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Title	Non-E-Overlapping, Weakly Shallow, and Non- Collapsing TRSs are Confluent
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Citation	Lecture Notes in Computer Science, 9195: 111-126
Issue Date	2015-07-25
Туре	Journal Article
Text version	author
URL	http://hdl.handle.net/10119/14220
Rights	This is the author-created version of Springer, Masahiko Sakai, Michio Oyamaguchi, Mizuhito Ogawa, Lecture Notes in Computer Science, 9195, 2015, 111–126. The original publication is available at www.springerlink.com, http://dx.doi.org/10.1007/978-3-319-21401-6_7
Description	25th International Conference on Automated Deduction, Berlin, Germany, August 1–7, 2015, Proceedings



Japan Advanced Institute of Science and Technology

Non-*E*-Overlapping, Weakly Shallow, and Non-Collapsing TRSs are Confluent^{*}

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Abstract. A term is *weakly shallow* if each defined function symbol occurs either at the root or in the ground subterms, and a term rewriting system is weakly shallow if both sides of a rewrite rule are weakly shallow. This paper proves that non-*E*-overlapping, weakly-shallow, and non-collapsing term rewriting systems are confluent by extending *reduction graph* techniques in our previous work [SO10] with *towers of expansions*.

1 Introduction

Confluence of term rewriting systems (TRSs) is undecidable, even for flat TRSs [MOJ06] or length-two string rewrite systems [SW08]. Two decidable subclasses are known: right-linear and shallow TRSs by tree automata techniques [GT05] and terminating TRSs by resolving to finite search [KB70]. Many sufficient conditions have been proposed, and they are classified into two categories.

- Local confluence for terminating TRSs [KB70]. It was extended to TRSs with relative termination [HM11,KH12]. Another criterion comes with the decomposition to linear and terminating non-linear TRSs [LDJ14]. It requires conditions for the existence of well-founded *ranking*.
- Peak elimination with an explicit well-founded measure. Lots of works explore left-linear TRSs under the non-overlapping condition and its extensions [Ros73,Hue80,Toy87,Oos95,Oku98,OO97]. For non-linear TRSs, there are quite few works [TO95,GOO98] under the non-*E*-overlapping condition (which coincides with non-overlapping if left-linear) and additional restrictions that allow to define such measures.

We have proposed a different methodology, called a *reduction graph* [SO10], and shown that "*weakly non-overlapping, shallow, and non-collapsing TRSs are confluent*". An original idea comes from observing that, when non-*E*-overlapping,

^{*} The results without proofs are orally presented at IWC 2014 [SOO14].

^{**} This work is supported by JSPS KAKENHI Grant Number 25540003.

peak-elimination uses only "*copies*" of reductions in an original rewrite sequences. Thus, if we focus on terms appearing in peak elimination, they are finitely many. We regard a rewrite relation over these terms as a directed graph, and construct a confluent directed acyclic graph (DAG) in a bottom-up manner, in which the shallowness assumption works. The keys are, such a DAG always has a unique normal form (if it is finite), and convergence is preserved if we add an arbitrary reduction starting from a normal form. Our reduction graph technique is carefully designed to preserve both acyclicity and finiteness.

This paper introduces the notion of towers of expansions, which extends a reduction graph by adding terms and edges expanded with function symbols in an on-demand way, and shows that "weakly shallow, non-E-overlapping, and non-collapsing TRSs are confluent". A term is weakly shallow if each defined function symbol appears either at the root or in the ground subterms, and a TRS is weakly shallow if the both sides of rules are weakly shallow. It is worth mentioning:

- A Turing machine is simulated by a weakly shallow TRS [Klo93] (see Remark 1), and many decision problems, such as the word problem, termination and confluence, are undecidable [MOM12]. Note that the word problem is decidable for shallow TRSs [CHJ94]. The fact distinguishes these classes.
- The non-*E*-overlapping property is undecidable for weakly shallow TRSs [MOM12]. A decidable sufficient condition is *strongly non-overlapping*, where a TRS is *strongly non-overlapping* if its linearization is non-overlapping [OO89]. Here, these conditions are the same when left-linear.
- Our result gives a new criterion for confluence provers of TRSs. For instance,

$$\{d(x,x) \to h(x), f(x) \to d(x, f(c)), c \to f(c), h(x) \to h(g(x))\}$$

is shown to be confluent only by ours.

Remark 1. Let Q, Σ and $\Gamma (\supseteq \Sigma)$ be finite sets of states, input symbols and tape symbols of a Turing machine M, respectively. Let $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{left}, \text{right}\}$ be the transition function of M. Each configuration $a_1 \cdots a_i q a_{i+1} \cdots a_n \in \Gamma^+ Q \Gamma^+$ (where $q \in Q$) is represented by a term $q(a_i \cdots a_1(\$), a_{i+1} \cdots a_n(\$))$ where arities of function symbols q, a_j $(1 \leq j \leq n)$ and \$ are 2, 1 and 0, respectively. The corresponding TRS R_M consists of rewriting rules below:

 $q(x, a(y)) \to p(b(x), y) \quad \text{if } \delta(q, a) = (p, b, \text{right}),$ $q(a'(x), a(y)) \to p(x, a'(b(y))) \quad \text{if } \delta(q, a) = (p, b, \text{left})$

2 Preliminaries

2.1 Abstract Reduction System

For a binary relation \rightarrow , we use \leftarrow , \leftrightarrow , \rightarrow^+ and \rightarrow^* for the inverse relation, the symmetric closure, the transitive closure, and the reflexive and transitive closure of \rightarrow , respectively. We use \cdot for the composition operation of two relations.

An abstract reduction system (ARS) is a directed graph $G = \langle V, \to \rangle$ with reduction $\to \subseteq V \times V$. If $(u, v) \in \to$, we write it as $u \to v$. An element u of Vis (\to) normal if there exists no $v \in V$ with $u \to v$. We sometimes call a normal element a normal form. For subsets V' and V'' of $V, \to |_{V' \times V''} = \to \cap (V' \times V'')$.

Let $G = \langle V, \to \rangle$ be an ARS. We say G is *finite* if V is finite, *confluent* if $\leftarrow^* \cdot \to^* \subseteq \to^* \cdot \leftarrow^*$, *Church-Rosser* (CR) if $\leftrightarrow^* \subseteq \to^* \cdot \leftarrow^*$, and *terminating* if it does not admit an infinite reduction sequence from a term. G is *convergent* if it is confluent and terminating. Note that confluence and CR are equivalent.

We refer standard terminology in graphs. Let $G = \langle V, \rightarrow \rangle$ and $G' = \langle V', \rightarrow' \rangle$ be ARSs. We use $V_{G'}$ and $\rightarrow_{G'}$ to denote V' and \rightarrow' , respectively. An edge $v \rightarrow u$ is an *outgoing-edge* of v and an *incoming-edge* of u, and v is the *initial vertex* of \rightarrow . A vertex v is \rightarrow -normal if it has no outgoing-edges. The union of graphs is defined as $G \cup G' = \langle V \cup V', \rightarrow \cup \rightarrow' \rangle$. We say

- G is connected if $(u, v) \in \leftrightarrow^*$ for each $u, v \in V$.
- -G' includes G, denoted by $G' \supseteq G$, if $V' \supseteq V$ and $\rightarrow' \supseteq \rightarrow$.
- -G' weakly subsumes G, denoted by $G' \supseteq G$, if $V' \supseteq V$ and ${\leftrightarrow'}^* \supseteq {\rightarrow}$.
- -G' conservatively extends G, if $V' \supseteq V$ and ${\leftrightarrow'}^*|_{V \times V} = {\leftrightarrow}^*$.

