A 4.31-Approximation for the Geometric Unique Coverage Problem on Unit Disks

Takehiro Ito[a], Shin-ichi Nakano[b], Yoshio Okamoto[c], Yota Otachi[d], Ryuhei Uehara[d], Takeaki Uno[e], Yushi Uno[f]

[a] *Graduate School of Information Sciences, Tohoku University, Aoba-yama 6–6–05, Sendai 980–8579, Japan*
[b] *Department of Computer Science, Gunma University, Kiryu 376–8515, Japan*
[c] *Department of Communication Engineering and Informatics, University of Electro-Communications, Chofugaoka 1–5–1, Chofu, Tokyo 182–8585, Japan*
[d] *School of Information Science, Japan Advanced Institute of Science and Technology, Asahidai 1–1, Nomi, Ishikawa 923–1292, Japan*
[e] *Principles of Informatics Research Division, National Institute of Informatics, 2–1–2 Hitotsubashi, Chiyoda-ku, Tokyo 101–8430, Japan*
[f] *Graduate School of Science, Osaka Prefecture University, 1–1 Gakuen-cho, Naka-ku, Sakai 599-8531, Japan*

Abstract

We give an improved approximation algorithm for the unique unit-disk coverage problem: Given a set of points and a set of unit disks, both in the plane, we wish to find a subset of disks that maximizes the number of points contained in exactly one disk in the subset. Erlebach and van Leeuwen (2008) introduced this problem as the geometric version of the unique coverage problem, and gave a polynomial-time 18-approximation algorithm. In this paper, we improve this approximation ratio 18 to $2 + 4/\sqrt{3} + \varepsilon$ $(< 4.3095 + \varepsilon)$ for any fixed constant $\varepsilon > 0$. Our algorithm runs in polynomial time which depends exponentially on $1/\varepsilon$. The algorithm can be generalized to the budgeted unique unit-disk coverage problem in which each point has a profit, each disk has a cost, and we wish to maximize the total profit of the uniquely covered points under the condition that the total cost is at most a given bound.

Keywords: approximation algorithm, computational geometry, unique coverage problem, unit disk

1. Introduction

Motivated by applications from wireless networks, Erlebach and van Leeuwen [4] study the following problem. Let $\mathcal{P}$ be a set of points and $\mathcal{D}$ a set of unit disks, both in the plane $\mathbb{R}^2$. For a subset $\mathcal{C} \subseteq \mathcal{D}$ of unit disks, we say that a point $p \in \mathcal{P}$ is *uniquely covered* by $\mathcal{C}$ if there is exactly one disk $D \in \mathcal{C}$ containing p. In the (maximum) *unique unit-disk coverage problem*, we

Email addresses: takehiro@ecei.tohoku.ac.jp (Takehiro Ito), nakano@cs.gunma-u.ac.jp (Shin-ichi Nakano), okamotoy@uec.ac.jp (Yoshio Okamoto), otachi@jaist.ac.jp (Yota Otachi), uehara@jaist.ac.jp (Ryuhei Uehara), uno@nii.ac.jp (Takeaki Uno), uno@mi.s.osakafu-u.ac.jp (Yushi Uno)

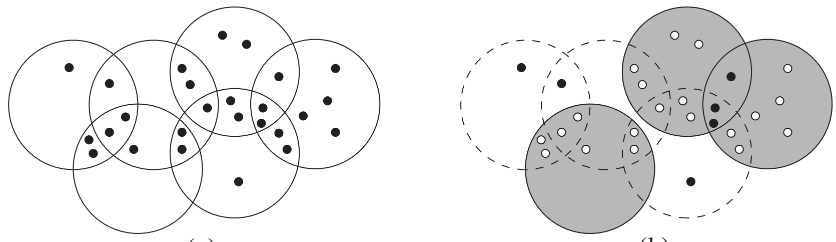

Figure 1: (a) An instance $\langle \mathcal{P}, \mathcal{D} \rangle$ of the unique unit-disk coverage problem, and (b) an optimal solution $\mathcal{C}^*$ to $\langle \mathcal{P}, \mathcal{D} \rangle$, where each disk in $\mathcal{C}^*$ is hatched and each uniquely covered point is drawn as a small white circle.

are given a pair $\langle \mathcal{P}, \mathcal{D} \rangle$ of a set $\mathcal{P}$ of points and a set $\mathcal{D}$ of unit disks as input, and we are asked to find a subset $\mathcal{C} \subseteq \mathcal{D}$ such that the number of points in $\mathcal{P}$ uniquely covered by $\mathcal{C}$ is maximized. An instance is shown in Figure 1(a), and an optimal solution $\mathcal{C}^*$ to this instance is illustrated in Figure 1(b).

In the context of wireless networks, as described by Erlebach and van Leeuwen [4], each point corresponds to a customer location, and the center of each disk corresponds to a place where the provider can build a base station. If several base stations cover a certain customer location, then the resulting interference might cause this customer to receive no service at all. Ideally, each customer should be serviced by exactly one base station, and service should be provided to as many customers as possible. This situation corresponds to the unique unit-disk coverage problem.

1.1. Past work and motivation

Demaine et al. [3] formulated the non-geometric unique coverage problem in a more general setting. They gave a polynomial-time $O(\log n)$-approximation algorithm[1] for the non-geometric unique coverage problem, where n is the number of elements (in the geometric version, n corresponds to the number of points). Guruswami and Trevisan [5] studied the same problem and its generalization, which they called 1-in-k SAT. The appearance of the unique coverage problem is not restricted to wireless networks. The previous papers [3, 5] provide a connection with unlimited-supply single-minded envy-free pricing and the maximum cut problem. We refer the reader to their papers for details.

The parameterized complexity of the unique coverage problem has also been studied by Misra et al. [10].

Erlebach and van Leeuwen [4] studied geometric versions of the unique coverage problem. They showed that the unique unit-disk coverage problem is

[1] For the sake of notational convenience, throughout the paper, we say that an algorithm for a maximization problem is *α-approximation* if it returns a solution with the objective value APX such that OPT $\leq \alpha$APX, where OPT is the optimal objective value, and hence $\alpha \geq 1$.

strongly NP-hard, and gave a polynomial-time 18-approximation algorithm. They also consider the problem on unit squares, and gave a polynomial-time $(4 + \varepsilon)$-approximation algorithm for any fixed constant $\varepsilon > 0$. Later, van Leeuwen [11] gave a proof that the unit-square version is strongly NP-hard, and improved the approximation ratio for the unit squares to $2 + \varepsilon$. In a sister paper, we exhibit a polynomial-time approximation scheme (PTAS) for the unique unit-square coverage problem [8].

1.2. Contribution of this paper

In this paper, we improve the approximation ratio 18 for the unique unit-disk coverage problem to $2 + 4/\sqrt{3} + \varepsilon$ ($< 4.3095 + \varepsilon$) for any fixed constant $\varepsilon > 0$. Our algorithm runs in polynomial time, but the dependency on $1/\varepsilon$ is exponential. The algorithm can be generalized to the *budgeted* unique unit-disk coverage problem, in which we are given a budget B, each point in $\mathcal{P}$ has a profit, each disk in $\mathcal{D}$ has a cost, and we wish to find $\mathcal{C} \subseteq \mathcal{D}$ that maximizes the total profit of the uniquely covered points by $\mathcal{C}$ under the condition that the total cost of $\mathcal{C}$ is at most B.

An extended abstract of this paper has been presented at ISAAC 2012 [7].

2. Preliminaries

An instance is denoted by $\langle \mathcal{P}, \mathcal{D} \rangle$, where $\mathcal{P}$ is a set of points in the plane, and $\mathcal{D}$ is a set of unit disks in the plane. A unit disk in this paper means a closed disk with radius $1/2$, and hence contains the boundary. Without loss of generality, we assume that any two points in $\mathcal{P}$ (resp., any two centers of disks in $\mathcal{D}$) have distinct x-coordinates and distinct y-coordinates. If not, we rotate the plane in polynomial time so that this condition is satisfied [11]. We also assume that no two disks in $\mathcal{D}$ touch, that is, there is no pair of two disks having exactly one point in common, and no point in $\mathcal{P}$ lies on the boundary of any disk in $\mathcal{D}$. If not, we increase the radii of the disks by a sufficiently small amount in polynomial time so that the number of uniquely covered points by any disk subset does not change [11]. For brevity, the x-coordinate of the center of a disk D is referred to as the x-coordinate of the disk, and denoted by $x(D)$. Similarly, the y-coordinate of a disk D means the y-coordinate of the center of D, and is denoted by $y(D)$.

3. Technique highlight

3.1. Comparison with the previous algorithm

We describe here how our approach differs from that of [4].

We use the following two techniques in common. (1) The shifting technique, first developed by Baker [1] for planar graphs, and later adapted to geometric settings by Hochbaum and Maass [6]: This subdivides the whole plane into some smaller pieces, and ignores some points so that the combination of approximate solutions to smaller pieces will yield an approximation solution to the whole plane. (2) A classification of disks: Namely, for each instance on a smaller piece, we partition the set of disks into a few classes so that the instance on a restricted set of disks can be handled in polynomial time. Taking the best solution in those classes yields a constant-factor approximation.

Erlebach and van Leeuwen [4] employed the techniques above in the following way. (1) Their smaller pieces are unit squares S with side length $1/2$. They look at instances on the points in S and the disks that intersect S. (2) For each unit square S, the disks intersecting S are classified into two classes: A disk is classified "vertical" if its overlap with vertical sides is larger than the overlap with horizontal sides; Otherwise, it is classified "horizontal." They give a polynomial-time exact algorithm for the instance with the points inside S and the disks in each of the two classes, with dynamic programming. At Step 1, they lose the approximation ratio of 9, and at Step 2, they lose the approximation ratio of 2. Thus, the overall approximation ratio of their algorithm is $9 \times 2 = 18$. The reader can refer to their paper for more details [4].

On the other hand, our algorithm in this work exploits the techniques above in the following way. (1) Our smaller pieces are stripes, which consists of some number of horizontal ribbons such that each ribbon is of height $h = \sqrt{3}/4$ and the gap between ribbons is $b = 1/2$, as illustrated in Figure 2. At this step, we lose the approximation ratio of $1+b/h = 1+2/\sqrt{3}$, as shown later in Lemma 4.2. (2) We classify the disks intersecting a stripe into two classes. The first class consists of the disks whose centers lie outside the ribbons in the stripe, and the second class consists of the disks whose centers lie inside the ribbons. It is important to notice that we will not solve the classified instances exactly, but rather we design a PTAS for each of them. Namely, we provide a polynomial-time algorithm for each of the classified instances with approximation ratio $1 + \varepsilon'$, where $\varepsilon' > 0$ is a fixed constant. Note that the polynomial running time depends exponentially on $1/\varepsilon'$. Then, since we have two classes, we only lose the approximation ratio of $2(1 + \varepsilon')$ at this step (Lemma 4.3). Thus, choosing ε' appropriately, we can achieve the overall approximation ratio of $(1+2/\sqrt{3}) \times 2(1 + \varepsilon') = 2 + 4/\sqrt{3} + \varepsilon$.

3.2. Comparison with the unit-square case

The PTAS in this paper for each of the classified instances uses an idea similar to our PTAS for unit squares [8]. However, there is a big difference, as explained below, that makes us unable to give a PTAS for the original instance on unit disks. Look at a horizontal ribbon. For the unit-square case, the intersection of the ribbon and a unit square is a rectangle. Then, its boundary is an x-monotone curve. The monotonicity enables us to provide a PTAS. However, for the unit-disk case, if we look at the intersection of the ribbon and a unit disk, then its boundary is not necessarily x-monotone. To make it x-monotone, we need to give a gap between ribbons and throw away the disks that have centers inside the ribbons; This is why we classified the disks into two classes, as mentioned above. It should be noted that, by this disk classification, we can get the x-monotonicity only for the disks whose centers lie outside the ribbons. To obtain the approximation ratio of $2+4/\sqrt{3}+\varepsilon$, we need to construct

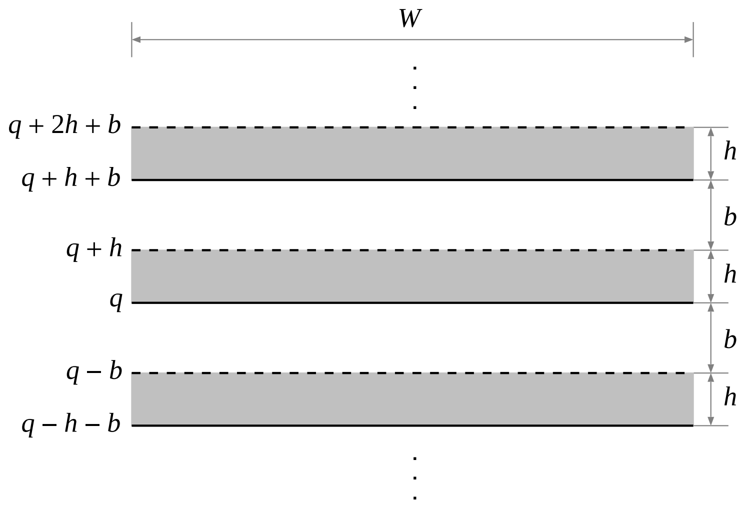

Figure 2: Stripe $R_W(q, h, b)$ consisting of ribbons with height h.

a PTAS for the classified instance in which the centers of disks lie inside the ribbons. We thus develop several new techniques to deal with such disks.

4. Main result and outline

The following is the main result of the paper.

Theorem 4.1. *For any fixed constant $\varepsilon > 0$, there is a polynomial-time $(2 + 4/\sqrt{3} + \varepsilon)$-approximation algorithm for the unique unit-disk coverage problem.*

In the remainder of the paper, we give a polynomial-time $2(1+\varepsilon')(1+2/\sqrt{3})$-approximation algorithm for the unique unit-disk coverage problem, where ε' is a fixed positive constant such that $2\varepsilon'(1 + 2/\sqrt{3}) = \varepsilon$.

