On Confluence Conditions of Left-Linear Term Rewriting Systems

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February 15, 2000

Keywords: Term Rewriting System, Left-Linear, Confluence, Critical Pair, Confluence Condition.

This thesis studies confluence conditions of left-linear Term Rewriting Systems (TRSs). Confluence conditions are based on the notion of critical pairs, which are generated from overlap between rewrite rules. Two kinds of critical pairs have been proposed, that is, the ordinary critical pairs and the extended ones. The ordinary critical pairs give simpler confluence conditions than the extended ones. On the other hand, the extended critical pairs give confluence conditions with weaker restrictions. The purpose of this study is to give a new confluence condition, which is simple like that based on the ordinary critical pairs and has weak restriction like that based on the extended ones.

1 Background

A TRS is a set of oriented equations called rewrite rules. In TRS the left-hand side can be replaced by the right-hand side, but not vice versa. This emphasizes the computation property of equations. For example, we can write $1 + 2 = 3$, or conversely $3 = 1 + 2$ in equational logic. However, when we see it as computations, the equation should be regarded as $1 + 2 \rightarrow 3$, which says $1 + 2$ becomes 3. It is just a rewrite rule which replaces $1 + 2$ with 3. Generally, in order to obtain a reduction from a term, we have to identify a part of it that matches the left-hand side of some rewrite rule. Then we can replace the matched part of the term with the right-hand side of the rule. Thus, we can get a natural computational model with oriented equations. TRSs are applicable in various fields, for example, program transformation, automatic theorem proving and functional programming languages.

Confluence is the important property of TRSs. It guarantees the uniqueness of answer in non-deterministic computations, and decides whether an equation holds or not. Hence, it plays an important role in functional programming languages, automatic theorem proving, and so on. Unfortunately, confluence is an undecidable property of TRSs. Thus, several decidable criteria of confluence have been proposed based on critical pairs. For terminating TRSs, confluence is equivalent to confluence of critical pairs. However, for non-terminating TRSs, confluence of critical pairs does not guarantee confluence.

In 1980, for left-linear non-terminating TRSs, Huet proposed a fundamental confluence condition, where each ordinary critical pair is closed by a parallel reduction. In 1988, Toyama extended Huet's confluence condition by analyzing overlap situation. In 1997, Oostrom extended them further with development, which includes parallel reduction. In 1998, Oyamaguchi and Ohta proposed a confluence condition, with some restriction of rewrite positions, based on ordinary critical pairs.

In 1981, Toyama proposed a condition based on parallel critical pairs. In 1998, Okui proposed a condition based on simultaneous critical pairs. Their closed conditions of extended critical pairs are weaker than those based on ordinary critical pairs. This fact leads us to a new condition based on ordinary critical pairs, which has a weak restriction similar to that based on extended critical pairs.

2 Results

In this study, we propose a confluence condition, called most independence condition, based on ordinary critical pairs for left-linear non-terminating TRSs. Since our confluence condition has a closed condition similar to parallel critical pairs, it has the weakest restriction among well-known confluence conditions based on ordinary critical pairs. The restriction of rewrite positions in our condition is essential and plays a very important role in the proof of the confluence property.

Considering several examples, it follows that we can apply our condition to some TRSs to which we cannot apply conditions based on extended critical pairs. Therefore, as the case may be, conditions based on ordinary critical pairs are effective. We display the following TRS R_1 (by Okui 1998). R_1 does not satisfy the condition based on simultaneous critical pairs, but does the condition based on parallel critical pairs or our condition.

$$R_1 = \left\{ \begin{array}{rcl} f(g(g(x))) & \to & a \\ f(g(h(x))) & \to & b \\ f(h(g(x))) & \to & b \\ f(h(h(x))) & \to & c \\ g(x) & \to & h(x) \\ a & \to & b \\ b & \to & c \\ c & \to & c \end{array} \right.$$

R_1 has a simultaneous critical pair $\langle a, f(h(h(x)))\rangle$, which is not closed by a development. Therefore, R_1 does not satisfy the condition based on simultaneous critical pairs.

Fortunately, when we apply our condition to R_1 we do not have to test simultaneous critical pairs. Since the arity of every function symbol in R_1 is 1, R_1 satisfies the restriction of our condition. Thus, from our confluence condition, we can show that R_1 is confluent.