Var} \to W$. We fix some f such that $f(x) = w_1$. Since $f(x) \notin \mathcal{D}(\mathfrak{M}^{\neg p})$ ($f(x)$ does not survive after $\neg p$), $\mathfrak{M}, f \not\Vdash x{:}^{p}\neg p$ by Proposition 5.7.1, as desired. $\square$

Proposition 5.7.2 is a counter-example of the soundness theorem with s-validity, and so it forces us to change the definition of validity, A key idea of finding another candidate here is that we read $\Gamma \Rightarrow \Delta$ as 'it is impossible that all of Γ hold and all of Δ fail.' We define the notion of failure for the labelled expressions explicitly by requiring survival of states as follows (we read '$\mathfrak{M}, f \Vdash \overline{\mathfrak{A}}$' by 'labelled expression $\mathfrak{A}$ fails under $\mathfrak{M}$ and f').

Definition 5.7.3. Let $\mathfrak{M}$ be an *IntK*-model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment.

$$\begin{array}{lll}
\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A} & \text{iff} & \mathfrak{M}^{\alpha}, f(x) \not\Vdash A \text{ and } f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}),\\
\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\epsilon}y} & \text{iff} & (f(x), f(y)) \notin R_a,\\
\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha;A}y} & \text{iff} & \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y} \text{ or } \mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A} \text{ or } \mathfrak{M}, f \Vdash \overline{y{:}^{\alpha}A}.
\end{array}$$

Note that the first item means that $f(x)$ survives at the domain of the restricted model $\mathfrak{M}^{\alpha}$ and A is false at the survived world $f(x)$ in $\mathfrak{M}^{\alpha}$.

Definition 5.7.4 (t-validity)**.** $\Gamma \Rightarrow \Delta$ is *t-valid* in $\mathfrak{M}$ if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$.

Let us denote by $Var(\Gamma \Rightarrow \Delta)$ the set of all labels occurring in $\Gamma \Rightarrow \Delta$. Then, we note that the domain Var of an assignment f in Definition 5.7.4 can be restricted to $Var(\Gamma \Rightarrow \Delta)$. The following proposition shows that the clauses for relational atoms and negated form of them characterize what they intend to capture.

Proposition 5.7.3. For any *IntK*-model $\mathfrak{M}$, assignment f, agent $a \in \mathsf{Agt}$ and $x, y \in \mathsf{Var}$,

$$\text{(i)} : \mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha}y \text{ iff } (f(x), f(y)) \in R_a^{\alpha}, \quad \text{(ii)} : \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y} \text{ iff } (f(x), f(y)) \notin R_a^{\alpha}.$$

[4]Note that $\neg p$ is an abbreviation of $p \to \bot$.

Proof. Both are easily shown by induction of α. Let us consider the case of $\alpha = (\alpha', A)$ in the proof of (ii). We show $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha,A}y}$ iff $(f(x), f(y)) \in R_a^{\alpha,A}$. $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha}y}$ is, by Definition 5.7.3 and the induction hypothesis, equivalent to $(f(x), f(y)) \in R_a^{\alpha}$ and $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha}, f(y) \Vdash A$. That is also equivalent to $(f(x), f(y)) \in R_a^{\alpha,A}$. □

For preparations of the soundness theorem, we show the following propositions.

Proposition 5.7.4. Let $\mathfrak{M} = (W, (R_a)_{a\in\mathsf{Agt}}, V)$ and $\mathfrak{M}^{\alpha} = (W^{\alpha}, (R_a^{\alpha})_{a\in\mathsf{Agt}}, V^{\alpha})$ be arbitrary Kripke models. If $wR_a^{\alpha}v$ and uR_at and $w \leqslant u$ and $v \leqslant t$, then $uR_a^{\alpha}t$ holds.

Proof. By induction on α, and the base case where $\alpha = \epsilon$ is trivial. Therefore, we show the case where $\alpha = \alpha', A$. Suppose $wR_a^{\alpha',A}v$ and uR_at and $w \leqslant u$ and $v \leqslant t$. By the supposition, we obtain $wR_a^{\alpha'}v$ and $\mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, v \Vdash A$. By induction hypothesis, $uR_a^{\alpha'}t$; and by Proposition 5.1.1, we obtain $\mathfrak{M}, u \Vdash A$ and $\mathfrak{M}, t \Vdash A$. Combining with $uR_a^{\alpha'}t$, we conclude $uR_a^{\alpha',A}t$. □

Proposition 5.7.5. For any Kripke model $\mathfrak{M}$, assignment f, $a \in \mathsf{Agt}$ and $x, y \in \mathsf{Var}$,

(i) $\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha}y$ iff $(f(x), f(y)) \in R_a^{\alpha}$,
(ii) $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$ iff $(f(x), f(y)) \notin R_a^{\alpha}$.

Proof. Both are easily shown by induction of α. Let us consider the case of $\overline{\alpha = \alpha'}, A$ in the proof of (ii). We show $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha,A}y}$ iff $(f(x), f(y)) \in R_a^{\alpha,A}$. $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}_a^{\alpha,\mathsf{A}}}y$ is, by Definition 3.3.3 and the induction hypothesis, equivalent to $(f(x), f(y)) \in R_a^{\alpha}$ and $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha}, f(y) \Vdash A$. That is also equivalent to $(f(x), f(y)) \in R_a^{\alpha,A}$.

□

In order to establish the soundness of **GIntPAL** for birelational Kripke semantics, we basically employ Simpson's argument [78, p.153-5] for the soundness of a natural deduction system for IntK with some modifications for the notion of public announcement[5]. Given any sequent $\Gamma \Rightarrow \Delta$, we may extract a directed graph with the help of the relational atoms in Γ as follows.

Definition 5.7.5. The derived graph $Gr(\Gamma \Rightarrow \Delta)$ from a sequent $\Gamma \Rightarrow \Delta$ is a (labelled) directed graph $(L, (E_a)_{a\in\mathsf{Agt}})$ where L is the set $Var(\Gamma \Rightarrow \Delta)$ of all labels in $\Gamma \Rightarrow \Delta$ and $E_a \subseteq V \times V$ is defined as follows: xE_ay iff $x\mathsf{R}_a^{\alpha}y \in \Gamma$ for some list α ($a \in \mathsf{Agt}$).

Next we recall the notion of tree for a finite directed graph.

Definition 5.7.6 (Tree)**.** Give any finite directed graph $(L, (E_a)_{a\in\mathsf{Agt}})$, we say that $(L, (E_a)_{a\in\mathsf{Agt}})$ is a *tree* if the graph is generated with the root x_0 and, for every node x, there is a unique sequence $(x_1, \ldots, x_m)$ from L such that, for all $0 \leq k < m$, there exists an agent $a_k \in \mathsf{Agt}$ such that $x_kE_{a_k}x_{x+1}$ and $x = x_m$.

In order to prove the soundness of the rules $(R\Box_a)$ and $(R[.])$, our attention must be restricted to the sequents whose derived graphs are trees. And, the following lemma (cf. [78, Lemma 8.1.3]) plays a key role in establishing the soundness of the above two rules, where we also note that the restrictions $(F1)$ and $(F2)$ in birelational Kripke semantics are necessary to prove the lemma.

[5]The author is grateful to the suggestion by Katsuhiko Sano. Because of that, he noticed the application of Simpson's lifting lemma to a coherent proof of the soundness theorem of **GIntPAL**.

Lemma 5.7.1 (Lifting lemma)**.** Let $\Gamma \Rightarrow \Delta$ be a sequent such that $Gr(\Gamma \Rightarrow \Delta)$ is a tree, $\mathfrak{M} = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ an IntK-model, and f an assignment from $Var(\Gamma \Rightarrow \Delta)$ to W such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$. Then, for all labels $x \in Var(\Gamma \Rightarrow \Delta)$ and $w \in W$ with $f(x) \leqslant w$, there exists an assignment f' from $Var(\Gamma \Rightarrow \Delta)$ to W such that $f'(x) = w$, $f(z) \leqslant f'(z)$ for all labels $z \in Var(\Gamma \Rightarrow \Delta)$ and $\mathfrak{M}, f' \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$.

Proof. We sketch the idea of its proof by an example. Consider a sequent $\Gamma \Rightarrow \Delta$ where $\Gamma = \{x_0 \mathsf{R}^{\alpha}_a x_1, x_1 \mathsf{R}^{\beta}_b x_2, x_0 \mathsf{R}^{\gamma}_c x_3\}$ and $\Delta = \varnothing$. Then, $Gr(\Gamma \Rightarrow \Delta) = (L, E_a, E_b, E_c) = (\{x_0, x_1, x_2, x_3\}, \{(x_0, x_1)\}, \{(x_1, x_2)\}, \{(x_0, x_3)\})$ is a tree. Let $\mathfrak{M} = (W, R_a, R_b, R_c, V)$ be an IntK-model, and $f{:}\{x_0, x_1, x_2, x_3\} \to W$ an assignment such that $\mathfrak{M}, f \Vdash x_0 \mathsf{R}^{\alpha}_a x_1$ and $\mathfrak{M}, f \Vdash x_1 \mathsf{R}^{\beta}_b x_2$ and $\mathfrak{M}, f \Vdash x_0 \mathsf{R}^{\gamma}_c x_3$ (they are, by Proposition 5.7.5, equivalent to $f(x_0) R^{\alpha}_a f(x_1)$ and $f(x_1) R^{\beta}_b f(x_2)$ and $f(x_0) R^{\gamma}_c f(x_3)$ respectively).