4.1. Restricting the problem to a stripe

A rectangle is *axis-parallel* if its boundary consists of horizontal and vertical line segments. Let R_W be an (unbounded) axis-parallel rectangle of width W and height ∞ which properly contains all points in $\mathcal{P}$ and all unit disks in $\mathcal{D}$. We fix the origin of the coordinate system on the left vertical boundary of R_W. For two positive real numbers h, b and a non-negative real number $q \in [0, h+b)$, we define a *stripe* $R_W(q, h, b)$ as follows (see also Figure 2):

$$R_W(q, h, b) = \{[0, W] \times [q + i(h + b), q + (i + 1)h + ib) \mid i \in \mathbb{Z}\},$$

that is, $R_W(q, h, b)$ is a set of rectangles with width W and height h; Each rectangle in $R_W(q, h, b)$ is called a *ribbon*. It should be noted that the upper boundary of each ribbon is not contained in the ribbon, while the lower boundary is contained. We denote by $\mathcal{P} \cap R_W(q, h, b)$ the set of all points in $\mathcal{P}$ contained in $R_W(q, h, b)$. We have the following lemma, by applying the well-known shifting technique [4, 6].

Lemma 4.2. *Suppose that there is a polynomial-time α-approximation algorithm for the unique unit-disk coverage problem on $\langle \mathcal{P} \cap R_W(q, h, b), \mathcal{D} \rangle$ for arbitrary constant q and fixed constants h, b. Then, there is a polynomial-time $\alpha(1 + b/h)$-approximation algorithm for the unique unit-disk coverage problem on $\langle \mathcal{P}, \mathcal{D} \rangle$.*

Proof. For a point set $\mathcal{P}$ and a subset $\mathcal{C}$ of a disk set $\mathcal{D}$, we denote by $\mathsf{profit}(\mathcal{P}, \mathcal{C})$ the number of points in $\mathcal{P}$ that are uniquely covered by $\mathcal{C}$.

Consider an arbitrary optimal solution $\mathcal{C}^* \subseteq \mathcal{D}$ for the problem on $\langle \mathcal{P}, \mathcal{D} \rangle$. Then, the optimal objective value for $\langle \mathcal{P}, \mathcal{D} \rangle$ is equal to $\mathsf{profit}(\mathcal{P}, \mathcal{C}^*)$. Pick a real number q uniformly at random from $[0, h + b)$, and fix the stripe $R_W(q, h, b)$. Let $\mathcal{P}_q = \mathcal{P} \cap R_W(q, h, b)$. The probability that a point of $\mathcal{P}$ is contained in the stripe $R_W(q, h, b)$ is $h/(h + b)$. Therefore, we have

$$\mathbf{E}\big[\mathsf{profit}(\mathcal{P}_q, \mathcal{C}^*)\big] = \frac{h}{h+b} \cdot \mathsf{profit}(\mathcal{P}, \mathcal{C}^*). \tag{1}$$

Let $\mathcal{C}_q^* \subseteq \mathcal{D}$ be an arbitrary optimal solution to $\langle \mathcal{P}_q, \mathcal{D} \rangle = \langle \mathcal{P} \cap R_W(q, h, b), \mathcal{D} \rangle$. Then, we have $\mathsf{profit}(\mathcal{P}_q, \mathcal{C}^*) \leq \mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q^*)$ because $\mathcal{C}^* \subseteq \mathcal{D}$ and $\mathcal{C}_q^*$ is an optimal solution to $\langle \mathcal{P}_q, \mathcal{D} \rangle$. By the assumption, we can find a subset $\mathcal{C}_q \subseteq \mathcal{D}$ in polynomial time such that $\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q^*) \leq \alpha \cdot \mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q)$. Therefore, we have $\mathsf{profit}(\mathcal{P}_q, \mathcal{C}^*) \leq \alpha \cdot \mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q)$. By Eq. (1), we thus have

$$\mathsf{profit}(\mathcal{P}, \mathcal{C}^*) = \frac{h+b}{h} \cdot \mathbf{E}\big[\mathsf{profit}(\mathcal{P}_q, \mathcal{C}^*)\big] \leq \alpha \cdot \left(1 + \frac{b}{h}\right) \cdot \mathbf{E}\big[\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q)\big].$$

This approach can be derandomized. The choices of q for which the same set of points is in the stripe $R_W(q, h, b)$ give an approximation of the same quality. Therefore, it suffices to look at the $O(|\mathcal{P}|)$ values of q for which a ribbon boundary hits a point in $\mathcal{P}$, and thus we can consider all values of q in polynomial time. As our approximate solution for the problem on $\langle \mathcal{P}, \mathcal{D} \rangle$, we output the solution with the highest $\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_q)$ among the $O(|\mathcal{P}|)$ values of q. Then, the solution is an $\alpha(1 + b/h)$-approximation, as required. □

For the sake of further simplification, we assume without loss of generality that no ribbon has a point of $\mathcal{P}$ or the center of a disk of $\mathcal{D}$ on its boundary (of the closure).

4.2. Approximating the problem on a stripe

Using Lemma 4.2, one can obtain a polynomial-time $\alpha(1 + 2/\sqrt{3})$-approximation algorithm by setting $h = \sqrt{3}/4$ and $b = 1/2$. To complete the proof of Theorem 4.1, for any fixed constant $\varepsilon' > 0$, we thus give a polynomial-time $2(1 + \varepsilon')$-approximation algorithm for the unique unit-disk coverage problem on $\langle \mathcal{P} \cap R_W(q, h, b), \mathcal{D} \rangle$.

We first partition the disk set $\mathcal{D}$ into two subsets $\mathcal{D}_O$ and $\mathcal{D}_I$ under a fixed stripe $R_W(q,h,b)$. Let $\mathcal{D}_O \subseteq \mathcal{D}$ be the set of unit disks whose centers are not contained in the stripe $R_W(q,h,b)$. Let $\mathcal{D}_I = \mathcal{D} \setminus \mathcal{D}_O$, that is, $\mathcal{D}_I$ is the set of unit disks whose centers are contained in $R_W(q,h,b)$. Let $\mathcal{P}_q = \mathcal{P} \cap R_W(q,h,b)$. In Sections 5 and 8, we will show that each of the problems on $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$ and $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$ admits a polynomial-time $(1+\varepsilon')$-approximation algorithm for any fixed constant $\varepsilon' > 0$, respectively. (Sections 6 and 7 will be devoted to prove the key lemmas of our algorithm for $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$.) We choose a better solution from $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$ and $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$ as our approximate solution. The following lemma shows that this choice gives rise to a $2(1+\varepsilon')$-approximation for the problem on $\langle \mathcal{P}_q, \mathcal{D} \rangle$.

Lemma 4.3. *Let $\langle \mathcal{P}, \mathcal{D} \rangle$ be an instance of the unique unit-disk coverage problem, and let $\mathcal{D}_1$ and $\mathcal{D}_2$ partition $\mathcal{D}$ (i.e., $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{D}$ and $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$). Let $\mathcal{C}_1 \subseteq \mathcal{D}_1$ and $\mathcal{C}_2 \subseteq \mathcal{D}_2$ be β-approximate solutions to the instances $\langle \mathcal{P}, \mathcal{D}_1 \rangle$ and $\langle \mathcal{P}, \mathcal{D}_2 \rangle$, respectively. Then, the set among $\mathcal{C}_1$ and $\mathcal{C}_2$ having $\max\{\mathsf{profit}(\mathcal{P},\mathcal{C}_1), \mathsf{profit}(\mathcal{P},\mathcal{C}_2)\}$ is a 2β-approximate solution to $\langle \mathcal{P}, \mathcal{D} \rangle$.*

Proof. Let $\mathcal{C}^* \subseteq \mathcal{D}$, $\mathcal{C}_1^* \subseteq \mathcal{D}_1$ and $\mathcal{C}_2^* \subseteq \mathcal{D}_2$ be optimal solutions to the instances $\langle \mathcal{P}, \mathcal{D} \rangle$, $\langle \mathcal{P}, \mathcal{D}_1 \rangle$ and $\langle \mathcal{P}, \mathcal{D}_2 \rangle$, respectively. Then, the optimal values for $\langle \mathcal{P}, \mathcal{D} \rangle$, $\langle \mathcal{P}, \mathcal{D}_1 \rangle$ and $\langle \mathcal{P}, \mathcal{D}_2 \rangle$ are $\mathsf{profit}(\mathcal{P},\mathcal{C}^*)$, $\mathsf{profit}(\mathcal{P},\mathcal{C}_1^*)$ and $\mathsf{profit}(\mathcal{P},\mathcal{C}_2^*)$, respectively. We have the following series of inequalities.

$$
\begin{aligned}
\mathsf{profit}(\mathcal{P},\mathcal{C}^*) &\leq \mathsf{profit}(\mathcal{P},\mathcal{C}^* \cap \mathcal{D}_1) + \mathsf{profit}(\mathcal{P},\mathcal{C}^* \cap \mathcal{D}_2) \\
&\leq \mathsf{profit}(\mathcal{P},\mathcal{C}_1^*) + \mathsf{profit}(\mathcal{P},\mathcal{C}_2^*) \\
&\leq \beta \cdot \mathsf{profit}(\mathcal{P},\mathcal{C}_1) + \beta \cdot \mathsf{profit}(\mathcal{P},\mathcal{C}_2) \\
&\leq 2\beta \cdot \max\{\mathsf{profit}(\mathcal{P},\mathcal{C}_1), \mathsf{profit}(\mathcal{P},\mathcal{C}_2)\}.
\end{aligned}
$$

The first inequality follows since $U(\mathcal{P},\mathcal{C}^*) \subseteq U(\mathcal{P},\mathcal{C}^* \cap \mathcal{D}_1) \cup U(\mathcal{P},\mathcal{C}^* \cap \mathcal{D}_2)$, where $U(\mathcal{P},\mathcal{C})$ is the set of all points in $\mathcal{P}$ that are uniquely covered by $\mathcal{C}$ for a point set $\mathcal{P}$ and a subset $\mathcal{C} \subseteq \mathcal{D}$. To see this, let $p \in U(\mathcal{P},\mathcal{C}^*)$. Then, p is contained in exactly one disk D in $\mathcal{C}^*$. If $D \in \mathcal{D}_1$, then p is contained in exactly one disk in $\mathcal{C}^* \cap \mathcal{D}_1$; Otherwise, $D \in \mathcal{D}_2$, and so p is contained in exactly one disk in $\mathcal{C}^* \cap \mathcal{D}_2$. The second inequality follows since $\mathcal{C}^* \cap \mathcal{D}_1 \subseteq \mathcal{D}_1$ and $\mathcal{C}_1^*$ is an optimal solution to $\langle \mathcal{P}, \mathcal{D}_1 \rangle$ (and the same applies to the second term). Thus, choosing the better of $\mathsf{profit}(\mathcal{P},\mathcal{C}_1)$ and $\mathsf{profit}(\mathcal{P},\mathcal{C}_2)$ gives a 2β-approximate solution. □

In the rest of the paper, we fix a stripe $R_W(q,h,b)$ for $h = \sqrt{3}/4$, $b = 1/2$ and some real number $q \in [0, h+b)$. We may assume without loss of generality that each ribbon in $R_W(q,h,b)$ contains at least one point in $\mathcal{P}$. (We can simply ignore the ribbons containing no points.) We thus deal with only a polynomial number of ribbons. Let $R_1, R_2, \ldots, R_t$ be the ribbons in $R_W(q,h,b)$ ordered from bottom to top.

5. PTAS for the problem on $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$

In this section, we give a PTAS for the problem on $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$.

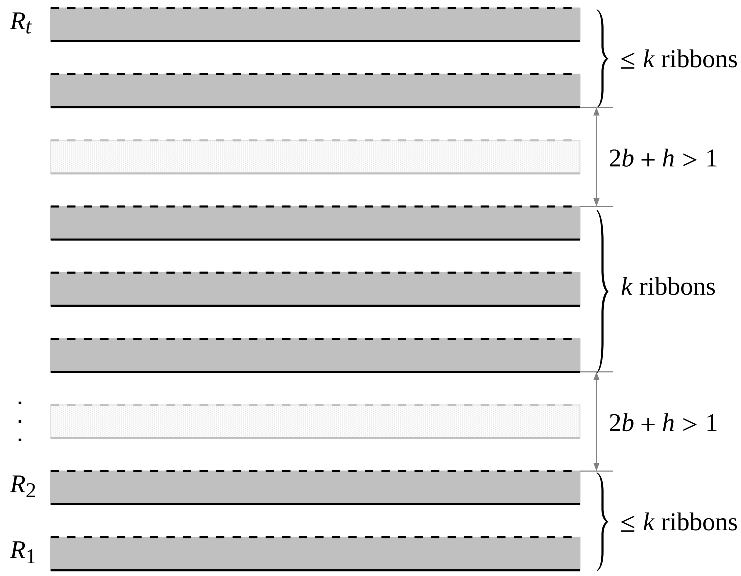

Figure 3: Sub-stripe R_W^j of a stripe $R_W(q, h, b)$.

Lemma 5.1. *For any fixed constant $\varepsilon' > 0$, there is a polynomial-time $(1+\varepsilon')$-approximation algorithm for the unique unit-disk coverage problem on $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$.*

Let $k = \lceil 1/\varepsilon' \rceil$. Lemma 5.1 is a direct consequence of the following two lemmas.

Lemma 5.2. *Suppose that we can obtain an optimal solution to $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ in polynomial time for every set G consisting of at most k ribbons. Then, we can obtain a $(1+\varepsilon')$-approximate solution to $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$ in polynomial time.*

PROOF. This is again done by the shifting technique.

Remember that the stripe $R_W(q, h, b)$ consists of t ribbons $R_1, R_2, \ldots, R_t$ ordered from bottom to top. For an index j, $0 \leq j \leq k$, let R_W^j be the *sub-stripe* obtained from $R_W(q, h, b)$ by deleting the ribbons R_i, $1 \leq i \leq t$, if and only if $i \equiv j \bmod k+1$. (See Figure 3.) We optimally solve the problem on $\langle \mathcal{P}_q \cap R_W^j, \mathcal{D}_O \rangle$ for each j, $0 \leq j \leq k$, as follows. We regard the remaining (at most) k consecutive ribbons in R_W^j as forming one *group*. Then, those groups have pairwise distance $2b + h = 1 + \sqrt{3}/4 > 1$, and hence no disk (with radius $1/2$) can cover points in two distinct groups. Therefore, we can independently solve the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$, where G is a group in R_W^j. (Indeed, it suffices to consider the disks in $\mathcal{D}_O$ which overlap the group G.) Combining the optimal solutions for all groups in R_W^j, we obtain an optimal solution $\mathcal{C}_O(j) \subseteq \mathcal{D}_O$ to $\langle \mathcal{P}_q \cap R_W^j, \mathcal{D}_O \rangle$.