The weak subsumption relation \square is transitive.

2.2 Term Rewriting System

Let F be a finite set of function symbols, and X be an enumerable set of variables with $F \cap X = \emptyset$. T(F, X) denotes the set of terms constructed from F and Xand Var(t) denotes the set of variables occurring in a term t. A ground term is a term in $T(F, \emptyset)$. The set of positions in t is Pos(t), and the root position is ε . For $p \in Pos(t)$, the subterm of t at position p is denoted by $t|_p$. The root symbol of t is root(t), and the set of positions in t whose symbols are in S is denoted by $Pos_S(t) = \{p \mid root(t|_p) \in S\}$. The term obtained from t by replacing its subterm at position p with s is denoted by $t[s]_p$. The size |t| of a term t is |Pos(t)|. As notational convention, we use s, t, u, v, w for terms, x, y for variables, a, b, c, f, gfor function symbols, p, q for positions, and σ, θ for substitutions.

We define $\operatorname{sub}(t)$ as $\operatorname{sub}(x) = \emptyset$ and $\operatorname{sub}(t) = \{t_1, \ldots, t_n\}$ if $t = f(t_1, \ldots, t_n)$. A rewrite rule is a pair (ℓ, r) of terms such that $\ell \notin X$ and $\operatorname{Var}(\ell) \supseteq \operatorname{Var}(r)$. We write it $\ell \to r$. A term rewriting system (TRS) is a finite set R of rewrite rules. The rewrite relation of R on $\operatorname{T}(F, X)$ is denoted by \to . We sometimes write $s \xrightarrow{p}_R t$ to indicate the rewrite step at the position p. Let $s \xrightarrow{p}_R t$. It is a top reduction if $p = \varepsilon$. Otherwise, it is an inner reduction, written as $s \xrightarrow{\varepsilon \leq} t$.

Given a TRS R, the set D of defined symbols is $\{\operatorname{root}(\ell) \mid \ell \to r^R \in R\}$. The set C of constructor symbols is $F \setminus D$. For $T \subseteq \operatorname{T}(F, X)$ and $f \in F$, we use $T|_f$ to denote $\{s \in T \mid \operatorname{root}(s) = f\}$. For a subset F' of F, we use $T|_{F'}$ to denote the union $\bigcup_{f \in F'} T|_f$.

A constructor term is a term in T(C, X), and a semi-constructor term is a term in which defined function symbols appear only in the ground subterms. A term is shallow if the length |p| is 0 or 1 for every position p of variables in the

term. A weakly shallow term is a term in which defined function symbols appear only either at the root or in the ground subterms (i.e., $p \neq \varepsilon$ and $\operatorname{root}(s|_p) \in D$ imply that $s|_p$ is ground). Note that every shallow term is weakly shallow.

A rewrite rule $\ell \to r$ is *weakly shallow* if ℓ and r are weakly shallow, and *collapsing* if r is a variable. A TRS is *weakly shallow* if each rewrite rule is weakly shallow. A TRS is *non-collapsing* if it contains no collapsing rules.

Example 2. A TRS R_1 is weakly shallow and non-collapsing.

 $R_1 = \{f(x, x) \to a, f(x, g(x)) \to b, c \to g(c)\}$ [Hue80]

Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ be rewrite rules in a TRS R. Let p be a position in ℓ_1 such that $\ell_1|_p$ is not a variable. If there exist substitutions θ_1, θ_2 such that $\ell_1|_p\theta_1 = \ell_2\theta_2$ (resp. $\ell_1|_p\theta_1 \stackrel{\xi \leq *}{\underset{R}{\leftarrow}} \ell_2\theta_2$), we say that the two rules are *overlapping* (resp. *E-overlapping*), except that $p = \varepsilon$ and the two rules are identical (up to renaming variables). A TRS R is *overlapping* (resp. *E-overlapping*) if it contains a pair of overlapping (resp. *E*-overlapping) rules. Note that TRS R_1 in Example 2 is *E*-overlapping since $f(c, c) \stackrel{\xi \leq *}{\underset{R}{\leftarrow}} f(c, g(c))$.

3 Extensions of Convergent Abstract Reduction Systems

This section describes a transformation system from a finite ARS to obtain a convergent (i.e., terminating and confluent) ARS that preserves the connectivity.

Let $G = \langle V, \rightarrow \rangle$ be an ARS. If G is finite and convergent, then we use a function \downarrow_G (called the choice mapping) that takes an element of V and returns the normal form [SO10]. We also use $v \downarrow_G$ instead of $\downarrow_G(v)$.

Definition 3. For ARSs $G_1 = \langle V_1, \rightarrow_1 \rangle$ and $G_2 = \langle V_2, \rightarrow_2 \rangle$, we say that $G_1 \cup G_2$ is the hierarchical combination of G_2 with G_1 , denoted by $G_1 > G_2$, if $\rightarrow_1 \subseteq (V_1 \setminus V_2) \times V_1$.

Proposition 4. $G_1 \ge G_2$ is terminating if both G_1 and G_1 are so.

Lemma 5. Let $G_1 > G_2$ be a confluent and hierarchical combination of ARSs. If a confluent ARS G_3 weakly subsumes G_2 and $G_1 > G_3$ is a hierarchical combination, then $G_1 > G_3$ is confluent.

Proof. We use $\langle V_i, \rightarrow_i \rangle$ to denote G_i . Let $\alpha : u' \leftarrow_{G_1 > G_3}^* u \rightarrow_{G_1 > G_3}^* u''$. If $u \in V_3$, only \rightarrow_3 appears in α , and hence $u' \rightarrow_3^* \cdot \leftarrow_3^* u''$ follows from the confluence of G_3 . Otherwise, α is represented as $u' \leftarrow_3^* v' \leftarrow_1^* u \rightarrow_1^* v'' \rightarrow_3^* u''$. Since $v' \rightarrow_1^* w' \rightarrow_2^* \cdot \leftarrow_2^* w'' \leftarrow_1^* v''$ for some w' and w'' (from the confluence of $G_1 > G_2$) and $G_2 \sqsubseteq G_3$, we obtain $u' \leftarrow_3^* v' \rightarrow_1^* w' \leftrightarrow_3^* w'' \leftarrow_1^* v'' \rightarrow_3^* u''$. Since $G_1 > G_3$ is a hierarchical combination, v' = w' if $v' \in V_3$, and v' = u' otherwise. Hence, $u' \rightarrow_1^* \cdot \leftrightarrow_3^* w'$. Similarly either v'' = w'' or v'' = u''. Thus, $u' \rightarrow_1^* \cdot \leftrightarrow_3^* \cdot \leftarrow_1^* u''$. The confluence of G_3 gives $u' \rightarrow_1^* \cdot \rightarrow_3^* \cdot \leftarrow_3^* \cdot \leftarrow_1^* u''$, and $u' \rightarrow_{G_1 > G_3}^* \cdot \leftarrow_{G_1 > G_3}^* u''$.