Fix any $w_1 \in W$ such that $f(x_1) \leqslant w$. By assumptions $f(x_0) R^{\alpha}_a f(x_1)$ and $f(x_1) \leqslant w_1$, we obtain $f(x_0)(R_a \circ \leqslant) w_1$. Then by $(FS2)$, we obtain $f(x_0) \leqslant w_0$ and $w_0 R_a w_1$ for some $w_0 \in W$. Fix such w_0. By Proposition 5.7.4, $w_0 R^{\alpha}_a w_1$. Next, since we have $f(x_1) R^{\beta}_b f(x_2)$ and $f(x_1) \leqslant w_1$, we also have $w_1 R_b w_2$ and $w_2 \geqslant f(x_2)$ for some $w_2 \in W$ by $(FS1)$. Fix such $w_2 \in W$. By Proposition 5.7.4, $w_1 R^{\beta}_b w_2$. Lastly, since we have $f(x_0) R^{\gamma}_c f(x_3)$ and $f(x_0) \leqslant w_0$, we also have $w_0 R_b w_3$ and $w_3 \geqslant f(x_3)$for some $w_3 \in W$ by $(FS1)$. Fix such $w_3 \in W$. By Proposition 5.7.4, $w_0 R^{\gamma}_c w_3$.

We define $f' : \{x_0, x_1, x_2, x_3\} \to W$ by $f'(x_i) = w_i$ ($i \in \{0, 1, 2, 3\}$). Function f' defined in this way satisfies the following requirements:

- $f'(x_1) = w_1$,
- $f(z) \leqslant f'(z)$ for all $z \in \{x_0, x_1, x_2, x_3\}$,
- $\mathfrak{M}, f' \Vdash x_0 \mathsf{R}^{\alpha}_a x_1$ and $\mathfrak{M}, f' \Vdash x_1 \mathsf{R}^{\beta}_b x_2$ and $\mathfrak{M}, f' \Vdash x_0 \mathsf{R}^{\gamma}_c x_3$ from $w_0 R^{\alpha}_a w_1, w_1 R^{\beta}_b w_2$ and $w_0 R^{\gamma}_c w_3$.

$\square$

Now, we are ready to prove a stronger form of the soundness theorem of **GIntPAL** with the notion of tree for derived graphs from sequents.

Theorem 5.7.1 (Soundness of **GIntPAL**)**.** Given any sequent $\Gamma \Rightarrow \Delta$ such that $Gr(\Gamma \Rightarrow \Delta)$ is a finite tree, if $\vdash_{\mathbf{GIntPAL}} \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is t-valid in every *IntK*-model $\mathfrak{M}$.

Proof. Suppose $\vdash_{\mathbf{GIntPAL}} \Gamma \Rightarrow \Delta$ such that $Gr(\Gamma \Rightarrow \Delta)$ is a finite tree. Then the proof is carried out by induction of the height of the derivation of $\Gamma \Rightarrow \Delta$ in **GIntPAL**. We confirm the following cases alone.

Base case: we show that $x \mathsf{R}^{\alpha}_a v \Rightarrow x \mathsf{R}^{\alpha}_a v$ is t-valid. Suppose for contradiction that $\mathfrak{M}, f \Vdash x \mathsf{R}^{\alpha}_a v$ and $\mathfrak{M}, f \Vdash \overline{x \mathsf{R}^{\alpha}_a v}$. By Proposition 5.7.3, this is impossible.

The case where the last applied rule is $(L\Box_a)$**:** In this case, we have a derivation of $\Gamma \Rightarrow \Delta, x \mathsf{R}^{\alpha}_a y$ and $y{:}^{\alpha} A, \Gamma \Rightarrow \Delta$ in **GIntPAL**. Both $Gr(\Gamma \Rightarrow \Delta, x \mathsf{R}^{\alpha}_a y)$ and $Gr(y{:}^{\alpha} A, \Gamma \Rightarrow \Delta)$ trivially keep the same structure of tree; therefore, the induction hypothesis may be applied to both derivations. And now we have $\Gamma \Rightarrow \Delta, x \mathsf{R}^{\alpha}_a y$ and $y{:}^{\alpha} A, \Gamma \Rightarrow \Delta$ are t-valid in any IntK-Kripke model $\mathfrak{M}$. Suppose for a contradiction that there is some $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and

$\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\Box_a A}$ and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Fix such f. Now it suffices to show $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^{\alpha}_a y}$ or $\mathfrak{M}, f \Vdash y{:}^{\alpha}A$. From our supposition $\mathfrak{M}, f \Vdash x{:}^{\alpha}\Box_a A$, we obtain $(f(x), f(y)) \notin (\leqslant \circ R_a)^{\alpha}$ or $\mathfrak{M}^{\alpha}, f(y) \Vdash A$. Suppose the former disjunct, which is equivalent to $f(x) \nleqslant v$ or $(v, f(y)) \notin R_a$ for any $v \in W$. Fix v as $f(x)$. Then we obtain $(f(x), f(y)) \notin R_a$, and by Proposition 5.7.3, $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^{\alpha}_a y}$. It contradicts $\Gamma \Rightarrow \Delta, x\mathsf{R}^{\alpha}_a y$ is t-valid. Next, suppose the latter disjunct $\mathfrak{M}^{\alpha}, f(y) \Vdash A$ which is equivalent to $\mathfrak{M}, f \Vdash y{:}^{\alpha}A$. It contradicts $y{:}^{\alpha}A, \Gamma \Rightarrow \Delta$ is t-valid. Therefore, we obtain contradictions in either case.

The case where the last applied rule is (Rat)**:** We show the contraposition. Suppose there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$. Fix such f. We suffice to show $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}p}$. By Definition 5.7.3, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$ is equivalent to $\mathfrak{M}^{\alpha,A}, f(x) \nVdash p$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$. By $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$, we obtain $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and $\mathfrak{M}^{\alpha}, f(x) \Vdash A$. It follows from $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha,A}, f(x) \nVdash p$ that $f(x) \notin V^{\alpha}(p)$, This is equivalent to $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}p}$. Then, the contraposition has been shown.

The case where the last applied rule is ($Lrel_a 3$)**:** In this case, we have a derivation of $x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta$ in **GIntPAL**. Since $Gr(x\mathsf{R}^{\alpha,A}_a y, \Gamma \Rightarrow \Delta)$ is a tree and any formula restricting relational atom does not affect the structure of the graph $Gr(x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta)$, it is also a tree. Then the induction hypothesis is applicable to the uppersequent of the derivation, and therefore, we obtain that $x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta$ is t-valid for any $\mathfrak{M}$. We must show $x\mathsf{R}^{\alpha,A}_a y, \Gamma \Rightarrow \Delta$ is t-valid for any $\mathfrak{M}$ Suppose there is some $f{:}L \to \mathcal{D}(\mathfrak{M}^{\alpha})$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, $\mathfrak{M}, f \Vdash x\mathsf{R}^{\alpha,A}_a y$ and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Fix such f. From $x\mathsf{R}^{\alpha,A}_a y$, we obtain $\mathfrak{M}, f \Vdash x\mathsf{R}^{\alpha}_a y$. This is what we want to show.

The case where the last applied rule is ($R \to$)**:** In this case, we have a derivation of $x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B$ in **GIntPAL**, and since $Gr(x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B) = Gr(\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B)$, it is trivially a tree. Let the graph be $(L, (E_a)_{a \in \mathsf{Agt}})$. By the application of the induction hypothesis, we obtain there is no $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$ and $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}B}$. Then it suffices to show that if there is $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A \to B}$, then there is $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$ and $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}B}$ $(*)$. Suppose the antecedent of $(*)$, and consider such $f{:}L \to \mathcal{D}(\mathfrak{M})$. Then we obtain $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A \to B}$ which is equivalent to $f(y) \leq^{\alpha} v$ and $\mathfrak{M}^{\alpha}, f(y) \Vdash A$ and $\mathfrak{M}^{\alpha}, f(y) \nVdash B$ for some $v \in \mathcal{D}(\mathfrak{M}^{\alpha})$, and $f(y) \in \mathcal{D}(\mathfrak{M}^{\alpha})$. Meanwhile, by Lemma 5.7.1, we obtain a function $f'{:}L \to \mathcal{D}(\mathfrak{M}^{\alpha})$ such that $f' \geq f$ and $\mathfrak{M}, f' \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$. Now, f' can be extend to $f''{:}L \to \mathcal{D}(\mathfrak{M}^{\alpha})$ such that $f''(x) = v$ and all others are the same as f'. Then we obtain the succedent of $(*)$ that is what we desired.

The case where the last applied rule is ($R[.]$)**:** In this case, we have a derivation of $x{:}^{\alpha}A, \Gamma \Rightarrow x{:}^{\alpha,A}B$ in **GIntPAL**, and since $Gr(x{:}^{\alpha}A, \Gamma \Rightarrow x{:}^{\alpha,A}B) = Gr(\Gamma \Rightarrow x{:}^{\alpha}[A]B)$, it is trivially a finite tree. Let us denote the graph by $(L, (E_a)_{a \in \mathsf{Agt}})$. Suppose for contradiction that there is an assignment $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that

$\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}[A]B}$. Fix such $f{:}L \to \mathcal{D}(\mathfrak{M})$. Then, it suffices to show that there is an assignment $f'{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f' \Vdash x{:}^{\alpha}A$ and $\mathfrak{M}, f' \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f' \Vdash \overline{x{:}^{\alpha,A}B}$, since this gives us a contradiction with our induction hypothesis to $x{:}^{\alpha}A, \Gamma \Rightarrow x{:}^{\alpha,A}B$. By the supposition, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}[A]B}$, which is equivalent to: $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and there is some $v \in \mathcal{D}(\mathfrak{M}^{\alpha})$ such that $f(x) \leqslant^{\alpha} v$ and $\mathfrak{M}^{\alpha}, v \Vdash A$ and $\mathfrak{M}^{\alpha,A}, v \nVdash B$. By Lemma 5.7.1 and the supposition that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, we obtain an assignment $f'{:}L \to \mathcal{D}(\mathfrak{M})$ such that $f'(x) = v$ and $f(z) \leqslant f'(z)$ for all $z \in L$ and $\mathfrak{M}, f' \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$. It also follows that $\mathfrak{M}, f' \Vdash x{:}^{\alpha}A$ and $\mathfrak{M}, f' \Vdash \overline{x{:}^{\alpha,A}B}$, as desired.