As our approximate solution $\mathcal{C}_O \subseteq \mathcal{D}_O$ to $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$, we choose the best one from $\mathcal{C}_O(j)$, $0 \leq j \leq k$, and hence we have

$$\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_O) \geq \max_{0 \leq j \leq k} \mathsf{profit}(\mathcal{P}_q \cap R_W^j, \mathcal{C}_O(j)). \tag{2}$$

Clearly, we can obtain the approximate solution $\mathcal{C}_O$ in polynomial time if the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ for each group G can be optimally solved in polynomial time.

We now show that the above algorithm is $(1+\varepsilon')$-approximation. Consider an arbitrary optimal solution $\mathcal{C}_O^* \subseteq \mathcal{D}_O$ for the problem on $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$. By applying the well-known shifting technique [6] with respect to the index j, it is easy to show that there exists an index j^* in $0, 1, \ldots, k$ such that

$$\frac{k}{k+1} \cdot \mathsf{profit}(\mathcal{P}_q, \mathcal{C}_O^*) \leq \mathsf{profit}(\mathcal{P}_q \cap R_W^{j^*}, \mathcal{C}_O^*).$$

Remember that $\mathcal{C}_O^* \subseteq \mathcal{D}_O$ and $\mathcal{C}_O(j^*)$ is an optimal solution to $\langle \mathcal{P}_q \cap R_W^{j^*}, \mathcal{D}_O \rangle$. Therefore, we have $\mathsf{profit}(\mathcal{P}_q \cap R_W^{j^*}, \mathcal{C}_O^*) \leq \mathsf{profit}(\mathcal{P}_q \cap R_W^{j^*}, \mathcal{C}_O(j^*))$. Since $k = \lceil 1/\varepsilon' \rceil$, we thus have

$$\begin{aligned} \mathsf{profit}(\mathcal{P}_q, \mathcal{C}_O^*) &\leq \left(1 + \frac{1}{k}\right) \cdot \mathsf{profit}(\mathcal{P}_q \cap R_W^{j^*}, \mathcal{C}_O^*) \\ &\leq (1 + \varepsilon') \cdot \mathsf{profit}(\mathcal{P}_q \cap R_W^{j^*}, \mathcal{C}_O(j^*)). \end{aligned}$$

By Inequality (2) we thus have $\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_O^*) \leq (1+\varepsilon')\mathsf{profit}(\mathcal{P}_q, \mathcal{C}_O)$, as required. □

Lemma 5.3. *We can obtain an optimal solution to $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ in polynomial time for every set G consisting of at most k ribbons.*

The proof of Lemma 5.3 is one of the cruxes in this paper, to which the rest of this section will be devoted. We give a constructive proof, namely, we give such an algorithm.

5.1. Basic ideas

Our algorithm employs a dynamic-programming approach based on the line-sweep paradigm. Namely, we look at points and disks from left to right, and extend the uniquely covered region sequentially. However, adding one disk D at the rightmost position can influence a lot of disks that were already chosen, and can change the situation drastically (we say that D *influences* a disk D' if the region uniquely covered by D' changes after the addition of D). We therefore need to keep track of the disks that are possibly influenced by a newly added disk. Unless the number of those disks is bounded by some constant (or the logarithm of the input size), this approach cannot lead to a polynomial-time algorithm. Unfortunately, new disks may influence a super-constant (or super-logarithmic) number of disks.

Instead of adding a disk at the rightmost position, we add a disk D such that the number of disks that were already chosen and influenced by D can be bounded by a constant. Lemmas 5.5 and 5.6 state that we can do this for any set of disks, as long as a trivial condition for the disk set to be an optimal solution is satisfied. Furthermore, such a disk can be found in polynomial time.

5.2. Basic definitions

We may assume without loss of generality that the set G consists of consecutive ribbons forming a *group*; otherwise we can simply solve the problem for each group, because those groups have pairwise distance more than 1. (See Figure 3.) Suppose that G consists of k consecutive ribbons $R_{j+1}, R_{j+2}, \ldots, R_{j+k}$ in $R_W(q, h, b)$, ordered from bottom to top, for some integer j. If a disk can cover points in $\mathcal{P}_q \cap G$, then its center lies between R_{j+i} and R_{j+i+1} for some $i \in \{0, \ldots, k\}$. For notational convenience, we assume $j = 0$ without loss of generality. Note that the two ribbons R_0 and R_{k+1} are not in G.

For each $i \in \{0, \ldots, k\}$, we denote by $\mathcal{D}_{i,i+1}$ the set of all disks in $\mathcal{D}_O$ with their centers lying between R_i and R_{i+1}, that is, each disk in $\mathcal{D}_{i,i+1}$ intersects R_i and R_{i+1}. Note that $\mathcal{D}_{0,1}, \mathcal{D}_{1,2}, \ldots, \mathcal{D}_{k,k+1}$ form a partition of the disks in $\mathcal{D}_O$ intersecting G. Since $h + b > 1/2$, we clearly have the following lemma.

Lemma 5.4. *If a disk D in $\mathcal{D}_{i,i+1}$ has a non-empty intersection in R_i (resp., in R_{i+1}) with another disk D', then $D' \in \mathcal{D}_{i-1,i} \cup \mathcal{D}_{i,i+1}$ (resp., $D' \in \mathcal{D}_{i,i+1} \cup \mathcal{D}_{i+1,i+2}$).* □

For a disk set $\mathcal{C} \subseteq \mathcal{D}$, let $A_0(\mathcal{C})$, $A_1(\mathcal{C})$, $A_2(\mathcal{C})$ and $A_{\geq 3}(\mathcal{C})$ be the areas covered by no disk, exactly one disk, exactly two disks, and three or more disks in $\mathcal{C}$, respectively. Then, each point contained in $A_1(\mathcal{C})$ is uniquely covered by $\mathcal{C}$.

5.3. Properties on disk subsets of $\mathcal{D}_{i,i+1}$

We first deal with the special case where disks are contained only in a set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$, and consider the region uniquely covered by them. Of course, disks in $\mathcal{D}_{i-1,i} \cup \mathcal{D}_{i+1,i+2}$ may influence disks in $\mathcal{C}$; This issue will be discussed later. We sometimes denote by $R_{i,i+1}$ the set of two consecutive ribbons R_i and R_{i+1}, namely $R_{i,i+1} = R_i \cup R_{i+1}$.

5.3.1. Upper and lower envelopes

Let $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ be a disk set. Since any two unit disks have distinct x-coordinates and distinct y-coordinates, we can partition the boundary of the closure of $A_1(\mathcal{C})$ into two types: The boundary between $A_0(\mathcal{C})$ and $A_1(\mathcal{C})$, and that between $A_1(\mathcal{C})$ and $A_2(\mathcal{C})$. The *upper envelope* of $\mathcal{C}$ is defined to be the boundaries between $A_0(\mathcal{C})$ and $A_1(\mathcal{C})$ that appear above the lower boundary of R_{i+1}, while the *lower envelope* of $\mathcal{C}$ is defined to be the ones that appear below the upper boundary of R_i. (See Figure 4.) We say that a disk D *forms* the boundary of an area A if a part of the boundary of D is a part of that of A. Let $UE(\mathcal{C})$ and $LE(\mathcal{C})$ be the sequences of disks that form the upper and lower

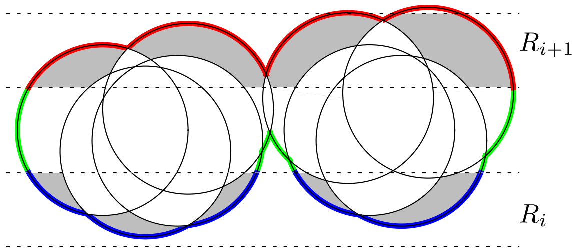

Figure 4: A set $\mathcal{C}$ of disks in $\mathcal{D}_{i,i+1}$, together with $A_1(\mathcal{C}) \cap R_{i,i+1}$ (gray), the upper envelope (red), the lower envelope (blue) and the other part of the outer boundary (green). The dotted lines show the boundaries of R_i and R_{i+1}.

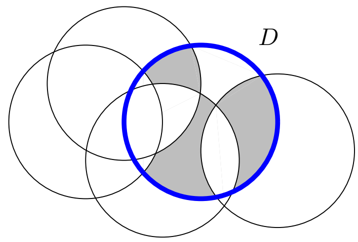

Figure 5: The gray region shows $A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})$ for the (blue) thick disk D.

envelopes of $\mathcal{C}$, from right to left, respectively. Note that a disk $D \in \mathcal{C}$ may appear in both $UE(\mathcal{C})$ and $LE(\mathcal{C})$.

Consider an arbitrary optimal solution $\mathcal{C}^* \subseteq \mathcal{D}_{i,i+1}$ to $\langle \mathcal{P}_q \cap R_{i,i+1}, \mathcal{D}_{i,i+1} \rangle$. If there is a disk $D \in \mathcal{C}^*$ that is not part of $A_1(\mathcal{C}^*)$, we can simply remove it from $\mathcal{C}^*$ without losing the optimality. Thus, hereafter we deal with a disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ such that every disk D in $\mathcal{C}$ forms the upper or lower envelopes of $\mathcal{C}$, that is, $D \in UE(\mathcal{C})$ or $D \in LE(\mathcal{C})$ holds. This property enables us to sweep the ribbons $R_{i,i+1}$, roughly speaking from left to right, and to extend the upper and lower envelopes sequentially.

5.3.2. Top disks and the key lemma

When we add a "new" disk D to the current disk set $\mathcal{C} \setminus \{D\}$, we need to know the symmetric difference between $A_1(\mathcal{C})$ and $A_1(\mathcal{C} \setminus \{D\})$: The area $A_1(\mathcal{C}) \setminus A_1(\mathcal{C} \setminus \{D\}) \subseteq A_1(\mathcal{C})$ is the uniquely covered area obtained newly by adding the disk D, and the area $A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C}) \subseteq A_2(\mathcal{C})$ is the non-uniquely covered area due to D. However, it suffices to know the area $A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})$ and its boundary, because the boundary of $A_1(\mathcal{C}) \setminus A_1(\mathcal{C} \setminus \{D\})$ is formed only by D and disks forming the boundary of $A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})$.

For a disk D in a set $\mathcal{C} \subseteq \mathcal{D}$, let $\Delta(\mathcal{C}, D)$ be the set of all disks in $\mathcal{C}$ that form the boundary of $A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})$. (See Figure 5.) Clearly, every disk in $\Delta(\mathcal{C}, D)$ has non-empty intersection with D. As we mentioned, $\Delta(\mathcal{C}, D)$ may contain a super-constant (or super-logarithmic) number of disks if we simply choose the rightmost disk D in $\mathcal{C}$. We will show that, for any disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$,

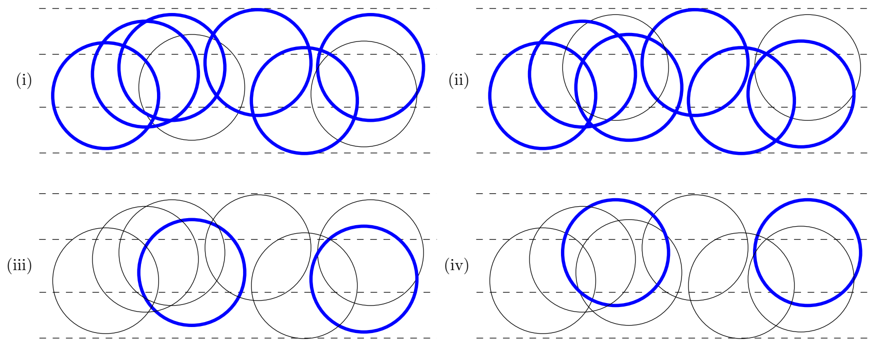

Figure 6: An example of top disks. The (blue) thick disks are top disks, and the numbers correspond to the conditions in the definition.

there always exists a disk $D \in \mathcal{C}$ such that $\Delta(\mathcal{C}, D)$ contains at most 16 disks, called top disks, and D itself is a top disk.

For a disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$, a disk $D \in \mathcal{C}$ is called a *top disk* of $\mathcal{C}$ if one of the following conditions (i)–(iv) holds:

(i) D is one of the six rightmost disks of $UE(\mathcal{C})$;
(ii) D is one of the six rightmost disks of $LE(\mathcal{C})$;
(iii) D is one of the two rightmost disks of $UE(LE(\mathcal{C}) \setminus UE(\mathcal{C}))$;
(iv) D is one of the two rightmost disks of $LE(UE(\mathcal{C}) \setminus LE(\mathcal{C}))$.

An example is given in Figure 6. Remember that the disks in $UE(\mathcal{C})$ and $LE(\mathcal{C})$ are ordered from right to left. We denote by $\mathsf{Top}(\mathcal{C})$ the set of top disks of $\mathcal{C}$. Note that a disk may satisfy more than one of the conditions above. A disk set $\mathcal{F} \subseteq \mathcal{D}_{i,i+1}$ is *feasible on* $\mathcal{D}_{i,i+1}$ if $\mathsf{Top}(\mathcal{F}) = \mathcal{F}$. For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$, we denote by $\mathfrak{C}_{i,i+1}(\mathcal{F})$ the set of all disk sets whose top disks are equal to $\mathcal{F}$, that is,

$$\mathfrak{C}_{i,i+1}(\mathcal{F}) = \{\mathcal{C} \subseteq \mathcal{D}_{i,i+1} \mid \mathsf{Top}(\mathcal{C}) = \mathcal{F}\}.$$

A top disk D in a feasible set $\mathcal{F}$ is said to be *stable in* $\mathcal{F}$ if $\Delta(\mathcal{C}, D)$ consists only of top disks in $\mathcal{F}$ for any disk set $\mathcal{C} \in \mathfrak{C}_{i,i+1}(\mathcal{F})$. It should be noted that, if a top disk D is stable in $\mathcal{F}$, then $\Delta(\mathcal{C}, D) \subseteq \mathcal{F}$ holds for *any* disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ such that $\mathsf{Top}(\mathcal{C}) = \mathcal{F}$. Therefore, we can compute $\Delta(\mathcal{C}, D)$ in polynomial time by keeping track of only top disks $\mathcal{F}$ which satisfies $|\mathcal{F}| \leq 16$. Thus, below is the key lemma, which ensures that stable top disks always exist for every feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$.