In the sequel, we generalize properties of ARSs obtained in [SO10].

Definition 6. Let $G = \langle V, \rightarrow \rangle$ be a convergent ARS. Let v, v' be vertices such that $v \neq v'$ and if $v \in V$ then v is \rightarrow -normal. Then G', denoted by $G \multimap (v \rightarrow v')$, is defined as follows (see Fig. 1):



Note that v' becomes a normal form of G' when the first or the third transformation is applied.

Proposition 7. For a convergent ARS G, the ARS $G' = G \multimap (v \rightarrow v')$ is convergent, and satisfies $G' \supseteq G$.

We represent $G \multimap (v_0 \to v_1) \multimap (v_1 \to v_2) \multimap \cdots \multimap (v_{n-1} \to v_n)$ as $G \multimap (v_0 \to v_1 \to \cdots \to v_n)$ (if Definition 6 can be repeatedly applied).

Proposition 8. Let $G = \langle V, \rightarrow \rangle$ be a convergent ARS. Let v_0, v_1, \ldots, v_n satisfy $v_i \neq v_j$ (for $i \neq j$), and one of the following conditions:

(1) $v_0 \in V$, v_0 is \rightarrow -normal, and $v_i \in V$ implies $v_i \leftrightarrow^* v_0$ for each i(< n), (2) $v_0, \cdots, v_{n-1} \notin V$.

Then, $G' = G \multimap (v_0 \to v_1 \to \cdots \to v_n)$ is well-defined and convergent, and $G' \supseteq G$ holds.

4 Reduction Graphs

From now on, we fix C and D as the sets of constructors and defined function symbols for a TRS R, respectively. We assume that there exists a constructor with a positive arity in C, otherwise all weakly shallow terms are shallow.

4.1 Reduction Graphs and Monotonic Extension

Definition 9 ([SO10]). An ARS $G = \langle V, \rightarrow \rangle$ is an *R*-reduction graph if *V* is a finite subset of T(F, X) and $\rightarrow \subseteq \xrightarrow{P}$.

For an *R*-reduction graph $G = \langle V, \rightarrow \rangle$, inner-edges, strict inner-edges, and top-edges are given by $\stackrel{\varepsilon_{\leq}}{\to} = \rightarrow \cap \stackrel{\varepsilon_{\leq}}{\underset{R}{\neq}}, \stackrel{\neq_{\varepsilon}}{\to} = \rightarrow \setminus \stackrel{\varepsilon}{\underset{R}{\Rightarrow}}, \text{ and } \stackrel{\varepsilon}{\to} = \rightarrow \cap \stackrel{\varepsilon}{\underset{R}{\Rightarrow}}, \text{ respectively. We use } G^{\varepsilon_{\leq}}, G^{\neq_{\varepsilon}}, \text{ and } G^{\varepsilon} \text{ to denote } \langle V, \stackrel{\varepsilon_{\leq}}{\to} \rangle, \langle V, \stackrel{\neq_{\varepsilon}}{\to} \rangle, \text{ and } \langle V, \stackrel{\varepsilon}{\to} \rangle,$

respectively. Remark that for $R = \{a \to b, f(x) \to f(b)\}$ $V = \{f(a), f(b)\}$, and $G = \langle V, \{(f(a), f(b))\} \rangle$, we have $G^{\varepsilon <} = G^{\varepsilon} = G$ and $G^{\neq \varepsilon} = \langle V, \emptyset \rangle$.

For an R-reduction graph $G = \langle V, \rightarrow \rangle$ and $F' \subseteq F$, we represent $G|_{F'} =$ $\langle V, \rightarrow |_{F'} \rangle$ where $\rightarrow |_{F'} = \rightarrow |_{V|_{F'} \times V}$. Note that $\rightarrow |_C = \rightarrow |_{V|_C \times V|_C}$ and $\rightarrow =$ $\rightarrow |_D \cup \rightarrow |_{V|_C \times V|_C}.$

Definition 10. Let $G = \langle V, \rightarrow \rangle$ be an *R*-reduction graph. The direct-subterm reduction-graph $\operatorname{sub}(G)$ of G is $(\operatorname{sub}(V), \operatorname{sub}(\rightarrow))$ where

 $\begin{cases} \operatorname{sub}(V) = \bigcup_{t \in V} \operatorname{sub}(t) \\ \operatorname{sub}(\to) = \{(s_i, t_i) \mid f(s_1, \dots, s_n) \xrightarrow{\varepsilon \leq} f(t_1, \dots, t_n), \ s_i \neq t_i, \ 1 \leq i \leq n \}. \end{cases}$ An *R*-reduction graph $G = \langle V, \to \rangle$ is subterm-closed if $\operatorname{sub}(G^{\neq \varepsilon}) \sqsubset G$.

Lemma 11. Let $G = \langle V, \rightarrow \rangle$ be a subterm-closed R-reduction graph. Assume that (1) $s[t]_p \leftrightarrow^* s[t']_p$, and (2) for any p' < p, if $(s[t]_p)|_{p'} \leftrightarrow^* (s[t']_p)|_{p'}$ then $(s[t]_p)|_{p'} \stackrel{\neq \varepsilon}{\leftrightarrow} (s[t']_p)|_{p'}$. Then $t \leftrightarrow t'$.

Proof. By induction on |p|. If $p = \varepsilon$, trivial. Let p = iq and $s = f(s_1, \ldots, s_n)$. Since $s[t]_p \stackrel{\neq \varepsilon}{\leftrightarrow} s[t']_p$ from the assumptions, the subterm-closed property of G implies $s_i[t]_q \leftrightarrow^* s_i[t']_q$. Hence, $t \leftrightarrow^* t'$ holds by induction hypothesis.

Definition 12. For a set $F' \subseteq F$ and an R-reduction graph $G = \langle V, \rightarrow \rangle$, the F'-monotonic extension $M_{F'}(G) = \langle V_1, \rightarrow_1 \rangle$ is

 $\begin{cases} V_1 = \{f(s_1, \dots, s_n) \mid f \in F', s_1, \dots, s_n \in V\}, \\ \rightarrow_1 = \{(f(\cdots s \cdots), f(\cdots t \cdots)) \in V_1 \times V_1 \mid s \to t\}. \end{cases}$

Example 13. As a running example, we use the following TRS, which is non-*E*-overlapping, non-collapsing, and weakly shallow with $C = \{g\}$ and $D = \{c, f\}$:

 $R_2 = \{ f(x, q(x)) \to q^3(x), c \to q(c) \}.$

Consider a subterm-closed R_2 -reduction graph $G = \langle \{c, g(c), g^2(c)\}, \{(c, g(c))\} \rangle$. In the sequel, we use a simple representation of graphs as $G = \{c \to g(c), g^2(c)\}$. The C-monotonic extension $M_C(G)$ of G is $M_C(G) = \{g(c) \to g^2(c), g^3(c)\}.$

Proposition 14. Let $M_{F'}(G) = \langle V', \to' \rangle$ be the F'-monotonic extension of an R-reduction graph $G = \langle V, \rightarrow \rangle$. Then,

- (1) if G is terminating (resp. confluent), then $M_{F'}(G)$ is.
- (2) If G is subterm-closed, then for $u, v \in V|_{F'}$, we have (a) $u, v \in V'$, and (b) $u \stackrel{\neq \varepsilon}{\rightarrow} v \text{ implies } u \leftrightarrow'^* v.$
- (3) $\operatorname{sub}(M_{F'}(G)) \subset G$ if F' contains a function symbol with a positive arity.