The case where the last applied rule is $(R\Box_a)$**:** In this case, we have a derivation of $x\mathsf{R}^{\alpha}_{a}y, \Gamma \Rightarrow y{:}^{\alpha}A$ in **GIntPAL**. Let us denote a tree $Gr(\Gamma \Rightarrow x{:}^{\alpha}\Box_a A)$ by $(L, (E_b)_{b \in \mathsf{Agt}})$. Since y is a fresh variable, $Gr(x\mathsf{R}^{\alpha}_{a}y, \Gamma \Rightarrow y{:}^{\alpha}A) = (L \cup \{y\}, E_a \cup \{(x,y)\}, (E_b)_{b \in \mathsf{Agt} \setminus \{a\}})$ is still a finite tree. Suppose for contradiction that there is an assignment $f{:}L \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\Box_a A}$. Fix such assignment $f{:}L \to \mathcal{D}(\mathfrak{M})$. It suffices to show that there is an assignment $g{:}L \cup \{y\} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, g \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, g \Vdash x\mathsf{R}^{\alpha}_{a}y$ and $\mathfrak{M}, g \Vdash \overline{y{:}^{\alpha}A}$, since this gives us a contradiction with our induction hypothesis to $x\mathsf{R}^{\alpha}_{a}y, \Gamma \Rightarrow y{:}^{\alpha}A$. Then, by the supposition of $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}\Box_a A}$, we have $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ and there are some $v, u \in \mathcal{D}(\mathfrak{M}^{\alpha})$ such that $f(x) \leqslant^{\alpha} u$, $uR^{\alpha}_{a}v$ and $\mathfrak{M}^{\alpha}, v \nVdash A$. By the supposition that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, we apply Lemma 5.7.1 to the sequent $\Gamma \Rightarrow x{:}^{\alpha}\Box_a A$ to find an assignment $f'{:}L \to \mathcal{D}(\mathfrak{M}^{\alpha})$ such that $f'(x) = u$, $f(z) \leqslant f'(z)$ for all $z \in L$ and $\mathfrak{M}, f' \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$. Now, f' can be extend to a new assignment $g{:}L \cup \{y\} \to \mathcal{D}(\mathfrak{M})$ such that g is the same as f' except $g(y) = v$. Then, we obtain $\mathfrak{M}, g \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, $\mathfrak{M}, g \Vdash x\mathsf{R}^{\alpha}_{a}y$ and $\mathfrak{M}, g \Vdash \overline{y{:}^{\alpha}A}$, as desired.

□

We have done to prove all theorems which are declared to be shown in the introduction. But the following last piece should be significant for an indirect proof of the completeness of **GIntPAL**.

Proposition 5.7.6. If $\Rightarrow x{:}^{\epsilon}A$ is t-valid in an *IntK*-model $\mathfrak{M}$, then A is valid in $\mathfrak{M}$.

Proof. Suppose that $\Rightarrow x{:}^{\epsilon}A$ is t-valid. So, it is not the case that there exists some assignment f such that $\mathfrak{M}, f \Vdash \overline{x{:}^{\epsilon}A}$. Equivalently, for all assignments f, $\mathfrak{M}, f \nVdash \overline{x{:}^{\epsilon}A}$. For any assignment f, $\mathfrak{M}, f \nVdash \overline{x{:}^{\epsilon}A}$ is equivalent to $\mathfrak{M}, f(x) \Vdash A$ because $f(x) \in \mathcal{D}(\mathfrak{M})$. So, it follows that $\mathfrak{M}, f(x) \Vdash A$ for all assignments f. Then, it is immediate to see that A is valid in $\mathfrak{M}$, as required. □

Finally, we may establish the completeness theorem as follows.

Corollary 5.7.1 (Completeness of **GIntPAL**)**.** Given any formula A and label $x \in \mathsf{Var}$, the following are equivalent:

(i) A is valid on all *IntK*-models;

(ii) $\vdash_{\mathbf{HIntPAL}} A$;

(iii) $\vdash_{\mathbf{GIntPAL}^{+}} \Rightarrow x{:}^{\epsilon}A$;

(iv) $\vdash_{\mathbf{GIntPAL}} \Rightarrow x{:}^{\epsilon}A$.

Proof. The direction from (i) to (ii) is established by Fact 5.3.1 and the direction from (ii) to (iii) is shown by Theorem 5.5.1. Then, the direction from (iii) to (iv) is established by the admissibility of cut, i.e., Theorem 5.6.1. Finally, the direction from (iv) to (i) is shown by Theorem 5.7.1 and Proposition 5.7.6, since $Gr(\Rightarrow x{:}^{\epsilon}A)$ is a tree (a single point-tree) and therefore Theorem 5.7.1 is applicable, and then Proposition 5.7.6 may be applied to its conclusion. □

Chapter 6

Conclusion

6.1 Summary of contributions

In Chapter 2, we introduced multi-modal logics and labelled sequent calculus which are main bases of this thesis. In Chapter 3, we found that inference rules for accessibility relations were missing in the existing labelled sequent calculus of **G3PAL**, and that (RA4) ($[A]\Box_a B \leftrightarrow A \rightarrow \Box_a[A]B$), one of the recursion axioms in **HPAL**, was not provable by the system, although it should be if it is complete for Kripke semantics. Therefore, we revised **G3PAL** by reformulating and adding some rules to it and named the first labelled system in this thesis **GPAL**. Additionally, we showed the cut-elimination theorem of **GPAL** (Theorem 3.2.2). During this revision, we also make the notion of *survival* explicit. According to this revision, we could show that **GPAL** is sound for Kripke semantics (Theorem 3.3.1). Moreover, by carefully considering the notion of survival, we found the link-cutting version of PAL's semantics is more suitable to our labelled sequent calculus than the standard semantics i.e., the world-deleting semantics, and then we showed **GPAL** is complete for the link-cutting semantics (Theorem 3.4.1). Then, the basis of **GPAL** was extended to be based on other basic modal logics including S5 which is the usual basis of epistemic logics.

In Chapter 4, we introduced the second labelled sequent calculus **GEAK**, and showed its cut-admissibility (Theorem 4.2.1) and the soundness theorem (Theorem 4.4.1). After that, we obtained as a corollary the semantic completeness (Corollary 4.4.1) through the completeness theorem of an existing Hilbert-system **HEAK**. Moreover, we also showed our system is sound for the standard Kripke semantics. In the proof of the soundness theorem, we also took into account the notion of survival of worlds in the restricted domain. Therefore, we demonstrated that it is critical especially in the case of labelled systems to carefully consider deleted (or restricted) world(s) in a modified Kripke model. EAK is not only a complicated logic but also the core of the field of DEL (we mentioned in the introduction EAK is called Dynamic Epistemic Logic in a narrower range of the meaning). Therefore, our labelled system is handled much easier than Hilbert-system **HEAK** and is beneficial for the study of DEL since it is often troublesome to construct a derivation of a theorem of it (formulas concerning a knowledge-state tend to be long and complicated, in fact). Then, the basis of **GEAK** was extended to be based on other basic modal logics including S5 which is the usual basis of epistemic logics.

In Chapter 5, we provided the third labelled sequent calculus **GIntPAL** for PAL within an intuitionistic framework, and as with previous labelled systems, we showed the cut-elimination theorem (Theorem 5.6.1), the soundness theorem (Theorem 5.7.1) and the completeness theorem as a corollary (Corollary 5.7.1) of the completeness of Hilbert-system **HEAK** and Theorem 5.7.1. A sequent calculus that is easy to handle may be particularly significant for intuitionistic epistemic logics that regard verification or evidence as important.

6.2 Future directions

We may consider some other tasks from our three labelled calculi. Firstly, although we employ Gentzen's traditional approach, there is another approach for an labelled system such as the G3-system introduced by Troelstra and Schwichtenberg [80] and Negri and von Plato [58, p.192] and for the case of intuitionistic logic Dyckhoff and Negri [23]. As we mentioned in Section 2.1.3, the G3-system is a sequent calculus in which all structural rules including contraction rules are height-preserving admissible. Since a G3-system has such outstanding features, the possibility of employing it is worth being considered in the future. Secondly, we have not given direct proofs of the completeness theorems of **GIntPAL** and **GEAK**; however, the idea of the link-cutting semantics in PAL can be applicable to also IntPAL and EAK to show the direct proofs, and we need to consider these possibilities. Thirdly, Although the extension of **GIntPAL** from K to other modal logics was not conducted in this thesis, the extension like the cases of **GPAL**, **GEAK** is desirable and should be done in the future. Fourthly, the cut-free labelled sequent calculi we have argued so far can become a stepping-stone to consider other logical problems such as the decision problem and the computational complexity. Especially. these two problems can be significant to the field of DEL which is related with Artificial Intelligence (i.e., implementing formalized knowledge on a computer), it is a good opportunity to put the proof-theoretic research of DEL forward based on the cut-free systems. Lastly, extensions of calculi do not end at an extension of them from **K** to **S5**, but we may consider adding finite subset B of Agt which is known as 'common knowledge' in languages of DELs. If we can successfully do this, our calculi will be more valuable for the study of DELs. These will be left to our future works.