Lemma 5.5. *For any feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$, at least one top disk $K(\mathcal{F})$ is stable in $\mathcal{F}$. Moreover, $K(\mathcal{F})$ can be found in polynomial time.*

We postpone the proof of Lemma 5.5 to Section 6.

5.4. Properties on disk subsets of $\mathcal{D}_O$

We finish the concentration on $\mathcal{D}_{i,i+1}$, and look at the whole set of $\mathcal{D}_O$.

A disk set $\mathcal{F} \subseteq \mathcal{D}_O$ is *feasible on* $\mathcal{D}_O$ if $\mathsf{Top}(\mathcal{F} \cap \mathcal{D}_{i,i+1}) = \mathcal{F} \cap \mathcal{D}_{i,i+1}$ for each $i \in \{0, \ldots, k\}$. For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$ and $i \in \{0, \ldots, k\}$, let $\mathcal{F}_{i,i+1} = \mathcal{F} \cap \mathcal{D}_{i,i+1}$, and let

$$\mathfrak{C}(\mathcal{F}) = \{\mathcal{C} \subseteq \mathcal{D}_O \mid \mathsf{Top}(\mathcal{C} \cap \mathcal{D}_{i,i+1}) = \mathcal{F}_{i,i+1} \text{ for each } i \in \{0, \ldots, k\}\}.$$

We say that $\mathcal{F}_{i,i+1}$ is *safe for* $\mathcal{F}$ if $\Delta(\mathcal{C}, K(\mathcal{F}_{i,i+1})) \subset \mathcal{F}$ for any disk set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, where $K(\mathcal{F}_{i,i+1})$ is a stable top disk in $\mathcal{F}_{i,i+1}$ which is selected as in the proof of Lemma 5.5.

Lemma 5.6. *For any feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$, there exists an index $s \in \{0, \ldots, k\}$ such that $\mathcal{F}_{s,s+1}$ is safe for $\mathcal{F}$.*

We postpone the proof of Lemma 5.6 to Section 7.

5.5. Algorithm for the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$

For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$, let $f(\mathcal{F})$ be the maximum number of points in $\mathcal{P}_q \cap G$ uniquely covered by a disk set in $\mathfrak{C}(\mathcal{F})$, that is,

$$f(\mathcal{F}) = \max\{\mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C}) \mid \mathcal{C} \in \mathfrak{C}(\mathcal{F})\},$$

where $\mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C})$ is the number of points in $\mathcal{P}_q \cap G$ that are uniquely covered by $\mathcal{C}$. Then, since every subset of $\mathcal{D}_O$ belongs to $\mathfrak{C}(\mathcal{F})$ for some feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$, the optimal value $\mathrm{OPT}(\mathcal{P}_q \cap G, \mathcal{D}_O)$ for $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ can be computed as

$$\mathrm{OPT}(\mathcal{P}_q \cap G, \mathcal{D}_O) = \max\{f(\mathcal{F}) \mid \mathcal{F} \text{ is feasible on } \mathcal{D}_O\}.$$

Since $|\mathcal{F}| < 16(k+1)$, this computation can be done in polynomial time if we have the values $f(\mathcal{F})$ for all feasible disk sets $\mathcal{F}$ on $\mathcal{D}_O$.

We here explain how to compute $f(\mathcal{F})$ in polynomial time for all feasible disk sets $\mathcal{F}$ on $\mathcal{D}_O$, and complete the proof of Lemma 5.3.

The values $f(\mathcal{F})$ can be computed according to the "parent-child relation." For a disk set $\mathcal{C} \subseteq \mathcal{D}_O$, we denote simply by $\mathsf{Top}(\mathcal{C}) = \bigcup_{0 \le i \le k} \mathsf{Top}(\mathcal{C} \cap \mathcal{D}_{i,i+1})$. For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$, let $K(\mathcal{F}) = K(\mathcal{F}_{s,s+1})$ where $\mathcal{F}_{s,s+1} = \mathcal{F} \cap \mathcal{D}_{s,s+1}$ is safe for $\mathcal{F}$; note that by Lemma 5.6 such an index s always exists. For two feasible disk sets $\mathcal{F}$ and $\mathcal{F}'$ on $\mathcal{D}_O$, we say that $\mathcal{F}'$ is a *child* of $\mathcal{F}$ if there exists a disk set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ such that $\mathsf{Top}(\mathcal{C} \setminus \{K(\mathcal{F})\}) = \mathcal{F}'$.

Lemma 5.7. *The parent-child relation for the feasible disk sets on $\mathcal{D}_O$ can be constructed in polynomial time. The parent-child relation is acyclic.*

Proof. We can enumerate all feasible disk sets on $\mathcal{D}_O$, as follows: We first generate all sets $\mathcal{C} \subseteq \mathcal{D}_O$ consisting of $16(k+1)$ disks, and then check whether $\mathsf{Top}(\mathcal{C} \cap \mathcal{D}_{i,i+1}) = \mathcal{C} \cap \mathcal{D}_{i,i+1}$ for each $i \in \{0, \ldots, k\}$. Since k is a constant, this enumeration can be done in polynomial time.

For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$, let $\mathcal{C}$ be any disk set in $\mathfrak{C}(\mathcal{F})$. Then, we have $|\mathsf{Top}(\mathcal{C} \setminus \{K(\mathcal{F})\}) \setminus \mathsf{Top}(\mathcal{C})| \le 3$ since the top disk $K(\mathcal{F}) = K(\mathcal{F}_{s,s+1})$ can appear in at most three sets among $UE(\mathcal{C}_{s,s+1})$, $LE(\mathcal{C}_{s,s+1})$, $UE(LE(\mathcal{C}_{s,s+1}) \setminus UE(\mathcal{C}_{s,s+1}))$ and $LE(UE(\mathcal{C}_{s,s+1}) \setminus LE(\mathcal{C}_{s,s+1}))$. Therefore, the number of candidates of children of $\mathcal{F}$ can be bounded by $O(|\mathcal{D}_O|^3)$. We can thus construct the parent-child relation in polynomial time.

Consider the sequence of the x-coordinates of top disks from right to left. Since all disks have distinct x-coordinates, any child $\mathcal{F}'$ has a sequence lexicographically smaller than its parent $\mathcal{F}$, or $\mathcal{F}' \subset \mathcal{F}$. This implies that the parent-child relation is acyclic. □

We finally give our algorithm to solve the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$.

For each $i \in \{0, \ldots, k\}$, let $\mathcal{F}^0_{i,i+1}$ be the disk set consisting of the 16 leftmost disks in $\mathcal{D}_{i,i+1}$ having the smallest x-coordinates. Let $\mathcal{F}^0 = \bigcup_{0 \le i \le k} \mathcal{F}^0_{i,i+1}$, then $|\mathcal{F}^0| \le 16(k+1)$. As the initialization, we first compute $f(\mathcal{F})$ for all feasible sets $\mathcal{F}$ on $\mathcal{F}^0$. Since $|\mathcal{F}^0|$ is a constant, the total number of feasible sets $\mathcal{F}$ on $\mathcal{F}^0$ is also a constant. Therefore, this initialization can be done in polynomial time.

We then compute $f(\mathcal{F})$ for a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$ from $f(\mathcal{F}')$ for all children $\mathcal{F}'$ of $\mathcal{F}$. Since the parent-child relation is acyclic, we can find a feasible disk set $\mathcal{F}$ such that $f(\mathcal{F}')$ are already computed for all children $\mathcal{F}'$ of $\mathcal{F}$. By Lemma 5.6 there always exists a feasible disk set $\mathcal{F}_{s,s+1} = \mathcal{F} \cap \mathcal{D}_{s,s+1}$ on $\mathcal{D}_{s,s+1}$ which is safe for $\mathcal{F}$, and hence by Lemma 5.5 we have a stable top disk $K(\mathcal{F}) = K(\mathcal{F}_{s,s+1})$ in polynomial time. For a disk set $\mathcal{C} \subseteq \mathcal{D}_O$ and a disk $D \in \mathcal{C}$, we denote by $z(\mathcal{C}, D)$ the difference of uniquely covered points in $\mathcal{P}_q \cap G$ caused by adding D to $\mathcal{C} \setminus \{D\}$, that is, the number of points in $\mathcal{P}_q \cap G$ that are included in $D \cap A_1(\mathcal{C})$ minus the number of points in $\mathcal{P}_q \cap G$ that are included in $D \cap A_1(\mathcal{C} \setminus \{D\})$. Since $\mathcal{F}_{s,s+1}$ is safe for $\mathcal{F}$ and $K(\mathcal{F}) = K(\mathcal{F}_{s,s+1})$, we have $z(\mathcal{F}, K(\mathcal{F})) = z(\mathcal{C}, K(\mathcal{F}))$ for all disk sets $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. Therefore, we can correctly update $f(\mathcal{F})$ by

$$f(\mathcal{F}) := \max\{f(\mathcal{F}') \mid \mathcal{F}' \text{ is a child of } \mathcal{F}\} + z(\mathcal{F}, K(\mathcal{F})). \tag{3}$$

This way, the algorithm correctly solves the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ in polynomial time.

This completes the proof of Lemma 5.3. □

6. Proof of Lemma 5.5

We now prove our key lemma, which ensures that stable top disks always exist for every feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$. In most cases, we choose the rightmost disk of $\mathcal{F}$ as the stable top disk $K(\mathcal{F})$ in $\mathcal{F}$. However, as we mentioned before, the rightmost disk may intersect too many other disks including non-top disks. Indeed, $K(\mathcal{F})$ will be one of the following five disks:

1. the rightmost disk of $\mathcal{F}$;
2. the rightmost disk of $LE(\mathcal{F}) \setminus UE(\mathcal{F})$;
3. the second rightmost disk of $LE(\mathcal{F}) \setminus UE(\mathcal{F})$;
4. the rightmost disk of $UE(\mathcal{F}) \setminus LE(\mathcal{F})$; and

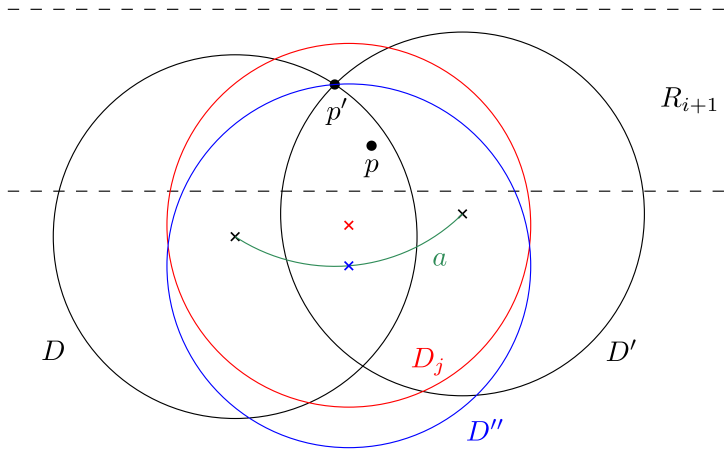

Figure 7: Proof of Lemma 6.2.

5. the second rightmost disk of $UE(\mathcal{F}) \setminus LE(\mathcal{F})$.

To prove Lemma 5.5, we need a thorough preparation.

6.1. Upper and lower envelopes

First, the following lemma clearly holds.

Lemma 6.1. *Let $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ be a disk set. If a disk $D \in \mathcal{C}$ is not in $UE(\mathcal{C})$, then any point in $D \cap R_{i+1}$ is covered by at least one disk in $UE(\mathcal{C})$. Similarly, if a disk $D \in \mathcal{C}$ is not in $LE(\mathcal{C})$, then any point in $D \cap R_i$ is covered by at least one disk in $LE(\mathcal{C})$.*

We then give the following lemma for the upper envelope.

Lemma 6.2. *Let D and D' be any two disks in a disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ with $x(D) < x(D')$. Suppose that there are q disks $D_1, D_2, \ldots, D_q$, $q \geq 1$, such that $D_j \in UE(\mathcal{C})$ and $x(D) < x(D_j) < x(D')$ for each index $j \in \{1, \ldots, q\}$. Then, any point in $D \cap D' \cap R_{i+1}$ is covered by at least $2 + q$ disks of $\mathcal{C}$.*

Proof. It suffices to show that every point p in $D \cap D' \cap R_{i+1}$ is covered by every disk D_j, $1 \leq j \leq q$.

We see that the intersection of $D \cap D'$ and the closed halfplane above the lower boundary of R_{i+1} is bounded by two arcs and one line, as illustrated in Figure 7: A part of the boundary of D, a part of the boundary of D', and a part of the lower boundary of R_{i+1}. Let p' be the intersection of the boundaries of D and D' that lies above (or on) the lower boundary of R_{i+1}. Consider the shorter arc a of the circle centered at p' that connects the centers of D and D'. Note that a lies outside of R_{i+1} since the centers of D and D' lie below the lower boundary of R_{i+1}, but p lies above it. Then, the point p is contained in every unit disk with its center on this arc a.