Constructor Expansion 4.2

Definition 15. For a subterm-closed R-reduction graph G, a constructor expansion $\overline{M_C}(G)$ is the hierarchical combination $G|_D > M_C(G)$ (= $G|_D \cup M_C(G)$). The k-times application of $\overline{M_C}$ to G is denoted by $\overline{M_C}^k(G)$.

Example 16. For G in Example 13, the constructor expansions $\overline{M_C}^i(G)$ of G (i = 1, 3) are

$$\begin{split} \overline{M_C}(G) &= \{c \to g(c) \to g^2(c), \ g^3(c)\}, \\ \overline{M_C}^3(G) &= \{c \to g(c) \to g^2(c) \to g^3(c) \to g^4(c), \ g^5(c)\} \end{split}$$

Lemma 17. Let G be a subterm-closed R-reduction graph. Then,

(1) $\operatorname{sub}(\overline{M_C}(G)^{\neq \varepsilon}) \sqsubseteq G$, and

(2) $\rightarrow_{G^{\neq \varepsilon}} \subseteq \leftrightarrow^*_{M_F(G)}$, that is, $G \sqsubseteq G^{\varepsilon} \cup M_F(G)$,

Proof. Let $G = \langle V, \to \rangle$. We refer $M_C(G)$ by $G' = \langle V', \to' \rangle$. Thus, for $v \in V'$, root $(v) \in C$. Note that $\overline{M_C}(G) = G|_D > M_C(G) = \langle V' \cup V, \to' \cup \to|_{V|_D \times V} \rangle$.

- (1) Due to $\operatorname{sub}(\overline{M_C}(G)^{\neq \varepsilon}) = \operatorname{sub}(G^{\neq \varepsilon}|_D) \cup \operatorname{sub}(M_C(G))$, it is enough to show $\operatorname{sub}(G^{\neq \varepsilon}|_D) \sqsubseteq G$ and $\operatorname{sub}(M_C(G)) \sqsubseteq G$. The former follows from the fact that $\operatorname{sub}(G^{\neq \varepsilon}|_D) \subseteq \operatorname{sub}(G^{\neq \varepsilon})$ and G is subterm-closed. The latter follows from $\operatorname{sub}(M_C(G)) \subseteq G$.
- (2) Obvious from Proposition 14 (2).

Lemma 18. For a subterm-closed R-reduction graph G,

- (1) $G \sqsubseteq \overline{M_C}(G)$,
- (2) $\overline{M_C}(G)$ is subterm-closed, and
- (3) $\overline{M_C}(G)$ is convergent if G is convergent.

Proof. Let $G = \langle V, \to \rangle$. Note that $\overline{M_C}(G) = (G|_D > M_C(G)) = \langle V \cup$ $V_{M_C(G)}, \rightarrow |_D \cup \rightarrow_{M_C(G)} \rangle.$

- (1) Since $\rightarrow|_{V|_C \times V|_C} \subseteq \stackrel{\neq \varepsilon}{\rightarrow}_G$, we have $\rightarrow|_{V|_C \times V|_C} \subseteq \leftrightarrow^*_{M_C(G)}$ (by Proposition 14 (2)), so that $G \sqsubseteq \overline{M_C}(G)$.
- (2) By Lemma 17 (1), $\operatorname{sub}(\overline{M_C}(G)^{\neq \varepsilon}) \sqsubseteq G$. Combining this with $G \sqsubseteq \overline{M_C}(G)$, we obtain $\operatorname{sub}(\overline{M_C}(G)^{\neq \varepsilon}) \sqsubseteq \overline{M_C}(G)$. Thus, $\overline{M_C}(G)$ is subterm-closed.
- (3) If we show $G' = \langle V|_C, \rightarrow|_{V|_C \times V|_C} \rangle \subseteq M_C(G)$, the confluence of $\overline{M_C}(G) =$ $G|_D > M_C(G)$ follows from Lemma 5, since $G = G|_D > G'$ and $M_C(G)$ is confluent by Proposition 14 (1). Since G is subterm-closed, we have $V|_C \subseteq$ $V_{M_C(G)}$ and $\rightarrow|_{V|_C \times V|_C} \subseteq \Leftrightarrow_{M_C(G)}^*$ by Proposition 14 (2). Hence, $G' \subseteq$ $M_C(G)$. The termination of $\overline{M_C}(G)$ follows from Proposition 4, since $G|_D$ and $M_C(G)$ are terminating. Π

Corollary 19. For a subterm-closed R-reduction graph G and k > 0, we have:

(1) $G \sqsubseteq \overline{M_C}^k(G)$. (2) $\overline{M_C}^k(G)$ is subterm-closed. (3) $\overline{M_C}^k(G)$ is convergent, if G is convergent.

Remark 20. When an R-reduction graph G is subterm-closed, we observe that $\leftrightarrow^*_{\overline{M_C}^k(G)} = \leftrightarrow^*_{G \cup M_C(G) \cup \dots \cup M_C^k(G)} \text{ from } \rightarrow_{G|_C} \subseteq \leftrightarrow^*_{M_C(G)} \text{ by Proposition 14 (2).}$

Proposition 21. Let G be a subterm-closed R-reduction graph. Then, $\overline{M_C}^k(G) \sqsubseteq \overline{M_C}^m(G) \text{ for } m > k \ge 0.$

Proof. By
$$\overline{M_C}^m(G) = \overline{M_C}^{m-k}(\overline{M_C}^k(G))$$
 and Corollary 19 (1) and (2).

5 Tower of Constructor Expansions

From now on, let G be a convergent and subterm-closed R-reduction graph. We call $M_F(\overline{M_C}^i(G))$ a tower of constructor expansions of G for $i \ge 0$. We use $G_{2_i} = \langle V_{2_i}, \rightarrow_{2_i} \rangle$ to denote $M_F(\overline{M_C}^i(G))$.

5.1 Enriching Reduction Graph

We show that there exists a convergent *R*-reduction graph G_1 with $M_F(G) \sqsubseteq G_1$ such that G_{2i} is a conservative extension of G_1 for large enough *i*.

Lemma 22. For a convergent and subterm-closed R-reduction graph G, there exist $k (\geq 0)$ and an R-reduction graph G_1 satisfying the following conditions.

- i) G_1 is convergent, and consists of inner-edges.
- *ii)* $G_1 \sqsubseteq G_{2_k}$.
- *iii)* $u \leftrightarrow_{2_i}^* v$ *implies* $u \leftrightarrow_1^* v$ *for each* $u, v \in V_1$ *and* $i \ (\geq 0)$ *.*
- *iv*) $M_F(G) \sqsubseteq G_1$.