Appendix A

Implementations for Dynamic Epistemic Logics

We have seen, based on Multi-modal Logic (ML), three Dynamic Epistemic Logics such as Public Announcement Logic (PAL), Logic of Epistemic Actions and Knowledge (EAK) and Intuitionistic PAL (IntPAL). In this chapter, we give a brief introduction of implementations of those semantics and labelled systems. Specifically, we implement the satisfaction relation of Kripke semantics with some examples (e.g., muddy children puzzle) for each DEL[1] and automated theorem provers based on our cut-free sequent calculi **GPAL**. Every implementation is written in the programming language Haskell. Our environment for the implementations are as follows.

- OS : Ubuntu 14.04 (64bit)
- ghc (Haskell compiler) : version 7.6.3
- graphviz : version 2.36

Let us warn the reader that, as a notice, since we cannot introduce the whole code of implementations (it is too long to put here), we introduce some core definitions and functions.

A.1 Semantic tools for DELs

A.1.1 Implementation for PAL

Language The language implemented is, of course, the same as the language of PAL $\mathcal{L}_{PAL}$ but all defined connectives are included in the language to avoid uselessly complicated computations. The code is as follows:

```
-----------------------------------------------
```

[1]As relational works, there exist automated semantic tools of EAK such as `DEMO` [84] and `Aximo` [73].

```
-- Language
------------------------------------------------
type Label = Int
type History =[Label]
type Agent = String

data Formula = Atom String              -- p
             | AnyF String              -- A
             | Top                      -- T
             | Bottom                   -- _|_
             | Neg  Formula             -- ~A
             | Box Agent History Formula  -- #a A
             | Dia Agent History Formula  -- $a A
             | Conj Formula Formula     -- A & B
             | Disj Formula Formula     -- A v B
             | Impl Formula Formula     -- A -> B
             | Equi Formula Formula     -- A <-> B
             | Announce Formula Formula   -- [A]B
             | Announce2 Formula Formula  -- <A>B
               deriving (Eq,Show,Ord)
```

where `#a A` and `$a A` mean $\Box_a A$ and $\Diamond_a A$ respectively and other connectives are the same as you can imagine, and `History` in the definitions of `#a A` and `$a A` is utilized in an automated theorem prover for PAL to keep a history of labels.

A.1.2 Kripke semantics

We now give definitions of the Kripke model and the satisfaction relation. They are almost the same as definitions in Section 2.2.1, except the definition of accessibility relation R is a ternary relation $R \subseteq \mathsf{Agt} \times W \times W$ where Agt where Agt is a finite set of agents and W is the domain of a Kripke model (in the code below, we write `type Relation = [(Agent,World,World)]`).

```
------------------------------------------------
-- Kripke model and satisfaction relation
------------------------------------------------
type World = String
type Agent = String
type Relation = [(Agent,World,World)]
type Valuation = [(String,[World])]
data Frame = Frame [World] Relation
               deriving (Eq,Show,Read)
data Model = Model Frame Valuation
               deriving (Eq,Show,Read)

frame1 (Frame w r) = w
frame2 (Frame w r) = r

valueF :: Frame -> Valuation -> World -> Formula -> Bool
valueF f v w Top = True
valueF f v w Bottom = False
valueF f v w (Atom p)       = w `elem`  rejectJust (lookup p v)
```

```
valueF f v w (Neg p)        = not $ valueF f v w p
valueF f v w (Conj p q)     = (valueF f v w p) && (valueF f v w q)
valueF f v w (Disj p q)     = (valueF f v w p) || (valueF f v w q)
valueF f v w (Impl p q)     = not (valueF f v w p) || (valueF f v w q)
valueF f v w (Equiv p q)    = (valueF f v w p) == (valueF f v w q)
valueF f v w (Box ag p)     = forall [z|(ag,u,z) <- (frame2 f),u == w]
                                (\z -> (valueF f v z p))
valueF f v w (Dia ag p)     = exists [z|(ag,u,z) <- (frame2 f),u == w]
                                (\z -> (valueF f v z p))
valueF f v w (Announce p q) = not (valueF f v w p)
         || (valueF (Frame lmtdW lmtdR) v w q)
        where lmtdW = [w|w <- (trueWorld f v p)]
              lmtdR = [(ag,x,y)|(ag,x,y)<-(trueRelation f v p)]

--- functions to construct a truth/false set.
trueWorld :: Frame -> Valuation -> Formula -> [World]
trueWorld f v p =  [x | x <- (frame1 f),(valueF f v x p) == True]

--- restricted relation
trueRelation :: Frame -> Valuation -> Formula -> Relation
trueRelation f v p =  frame2 f `difference`
    (union [(ag,u,w)|(ag,u,w) <- (frame2 f), (w `elem` (trueWorld f v (Neg p)))]
           [(ag,w,u)|(ag,w,u) <- (frame2 f), (w `elem` (trueWorld f v (Neg p)))])
```

These definitions, we make note of the functions `trueWorld`,`trueRelation`.
`trueWorld :: Frame -> Valuation -> Formula -> [World]` and
`trueRelation :: Frame -> Valuation -> Formula -> Relation`.
The former is a function to construct a restricted domain W^A such that $\{w \mid \mathfrak{M}, w \Vdash A\}$ from given Kripke model $\mathfrak{M}$ and formula A, and the latter is a function to restrict a given relation as $R_a \cap (W^A \times W^A)$.

Muddy children puzzle Below is the definition of a Kripke model (`muddyWorld`, `muddyRelation`,`muddyValues`) which represents the Muddy children puzzle 2.2.3 in Chapter2.

```
-------------------------------------------------
-- Frame and valuation of muddy children
-------------------------------------------------

muddyAgent = ["a","b","c"]
muddyWorld = [x ++ y ++ z | x<-["1","0"],y<-["1","0"],z<-["1","0"]]
muddyRelation =
 [("a",x++y++z,x'++y++z)|x<-["1","0"],x'<-["1","0"],y<-["1","0"],z<-["1","0"] ]++
 [("b",x++y++z,x++y'++z)|x<-["1","0"],y<-["1","0"],y'<-["1","0"],z<-["1","0"] ]++
 [("c",x++y++z,x++y++z')|x<-["1","0"],y<-["1","0"],z<-["1","0"],z'<-["1","0"] ]

valueP :: Formula -> [World]
valueP (Atom "0a") = ["011","001","000","010"]
valueP (Atom "1a") = ["111","101","100","110"]
valueP (Atom "0b") = ["000","001","100","101"]
valueP (Atom "1b") = ["010","011","110","111"]
valueP (Atom "0c") = ["000","100","010","110"]
```

```
valueP (Atom "1c") = ["001","011","101","111"]

muddyFrame =  Frame muddyWorld muddyRelation

muddyValues = [
  ("0a" ,["011","001","000","010"]),
  ("1a" ,["111","101","100","110"]),
  ("0b" ,["000","001","100","101"]),
  ("1b" ,["010","011","110","111"]),
  ("0c" ,["000","100","010","110"]),
  ("1c" ,["001","011","101","111"])
              ]
```

Then we define the first to third public announcements as follows.

```
-------------------------------------------------
-- Public announcements for muddy children
-------------------------------------------------
announceMuddy1 =
  (Atom "0a") `Disj` (Atom "0b") `Disj` (Atom "0c")

announceMuddy2 =
 ((Neg (Box "a" (Atom "0a"))) `Conj` (Neg (Box "a" (Neg (Atom "0a"))))) `Conj`
 ((Neg (Box "b" (Atom "0b"))) `Conj` (Neg (Box "b" (Neg (Atom "0b"))))) `Conj`
 ((Neg (Box "c" (Atom "0c"))) `Conj` (Neg (Box "c" (Neg (Atom "0c")))))

announceMuddy3 =
 ((Box "b" (Atom "0b")) `Disj` (Box "b" (Neg (Atom "0b")))) `Conj`
 ((Box "a" (Atom "0a")) `Disj` (Box "a" (Neg (Atom "0a"))))
```

For modifying Kripke models by an announcement ($\mathfrak{M}$ may be modified by an announcement A, and the modification yields $\mathfrak{M}^A$) we use the following function.

```
-------------------------------------------------
-- Function for modifying frames
-------------------------------------------------
modifyFrame :: Frame -> Formula -> Frame
modifyFrame f p =
 Frame (trueWorld f muddyValues p) (trueRelation f muddyValues p)
```

By these settings, we can solve the puzzle as follows. Here are the results of the Muddy children puzzle

```
*Main> modifyFrame muddyFrame announceMuddy1
Frame ["110","101","100","011","010","001","000"] [("a","110","110"),
("a","101","101"),("a","100","100"),("a","110","010"),("a","101","001"),
("a","100","000"),("a","010","110"),("a","001","101"),("a","000","100"),
("a","011","011"),("a","010","010"),("a","001","001"),("a","000","000"),
```

```
("b","110","110"),("b","110","100"),("b","100","110"),("b","101","101"),
("b","100","100"),("b","011","011"),("b","010","010"),("b","011","001"),
("b","010","000"),("b","001","011"),("b","000","010"),("b","001","001"),
("b","000","000"),("c","110","110"),("c","101","101"),("c","101","100"),
("c","100","101"),("c","100","100"),("c","011","011"),("c","011","010"),
("c","010","011"),("c","010","010"),("c","001","001"),("c","001","000"),
("c","000","001"),("c","000","000")]

*Main> modifyFrame it  announceMuddy2
Frame ["100","010","001","000"] [("a","100","100"),("a","100","000"),
("a","000","100"),("a","010","010"),("a","001","001"),("a","000","000"),
("b","100","100"),("b","010","010"),("b","010","000"),("b","000","010"),
("b","001","001"),("b","000","000"),("c","100","100"),("c","010","010"),
("c","001","001"),("c","001","000"),("c","000","001"),("c","000","000")]

*Main> modifyFrame it  announceMuddy3
Frame ["001"] [("a","001","001"),("b","001","001"),("c","001","001")]
```

As you can see above, after the third announcement, we obtain only one world `001` which is the answer.