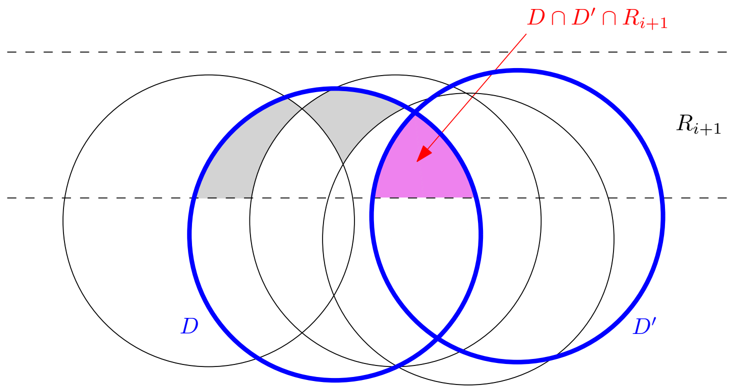

Figure 8: Example of Lemma 6.4 for $U\Delta(\mathcal{C}, D)$. The gray region depicts $(A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})) \cap R_{i+1}$ for the disk D, and hence $U\Delta(\mathcal{C}, D)$ consists of D and the three black disks.

Let D'' be a disk (not necessarily in $\mathcal{C}$) with its center on the arc a and $x(D'') = x(D_j)$. Then, $p \in D''$ by the observation above. Since $D_j \in UE(\mathcal{C})$, we see $y(D_j) \geq y(D'')$. Since the center of D'' lies below the lower boundary of R_{i+1}, it follows that $p \in D_j$. □

Similar arguments establish the counterpart for the lower envelope, as follows.

Lemma 6.3. *Let D and D' be any two disks in a disk set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ with $x(D) < x(D')$. Suppose that there are q disks $D_1, D_2, \ldots, D_q$, $q \geq 1$, such that $D_j \in LE(\mathcal{C})$ and $x(D) < x(D_j) < x(D')$ for each index $j \in \{1, \ldots, q\}$. Then, any point in $D \cap D' \cap R_i$ is covered by at least $2 + q$ disks of $\mathcal{C}$.*

6.2. Top disks

For a disk D in a set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$, we denote by $U\Delta(\mathcal{C}, D)$ the set of all disks that form the boundary of $(A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})) \cap R_{i+1}$, and by $L\Delta(\mathcal{C}, D)$ the set of all disks that form the boundary of $(A_1(\mathcal{C} \setminus \{D\}) \setminus A_1(\mathcal{C})) \cap R_i$. By the definition, we clearly have the following lemma. (See Figure 8.)

Lemma 6.4. *Let D and D' be two disks in a set $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$. Then, D' is not in $U\Delta(\mathcal{C}, D)$ if any point in $D' \cap D \cap R_{i+1}$ is contained in $A_{\geq 3}(\mathcal{C} \setminus \{D\})$. Similarly, D' is not in $L\Delta(\mathcal{C}, D)$ if any point in $D' \cap D \cap R_i$ is contained in $A_{\geq 3}(\mathcal{C} \setminus \{D\})$.*

The following lemma implies that, for a feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$, we can check in linear time whether each top disk $D \in \mathcal{F}$ is stable in $\mathcal{F}$.

Lemma 6.5. *Let D be any (top) disk in a feasible set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$. Then, D is stable in $\mathcal{F}$ if and only if $D' \notin \Delta(\mathcal{F} \cup \{D'\}, D)$ for every disk $D' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{D'\}) = \mathcal{F}$.*

PROOF. By the definition of stable disks, the necessity clearly holds. We thus show the sufficiency, i.e., we will show that, if D is not stable in $\mathcal{F}$, then there exists a non-top disk $D' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $D' \in \Delta(\mathcal{F} \cup \{D'\}, D)$ and $\mathsf{Top}(\mathcal{F} \cup \{D'\}) = \mathcal{F}$.

Since D is not stable in $\mathcal{F}$, there exists a disk set $\mathcal{C} \in \mathfrak{C}_{i,i+1}(\mathcal{F})$ such that $\Delta(\mathcal{C}, D)$ contains non-top disks of $\mathcal{C}$. Let D' be an arbitrary non-top disk in $\Delta(\mathcal{C}, D) \setminus \mathcal{F}$. Then, we have $D' \in \Delta(\mathcal{F} \cup \{D'\}, D)$. □

For a feasible disk set $\mathcal{F} \subseteq \mathcal{D}_{i,i+1}$, let $UE(\mathcal{F}) = (K_1^\top, K_2^\top, \dots, K_\alpha^\top)$ with

$$x(K_\alpha^\top) < x(K_{\alpha-1}^\top) < \dots < x(K_1^\top), \tag{4}$$

and let $LE(\mathcal{F}) = (K_1^\perp, K_2^\perp, \dots, K_\beta^\perp)$ with

$$x(K_\beta^\perp) < x(K_{\beta-1}^\perp) < \dots < x(K_1^\perp). \tag{5}$$

Note that some disks may appear in both $UE(\mathcal{F})$ and $LE(\mathcal{F})$. Then, we have the following lemma.

Lemma 6.6. *Let D_1 be the disk in $\mathcal{F}$ whose x-coordinate is largest. Suppose that there exists a disk $Q \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $Q \in U\Delta(\mathcal{F} \cup \{Q\}, D_1)$ and $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$. Then, $Q \in LE(\mathcal{F} \cup \{Q\})$, $|LE(\mathcal{F})| \geq 6$, and either $|UE(\mathcal{F})| \leq 2$ or $x(K_3^\top) < x(K_6^\perp)$ holds.*

PROOF. Note that $D_1 = K_1^\top$ or $D_1 = K_1^\perp$, and that $x(K_1^\top) \leq x(D_1)$ and $x(K_1^\perp) \leq x(D_1)$ hold. Since $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$ and $Q \notin \mathcal{F}$, Q is a non-top disk.

We first claim that there exists at most one disk $K^\top \in UE(\mathcal{F} \cup \{Q\})$ such that $x(Q) < x(K^\top) < x(D_1)$. Suppose for a contradiction that there exist two disks $K, K' \in UE(\mathcal{F} \cup \{Q\})$ such that $x(Q) < x(K) < x(K') < x(D_1)$. Then, by Lemma 6.2 every point in $Q \cap D_1 \cap R_{i+1}$ is covered by at least four disks and hence is contained in $A_{\geq 3}((\mathcal{F} \cup \{Q\}) \setminus \{D_1\})$. By Lemma 6.4 we then have $Q \notin U\Delta(\mathcal{F} \cup \{Q\}, D_1)$, a contradiction.

This claim implies that $Q \notin UE(\mathcal{F} \cup \{Q\})$; Otherwise, since $x(K_1^\top) \leq x(D_1)$, we have $Q \in \{K_1^\top, K_2^\top, K_3^\top\}$ and hence Q is a top disk in $\mathcal{F}$. Remember that each disk in $\mathcal{F} \cup \{Q\}$ appears in $UE(\mathcal{F} \cup \{Q\})$ or $LE(\mathcal{F} \cup \{Q\})$, and hence we have $Q \in LE(\mathcal{F} \cup \{Q\})$. Then, since Q is a non-top disk, we have $|LE(\mathcal{F})| \geq 6$ and

$$x(Q) < x(K_6^\perp). \tag{6}$$

The claim also implies that either $|UE(\mathcal{F})| \leq 2$ or

$$x(K_3^\top) < x(Q) \tag{7}$$

holds. By Inequalities (6) and (7) we have $x(K_3^\top) < x(K_6^\perp)$, as required. □

Similar arguments establish the counterpart of Lemma 6.6, as follows.

Lemma 6.7. *Let D_1 be the disk in $\mathcal{F}$ whose x-coordinate is largest. Suppose that there exists a disk $Q \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $Q \in L\Delta(\mathcal{F} \cup \{Q\}, D_1)$ and $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$. Then, $Q \in UE(\mathcal{F} \cup \{Q\})$, $|UE(\mathcal{F})| \geq 6$, and either $|LE(\mathcal{F})| \leq 2$ or $x(K_3^\perp) < x(K_6^\top)$ holds.*

Using Lemmas 6.6 and 6.7, we have the following lemma.

Lemma 6.8. *For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_{i,i+1}$, let D_1 be the disk in $\mathcal{F}$ whose x-coordinate is largest. Suppose that there exists a disk $Q \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $Q \in \Delta(\mathcal{F} \cup \{Q\}, D_1)$ and $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$. Then, the following (a) and (b) hold:*

(a) *If $Q \in U\Delta(\mathcal{F} \cup \{Q\}, D_1)$, then $Q' \notin L\Delta(\mathcal{F} \cup \{Q'\}, D_1)$ holds for any disk $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$;*

(b) *If $Q \in L\Delta(\mathcal{F} \cup \{Q\}, D_1)$, then $Q' \notin U\Delta(\mathcal{F} \cup \{Q'\}, D_1)$ holds for any disk $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$.*

Proof. We show that (a) holds. (The proof for (b) is similar.)

Suppose that $Q \in U\Delta(\mathcal{F} \cup \{Q\}, D_1)$. Then, by Lemma 6.6 we have $Q \in LE(\mathcal{F} \cup \{Q\})$ and

$$|LE(\mathcal{F})| \geq 6. \tag{8}$$

Furthermore, either $|UE(\mathcal{F})| \leq 2$ or

$$x(K_3^\top) < x(K_6^\perp) \tag{9}$$

holds.

Suppose for a contradiction that there exists a disk $Q' \in L\Delta(\mathcal{F} \cup \{Q'\}, D_1)$ such that $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ and $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$. Then, by Lemma 6.7 we have $Q' \in UE(\mathcal{F} \cup \{Q'\})$ and $|UE(\mathcal{F})| \geq 6$. Thus, Inequality (9) holds. Moreover, by Inequality (8) we have

$$x(K_3^\perp) < x(K_6^\top). \tag{10}$$

Therefore, by Inequalities (5), (9) and (10) we have $x(K_3^\top) < x(K_6^\top)$. This contradicts Inequality (4). □

Lemma 6.8 implies that, for every disk $Q \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$ and $Q \in \Delta(\mathcal{F} \cup \{Q\}, D_1)$, exactly one of $Q \in U\Delta(\mathcal{F} \cup \{Q\}, D_1)$ and $Q \in L\Delta(\mathcal{F} \cup \{Q\}, D_1)$ holds.

6.3. Finalizing the proof of Lemma 5.5

Proof (of Lemma 5.5). We consider the following cases, and prove that there is a stable top disk $K(\mathcal{F})$ in each case. Let D_1 be the disk in $\mathcal{F}$ whose x-coordinate is largest. Note that $D_1 = K_1^\top$ or $D_1 = K_1^\perp$.

Case 1: D_1 is stable in $\mathcal{F}$.

In this case, we set $K(\mathcal{F}) = D_1$. Note that by Lemma 6.5 we can check whether D_1 is stable in $\mathcal{F}$ in linear time.

Case 2: D_1 is not stable in $\mathcal{F}$.

Since D_1 is not stable in $\mathcal{F}$, by Lemma 6.5 there exists a non-top disk $Q \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $Q \in \Delta(\mathcal{F} \cup \{Q\}, D_1)$ and $\mathsf{Top}(\mathcal{F} \cup \{Q\}) = \mathcal{F}$. Lemma 6.8 allows us to assume $Q \in U\Delta(\mathcal{F} \cup \{Q\}, D_1)$ without loss of generality. (The case for $Q \in L\Delta(\mathcal{F} \cup \{Q\}, D_1)$ is symmetric.) Then, by Lemma 6.6 we have

$$|LE(\mathcal{F})| \geq 6 \tag{11}$$

and either $|UE(\mathcal{F})| \leq 2$ or

$$x(K_3^{\top}) < x(K_6^{\perp}) \qquad (12)$$

holds.

Consider an arbitrary non-top disk $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$. We claim that

$$x(Q') < x(K_6^{\perp}). \qquad (13)$$

Note that Inequality (11) ensures that the disk $K_6^{\perp}$ exists. Since Q' is a non-top disk, we clearly have $x(Q') < x(K_6^{\perp})$ if $Q' \in LE(\mathcal{F} \cup \{Q'\})$. We thus consider the case where $Q' \in UE(\mathcal{F} \cup \{Q'\})$. Then, since Q' is a non-top disk, $|UE(\mathcal{F})| \geq 6$ and $x(Q') < x(K_6^{\top})$ hold. Furthermore, $|UE(\mathcal{F})| \geq 6$ implies that Inequality (12) holds, and hence by Inequality (4) we have $x(Q') < x(K_6^{\perp})$. Therefore, in either case, Inequality (13) holds.

Let D_2 and D_3 be the rightmost and the second rightmost disks in $LE(\mathcal{F}) \setminus UE(\mathcal{F})$, respectively. Since either $|UE(\mathcal{F})| \leq 2$ or $x(K_3^{\top}) < x(K_6^{\perp})$ holds, at most two disks in $UE(\mathcal{F})$ can appear also in $K_1^{\perp}, K_2^{\perp}, \ldots, K_6^{\perp}$. Therefore, we have $D_2 \in \{K_1^{\perp}, K_2^{\perp}, K_3^{\perp}\}$ and $D_3 \in \{K_2^{\perp}, K_3^{\perp}, K_4^{\perp}\}$. We consider the following two sub-cases.

Case 2-1: D_3 is in $UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$.

In this case, we show that D_2 is stable in $\mathcal{F}$, and hence we set $K(\mathcal{F}) = D_2$. By Lemma 6.5 it suffices to show that $Q' \notin \Delta(\mathcal{F} \cup \{Q'\}, D_2)$ for every disk $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$.

We first show that $Q' \notin L\Delta(\mathcal{F} \cup \{Q'\}, D_2)$. Since $D_2 \in \{K_1^{\perp}, K_2^{\perp}, K_3^{\perp}\}$, by Inequality (13) we have

$$x(Q') < x(K_6^{\perp}) < x(K_5^{\perp}) < x(K_4^{\perp}) < x(D_2).$$

By Lemma 6.3 every point in $Q' \cap D_2 \cap R_i$ is covered by at least five disks, and hence is contained in $A_{\geq 3}((\mathcal{F} \cup \{Q'\}) \setminus \{D_2\})$. By Lemma 6.4 we thus have $Q' \notin L\Delta(\mathcal{F} \cup \{Q'\}, D_2)$, as required.