Proof. Let $G_1 := M_F(G)$ and k := 0. We define a condition iii)' as "iii) holds for all $i \ (< k)$ ". Initially, i) holds by Proposition 14 (1) since G is convergent. ii) and iv) hold from $G_1 = M_F(G) = G_{2_0}$, and iii)' holds from k = 0.

We transform G_1 so that i), ii), iii)' and iv) are preserved and the number $|V_1/\leftrightarrow_1^*|$ of connected components of G_1 decreases. This transformation $(G_1, k) \vdash (G'_1, k')$ continues until iii) eventually holds, since $|V_1/\leftrightarrow_1^*|$ is finite.

For current G_1 and k, we assume that i), ii), iii)' and iv) hold. If G_1 fails iii), there exist i with $i \ge k$ and $u, v \in V_1$ such that $u \ne v$ and $(u, v) \in \leftrightarrow_{2_i}^* \setminus \leftrightarrow_1^*$. We choose such k' as the least i. Remark that G_1 is convergent from i), and $G_{2_{k'}}$ is convergent from Corollary 19 (3) and Proposition 14 (1). Let \downarrow_1 and $\downarrow_{2_{k'}}$ be the choice mappings of G_1 and $G_{2_{k'}}$, respectively. Since $G_1 \sqsubseteq G_{2_{k'}}$ from ii) and Proposition 21, we have $(u \downarrow_1, v \downarrow_1) \in \leftrightarrow_{2_{k'}}^*$ and $u \downarrow_1 \ne v \downarrow_1$. From the convergence of $G_{2_{k'}}$, we have

$$\begin{cases} u \downarrow_1 = u_0 \to_{2_{k'}} u_1 \to_{2_{k'}} \dots \to_{2_{k'}} u_{n'} \to_{2_{k'}} \dots \to_{2_{k'}} u_n = (u \downarrow_1) \downarrow_{2_{k'}} \\ \vdots \\ v \downarrow_1 = v_0 \to_{2_{k'}} v_1 \to_{2_{k'}} \dots \to_{2_{k'}} v_{m'} \to_{2_{k'}} \dots \to_{2_{k'}} v_m = (v \downarrow_1) \downarrow_{2_{k'}} \end{cases}$$

where (n', m') is the smallest pair under the lexicographic ordering such that $u_{n'} = v_{m'}$. Note that u_j 's and v_j 's do not necessarily belong to V_1 . We define a transformation $(G_1, k) \vdash (G'_1, k')$ with G'_1 to be

$$\begin{cases} G_1 \multimap (u_0 \to \dots \to u_j) & \text{if there exists (the smallest) } j \text{ such that} \\ 0 < j \le n', u_j \in V_1, \text{ and } u_j \not\leftrightarrow_1^* u \\ G_1 \multimap (v_0 \to \dots \to v_{j'}) & \text{if there exists (the smallest) } j' \text{ such that} \\ 0 < j' \le m', v_{j'} \in V_1, \text{ and } v_{j'} \not\leftrightarrow_1^* v \\ G_1 \multimap (u_0 \to \dots \to u_{n'}) \multimap (v_0 \to \dots \to v_{m'}) & \text{otherwise.} \end{cases}$$

Since the condition (1) of Proposition 8 holds, i) is preserved. From $G_1 \sqsubseteq G'_1$ iv) holds, and ii) $G'_1 \sqsubseteq G_{2_{k'}}$ by Proposition 21. If k' = k, iii)' does not change. If k' > k, then $u \leftrightarrow_{2_i}^* v$ implies $u \leftrightarrow_1^* v$ for i with $k \le i < k'$, since we chose k' as the least. Hence iii)' holds. In either case, $|V_1/\leftrightarrow_1^*|$ decreases.

Example 23. For G in Example 13, Lemma 22 starts from $M_F(G)$, which is displayed by the solid edges in Fig. 2. G_1 is constructed by augmenting the dashed edges with k = 1.

$$\begin{array}{cccc} c & f(c,c) \rightarrow f(g(c),c) & f(g^{2}(c),c) \\ g(c) & \downarrow & \downarrow \\ g^{2}(c) & f(c,g^{2}(c)) \rightarrow f(g(c),g^{2}(c)) & \dashrightarrow f(g^{2}(c),g^{2}(c)) \\ \downarrow & \downarrow \\ g^{3}(c) & g^{3}(c) \end{array}$$

Fig. 2. G_1 constructed by Lemma 22 from G in Example 13

Corollary 24. Assume that $G_1 = \langle V_1, \rightarrow_1 \rangle$ and $h \geq 0$ satisfy the conditions i) to iv) in Lemma 22. Let v_0, v_1, \ldots, v_n satisfy $v_j \neq v_{j'}$ for $j \neq j'$ and $v_{j-1} (\Leftrightarrow_{2_k}^* \cap \stackrel{\varepsilon \leq}{R}) v_j \text{ for } 1 \leq j \leq n. \text{ If either } (1) v_0 \in V_1 \text{ and } v_0 \text{ is } \rightarrow_1\text{-normal,}$ or (2) $v_0, \dots, v_{n-1} \notin V_1$ and $v_n \in V_1$, then the conditions i) to iv) hold for $G_{1'} = G_1 \multimap (v_0 \to v_1 \to \dots \to v_n)$ and $k' = \max(k, h)$.

Proof. For (1), from iii) of $G_1, v_i \in V_1$ implies $v_i \leftrightarrow_1^* v_0$. For either case, from i) and iv) of G_1 and Proposition 8, $G_{1'}$ satisfies i) and iv). Since $v_{j-1} \leftrightarrow_{2_k}^* v_j$, $G_{1'}$ immediately satisfies ii). Since $v_0 \in V_1$ or $v_n \in V_1$, $G_{1'}$ satisfies iii).

5.2Properties of Tower of Expansions on Weakly Shallow Systems

Lemma 25. Let R be a non-E-overlapping and weakly shallow TRS. Let G = $\langle V, \rightarrow \rangle$ be a convergent and subterm-closed R-reduction graph, and let $\ell \rightarrow r \in R$.

- (1) If $\ell \sigma \leftrightarrow_{2_i}^* \ell \theta$, then $x \sigma \leftrightarrow_{\overline{M_C}^i(G)}^* x \theta$ for each variable $x \in \operatorname{Var}(\ell)$. (2) For a weakly shallow term s with $s \notin X$, assume that $x \sigma \leftrightarrow_{\overline{M_C}^i(G)}^* x \theta$ for
- each variable $x \in \operatorname{Var}(s)$. If $s\sigma \in V_{2_i}$, then $s\sigma \leftrightarrow_{2_k}^* s\theta$ for some $k \ (\geq i)$. (3) If $\ell\sigma \leftrightarrow_{2_i}^* u$, then there exist a substitution θ and $k \ (\geq i)$ such that $u \ (\stackrel{\varepsilon \leq}{R} \cap \leftrightarrow_{2_k}^*)^* \ \ell\theta$ and $x\sigma \rightarrow_{\overline{M_C}^i(G)}^* x\theta$ for each variable $x \in \operatorname{Var}(\ell)$.

Proof. Note that G_{2_i} is convergent by Corollary 19 (3) and Proposition 14 (1).