A.1.3 Implementation for EAK

Similar to the previous section, we implement the language of EAK at first and its definition is given in the same manner as Definition 2.3.1 in Chapter 2.3, but all defined logical connectives (e.g., $\wedge$, $\vee$ etc.) are added to the definition.

```
------------------------------------------------
-- Language of EAK
------------------------------------------------

type Agent = String
type Name = String
type State = String
type RelAM = [(Agent,State,State)]
type Pre =  State -> Formula --[(State,Formula)]
data Action =    PointAM (AM,State)
                 | Cup Action Action

data AM = AM Name [State] RelAM Pre

data Formula =
               Top
             | Bottom
             | Atom  String              -- A
             | Neg   Formula             -- ~A
             | Conj  Formula Formula     -- A & B
             | Disj  Formula Formula     -- A v B
             | Impl  Formula Formula     -- A -> B
             | Equiv Formula Formula     -- A <-> B
             | Box Agent Formula         -- #aA
             | Diamond Agent Formula     -- $aA
```

```
            | AfterAction  Action Formula     -- [α]A
            | HutAfterAction  Action Formula  --  ⟨α⟩ A
```

Subsequently, we define the Kripke semantics and the satisfaction relation of EAK (which are also implementations of Section 2.3.1) as follows.

```
-------------------------------------------------
-- Semantics of EAK
-------------------------------------------------

type World = String
type Valuation = Formula -> [World]
type RelKM = [(Agent,World,World)]
data KM = KM Name [World] RelKM Valuation
data PointKM   =  PointKM (KM,World)

modelKM  (PointKM (mo,wo)) = mo
modelKM2 (PointKM (mo,wo)) = wo
modelAM  (PointAM (mo,wo)) = mo
modelAM2 (PointAM (mo,wo)) = wo

km0 (KM name  _ _ _) = name
km1 (KM _ world _ _) = world
km2 (KM _ _ relat _) = relat
km3 (KM _ _ _ valua) = valua

am0 (AM name  _ _ _) = name
am1 (AM _ state _ _) = state
am2 (AM _ _ relat _) = relat
am3 (AM _ _ _ preco) = preco

valueF :: PointKM -> Formula -> Bool
valueF (PointKM (m,w)) Top
  = True
valueF (PointKM (m,w)) Bottom
  = False
valueF (PointKM (m,w)) (Atom p)
  =  if '<' `notElem` w
     then  w `elem`  ((km3 m) (Atom p))
     else  (takeW w) `elem`  ((km3 m) (Atom p))
valueF (PointKM (m,w)) (Neg p)
  = not $ valueF (PointKM (m,w)) p
valueF (PointKM (m,w)) (Conj p q)
  = (valueF (PointKM (m,w)) p) && (valueF (PointKM (m,w)) q)
valueF (PointKM (m,w)) (Disj p q)
  = (valueF (PointKM (m,w)) p) || (valueF (PointKM (m,w)) q)
valueF (PointKM (m,w)) (Impl p q)
  = (valueF (PointKM (m,w)) p) ==> (valueF (PointKM (m,w)) q)
valueF (PointKM (m,w)) (Equiv p q)
  = (valueF (PointKM (m,w)) p) == (valueF (PointKM (m,w)) q)
valueF (PointKM (m,w)) (Box ag  p)
  = forall [z | (ag',w',z) <- (km2 m),w'==w, ag==ag'] (\v -> (valueF (PointKM (m,v)) p))
valueF (PointKM (m,w)) (Diamond ag p)
  = exists [z | (ag',u,z) <- (km2 m),u==w,ag==ag']  (\v -> (valueF (PointKM (m,v)) p))
```

```
valueF (PointKM (m,w)) (AfterAction a p)
  = case a of
     Cup b b' -> (valueF (PointKM (m,w)) (AfterAction b p))
                 && (valueF (PointKM (m,w)) (AfterAction b' p))
     PointAM (e,s) ->   (valueF (PointKM (m,w)) ((am3 e) s))
                        ==>  valueF (PointKM (m,w) *** PointAM (e,s)) p
valueF (PointKM (m,w)) (HutAfterAction a p)
  = valueF (PointKM (m,w)) (Neg (AfterAction a (Neg p)))
```

Functions `takeW`, `***` (the function for updating a model by an action $\mathfrak{M}\otimes\mathsf{M}'$) and +++ (the function for composing action models $\mathsf{M};\mathsf{M}'$) for the definition of the satisfaction relation of action operators are defined as follows.

```
-------------------------------------------------
-- Functions for action models
-------------------------------------------------

makeW :: String-> String -> String -- construct <1,p> from "1" and "a"
makeW w s = "<"++w ++","++ s++ ">"

takeW :: String -> String   -- extract 1 from <<1,q>,p>
takeW  = takeW2.words.takeW1

takeW1 x = takeW1' x []
  where takeW1' [] ls =  reverse ls
        takeW1' (x:xs) ls = case x of '<' -> takeW1' xs (" < "++ls)
                                      '>' -> takeW1' xs (" > "++ls)
                                      ',' -> takeW1' xs (" , "++ls)
                                      x   -> takeW1' xs ([x]++ls)

takeW2 (x:xs) =  case x of "<" ->  takeW2 xs
                           "," ->  takeW2 xs
                           x     -> x

(***) :: PointKM -> Action -> PointKM   -- def of M*E
(PointKM (m, w)) ***  (PointAM (e,s))
  = PointKM ((KM name ws' rel1' v'), (makeW w s))
   where
   name = "("++(km0 m)++"*"++(am0 e)++")"
   ws'   = nub ["<"++ w++ ","++ s++ ">"
     |w <-(km1 m),s <- (am1 e) , valueF (PointKM (m,w)) ((am3 e) s)]
   rel1' = nub [(ag, "<"++w1++ ","++ s1++ ">", "<"++w2++","++s2++ ">")
     | (ag, w1, w2)<-(km2 m),(ag', s1, s2)<-(am2 e),
       valueF (PointKM (m,w1)) ((am3 e) s1),
      valueF (PointKM (m,w2)) ((am3 e) s2) ,ag==ag']
   v'    = km3 m

(+++) :: Action -> Action -> Action   -- def of Action composition
(PointAM (e1,s1)) +++  (PointAM (e2,s2))
  = (PointAM ((AM name state rel pre), (makeW s1 s2)))
   where
   name = "("++(am0 e1)++";"++(am0 e2)++")"
```

```
   state  = nub ["<"++ s ++ ","++ s' ++ ">"
      | s<-(am1 e1), s'<-(am1 e2)]
   rel = nub [(ag, "<"++s1++ ","++ t1++ ">", "<"++s2++","++t2++ ">")
      | (ag,s1,s2)<-(am2 e1),(ag2,t1,t2)<-(am2 e2),ag==ag2{--} ]
   pre  x  = makePre (parseState x) [] ((am3 e1) (fst( sepState x)))
       where makePre [] stack fo = fo
             makePre (">":xs) (y1:y2:ys) fo
              = makePre xs ys  (fo `Conj`
                     AfterAction (PointAM (e1,(fst( sepState x))))
                                 ((am3 e2) (snd( sepState x))))
             makePre (x:xs) stack fo   =  makePre xs (x:stack) fo
```

Then we define functions for outputting Kripke models by Graphviz as follows.

```
------------------------------------------------
-- Functions for graphviz
------------------------------------------------

draw :: [Formula]-> PointKM -> IO ()
draw st (PointKM (m,w)) =
  do writeFile "model.dot"
    ("graph model {\n graph [size = \"1.2, 2.3\", label = "++ "\""++(km0 m)
    ++ "\""++"];\n graph[rankdir =LR];\n" ++"\""++  w ++ "\""++ "
    [peripheries = 2];\n"++ (drawRel (km2 m)) ++ (drawVar st m)++ "}")
    writeFile "model.txt"
     ("graph model {\n graph [size = \"1.2, 2.3\",
       label = "++ "\""++(km0 m) ++ "\""++"];\n graph[rankdir =LR];
       \n" ++"\""++  w ++ "\""++ " [peripheries = 2];\n"++ (drawRel (km2 m))
         ++ (drawVar st m)++ "}")

graphviz :: (Show a) =>   IO a -> IO ProcessHandle
graphviz x =    do x
                   runCommand $ "dot -Tpdf model.dot"
                           ++   " -o       model.pdf;"
                           ++   " gnome-open     model.pdf"
```

where `gnome-open` is a terminal command of Unbuntu 14.04 to open a pdf-file.

Example 1 (Muddy children puzzle) Let us trace here the Muddy children puzzle by these settings of EAK. First, we define `publicAnnouncement` which is an action to simulate a public announcement of PAL as we already have seen in Example 2.3.2.

```
------------------------------------------------
-- Kripke model of Muddy children puzzle
------------------------------------------------

publicAnnouncement :: [Agent] ->String -> Formula -> Action ((AM name pubS pubR pubPre),y)
    where
    name = "pubAnn(" ++ y ++ ")"
```

```
    pubS = [y]
    pubR = (refl ags [y])
    pubPre y = x
```

Subsequently, we define the Kripke model for the puzzle as follows.