We then show that $Q' \notin U\Delta(\mathcal{F} \cup \{Q'\}, D_2)$. Since $D_3 \in \{K_2^{\perp}, K_3^{\perp}, K_4^{\perp}\}$ and $x(D_3) < x(D_2)$, by Inequality (13) we have $x(Q') < x(D_3) < x(D_2)$. Since $D_3 \in UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$, by Lemma 6.2 every point in $Q' \cap D_2 \cap R_{i+1}$ is covered by at least three disks. Moreover, since $D_2 \notin UE(\mathcal{F})$, by Lemma 6.1 every point in $Q' \cap D_2 \cap R_{i+1}$ is covered by at least one disk in $UE(\mathcal{F})$. Thus, in total, every point in $Q' \cap D_2 \cap R_{i+1}$ is covered by at least four disks in $\mathcal{F}$, and hence is contained in $A_{\geq 3}((\mathcal{F} \cup \{Q'\}) \setminus \{D_2\})$. By Lemma 6.4 we thus have $Q' \notin U\Delta(\mathcal{F} \cup \{Q'\}, D_2)$, as required.

Case 2-2: D_3 is not in $UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$.

In this case, we show that D_3 is stable in $\mathcal{F}$, and hence we set $K(\mathcal{F}) = D_3$. By Lemma 6.5 it suffices to show that $Q' \notin \Delta(\mathcal{F} \cup \{Q'\}, D_3)$ for every disk $Q' \in \mathcal{D}_{i,i+1} \setminus \mathcal{F}$ such that $\mathsf{Top}(\mathcal{F} \cup \{Q'\}) = \mathcal{F}$.

We first show that $Q' \notin L\Delta(\mathcal{F} \cup \{Q'\}, D_3)$. Since $D_3 \in \{K_2^{\perp}, K_3^{\perp}, K_4^{\perp}\}$, by Inequality (13) we have

$$x(Q') < x(K_6^{\perp}) < x(K_5^{\perp}) < x(D_3).$$

By Lemma 6.3 every point in $Q' \cap D_3 \cap R_i$ is covered by at least four disks, and hence is contained in $A_{\geq 3}((\mathcal{F} \cup \{Q'\}) \setminus \{D_3\})$. By Lemma 6.4 we thus have $Q' \notin L\Delta(\mathcal{F} \cup \{Q'\}, D_3)$, as required.

We then show that $Q' \notin U\Delta(\mathcal{F} \cup \{Q'\}, D_3)$. Since $D_3 \notin UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$, by applying Lemma 6.1 to $LE(\mathcal{F}) \setminus UE(\mathcal{F})$, every point in $Q' \cap D_3 \cap R_{i+1}$ is covered by at least one disk X in $UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$. Moreover, since $D_3 \notin UE(\mathcal{F})$, by applying Lemma 6.1 to $\mathcal{F}$, every point in $Q' \cap D_3 \cap R_{i+1}$ is covered by at least one disk Y in $UE(\mathcal{F})$. Note that $X \neq Y$ since $X \in UE(LE(\mathcal{F}) \setminus UE(\mathcal{F}))$ and $Y \in UE(\mathcal{F})$. Thus, in total, every point in $Q' \cap D_3 \cap R_{i+1}$ is covered by at least four disks (Q', D_3, X, Y) in $\mathcal{F}$, and hence is contained in $A_{\geq 3}((\mathcal{F} \cup \{Q'\}) \setminus \{D_3\})$. By Lemma 6.4 we thus have $Q' \notin U\Delta(\mathcal{F} \cup \{Q'\}, D_3)$, as required. ☐

7. Proof of Lemma 5.6

We then prove another key lemma, which ensures that every feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$ has at least one $\mathcal{F}_{s,s+1} = \mathcal{F} \cap \mathcal{D}_{s,s+1}$, $s \in \{0, \dots, k\}$, which is safe for $\mathcal{F}$. Recall that the stable top disk $K(\mathcal{F}_{i,i+1}) \in \mathcal{D}_{i,i+1}$ intersects disks only in $\mathcal{D}_{i-1,i} \cup \mathcal{D}_{i,i+1} \cup \mathcal{D}_{i+1,i+2}$ for each $i \in \{1, \dots, k-1\}$. Since $K(\mathcal{F}_{i,i+1})$ is stable in $\mathcal{F}_{i,i+1}$, our concern is only the intersections with disks in $\mathcal{D}_{i-1,i} \cup \mathcal{D}_{i+1,i+2}$. Therefore, we give a sufficient condition for which $K(\mathcal{F}_{i,i+1})$ has no intersection with disks in $(\mathcal{D}_{i-1,i} \cup \mathcal{D}_{i+1,i+2}) \setminus (\mathcal{F}_{i-1,i} \cup \mathcal{F}_{i+1,i+2})$, and show that there exists an index $s \in \{0, \dots, k\}$ such that $\mathcal{F}_{s,s+1}$ satisfies the sufficient condition.

A proof of Lemma 5.6 needs preparation. We first give an auxiliary lemma which states that at least one of $\mathcal{F}_{i,i+1}$ and $\mathcal{F}_{i+1,i+2}$ is safe for the other for each $i \in \{0, \dots, k-1\}$.

Remember that the ribbons $R_0, R_1, \dots, R_{k+1}$ are ordered from bottom to top, and that $\mathcal{D}_{i,i+1}$ is the set of all disks in $\mathcal{D}_O$ with their centers lying between R_i and R_{i+1} for each $i \in \{0, \dots, k\}$. For a disk set $\mathcal{C} \subseteq \mathcal{D}_O$, let $\mathcal{C}_{i,i+1} = \mathcal{C} \cap \mathcal{D}_{i,i+1}$ for each $i \in \{0, \dots, k\}$. Then, $\mathcal{C}_{0,1}, \mathcal{C}_{1,2}, \dots, \mathcal{C}_{k,k+1}$ form a partition of $\mathcal{C}$.

Let $\mathcal{F}$ be a feasible disk set on $\mathcal{D}_O$. Then, for each $i \in \{1, \dots, k-1\}$, $\mathcal{F}_{i-1,i}$, $\mathcal{F}_{i,i+1}$ and $\mathcal{F}_{i+1,i+2}$ are feasible disk sets on $\mathcal{D}_{i-1,i}$, $\mathcal{D}_{i,i+1}$ and $\mathcal{D}_{i+1,i+2}$, respectively. We say that $\mathcal{F}_{i,i+1}$ is *safe for* $\mathcal{F}_{i+1,i+2}$ if $\Delta(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}, K(\mathcal{F}_{i,i+1})) \subset \mathcal{F}_{i,i+1} \cup \mathcal{F}_{i+1,i+2}$ for any disk set $\mathcal{C}$ in $\mathfrak{C}(\mathcal{F})$. Similarly, we say that $\mathcal{F}_{i,i+1}$ is *safe for* $\mathcal{F}_{i-1,i}$ if $\Delta(\mathcal{C}_{i-1,i} \cup \mathcal{C}_{i,i+1}, K(\mathcal{F}_{i,i+1})) \subset \mathcal{F}_{i-1,i} \cup \mathcal{F}_{i,i+1}$ for any disk set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. For notational convenience, let $\mathcal{D}_{-1,0} = \emptyset$ and $\mathcal{D}_{k+1,k+2} = \emptyset$; $\mathcal{F}_{0,1}$ is always safe for $\mathcal{F}_{-1,0}$, and $\mathcal{F}_{k,k+1}$ is always safe for $\mathcal{F}_{k+1,k+2}$. By Lemma 5.4 the disk $K(\mathcal{F}_{i,i+1}) \in \mathcal{D}_{i,i+1}$ intersects disks only in $\mathcal{D}_{i-1,i} \cup \mathcal{D}_{i,i+1}$ on R_i and disks only in $\mathcal{D}_{i,i+1} \cup \mathcal{D}_{i+1,i+2}$ on R_{i+1}. Therefore, for $i \in \{0, \dots, k\}$, $\mathcal{F}_{i,i+1}$ is safe for $\mathcal{F}$ if and only if $\mathcal{F}_{i,i+1}$ is safe for both $\mathcal{F}_{i-1,i}$ and $\mathcal{F}_{i+1,i+2}$.

Let $\mathcal{F}$ be a feasible disk set on $\mathcal{D}_O$, and let $\mathcal{C}$ be a disk set in $\mathfrak{C}(\mathcal{F})$. For each $i \in \{0, \dots, k\}$, let $ux(\mathcal{C}_{i,i+1})$ be the x-coordinate of the leftmost point of the area $R_{i+1} \cap K(\mathcal{F}_{i,i+1}) \cap (A_1(\mathcal{C}_{i,i+1}) \cup A_2(\mathcal{C}_{i,i+1}))$, while let $lx(\mathcal{C}_{i,i+1})$ be the x-coordinate of the leftmost point of the area $R_i \cap K(\mathcal{F}_{i,i+1}) \cap (A_1(\mathcal{C}_{i,i+1}) \cup A_2(\mathcal{C}_{i,i+1}))$. Note that $A_1(\mathcal{C}_{i,i+1}) \cap D \neq \emptyset$ for every disk $D \in \mathcal{C}_{i,i+1}$, because we deal with only a disk set such that every disk in the set is part of the uniquely covered region of the set. Therefore, both $ux(\mathcal{C}_{i,i+1})$ and $lx(\mathcal{C}_{i,i+1})$ are well-defined. Since $K(\mathcal{F}_{i,i+1})$ is stable in $\mathcal{F}_{i,i+1}$, we see that $ux(\mathcal{C}_{i,i+1})$ is invariant under the choice of $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. Thus, we also write $ux(\mathcal{F}_{i,i+1})$ to mean $ux(\mathcal{C}_{i,i+1})$ for any $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. The same applies to $lx(\mathcal{F}_{i,i+1})$.

We first give the following lemma.

Lemma 7.1. *Let $\mathcal{F}_{i,i+1}$ be a feasible disk set on $\mathcal{D}_{i,i+1}$. Let $\mathcal{C} \subseteq \mathcal{D}_{i,i+1}$ be any disk set in $\mathfrak{C}_{i,i+1}(\mathcal{F}_{i,i+1})$, and Q be a non-top disk of $\mathcal{C}$. Then,*

(a) every point $(x, y) \in Q \cap R_{i+1} \cap \big(A_1(\mathcal{C}) \cup A_2(\mathcal{C})\big)$ satisfies $x < ux(\mathcal{C})$, and

(b) every point $(x, y) \in Q \cap R_i \cap \big(A_1(\mathcal{C}) \cup A_2(\mathcal{C})\big)$ satisfies $x < lx(\mathcal{C})$.

Proof. We show that (a) holds; The proof for (b) is symmetric.

Suppose for a contradiction that there exists a point $p' = (x', y') \in Q \cap R_{i+1} \cap \big(A_1(\mathcal{C}) \cup A_2(\mathcal{C})\big)$ which satisfies $x' \geq ux(\mathcal{C})$. Since the disk $K(\mathcal{F}_{i,i+1})$ is stable in $\mathcal{F}_{i,i+1}$, no point in $K(\mathcal{F}_{i,i+1}) \cap Q$ is contained in $A_1(\mathcal{C}) \cup A_2(\mathcal{C})$. Therefore, we have

$$K(\mathcal{F}_{i,i+1}) \cap Q \cap R_{i+1} \cap \big(A_1(\mathcal{C}) \cup A_2(\mathcal{C})\big) = \emptyset, \tag{14}$$

and hence p' is not contained in $K(\mathcal{F}_{i,i+1})$.

We now claim that $x(Q) < x(K(\mathcal{F}_{i,i+1}))$ holds. Recall the choice of $K(\mathcal{F}_{i,i+1})$ in Lemma 5.5. If $K(\mathcal{F}_{i,i+1}) = D_1$ for the disk D_1 in $\mathcal{F}_{i,i+1}$ whose x-coordinate is largest, then $K(\mathcal{F}_{i,i+1})$ has the largest x-coordinate in $\mathcal{C}$ and hence we have $x(Q) < x(K(\mathcal{F}_{i,i+1}))$. Otherwise $K(\mathcal{F}_{i,i+1}) \in \{K_1^\perp, K_2^\perp, K_3^\perp, K_4^\perp\}$, where $LE(\mathcal{F}_{i,i+1}) = (K_1^\perp, K_2^\perp, \ldots, K_\beta^\perp)$; we here omit the symmetric case. Then, since Q is a non-top disk of $\mathcal{C}$, by Inequality (13) we have $x(Q) < x(K_6^\perp)$. By Inequality (5) we thus have $x(Q) < x(K(\mathcal{F}_{i,i+1}))$. Therefore, in either case, we have $x(Q) < x(K(\mathcal{F}_{i,i+1}))$ as claimed.

Since the centers of Q and $K(\mathcal{F}_{i,i+1})$ lie between R_i and R_{i+1}, and $x(Q) < x(K(\mathcal{F}_{i,i+1}))$, we may observe the following: every point in $(Q \setminus K(\mathcal{F}_{i,i+1})) \cap R_{i+1}$ lies to the left of every point in $(K(\mathcal{F}_{i,i+1}) \setminus Q) \cap R_{i+1}$.

By the definition of $ux(\mathcal{C})$, there exists a number y'' such that $p'' = (ux(\mathcal{C}), y'')$ belongs to $R_{i+1} \cap K(\mathcal{F}_{i,i+1}) \cap \big(A_1(\mathcal{C}) \cup A_2(\mathcal{C})\big) \subseteq K(\mathcal{F}_{i,i+1})$. We now claim that $p'' \in Q$, thus contradicting Eq. (14).

From the discussion above, we know that $p' \in (Q \setminus K(\mathcal{F}_{i,i+1})) \cap R_{i+1}$, and $p'' \in K(\mathcal{F}_{i,i+1}) \cap R_{i+1}$. If $p'' \notin Q$, then by the observation above, p' lies to the left of p''. This means that $x' < ux(\mathcal{C})$, which contradicts the assumption that $x' \geq ux(\mathcal{C})$. Therefore, $p'' \in Q$; this contradicts Eq. (14), and hence the claim is verified. □

Lemma 7.1 gives the following lemma.