(1) Let $\ell = f(\ell_1, \dots, \ell_n)$. For each j $(1 \le j \le n), \ell_j \sigma \leftrightarrow^*_{\overline{M_C}^i(G)} \ell_j \theta$. Since $\overline{M_C}^i(G)$ is convergent by Corollary 19 (3), there exists v_j such that $\ell_j \sigma \to^*_{\overline{M_C}^i(G)}$ $v_j \leftarrow_{\overline{M_C}^i(G)}^* \ell_j \theta$. Since $\overline{M_C}^i(G)$ is subterm-closed by Corollary 19 (2) and ℓ_j is semi-constructor, we have $x\sigma \leftrightarrow_{\overline{M_C}^i(G)}^* x\theta$ for every $x \in \operatorname{Var}(\ell)$ by Lemma 11.

(2) First, we show that for a semi-constructor term t if $t\sigma \in V_{\overline{M_C}{}^i(G)}$, there exists $k \ (\geq i)$ such that $t\sigma \leftrightarrow_{\overline{M_C}{}^k(G)}^* t\theta$ by induction on the structure of t. If t is either a variable or a ground term, immediate. Otherwise, let $t = f(t_1, \ldots, t_n)$ for $f \in C$. Since $\overline{M_C}{}^i(G)$ is subterm-closed, $t_j\sigma \in V_{\overline{M_C}{}^i(G)}$ for each j. Hence, induction hypothesis ensures $t_j\sigma \leftrightarrow_{\overline{M_C}{}^{k_j}(G)}^* t_j\theta$ for some $k_j \geq i$. Since $M_C(\overline{M_C}{}^i(G)) \subseteq \overline{M_C}{}^{i+1}(G)$ and Proposition 21, we have $t\sigma \leftrightarrow_{\overline{M_C}{}^k(G)}^* t\theta$ for $k = 1 + \max\{k_1, \ldots, k_n\}$. We show the statement (2). Since $s \notin X$, s is represented as $f(s_1, \ldots, s_n)$

where each s_i is a semi-constructor term in $V_{\overline{M_C}^i(G)}$. Since there exists k $(\geq i)$ such that $s_j\sigma \leftrightarrow^*_{\overline{M_C}^k(G)} s_j\theta$, we have $s\sigma \leftrightarrow^*_{M_F(\overline{M_C}^k(G))} s\theta$. (3) Since G_{2_i} is convergent, there exists v with $\ell\sigma \rightarrow^*_{2_i} v \leftarrow^*_{2_i} u$. Here, $u \rightarrow^*_{2_i} v$

(3) Since G_{2_i} is convergent, there exists v with $\ell\sigma \to_{2_i}^* v \leftarrow_{2_i}^* u$. Here, $u \to_{2_i}^* v$ and $\ell\sigma \to_{2_i}^* v$ imply $u (\to_{2_i} \cap \stackrel{\varepsilon \leq}{\stackrel{}{\xrightarrow{}}})^* v$ and $\ell\sigma (\to_{2_i} \cap \stackrel{\varepsilon \leq}{\stackrel{}{\xrightarrow{}}})^* v$, respectively. Since R is non-E-overlapping, $\ell\sigma \to_{2_i}^* v$ has no reductions at $\operatorname{Pos}_F(\ell)$. By a similar argument to that of (1), we have $\ell|_p\sigma \leftrightarrow_{\overline{M_C}^i(G)}^* v|_p$ for each $p \in \operatorname{Pos}_X(\ell)$.

Let $x \in \operatorname{Var}(\ell)$. Since $\overline{M_C}^{\iota}(G)$ is convergent from Corollary 19 (3), we have $x\sigma = \ell\sigma|_p \to_{\overline{M_C}^{\iota}(G)}^* x\theta \leftarrow_{\overline{M_C}^{\iota}(G)}^* v|_p$ for each $p \in \operatorname{Pos}_{\{x\}}(\ell)$ by taking θ as $x\theta = x\sigma\downarrow_{\overline{M_C}^{\iota}(G)}$. Since ℓ is weakly shallow, by repeating (2) to each step in $v|_p \to_{\overline{M_C}^{\iota}(G)}^* x\theta$, there exists k with $v\leftrightarrow_{2_k}^*\ell\theta$. We have $u \ (\stackrel{\varepsilon\leq}{R} \cap \leftrightarrow_{2_k}^*)^*$ $v \ (\stackrel{\varepsilon\leq}{R} \cap \leftrightarrow_{2_k}^*)^* \ell\theta$ by Proposition 21.

6 Bottom-Up Construction of Convergent Reduction Graph

From now on, we assume that a TRS R is non-E-overlapping, non-collapsing, and weakly shallow. We show that R is confluent by giving a transformation of any R-reduction graph G_0 (possibly) containing a divergence into a convergent and subterm-closed R-reduction graph G_4 with $G_0 \sqsubseteq G_4$. The non-collapsing condition is used only in Lemma 27. Note that non-overlapping is not enough to ensure confluence as R_1 in Example 2. Now, we see an overview by an example.

Example 26. Consider R_2 in Example 13. Given $G_0 = \{f(g(c), c) \leftarrow f(c, c) \rightarrow f(c, g(c)) \xrightarrow{\varepsilon} g^3(c)\}$, we firstly take the subterm graph $\operatorname{sub}(G_0)$ and apply the transformation on it recursively to obtain a convergent and subterm-closed reduction graph G. In the example case, $\operatorname{sub}(G_0)$ happens to be equal to G in Example 13, and already satisfies the conditions. Secondly, we apply Lemma 22 on $M_F(G)$ and obtain G_1 in Example 2. As the next steps, we will merge the top edges T_1 in $G_0 \cup G$ into G_1 , where $T_1 = \{f(c, g(c)) \xrightarrow{\varepsilon} g^3(c), c \xrightarrow{\varepsilon} g(c)\}$. Note that top edges in G is necessary for subterm-closedness. The union $G_1 \cup T_1$ is not, however, confluent in general. Thirdly, we remove unnecessary edges from T_1 by Lemma 27, and obtain T (in the example $T = T_1$). Finally, by

Lemma 28, we transform edges in T into S with modifying G_1 into $G_{1'}$ so that $G_4 = G_{1'|D} \cup S \cup M_C(\overline{M_C}^{k'}(G))$ is confluent $(k' \ge k)$. The resultant reduction graph G_4 is shown in Fig. 3, where the dashed edges are in S and some garbage vertices are not presented. (See Example 30 for details of the final step.)

Fig. 3. G_4 constructed by Lemma 29 from G_0 in Example 26

6.1 Removing Redundant Edges and Merging Components

For *R*-reduction graphs $G_1 = \langle V_1, \rightarrow_1 \rangle$ and $T_1 = \langle V_1, \rightarrow_{T_1} \rangle$, the component graph (denoted by T_1/G_1) of T_1 with G_1 is the graph $\langle \mathcal{V}, \rightarrow_{\mathcal{V}} \rangle$ having connected components of G_1 as vertices and \rightarrow_{T_1} as edges such that

 $\mathcal{V} = \{ [v]_{\leftrightarrow_1^*} \mid v \in V_1 \}, \quad \to_{\mathcal{V}} = \{ ([u]_{\leftrightarrow_1^*}, [v]_{\leftrightarrow_1^*}) \mid (u, v) \in \to_{T_1} \}.$

Lemma 27. Let $G_1 = \langle V_1, \rightarrow_1 \rangle$ be an *R*-reduction graph obtained from Lemma 22, and let $T_1 = \langle V_1, \rightarrow_{T_1} \rangle$ be an *R*-reduction graph with $\rightarrow_{T_1} = \stackrel{\varepsilon}{\rightarrow}_{T_1}$. Then, there exists a subgraph $T = \langle V_1, \rightarrow_T \rangle$ of T_1 with $\rightarrow_T \subseteq \rightarrow_{T_1}$ that satisfies the following conditions.