[l]Kripke model of Muddy children puzzle

```
muddyProp =
 [Atom (x ++ y) | x <- ["a","b","c"], y <- ["x","o"]]
muddyAgent = ["a","b","c"]

muddy :: PointKM
muddy  = PointKM ((KM name muddyS muddyR muddyV),"100")
    where
    name  = "muddy"
    muddyS = [(x ++ y ++z ) | x <- ["x","o"],
                y <- ["x","o"], z <- ["x","o"]]
    muddyR = (refl ["a","b","c"] muddyS)
     ++ (symm "a" ["000","100"]) ++ (symm "a" ["010","110"])
     ++ (symm "a" ["001","101"]) ++ (symm "a" ["011","111"])
     ++ (symm "b" ["000","010"]) ++ (symm "b" ["100","110"])
     ++ (symm "b" ["001","011"]) ++ (symm "b" ["101","111"])
     ++ (symm "c" ["000","001"]) ++ (symm "c" ["010","011"])
     ++ (symm "c" ["100","101"]) ++ (symm "c" ["110","111"])
    muddyV (Atom x) = case x of
       "a1" -> ["111","101","100","110"]
       "a0" -> ["011","001","000","010"]
       "bo" -> ["010","111","110","011"]
       "b0" -> ["000","101","100","001"]
       "c1" -> ["001","011","101","111"]
       "c0" -> ["000","100","010","110"]
```

Then we define the three public announcements (defined by the action) such as `muddyAnn1`, `muddyAnn2`, and `muddyAnn3` as follows. Of course, each corresponds to the formula in Example 2.2.3.

```
-------------------------------------------------
-- Public announcements by EAK
-------------------------------------------------


muddyAnn1 :: Action
muddyAnn1 =
 publicAnnouncement muddyAgent "ann1"
  ((Atom "a0") `Disj` (Atom "b0") `Disj` (Atom "c0"))

muddyAnn2 :: Action
muddyAnn2 =
 publicAnnouncement muddyAgent "ann2"
 (((Neg (Box "a" (Atom "a0"))) `Conj` (Neg (Box "a" (Neg (Atom "a0"))))) `Conj`
  ((Neg (Box "b" (Atom "b0"))) `Conj` (Neg (Box "b" (Neg (Atom "b0"))))) `Conj`
  ((Neg (Box "c" (Atom "c0"))) `Conj` (Neg (Box "c" (Neg (Atom "c0"))))))
muddyAnn3 :: Action
muddyAnn3 =
```

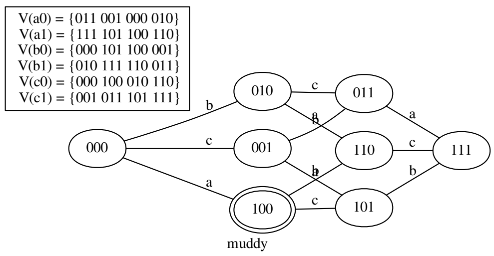
```
publicAnnouncement muddyAgent "ann3"
(((Neg (Box "a" (Atom "a0"))) `Conj` (Neg (Box "a" (Neg (Atom "a0"))))) `Conj`
 ((Box "b" (Atom "b0")) `Disj` (Box "b" (Neg (Atom "b0")))) `Conj`
 ((Box "c" (Atom "c0")) `Disj` (Box "c" (Neg (Atom "c0")))))
```

Then we define the next three functions (`drawMuddy0` to `3`) and obtain the following graphs yielded by graphviz which correspond to the Muddy children puzzle of PAL.

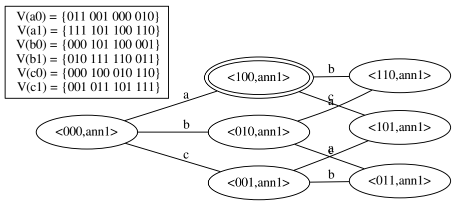
```
-------------------------------------------------
-- Muddy children puzzle by EAK
-------------------------------------------------

drawMuddy0 = graphviz $ draw2 muddyProp     muddy
drawMuddy1 = graphviz $ draw2 muddyProp $   muddy***muddyAnn1
drawMuddy2 = graphviz $ draw2 muddyProp $  (muddy***muddyAnn1)***muddyAnn2
drawMuddy3 = graphviz $ draw2 muddyProp $ ((muddy***muddyAnn1)***muddyAnn2)***muddyAnn3
```

Result of drawMuddy0 (Initial Kripke model)

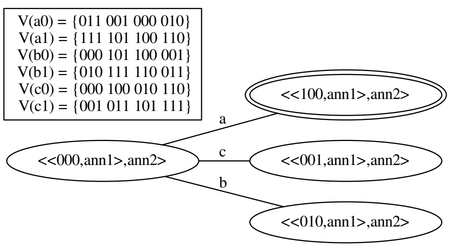


Result of drawMuddy0 (Kripke model after 1st announcement)



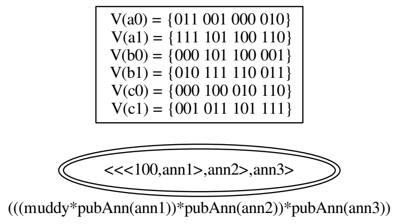
(muddy*pubAnn(ann1))

Result of drawMuddy0 (Kripke model after 2nd announcement)



((muddy*pubAnn(ann1))*pubAnn(ann2))

Result of drawMuddy0 (Kripke model after 3rd announcement)

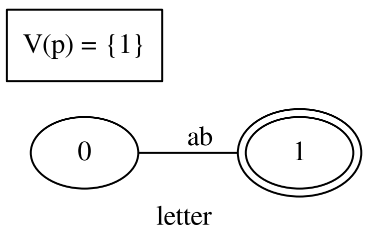


As you can see the last figure, world `<<<100,ann1>,ann2>,ann3>` remains, which means after the three actions (public announcements), only world 100 remains, and this is exactly the answer.

Next, let us implement Example 2.3.2 in which EAK is utilized to express a private announcement (reading a letter). So, we define the Kripke model **letter** (this is the same as Kripke model $\mathfrak{M}$ in Example 2.3.2) and action model Read.

```
------------------------------------------------
-- Kripke model letter
------------------------------------------------
letter :: PointKM
letter = PointKM ((KM name letterS letterR letterV),"1")
    where
    name = "letter"
    letterS = ["0","1"]
    letterR = (refl ["a","b"] letterS) ++ (symm "a" letterS) ++ (symm "b" letterS)
    letterV (Atom "p") = ["1"]
```

Graphic form of **letter**



```
-------------------------------------------------
-- Action model Read
-------------------------------------------------

reada :: Action
reada = PointAM((AM name readaS readaR readaPre),"pa")
    where
    name = "reada"
    readaS = ["npa","pa"]
    readaR = (refl ["a","b"] readaS) ++ [("b","npa","pa"),("b","pa","npa")]
    readaPre x  = case x of "pa" -> Atom "p"
                            "npa" -> Neg (Atom "p")
```

So, we trace Example 2.3.2 as follows.

A.2 Automated theorem prover for DELs : `Kripkenstein`

Let us now move onto the section for implementations of labelled sequent calculi that have been introduced so far. Each labelled system is implemented as an automated theorem prover, and we name the group of provers `Kripkenstein`.[2] We introduce one prover which is for **GPAL**. We notice that we referred to an existing theorem prover called `PESCA` [71],[3] as the basis of the prover.

A.2.1 GPAL of `Kripkenstein`

Let us define the language of **GPAL** for the prover, where we add all logical connectives in the language of PAL as follows, and of course each added connective is semantically equivalent to a formula of the original language.

$$A ::= \bot \mid \top \mid p \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \leftrightarrow A) \mid \Box_a A \mid \Diamond_a A.$$

An inference rule corresponding to each added connective is also included. Further, for automatizing the construction of a derivation, we revise the initial sequents and inference rules of PAL given in Table 3.2. For example, we add a list β of labels as a *history* of instantiated labels, and a label is added to the history by the application of inference rules $(L\Box_a)$ and $(R\Diamond_a)$ to avoid instantiating the same label as before. Let us look at the following example.

$$x{:}^{\epsilon}\Box_a^{y} B \Rightarrow y^{\epsilon}{:}C$$

In this case, we cannot apply $(L\Box_a)$ with instanating y since $\Box_a B$ has already been instantiated with y which is recorded in the history. This restriction of rules for the knowledge operators prevents to infinitally instantiate the same label.

The revised initial sequents and inference rules are shown in Table A.1. Here is the code of definitions for labelled expression, sequent, inference rule and proof (derivation).

[2]Kripkenstein is a fictional nickname that a philosopher coined to combine Saul Kripke and Ludwig Wittgenstein. Kripke [45] interpreted Wittgenstein's *Philosophische Untersuchungen* (Philosophical Investigations) [88] in a unique and peculiar way, but since his reading was considered far from Wittgenstein's original thought and controversial, the reading Kripke interpreted is sometimes called such a nickname. In [45], Kripke reads Wittgenstein as a skeptic with a well-known argument of the distinction between 'plus' and 'quus' which is a paradox regarding private language. In brief, how we can choose 'plus' (a mathematical function defined in a usual way) or 'quus' (another function defined in an unusual and odd way). Kripke's response to the paradox is called a *skeptical solution* inspired by David Hume. According to that, we cannot justify, by any objective fact, the link between a person's private use of 'plus' rather than 'quus' in a particular case and the putative rule itself; however, the use of 'plus' is justified by the fact that 'plus' has been used in a community and everyone there behaves, without any doubt, as it is correct (see more details in [16]). We put the meaning of 'everyone there behaves, without any doubt, as it is correct' into the name of `Kripkenstein` since computers seem to behave so in fact, and that is why we can usually believe the result yielded by them.