Lemma 7.2. *Let $\mathcal{F}$ be a feasible disk set on $\mathcal{D}$. Then, for each $i \in \{0, \ldots, k-1\}$, the following (a) and (b) hold:*

(a) $\mathcal{F}_{i,i+1}$ is safe for $\mathcal{F}_{i+1,i+2}$ if $lx(\mathcal{F}_{i+1,i+2}) < ux(\mathcal{F}_{i,i+1})$;

(b) $\mathcal{F}_{i+1,i+2}$ is safe for $\mathcal{F}_{i,i+1}$ if $ux(\mathcal{F}_{i,i+1}) < lx(\mathcal{F}_{i+1,i+2})$.

PROOF. We show that (a) holds: If $lx(\mathcal{F}_{i+1,i+2}) < ux(\mathcal{F}_{i,i+1})$, then $\Delta(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}, K(\mathcal{F}_{i,i+1})) \subset \mathcal{F}_{i,i+1} \cup \mathcal{F}_{i+1,i+2}$ for any disk set $\mathcal{C}$ in $\mathfrak{C}(\mathcal{F})$. (The proof for (b) is symmetric.)

Consider an arbitrary disk set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, and let Q be a disk in $\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}$ such that $Q \notin \mathcal{F}_{i,i+1} \cup \mathcal{F}_{i+1,i+2}$. We will show that $Q \notin \Delta(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}, K(\mathcal{F}_{i,i+1}))$. Note that, however, we have $Q \notin \Delta(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}, K(\mathcal{F}_{i,i+1}))$ if $Q \in \mathcal{C}_{i,i+1}$, because the disk $K(\mathcal{F}_{i,i+1})$ is stable in $\mathcal{F}_{i,i+1}$.

We thus consider the case where $Q \in \mathcal{C}_{i+1,i+2}$. Since $K(\mathcal{F}_{i,i+1}) \in \mathcal{C}_{i,i+1}$, the intersection $K(\mathcal{F}_{i,i+1}) \cap Q$ is contained in R_{i+1}. Therefore, similarly to Lemma 6.4, we have $Q \notin \Delta(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2}, K(\mathcal{F}_{i,i+1}))$ if any point in $Q \cap K(\mathcal{F}_{i,i+1}) \cap R_{i+1}$ is contained in $A_{\geq 3}(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2} \setminus \{K(\mathcal{F}_{i,i+1})\})$.

Since $K(\mathcal{F}_{i,i+1}) \in \mathcal{C}_{i,i+1}$, if a point in $Q \cap K(\mathcal{F}_{i,i+1}) \cap R_{i+1}$ is contained in $A_{\geq 3}(\mathcal{C}_{i+1,i+2})$, then the point is contained in $A_{\geq 3}(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2} \setminus \{K(\mathcal{F}_{i,i+1})\})$. Therefore, we consider a point (x', y') in $Q \cap K(\mathcal{F}_{i,i+1}) \cap R_{i+1}$ which is contained in $A_1(\mathcal{C}_{i+1,i+2}) \cup A_2(\mathcal{C}_{i+1,i+2})$; and hence (x', y') is contained in at least one disk in $\mathcal{C}_{i+1,i+2}$. Then, by Lemma 7.1 we have $x' < lx(\mathcal{F}_{i+1,i+2})$ and hence $x' < ux(\mathcal{F}_{i,i+1})$. Recall that $ux(\mathcal{F}_{i,i+1})$ is the x-coordinate of the leftmost point of the area $R_{i+1} \cap K(\mathcal{F}_{i,i+1}) \cap \big(A_1(\mathcal{C}_{i,i+1}) \cup A_2(\mathcal{C}_{i,i+1})\big)$. Therefore, since $x' < ux(\mathcal{F}_{i,i+1})$, the point (x', y') is contained in at least three disks in $\mathcal{C}_{i,i+1}$ (one of which is $K(\mathcal{F}_{i,i+1})$). Thus, the point (x', y') is contained in $A_{\geq 3}(\mathcal{C}_{i,i+1} \cup \mathcal{C}_{i+1,i+2} \setminus \{K(\mathcal{F}_{i,i+1})\})$. $\square$

We then finalize the proof of Lemma 5.6.

PROOF (OF LEMMA 5.6). Since the centers of any two unit disks have distinct x-coordinates, $ux(\mathcal{F}_{i,i+1}) \neq lx(\mathcal{F}_{i+1,i+2})$ for each $i \in \{0, \dots, k-1\}$. Therefore, by Lemma 7.2 at least one of $\mathcal{F}_{i,i+1}$ and $\mathcal{F}_{i+1,i+2}$ is safe for the other. Remember that $\mathcal{F}_{0,1}$ is always safe for $\mathcal{F}_{-1,0}$, and that $\mathcal{F}_{k,k+1}$ is always safe for $\mathcal{F}_{k+1,k+2}$. Therefore, there exists at least one index $s \in \{0, \dots, k\}$, such that $\mathcal{F}_{s,s+1}$ is safe for both $\mathcal{F}_{s-1,s}$ and $\mathcal{F}_{s+1,s+2}$. Then, $\mathcal{F}_{s,s+1}$ is safe for $\mathcal{F}$. $\square$

8. PTAS for the problem on $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$.

Having finished the description of our PTAS for $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$, we turn to a PTAS for $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$. Namely, we give the following lemma, which completes the proof of Theorem 4.1.

Lemma 8.1. *For any fixed constant $\varepsilon' > 0$, there is a polynomial-time $(1+\varepsilon')$-approximation algorithm for the unique unit-disk coverage problem on $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$.*

Remember that the upper boundary of each ribbon R_i in the stripe $R_W(q, h, b)$ is open. Therefore, the ribbons in $R_W(q, h, b)$ have pairwise distance strictly greater than $b = 1/2$. (See Figure 2.) Since $\mathcal{D}_I$ consists of unit disks (with radius $1/2$) whose centers are contained in ribbons, no disk in $\mathcal{D}_I$ can cover points in two distinct ribbons. Therefore, we can independently solve the problem on $\langle \mathcal{P}_q \cap R_i, \mathcal{D}_I \rangle$ for each ribbon R_i in $R_W(q, h, b)$. Thus, if there

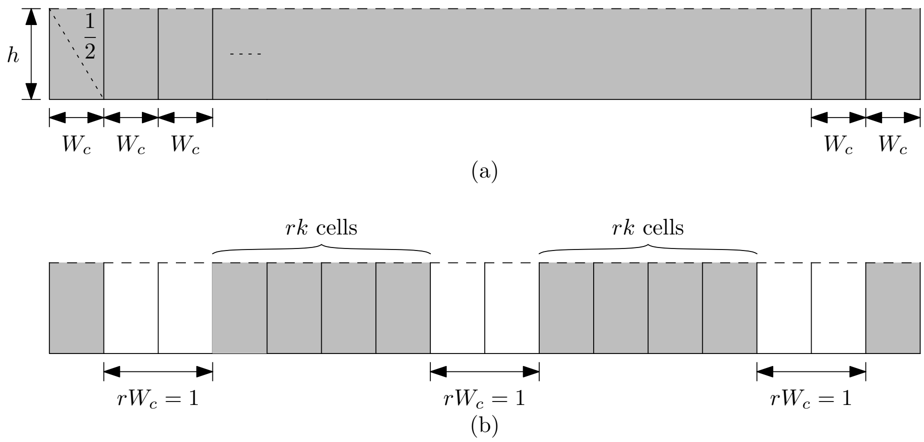

Figure 9: (a) Cells with diagonal 1/2 in a ribbon R_i, and (b) sub-ribbon of R_i in which every (at most) rk gray cells form a group.

is a PTAS for the problem on $\langle \mathcal{P}_q \cap R_i, \mathcal{D}_I \rangle$, then we can obtain a PTAS for the problem on $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$; We combine the approximate solutions to $\langle \mathcal{P}_q \cap R_i, \mathcal{D}_I \rangle$, and output it as our approximate solution to $\langle \mathcal{P}_q, \mathcal{D}_I \rangle$.

We now give a PTAS for the problem on $\langle \mathcal{P}_q \cap R_i, \mathcal{D}_I \rangle$ for each ribbon R_i. We first vertically divide R_i into rectangles, called *cells*, so that the diagonal of each cell is of length exactly 1/2. (See Figure 9(a).) Let W_c be the width of each cell, that is, $W_c = 1/4$ since $h = \sqrt{3}/4$. We may assume that, in each cell, the left boundary is closed and the right boundary is open. Let $r = 4$, then $rW_c = 1$.

Let $k = \lceil 1/\varepsilon' \rceil$. Similarly as in the PTAS for $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$, we remove r consecutive cells from every $r(1+k)$ consecutive cells, and obtain the "sub-ribbon" consisting of "groups," each of which contains at most rk consecutive cells. (See Figure 9(b).) Then, these groups have pairwise distance more than one, and hence no unit disk (with radius 1/2) can cover points in two distinct groups. (Remember that we have removed r cells of total width $rW_c = 1$, and the left boundary of a cell is closed and the right boundary is open.) Therefore, we can independently solve the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I \rangle$ for each group G in the sub-ribbon. The similar arguments in Lemma 5.2 establish that the problem on $\langle \mathcal{P}_q \cap R_i, \mathcal{D}_I \rangle$ admits a PTAS if there is a polynomial-time algorithm which optimally solves the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I \rangle$ for each group G. Therefore, the following lemma completes the proof of Lemma 8.1.

Lemma 8.2. *There is a polynomial-time algorithm which optimally solves the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I \rangle$ for a group G consisting of at most rk consecutive cells.*

We give a polynomial-time algorithm which optimally solves the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I \rangle$ for a group G consisting of at most rk consecutive cells. Let $S_{i,1}, S_{i,2}, \ldots, S_{i,m}$ be the cells in G ordered from left to right. (See Figure 11(a).) Remember that rk (and hence m) is a fixed constant. We denote by $\mathcal{D}_I(S_{i,j})$

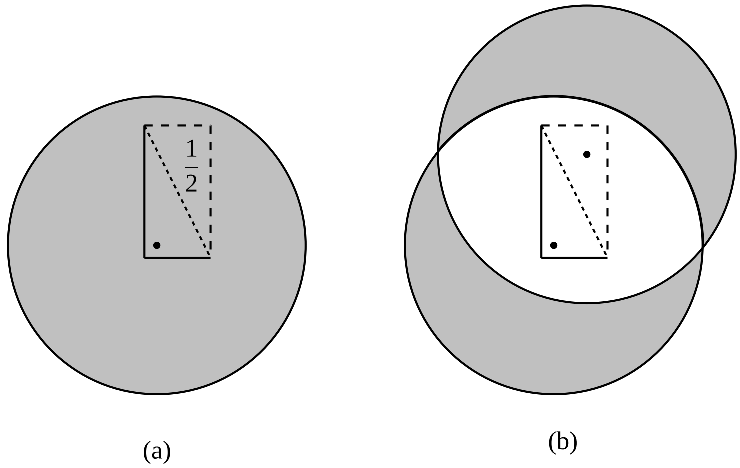

Figure 10: (a) Cell $S_{i,j}$ covered by one disk in $\mathcal{D}_I(S_{i,j})$, and (b) $S_{i,j}$ covered by more than one disks in $\mathcal{D}_I(S_{i,j})$, where the uniquely covered region is hatched.

the set of disks whose centers are contained in $S_{i,j}$. (Remember that, in each cell $S_{i,j}$, the left boundary is closed and the right boundary is open.) Notice that any disk in $\mathcal{D}_I(S_{i,j})$ covers all the points in $S_{i,j}$ since the diagonal of each cell is of length $1/2$. (See Figure 10(a).)

We have the following lemma, which is another crux of this paper.

Lemma 8.3. *Consider an arbitrary subset $\mathcal{C} \subseteq \mathcal{D}_I$ and let $S_{i,j}$ be a cell.*

(i) *If $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| \geq 2$, then no point in $S_{i,j}$ is uniquely covered by $\mathcal{C}$.*

(ii) *If $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| = 1$, then a point in $S_{i,j}$ is uniquely covered by $\mathcal{C}$ if and only if no disk in $\mathcal{C} \setminus \mathcal{D}_I(S_{i,j})$ covers the point.*

(iii) *If $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| = 0$, then a point in $S_{i,j}$ is uniquely covered by $\mathcal{C}$ if and only if exactly one disk in $\mathcal{C} \setminus \mathcal{D}_I(S_{i,j})$ covers the point.*

Proof. The lemma holds because any disk in $\mathcal{D}_I(S_{i,j})$ covers all the points in $S_{i,j}$. (See Figure 10(a) and (b).) □

Lemma 8.3 motivates us to classify all the subsets $\mathcal{C} \subseteq \mathcal{D}_I$ into $O(3^{rk} \cdot |\mathcal{D}_I|^{rk})$ types, as follows. Let $a_j \in \{0, 1, 2\}$ for each index j, $1 \leq j \leq m$. Then, a subset $\mathcal{C} \subseteq \mathcal{D}_I$ is called an $(a_1, a_2, \ldots, a_m)$*-cover using the set* $\mathcal{C}' \subseteq \mathcal{C}$ if the following three conditions (i)–(iii) hold:

(i) If $a_j = 2$, then $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| \geq 2$;

(ii) If $a_j = 1$, then $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| = 1$ and the disk $D \in \mathcal{C} \cap \mathcal{D}_I(S_{i,j})$ is contained in $\mathcal{C}'$;

(iii) If $a_j = 0$, then $|\mathcal{C} \cap \mathcal{D}_I(S_{i,j})| = 0$.