- (1) $(\leftrightarrow_1 \cup \leftrightarrow_{T_1})^* = (\leftrightarrow_1 \cup \leftrightarrow_T)^*.$
- (2) The component graph T/\overline{G}_1 is acyclic in which each vertex has at most one outgoing-edge.

Proof. We transform the component graph T_1/G_1 by removing edges in cycles and duplicated edges so that preserving its connectivity. This results in an acyclic directed subgraph $T = \langle V_1, \rightarrow_T \rangle$ without multiple edges.

Suppose some vertex in T/G_1 has more than one outgoing-edges, say $\ell \sigma \to_T r\sigma$ and $\ell' \theta \to_T r' \theta$, where $\ell \sigma \leftrightarrow_1^* \ell' \theta$, $r\sigma, r\theta \in V_1$ and $\ell \to r, \ell' \to r' \in R$. Since R is non-E-overlapping, we have $\ell = \ell'$ and r = r'. By the condition ii) of Lemma 22, $\ell \sigma \leftrightarrow_{2_k}^* \ell \theta$ holds. Since R is non-collapsing, Lemma 25 (1) and (2) ensure $r\sigma \leftrightarrow_{2_j}^* r\theta$ for some $j (\geq k)$. By the condition iii) of Lemma 22, $r\sigma \leftrightarrow_1^* r\theta$. These edges duplicate, contradicting to the assumption.

In Lemma 27, if \rightarrow_T is not empty, there exists a vertex of T/G_1 that has outgoing-edges, but no incoming-edges. We call such an outgoing-edge a *source edge*. Lemma 28 converts T to S in a source to sink order (by repeatedly choosing source edges) such that, for each edge in S, the initial vertex is \rightarrow_1 -normal.

Lemma 28. Let G_1 , S, and T be R-reduction graphs, where G_1 and k satisfy the conditions i) to iv) of Lemma 22. Assume that the following conditions hold.

- $v) \ V_S = V_T = V_{G_1}, \ \rightarrow_S = \stackrel{\varepsilon}{\rightarrow}_S, \ \rightarrow_T = \stackrel{\varepsilon}{\rightarrow}_T, \ and \ \rightarrow_S \cap \rightarrow_T = \emptyset.$
- vi) The component graph $(S \cup T)/G_1$ is acyclic, where outgoing-edges are at most one for each vertex. Moreover, if $[u]_{\leftrightarrow_1^*}$ has an incoming-edge in T/G_1 then it has no outgoing-edges in S/G_1 .
- vii) u is \rightarrow_1 -normal and $u \not\leftrightarrow_1^* v$ for each $(u, v) \in \rightarrow_S$.

When $\rightarrow_T \neq \emptyset$, there exists a conversion $(S, T, G_1, k) \vdash (S', T', G_{1'}, k')$ that preserves the conditions i) to iv) of Lemma 22, and conditions v) to vii), and satisfies the following conditions (1) to (3).

- (1) $G_{1'}$ is a conservative extension of G_1 .
- $(2) \ (\leftrightarrow_T \cup \leftrightarrow_S)^* \subseteq (\leftrightarrow_{T'} \cup \leftrightarrow_{S'} \cup \leftrightarrow_{1'})^*.$
- $(3) |\to_T| > |\to_{T'}|$

Proof. We design \vdash as sequential applications of $\vdash_{\ell}, \vdash_{r}$, and \vdash_{e} in this order. We choose a source edge $(\ell\sigma, r\sigma)$ (of T/G_1) from T. We will construct a substitution θ such that $(\ell\sigma)\downarrow_1 (\stackrel{\varepsilon\leq}{\xrightarrow{}} \cap \leftrightarrow_{2_{k'}}^*)^* \ell\theta$ and $(r\sigma)\downarrow_1 (\stackrel{\varepsilon\leq}{\xrightarrow{}} \cap \leftrightarrow_{2_{k'}}^*)^* \cdot (\stackrel{\varepsilon\leq}{\xrightarrow{}} \cap \leftrightarrow_{2_{k'}}^*)^* r\theta$ for enough large k'. The former sequence is added to G_1 by \vdash_{ℓ} , the latter is added to G_1 by \vdash_{r} , and \vdash_{e} removes $(\ell\sigma, r\sigma)$ from T and adds $(l\theta, r\theta)$ to S.

We have $\ell \sigma \to_1^* (\ell \sigma) \downarrow_1$ by i), and $\ell \sigma \leftrightarrow_{2_k}^* (\ell \sigma) \downarrow_1$ by ii). From Lemma 25 (3), there are $k^{\ell} \geq k$ and a substitution θ such that $x\sigma \to_{\overline{M_C}^k(G)}^* x\theta$ for each $x \in$ $\operatorname{Var}(\ell), (\ell \sigma) \downarrow_1 = u_0 \stackrel{\varepsilon \leq}{\underset{R}{\to}} u_1 \stackrel{\varepsilon \leq}{\underset{R}{\to}} \cdots \stackrel{\varepsilon \leq}{\underset{R}{\to}} u_n = \ell \theta$, and $u_{j-1} \leftrightarrow_{2_{k^{\ell}}}^* u_j$ for each $j \leq n$).

- (⊢_ℓ) We define $(S, T, G_1, k) \vdash_{\ell} (S, T, G_{1^{\ell}}, k^{\ell})$ by $G_{1^{\ell}} = G_1 \multimap (u_0 \to \cdots \to u_n)$ to satisfy $(\ell \sigma) \downarrow_1 \leftrightarrow_{1^{\ell}}^* \ell \theta$ such that $\ell \theta$ is $G_{1^{\ell}}$ -normal. Since u_0 is \to_1 -normal, the case (1) of Corollary 24 holds, so that \vdash_{ℓ} preserves i) to iv) for $G_{1^{\ell}}$ and k^{ℓ} . (1) and (2) are immediate. From (1), vi) is preserved. Since $[\ell \sigma]_{\leftrightarrow_1^*}$ does not have outgoing edges in S by vi), vii) is preserved.
- (\vdash_r) We define $(S, T, G_{1^\ell}, k^\ell) \vdash_r (S, T, G_{1'}, k')$. Let $G_{1^\ell} = \langle V_{1^\ell}, \rightarrow_{1^\ell} \rangle$. Since $x\sigma \leftrightarrow^*_{\overline{M_C}^{k^\ell}(G)} x\theta$ by Proposition 21 and $r\sigma \in V_{2_{k^\ell}}$, we obtain $r\sigma \leftrightarrow^*_{2_{k'}} r\theta$ for some $k' \geq k^\ell$ by Lemma 25 (2). We construct $G_{1'}$ to satisfy $(r\sigma)\downarrow_{1^\ell} \leftrightarrow^*_{1'} r\theta$. Since the confluence of $G_{2_{k'}}$ follows from Corollary 19 (3) and Proposition 14 (1), we have the following sequences.

$$\begin{cases} (r\sigma)\downarrow_{1^{\ell}} = u_0 \rightarrow_{2_{k'}} u_1 \rightarrow_{2_{k'}} \cdots \rightarrow_{2_{k'}} u_n = v, \\ r\theta = v_0 \rightarrow_{2_{k'}} v_1 \rightarrow_{2_{k'}} \cdots \rightarrow_{2_{k'}} v_m = v, \end{cases}$$

where we choose the least n satisfying $u_n = v_m$. There are two cases according to the second sequence.