[3]`PESCA` is also introduced in Negri et al. [57]

Table A.1: Labelled sequent calculus for `Kripkenstein` : **GPAL**′

(Initial sequents)

$$\frac{}{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}A}\ (init) \qquad \frac{}{x\mathsf{R}_a^{\alpha}v, \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}v}\ (init)$$

$$\frac{}{x{:}^{\alpha}\bot, \Gamma \Rightarrow \Delta}\ (initBot) \qquad \frac{}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\top}\ (initTop)$$

(Rules for propositional connectives)

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L{\to}) \qquad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B}\ (R{\to})$$

$$\frac{x{:}^{\alpha}A, x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \wedge B, \Gamma \Rightarrow \Delta}\ (L\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \wedge B}\ (R\wedge)$$

$$\frac{x{:}^{\alpha}A \to B, x{:}^{\alpha}B \to A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \leftrightarrow B, \Gamma \Rightarrow \Delta}\ (L{\leftrightarrow}) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}B \to A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \leftrightarrow B}\ (R{\leftrightarrow})$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \vee B, \Gamma \Rightarrow \Delta}\ (L\vee) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \vee B}\ (R\vee)$$

(Rules for knowledge operators)

$$\frac{\Gamma x{:}^{\alpha}\Box_a^{\beta,y}A, \Rightarrow x\mathsf{R}_a^{\alpha}y \quad y{:}^{\alpha}A, \Box_a^{\beta,y}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Box_a^{\beta}A, \Gamma \Rightarrow \Delta}\ (L\Box_a\ddagger) \qquad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\Box_a^{\beta}A}\ (R\Box_a)\dagger$$

$$\frac{x\mathsf{R}_a^{\alpha}y, y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\Diamond_a^{\beta}A, \Gamma \Rightarrow \Delta}\ (L\Diamond_a)\dagger \qquad \frac{\Gamma \Rightarrow \Delta, \Diamond_a^{\beta,y}A, x\mathsf{R}_a^{\alpha}y \quad \Gamma \Rightarrow \Delta, \Diamond_a^{\beta,y}A, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\Diamond_a^{\beta}A}\ (R\Diamond_a\ddagger)$$

† y does not appear in the lowersequent.
‡ y does not appear in β.

(Rules for IntPAL)

$$\frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, \Gamma \Rightarrow \Delta}\ (L[.]) \qquad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha}A, x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\langle A\rangle B, \Gamma \Rightarrow \Delta}\ (L\langle.\rangle) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\langle A\rangle B}\ (R\langle.\rangle)$$

$$\frac{x{:}^{\alpha}A, y{:}^{\alpha}A, x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a) \qquad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A}y}\ (Rrel_a)$$

```
------------------------------------------------
-- Basic definitions for the prover
------------------------------------------------

type Label = Int
type History =[Label]
data LabelExp = LabelForm ([Formula],Label,Formula)
              | RelAtom (Agent,[Formula],Label,Label)
                deriving (Eq,Show,Ord)
type Sequent = ([LabelExp],[LabelExp])
data Proof =  Proof String Sequent [Proof]
              deriving (Eq,Show,Ord)

data Rule = Rule (Int, String, Sequent -> Maybe [Sequent])
```

We now introduce inference rules of `Kripkenstein` for PAL.

```
------------------------------------------------
-- Initial sequents
------------------------------------------------
axiomRule  :: String->[Rule]
axiomRule mname =[
  Rule (initN, "init", \ (left,right) ->  if exists left (\x-> exists right (\y-> x==y))
                                         then Just []
                                         else Nothing),

  Rule (initN, "initTop", \ (left,right) ->
                    if (Top) `elem`  [ f | LabelForm (labs,lab,f) <-right ]
                    then Just []
                    else Nothing),
  Rule (initN, "initBot", \ (left,right) ->
                    if (Bottom) `elem`  [ f | LabelForm (labs,lab,f) <-left ]
                    then Just []
                    else Nothing),
  Rule  (initN, "end", \ (left,right) ->
                    if noApplicableRules mname (left,right) && notInit (left,right)
                    then Just []
                    else Nothing)]

noApplicableRules mname (left,right)
      =  let e1 = [(left,c)|c<-rotate right]
             e2 = [(b,right)|b<-rotate left]
         in and [(rule3 x) y == Nothing | x <-psys mname, y<-e1++e2]
notInit (left,right) =  and[x/=y|x<-left,y<-right]
```

Here we have the rule `(end)` defined as

$$\frac{}{\Gamma \Rightarrow \Delta}\ (end)\dagger$$

† There is no inference rule which can be applicable to the sequent $\Gamma \Rightarrow \Delta$. Actually `(end)` is not a rule but the sign of a dead-end of the construction of a derivation. Subsequently, we give inference rules for classical Boolean connectives.

```
-------------------------------------------------
-- Rules for classical connectives
-------------------------------------------------

ruleClassic  :: [Rule]
ruleClassic =[
 Rule (negLN, "L~", \ (left,right) -> case left of
  LabelForm (annf, la, Neg p):rest -> Just [{-1-}(rest,(LabelForm (annf, la, p)):right)]
  otherwise             -> Nothing),
 Rule (negRN, "R~", \ (left,right) -> case right of
  LabelForm(annf, la, Neg p):rest -> Just [{-1-}(LabelForm(annf, la, p):left,rest)]
  otherwise             -> Nothing),
 Rule (conjLN, "L&", \ (left,right) -> case left of
  LabelForm (annf, la, (Conj p q)):rest -> Just [({-1-} LabelForm (annf, la, p)
                                                  :LabelForm (annf, la, q):rest,right)]
  otherwise             -> Nothing),
 Rule (conjRN, "R&", \ (left,right) -> case right of
  LabelForm(annf, la, (Conj p q)):rest -> Just [{-1-}(left,LabelForm(annf, la, p):rest),
                                            {-2-}(left,LabelForm(annf, la, q):rest)]
  otherwise             -> Nothing),
 Rule (disjLN, "Lv", \ (left,right) -> case left of
  LabelForm (annf, la, (Disj p q)):rest -> Just [{-1-}(LabelForm(annf, la, p):rest,right),
                                             {-2-}(LabelForm(annf, la, q):rest,right)]
  otherwise             -> Nothing),
 Rule (disjRN, "Rv", \ (left,right) -> case right of
  LabelForm(annf, la, (Disj p q)):rest -> Just [{-1-} (left, LabelForm(annf, la, p)
                                                        :LabelForm(annf, la, q):rest)]
  otherwise             -> Nothing),
 Rule (implLN, "L->", \ (left,right) -> case left of
  LabelForm(annf, la, (Impl p q)):rest -> Just [{-1-}(rest,LabelForm(annf, la, p):right),
                                            {-2-}(LabelForm(annf, la, q):rest,right)]
  otherwise             -> Nothing),
 Rule (implRN, "R->", \ (left,right) -> case right of
  LabelForm(annf, la, (Impl p q)):rest
             -> Just [{-1-}(LabelForm(annf, la, p):left, LabelForm(annf, la, q):rest)]
  otherwise  -> Nothing),
 Rule (equiLN, "L<->", \ (left,right) -> case left of
  LabelForm(annf, la, (Equi p q)):rest
             -> Just [{-1-}(LabelForm(annf, la, Conj (Impl p q)  (Impl q p) ):rest,right)]
  otherwise  -> Nothing),
 Rule (equiRN, "R<->", \ (left,right) -> case right of
  LabelForm(annf, la, (Equi p q)):rest
             -> Just [{-1-}(left,LabelForm(annf, la, Conj (Impl p q)  (Impl q p) ):rest)]
  otherwise  -> Nothing)]
```