Then, the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I \rangle$ can be solved optimally in polynomial time if there is a polynomial-time algorithm to find an $(a_1, a_2, \ldots, a_m)$-cover using the set $\mathcal{C}'$ that maximizes the number of uniquely covered points in $\mathcal{P}_q \cap G$ for each m-tuple $(a_1, a_2, \ldots, a_m)$ with $a_j \in \{0, 1, 2\}$ and a set $\mathcal{C}' \subseteq \mathcal{D}_I$. We denote this

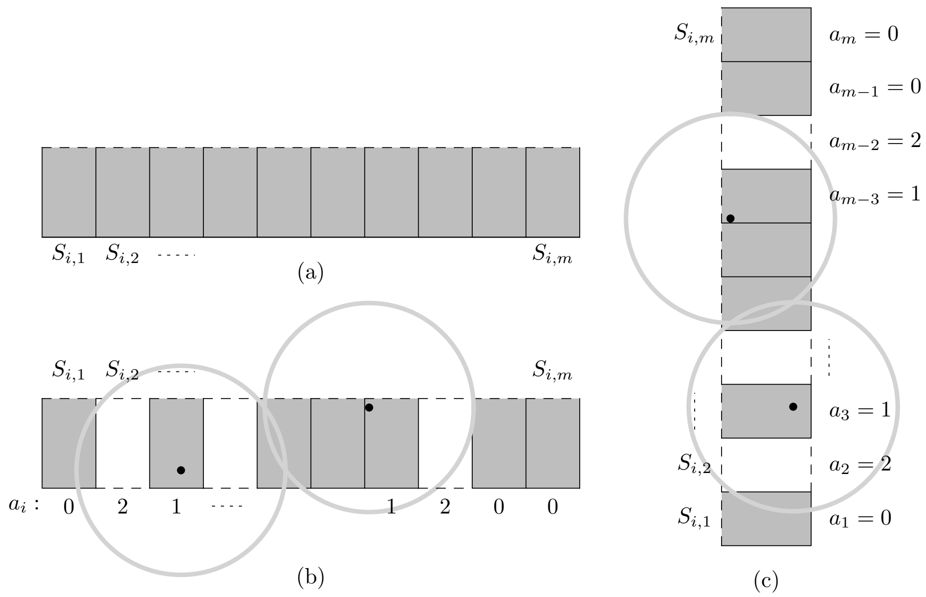

Figure 11: (a) Group of m consecutive cells, (b) an instance $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$, and (c) the "rotated" instance $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$.

instance by $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$. Figure 11(b) illustrates an instance $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$, where each a_j is written below the cell $S_{i,j}$, the two disks are contained in $\mathcal{C}'$, and the cells $S_{i,j}$ with $a_j = 2$ are colored white because we know that there is no uniquely covered point in the cells. We solve the instances $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$ for all m-tuples $(a_1, a_2, \ldots, a_m)$ and all "meaningful" sets $\mathcal{C}' \subseteq \mathcal{D}_I$, and output the best solution among them. Remember that m $\left(\leq rk\right)$ is a fixed constant, and hence the number of all possible m-tuples, $O(3^m)$, is also bounded by a constant. Furthermore, we do not need to solve the problem for all sets $\mathcal{C}' \subseteq \mathcal{D}_I$; The *meaningful* sets $\mathcal{C}' \subseteq \mathcal{D}_I$ can be obtained by choosing exactly one disk from each $\mathcal{D}_I(S_{i,j})$ with $a_j = 1$. Since an m-tuple $(a_1, a_2, \ldots, a_m)$ has at most m elements such that $a_j = 1$, the number of meaningful sets $\mathcal{C}' \subseteq \mathcal{D}_I$ can be bounded by $O(|\mathcal{D}_I|^m)$.

The problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$ can be optimally solved in polynomial time by slightly modifying the polynomial-time (exact) algorithm in Section 5 for the problem on $\langle \mathcal{P}_q \cap G', \mathcal{D}_O \rangle$, where G' is a group consisting of a constant number of ribbons. Remember that no point in $S_{i,j}$ with $a_j = 2$ is uniquely covered, and hence we can ignore the points in $S_{i,j}$ with $a_j = 2$. Therefore, we can treat the disks in $\mathcal{D}_I(S_{i,j})$ with $a_j = 2$ as if they form the set $\mathcal{D}_O$ from the viewpoint of the cells $S_{i,j'}$ with $a_{j'} \in \{0, 1\}$. Notice that the y-monotonicity is ensured for the intersection of any disk in $\mathcal{D}_I(S_{i,j})$ with $a_j = 2$ and the cells $S_{i,j'}$ with $a_{j'} \in \{0, 1\}$. Furthermore, because $2W_c = 1/2$ and the left boundary is closed and the right boundary is open in each cell, we have the following lemma which is the counterpart of Lemma 5.4.

Lemma 8.4. *Let D and D' be disks in $\mathcal{D}_I(S_{i,j})$ and $\mathcal{D}_I(S_{i,j'})$, respectively, such that $a_j = a_{j'} = 2$. If $D \cap D' \cap S_{i,j''} \neq \emptyset$ for a cell $S_{i,j''}$ with $a_{j''} \in \{0, 1\}$, then $j, j' \in \{j'' - 2, j'' - 1, j'' + 1, j'' + 2\}$.*

Recall that the choices of disks for the cells $S_{i,j'}$ with $a_{j'} = 1$ are fixed by the set $\mathcal{C}'$. Thus, our task is to choose disks from $\mathcal{D}_I(S_{i,j})$ with $a_j = 2$ which forms an optimal solution to $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$. Therefore, the polynomial-time algorithm in Section 5 can be easily modified so that it solves the problem on $\langle \mathcal{P}_q \cap G, \mathcal{D}_I; a_1, a_2, \ldots, a_m; \mathcal{C}' \rangle$, by rotating the plane to the horizontal direction. (See Figure 11(b) and (c).) Along the dynamic programming, when we delete a top disk, we also take the effect of $\mathcal{C}'$ into account, and update the function accordingly; because $\mathcal{C}'$ is fixed and we keep track of all top disks, the update formula (3) can be easily modified. Since the number of disks in $\mathcal{C}'$ is constant, this modification keeps the running time polynomially bounded.

This completes the proof of Lemma 8.2. □

9. Budgeted version

In this section, we consider the budgeted version and give the following theorem.

Theorem 9.1. *For any fixed constant $\varepsilon > 0$, there is a polynomial-time $(2 + 4/\sqrt{3} + \varepsilon)$-approximation algorithm for the budgeted unique unit-disk coverage problem.*

We give a sketch of how to adapt the algorithm for $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ in Section 5.5 to the budgeted unique unit-disk coverage problem. To this end, we first describe the adaptation to give an optimal solution to $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ in pseudo-polynomial time when budget, cost, and profit are all integers.

We keep the same strategy, but for the dynamic programming, we slightly change the definition of f. In the budgeted version, $\mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C})$ means the total profit of the points in $\mathcal{P}_q \cap G$ that are uniquely covered by a subset $\mathcal{C} \subseteq \mathcal{D}_O$, and $\mathsf{cost}(\mathcal{C})$ means the total cost of the disks in $\mathcal{C}$. Let $X = \sum_{p \in \mathcal{P}} \mathsf{profit}(p)$, then $\mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C}) \leq X$ for any disk set $\mathcal{C} \subseteq \mathcal{D}_O$. For a feasible disk set $\mathcal{F}$ on $\mathcal{D}_O$ and an integer $x \in \{0, 1, \ldots, X\}$, let $g(\mathcal{F}, x)$ be the minimum total cost of disks in a set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ such that the total profit of uniquely covered points in $\mathcal{P}_q \cap G$ by $\mathcal{C}$ is at least x, that is,

$$g(\mathcal{F}, x) = \min\{\mathsf{cost}(\mathcal{C}) \mid \mathcal{C} \in \mathfrak{C}(\mathcal{F}) \text{ and } \mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C}) \geq x\}.$$

If there is no disk set $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ such that $\mathsf{profit}(\mathcal{P}_q \cap G, \mathcal{C}) \geq x$, then let $g(\mathcal{F}, x) = +\infty$. Then, the optimal value $\mathrm{OPT}(\mathcal{P}_q \cap G, \mathcal{D}_O)$ for the budgeted version on $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ can be computed as

$$\mathrm{OPT}(\mathcal{P}_q \cap G, \mathcal{D}_O) = \max\{x \mid 0 \leq x \leq X, g(\mathcal{F}, x) \leq B\}.$$

We proceed along the same way as the algorithm in Section 5.5, except for the update formula (3) that should be replaced by

$$g(\mathcal{F}, x) := \min\{g(\mathcal{F}', y) \mid \mathcal{F}' \text{ is a child of } \mathcal{F},\ y + z(\mathcal{F}, K(\mathcal{F})) \geq x\} + \mathsf{cost}(K(\mathcal{F})),$$

where $z(\mathcal{F}, K(\mathcal{F}))$ means the difference of the total profit of uniquely covered points in $\mathcal{P}_q \cap G$ caused by adding the disk $K(\mathcal{F})$ to $\mathcal{F} \setminus \{K(\mathcal{F})\}$. This way, we obtain an optimal solution to $\langle \mathcal{P}_q \cap G, \mathcal{D}_O \rangle$ for a group G consisting of at most k consecutive ribbons. Note that the blowup in the running time is only polynomial in X.

We now explain how to obtain a solution to the problem on $\langle \mathcal{P}_q \cap R^j_W, \mathcal{D}_O \rangle$ for each sub-stripe R^j_W, $0 \leq j \leq k$. (See Figure 3.) The adapted algorithm above can solve the problem on each group G_l in R^j_W, and hence suppose that we have computed $g(\mathcal{F}, x)$ for each group G_l and all integers $x \in \{0, 1, \ldots, X\}$. Then, obtaining a solution to $\langle \mathcal{P}_q \cap R^j_W, \mathcal{D}_O \rangle$ can be regarded as solving an instance of the multiple-choice knapsack problem [2, 9], as follows: The capacity of the knapsack is equal to the budget B; Each $g(\mathcal{F}, x)$ in G_l and $x \in \{0, 1, \ldots, X\}$ have a corresponding item with profit x and cost $g(\mathcal{F}, x)$; The items corresponding to G_l form a class, from which at most one item can be packed into the knapsack. The multiple-choice knapsack problem can be solved in pseudo-polynomial time which polynomially depends on X [2, 9], and hence we can obtain an optimal solution to $\langle \mathcal{P}_q \cap R^j_W, \mathcal{D}_O \rangle$, $0 \leq j \leq k$, in pseudo-polynomial time.

We apply the standard scale-and-round technique to the profit (as used for the ordinary knapsack problem [9, 12]), that is, the profit of each point p is scaled down to $\lfloor \mathsf{profit}(p)/t \rfloor$ by some appropriate scaling factor t which depends on a fixed constant $\varepsilon'' > 0$. Then, for any fixed constant $\varepsilon'' > 0$, we obtain a $(1+\varepsilon'')$-approximate solution to $\langle \mathcal{P}_q \cap R^j_W, \mathcal{D}_O \rangle$ for each $j \in \{0, \ldots, k\}$. Overall, such an approximate solution to each of the $k+1$ subinstances $\langle \mathcal{P}_q \cap R^j_W, \mathcal{D}_O \rangle$, $0 \leq j \leq k$, can be obtained in polynomial time. By taking the best one, we can obtain a $(1+\varepsilon')$-approximate solution to $\langle \mathcal{P}_q, \mathcal{D}_O \rangle$ for any fixed constant $\varepsilon' > 0$, by choosing ε'' appropriately. Then, the similar arguments give $(2 + 4/\sqrt{3} + \varepsilon)$-approximate solution to the budgeted unique unit-disk coverage problem on $\langle \mathcal{P}, \mathcal{D} \rangle$. □

10. Conclusion

In this paper, we gave a polynomial-time $(2+4/\sqrt{3}+\varepsilon)$-approximation algorithm, for any fixed constant $\varepsilon > 0$, for the unique unit-disk coverage problem. Our algorithm combines the well-known shifting strategy [6] and a novel dynamic programming algorithm to solve the problem restricted to regions of constant height. It is not clear how we can adapt the method in this paper to other shapes such as disks with different radii. This remains an open question. The generality of the approach enables us to give a polynomial-time $(2 + 4/\sqrt{3} + \varepsilon)$-approximation algorithm, for any fixed constant $\varepsilon > 0$, for the budgeted version, too.

Acknowledgments

The authors thank anonymous referees of the preliminary version and of this journal version for their helpful suggestions. This work is partially supported by

Grant-in-Aid for Scientific Research, and by the Funding Program for World-Leading Innovative R&D on Science and Technology, Japan.

References

[1] Baker, B.S.: Approximation algorithms for NP-complete problems on planar graphs. Journal of the ACM 41, 153–180 (1994)

[2] Chandra, A.K., Hirschberg, D.S., Wong, C.K.: Approximate algorithms for some generalized knapsack problems. Theoretical Computer Science 3, 293–304 (1976)

[3] Demaine, E.D., Hajiaghayi, M.T., Feige, U., Salavatipour, M.R.: Combination can be hard: approximability of the unique coverage problem, SIAM J. on Computing 38, pp. 1464–1483 (2008)

[4] Erlebach, T., van Leeuwen, E.J.: Approximating geometric coverage problems, Proc. of SODA 2008, pp. 1267–1276 (2008)

[5] Guruswami, V., Trevisan, L.: The complexity of making unique choices: Approximating 1-in-k SAT. Proc. of APPROX-RANDOM 2005, pp. 99–110 (2005)

[6] Hochbaum, D.S., Maass, W.: Approximation schemes for covering and packing problems in image processing and VLSI, J. ACM 32, pp. 130–136 (1985)

[7] Ito, T., Nakano, S.-i., Okamoto, Y., Otachi, Y., Uehara, R., Uno, T., Uno, Y.: A 4.31-approximation for the geometric unique coverage problem on unit disks, Proc. of ISAAC 2012, pp. 372–381 (2012)

[8] Ito, T., Nakano, S.-i., Okamoto, Y., Otachi, Y., Uehara, R., Uno, T., Uno, Y.: A polynomial-time approximation scheme for the geometric unique coverage problem on unit squares, Proc. of SWAT 2012, pp. 24–35 (2012); A journal version is submitted

[9] Kellerer, H., Pferschy, U., Pisinger, D.: Knapsack Problems. Springer-Verlag, 2004.

[10] Misra, N., Moser, H., Raman, V., Saurabh, S., Sikdar, S.: The parameterized complexity of unique coverage and its variants. Algorithmica 65, pp. 517–544 (2013)

[11] van Leeuwen, E. J.: Optimization and approximation on systems of geometric objects. Ph.D. Thesis, University of Amsterdam, 2009

[12] Vazirani, V. V.: Approximation Algorithms. Springer-Verlag, Berlin, 2001.