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Optimal Packings of 8, 9, and 10 Equal Circles in an Isosceles Right Triangle

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1 Densest Circles Packings

Densest circle packing problem is one of the interesting open questions in combinatorial geometry as well as other possible variations of packing. We try to place n equal circles inside a given convex region in $\mathbf{R^2}$ like square, triangle, circle etc., such that the radius is maximized without overlapping.

The packing problem is usually regarded as distributing n points uniformly in the region P in such a way that the minimum distance d_n among the points is maximized, which we call *maximum point separation* problem. Suppose we define the *separation distance* as

$$d_n = \max_{S \subset P,\ |S|=n} \ \min_{p,q \in S,\ p \neq q} \ d(p,q) \tag{1}$$

($d(\ ,\ )$ is Euclidean distance), then the densest packing problem corresponds to that of optimizing the separation distance.

For example, let P be a triangle whose inscribed circle has radius r_{in} and r_n be the packing radius of n circles in P. Because the inward parallel body $(1 - \frac{r_n}{r_{in}})P$ contains n center points of the packing, by easy calculation the configuration of centers has the maximum separation distance $d = 2r_n$ inside $(1 - \frac{r_n}{r_{in}})P$.

On the other hand, if we have the optimal configuration of n points S inside P with separation distance d_n, then S consists of the center points of a packing in the outer parallel body $(1 + \frac{d_n}{2r_{in}})P$ whose radius is $r = \frac{d_n}{2}$. Therefore we obtain the relation between r_n and d_n in P s.t.

$$d_n = \frac{2r_{in}r_n}{r_{in} - r_n}, \quad r_n = \frac{r_{in}d_n}{2r_{in} + d_n}. \tag{2}$$

A configuration of n points with the maximum separation distance is called an *optimal point configuration*.

As one of the features of this finite circle packing it is often said " progress in proving lags that of conjecturings" [4]. That is, we need good packings in advance to compute the optimal packings.

Currently, the optimal packings in a unit square are known for $n \leq 9$ in [2, 13], $n = 14, 16, 25, 36$ in [7, 22, 23, 24]. The optimal packings for $10 \leq n \leq 20$ had been obtained by C. de Groot et al. in [6] by using the computer searching and recently $21 \leq n \leq 27$ were presented by K. J. Nurmela et al. in [13, 15]. Other best packings in a square are known for up to 50 in [14], and $n \leq 50$, $n = 51, 52, 54, 56, 60, 61$ including partial improvements of the conjectures in [14].

On the other hand, in a case of isosceles right triangle, the optimal packings were previously determined for $n \leq 7$ in [25] and the best packings are up to 16 in [10].

2 Method

One of the common techniques in computing the optimal packings is that we try to restrict the possible area in the bounding region only in which each of n points can reside without violating the conjectures. That is, when we are maximizing the separation distance, the conjectured distance is regarded as the lower bound of the optimal distance. So we can eliminate such areas in the region as will violate the lower bound if the points of optimal configuration are placed in. Computer-aided proof can simulate this reducing procedure to narrow the possible areas in the region, and can prove both the existence and optimality of the conjectured packings. The following is the general description of the computer-aided proof in this kind of problem.

1. Consider all the n-tile representatives from a given tiling (initial combinations).
2. Apply polygon reducing procedures to each n-tile, and obtain the corresponding rest regions (polygon reducing).
3. Guess the adjacency among n points which belong(s) to the rest combination.
4. Draw n approximate error squares.
   Check whether or not the approximate error squares are shrunk into a constant factor smaller squares after a finitely many number of rounds of reducings (proof of optimality and uniqueness).

3 Results

In our implementations, we have applied $E = 10^{-15}$, $e_0 = 10^{-7}$, $e_1 = 10^{-9}$, $e_2 = 10^{-13}$, $e_{ad} = 10^{-10}$ and $e_{eq} = E = 10^{-15}$. As $e_0 = 10^{-7}$ we used the truncated value of separation distance for d_{low}. Table 3.1 shows the initial, rest and optimal combinations in polygon reducings. The initial combinations are described as the number of orbits, and rest and optimal combinations as n-tile which happened to coincide in this range of n. Table 3.2 is the maximum separation distance for $2 \leq n \leq 10$.

n	Tiling	d_{low}	N_n	Rest and optimal combinations
5	(3,3)	0.5358983	4	{0,1,2,4,5}
6	(3,3)	0.5	1	{0,1,2,3,4,5}
7	(4,4)	0.4195420	64	{0,1,3,4,6,8,9}
8	(4,4)	0.3789373	25	{0,2,3,5,6,7,8,9} for 8a
				{0,1,3,4,5,6,8,9} for 8b
9	(5,5)	0.3535533	2535	{0,2,4,6,8,9,11,13,14}
10	(5,5)	0.3333333	1527	{0,1,3,4,5,6,8,12,13,14}

Table 1: Experimental data.

n	d_n	
2	$\sqrt{2}$	$\approx 1.414213562373095\ldots$
3	1	$= 1.0$
4	$\sqrt{2}/2$	$\approx 0.707106781186547\ldots$
5	$4 - 2\sqrt{3}$	$\approx 0.535898384862246\ldots$
6	$1/2$	$= 0.5$
7	$(\sqrt{44\sqrt{2}+50} - 2 - 4\sqrt{2})/7$	$\approx 0.419542091095306\ldots$
8	$2\sqrt{2} - \sqrt{6}$	$\approx 0.378937381963012\ldots$
9	$\sqrt{2}/4$	$\approx 0.353553390593274\ldots$
10	$1/3$	$\approx 0.333333333333333\ldots$

Table 2: Maximum separation distance.

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