Next, we give inference rules for announcement operators.

```
-------------------------------------------------
-- Rules for announcement operators
-------------------------------------------------

ruleGPAL  :: [Rule]
ruleGPAL =[
```

```
Rule (atLN, "Lat", \ (left,right)  -> case left of
 LabelForm (k:restw, la, Atom p):restl
            -> Just [{-1-}(LabelForm (restw,la, Atom p):restl, right)]
 otherwise  -> Nothing ) ,
Rule (atRN, "Rat", \ (left,right) -> case right of
  LabelForm(k:restw, la, Atom p):restr
            -> Just [{-1-}(left, (LabelForm (restw,la, Atom p)):restr)]
  otherwise -> Nothing),
Rule (annLN, "L[.]", \ (left,right)  -> case left of
  LabelForm(annf, la, (Announce p q)):rest
            -> Just [{-1-}(rest,LabelForm(annf, la, p):right),
                     {-2-} (LabelForm(annf++[p], la, q):rest,right)]
  otherwise -> Nothing),
Rule (annRN, "R[.]", \ (left,right)  -> case right of
  LabelForm(annf, la, (Announce p q)):rest
            -> Just [{-1-}(LabelForm(annf, la, p):left,
                           LabelForm(annf++[p], la, q):rest)]
  otherwise -> Nothing),
Rule (ann2LN, "L<.>", \ (left,right)  -> case left of
  LabelForm(annf, la, (Announce2 p q)):restl
             -> Just [{-1-}(LabelForm(annf, la, p)
                              :LabelForm(annf++[p], la, q):restl, right)]
  otherwise  -> Nothing),
Rule (ann2RN, "R<.>", \ (left,right)  -> case right of
  LabelForm(annf, la, (Announce2 p q)):restr
             -> Just [{-1-}( left,LabelForm(annf, la, p):restr),
                      {-2-} (left,LabelForm(annf++[p], la, q):restr)]
  otherwise  -> Nothing),
Rule (relLN, "Lrel", \ (left,right) -> case left of
   RelAtom (ag,{-hist,-}(x:annf), w1, w2):restl
             -> Just [{-1-}( (rel' (LabelForm (annf,w1, x)) restl [])
                          :(rel' (LabelForm (annf,w2, x)) restl [])
                          :(RelAtom (ag,{-hist,-} annf, w1, w2)):restl, right)]
   otherwise -> Nothing),
Rule (relRN, "Rrel", \ (left,right) -> case right of
   RelAtom (ag, (x:annf), w1, w2):restr
             -> if w1 /= w2 then Just [ {-1-}(left, (LabelForm (annf,w1, x)):restr),
                                        {-2-}(left, (LabelForm (annf,w2, x)):restr),
                                        {-3-}(left, RelAtom (ag, annf, w1, w2):restr)]
                else Just [ {-1-}(left, (LabelForm (annf,w1, x)):restr),
                           {-2-}(left, RelAtom (ag, annf, w1, w2):restr)]
   otherwise -> Nothing),
Rule (cmpLN, "Lcmp", \ (left,right)  -> case left of
              LabelForm(p `Conj` (Announce p' q):annf, w,r):restl| p==p'
                             -> Just [(LabelForm( (annf++[p]++[q]), w,r):restl,right)]
              otherwise              -> Nothing),
Rule (cmpRN, "Rcmp", \ (left,right)  -> case right of
              LabelForm( (p `Conj` (Announce p' q)):annf,w,r):restr| p==p'
                             -> Just [(left,LabelForm( annf++[p]++[q],w,r):restr )]
              otherwise              -> Nothing)]
```

Then, inference rules for knowledge operators are given as follows.

```
-------------------------------------------------
-- Rules for announcement operators
-------------------------------------------------
ruleK  :: [InferenceRule]
ruleK =[
  Rule (boxRN, "R#", \  (left,right) ->
   let label1 = freshLabel (left,right)
   in case right of
      LabelForm (annf,la, Box ag hist p):restr
                -> Just [({-1-}RelAtom (ag,annf,la, label1):left,
                               LabelForm (annf, label1, p):restr)]
      otherwise -> Nothing),

  Rule (boxLN, "L#", \  (left,right) -> case left of
   LabelForm (annf, la, Box ag hist p):restl ->
     let labels = difference (wholeLabel (left,right)) hist
          selectedlabels =
           [ a  |a <-labels,
              or[((rule3 initr) (left,RelAtom (ag,annf, la, a):right))==Just []
                                                        |  initr<-axiomRule "K"]
           || or[((rule3 initr) (LabelForm (annf, a, p):restl, right))==Just []
                                                        |  initr<-axiomRule "K"]]
          label2 =  head  $reverse labels
      in case  labels of
       [] ->  Nothing
       otherwise
            -> Just [{-1-} (LabelForm (annf, la, Box ag (snub (label2:hist)) p):restl,
                             RelAtom (ag,annf, la, label2):right),
                     {-2-} (LabelForm (annf, la, Box ag (snub (label2:hist)) p):
                             LabelForm (annf, label2, p):restl,right)]
   otherwise -> Nothing),

  Rule (diaRN, "R$", \  (left,right) -> case right of
   LabelForm (annf, la, Dia ag hist p):restr
            -> Just [{-1-}(left,LabelForm (annf, la, Neg (Box ag hist (Neg p))):restr)]
   otherwise -> Nothing),

  Rule (diaLN, "L$", \  (left,right) -> case left of
   LabelForm (annf, la, Dia ag hist p):restl
            -> Just [{-1-}(LabelForm (annf, la, Neg (Box ag hist (Neg p))):restl, right)]
   otherwise -> Nothing)]


-- freshLabel :: Sequent->Label
freshLabel sq = head[x | x<-[1..],x `notElem` (wholeLabel sq)]

-- wholeLabel :: Sequent->[Int]
wholeLabel sq =   ( nub [w | LabelForm (_,w,_) <-(fst sq)++(snd sq)]
                     ++ [w | RelAtom (_,_,w,v) <-(fst sq)++(snd sq)]
                     ++ [v | RelAtom (_,_,w,v) <-(fst sq)++(snd sq)] )


wholeAgent (left,right) = nub $ concat[ agentL x [] | x  <-left++right]

agentF x li = case  x of
```

```
      Box ag hist p ->  agentF p (ag:li)
      Dia ag hist p -> agentF p (ag:li)
      Neg p -> (agentF p li)
      Conj p q -> (agentF p li) ++ (agentF q li)
      Disj p q -> (agentF p li) ++ (agentF q li)
      Impl p q -> (agentF p li) ++ (agentF q li)
      Impl2 p q -> (agentF p li) ++ (agentF q li)
      Equi p q -> (agentF p li) ++ (agentF q li)
      Announce p q -> (agentF p li) ++ (agentF q li)
      Announce2 p q -> (agentF p li) ++ (agentF q li)
      otherwise -> li

agentL x li = case  x of
      LabelForm (annf,la, y)      -> agentF y li
      RelAtom (ag,annf, w1, w2)  -> (ag:li)
```

A.2.2 Core functions for automated theorem proving

We have define **GPAL** by Haskell so far, but this this not enough to construct a derivation of a theorem of PAL. What we need here is to define functions for automatically applying inference rule(s) to a sequent, and judge whether this is a derivable or not. The following is the central part of such automation.

```
-------------------------------------------------
-- automation for a derivation
-------------------------------------------------
sortRule []= []
sortRule (x:xs) =
   let smallerOrEqual = [a | a <- xs , fst (fst a) <= fst (fst x)]
       larger = [a | a <- xs , fst (fst a) > fst (fst  x)]
   in sortRule smallerOrEqual ++ [x] ++ sortRule larger

applicableRules mname sequ=
          let sys = psys mname ++ axiomRule mname
          in [ ((n, nm),map sortSeq(justList ru sequ))
            | Rule (n, nm,ru) <- sys, justTrue ru sequ]

sortSeq (l,r) = (snub l,snub r)

applyRule :: String->Sequent -> [Proof]
applyRule mname  (l1,r1) =
        let e1 = [applicableRules mname (l2, r1)|l2<-rotate l1]
            e2 = [applicableRules mname (l1, r2)|r2<-rotate r1]
            e = sortRule$concat(e1 ++e2)
            ((_,rule),seqs) = head e
            prfs =  head$ combinations $map (applyRule mname) seqs
        in [Proof rule (sortSeq (l1,r1)) prfs]
```

It is quite simple to understand what the functions work. At first, if we input a sequent to function `applicableRules`, it outputs a list of inference rules which can be applicable to the sequent. For example, sequent $A \wedge B, \neg C \Rightarrow, C \rightarrow B$ is inputted to the function, it outputs $((L\wedge), (L\neg), (R \rightarrow))$. After that, function `applyRule` choose a rule which has higher priority than others (an order of rules is defined but omitted here). Let us say rule $(L \rightarrow)$ is applied. Then, sequent $A \wedge B, \neg C, C \Rightarrow, B$ is yielded by the application of the rule. The process is repeated by the sequent which is an initial sequent or the sequent in which there is no applicable rule. The prover judges a given sequent as derivable (provable) if every branch of a sequent reaches to initial sequents, and as unprovable if there are some branches which are not initial sequents and in which there is no applicable rule. The below screen-shot is an example of deriving a theorem (one direction of $(RA4)$) of PAL.

Example of `Kripkenstein`

```
Input Formula A
A ::= p | ~A | (A & A) | (A v A) | (A -> A) | (A <-> A) | #aA | $aA | [A]A | <A>A
ex1. A->(B->A)
ex2. [A]#a B->(A->#a[A]B)

[A]#aB -> (A->#a[A]B)
 = "==>[A]#aB -> (A->#a[A]B)"
(R->)   ==> 0():[A]#a[]B -> (A -> #a[][A]B)
|
`- (R->)  0():[A]#a[]B ==> 0():A -> #a[][A]B
    |
    `- (L[.])  0():A, 0():[A]#a[]B ==> 0():#a[][A]B
        |
        +- (init)  0():A ==> 0():A, 0():#a[][A]B
        |
        `- (R#)  0():A, 0(A):#a[]B ==> 0():#a[][A]B
            |
            `- (R[.])  0():A, 0(A):#a[]B, 0Ra()1 ==> 1():[A]B
                |
                `- (L#)  0():A, 1():A, 0(A):#a[]B, 0Ra()1 ==> 1(A):B
                    |
                    +- (Rrel)  0():A, 1():A, 0(A):#a[1]B, 0Ra()1 ==> 1(A):B, 0Ra(A)1
                    |   |
                    |   +- (init)  0():A, 1():A, 0(A):#a[1]B, 0Ra()1 ==> 0():A, 1(A):B
                    |   |
                    |   +- (init)  0():A, 1():A, 0(A):#a[1]B, 0Ra()1 ==> 1():A, 1(A):B
                    |   |
                    |   `- (init)  0():A, 1():A, 0(A):#a[1]B, 0Ra()1 ==> 1(A):B, 0Ra()1
                    |
                    `- (init)  0():A, 1():A, 0(A):#a[1]B, 1(A):B, 0Ra()1 ==> 1(A):B

Output its LaTeX file? yes/no
```

The prover can be found at the following URL.

https://github.com/NomuraS/GPAL

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