- (a) If $v_i \in V_{1^{\ell}}$ for some *i*, we choose *i* as the least. If i = 0, then $G_{1'} = G_{1^{\ell}}$. Otherwise, let $G_{1'} := G_{1^{\ell}} \multimap (v_0 \to v_1 \to \cdots \to v_i)$. Since $G_{1^{\ell}}$ satisfies the case (2) of Corollary 24, \vdash_r preserves i) to iv). Since $u_0 \leftrightarrow^*_{2_{k'}} v_i$ and $u_0, v_i \in V_{1^{\ell}}, u_0 \leftrightarrow^*_{1^{\ell}} v_i$ by iii). Thus, $(r\sigma) \downarrow_{1^{\ell}} \leftrightarrow^*_{1'} r\theta$.
- (b) Otherwise (i.e., $v_i \notin V_{1^{\ell}}$ for each i), let
 - $\begin{cases} G_{1''} := G_{1^{\ell}} \multimap (u_0 \to u_1 \to \dots \to u_n) \\ G_{1'} := G_{1''} \multimap (v_0 \to v_1 \to \dots \to v_m). \end{cases}$

Since u_0 is $G_{1^{\ell}}$ -normal and $u_j \in V_{1^{\ell}}$ implies $u_0 \leftrightarrow_{1^{\ell}}^* u_j$ (by iii) of $G_{1^{\ell}}$, $G_{1''}$ and k' satisfy i) to iv) by Corollary 24. Let $G_{1''} = \langle V_{1''}, \rightarrow_{1''} \rangle$. Since $v_i \notin V_{1''}$ for each $i \ (< m)$ and $v_m = u_n = v \in V_{1''}, G_{1'}$ and k' also satisfy i) to iv) by Corollary 24. By construction, $(r\sigma)\downarrow_{1^{\ell}} \leftrightarrow_{1'}^* r\theta$ holds. Since S and T do not change, \vdash_r keeps v), (1), and (2). Lastly, vi) and vii) follows from (1).

 $(\vdash_{e}) \text{ We define } (S, T, G_{1'}, k') \vdash_{e} (S', T', G_{1'}, k'), \text{ where } V_{S'} = V_{G_{1'}}, V_{T'} = V_{G_{1'}}, \\ \rightarrow_{S'} = \rightarrow_{S} \cup \{(\ell\theta, r\theta)\}, \text{ and } \rightarrow_{T'} = \rightarrow_{T} \setminus \{(\ell\sigma, r\sigma)\}. \text{ Since } (\ell\sigma, r\sigma) \text{ is a source edge of } T/G_{1}, \vdash_{e} \text{ preserves vi}). \text{ Conditions i) to v}, (1) \text{ and } (3) \text{ are trivial. Since } \ell\sigma \leftrightarrow^{*}_{G_{1'}} (\ell\sigma) \downarrow_{1} \leftrightarrow^{*}_{G_{1'}} \ell\theta \rightarrow_{S'} r\theta \leftrightarrow^{*}_{G_{1'}} (r\sigma) \downarrow_{1^{\ell}} \leftrightarrow^{*}_{G_{1'}} r\sigma \text{ implies } (\ell\sigma, r\sigma) \in \leftrightarrow^{*}_{S' \cup G_{1'}}, \text{ we have } (2). \text{ vii) holds from vi}.$

6.2 Construction of a Convergent and Subterm-Closed Graph

Lemma 29. Let $G_0 = \langle V_0, \rightarrow_0 \rangle$ be an *R*-reduction graph. Then, there exists a convergent and subterm-closed *R*-reduction graph G_4 with $G_0 \sqsubseteq G_4$.

Proof. By induction on the sum of the size of terms in V_0 , i.e., $\Sigma_{v \in V_0} |v|$. If G_0 has no vertex, we set $G_4 = G_0$, which is the base case. Otherwise, by induction hypothesis, we obtain a convergent and subterm-closed *R*-reduction graph *G* with sub $(G_0) \sqsubseteq G$. We refer to the conditions i) to vii) in Lemma 28.

Let $G_1 = \langle V_1, \rightarrow_1 \rangle$ and k be as in Lemma 22. Let T be obtained from G_1 and $T_1 = \langle V_1, \rightarrow_{G^{\varepsilon}} \cup \rightarrow_{G^{\varepsilon}_0} \rangle$ by applying Lemma 27.

Let $S = \langle V_1, \emptyset \rangle$. For G_1 and k, i) to iv) hold by Lemma 22. vi) holds by Lemma 27 (2) and $\rightarrow_S = \emptyset$, and vii) trivially holds. Starting from (S, T, G_1, k) , we repeatedly apply \vdash (in Lemma 28), which moves edges in T to S until $\rightarrow_T = \emptyset$. Finally, we obtain $(S', \langle V_{1'}, \emptyset \rangle, G_{1'}, k')$ that satisfies i) to vii) and (1) to (3) in Lemma 28, where $G_{1'} = \langle V_{1'}, \rightarrow_{1'} \rangle$ and $V_{S'} = V_{1'}$. From Lemmas 27 and 28 (1) and (2), $(\leftrightarrow_1 \cup \leftrightarrow_{G^{\varepsilon}} \cup \leftrightarrow_{G_0^{\varepsilon}})^* = (\leftrightarrow_1 \cup \leftrightarrow_T)^* \subseteq (\leftrightarrow_{1'} \cup \leftrightarrow_{S'})^*$. Note that $G_{1'}$ is convergent by i).

Let $G_3 = \langle V_3, \rightarrow_3 \rangle$ be $S' \cup G_{1'}$. This is obtained by repeatedly extending $G_{1'}$ by $G_{1'} \multimap (u \rightarrow v)$ for each $(u, v) \in \rightarrow_{S'}$, since in each step vii) is preserved; u is $\rightarrow_{1'}$ -normal and $u \not\leftrightarrow_{1'}^* v$. Thus, the convergence of G_3 follows from Proposition 7.

We show $G_0 \sqsubseteq G_3$. Since $G_0^{\varepsilon} \subseteq T_1 \sqsubseteq G_1 \cup T \sqsubseteq G_{1'} \cup S'$ (by Lemmas 27 and 28) and $M_F(\operatorname{sub}(G_0)) \sqsubseteq M_F(G) \sqsubseteq G_1 \sqsubseteq G_{1'}$ (by $\operatorname{sub}(G_0) \sqsubseteq G$ and iv)), $G_0 \subseteq G_0^{\varepsilon} \cup M_F(\operatorname{sub}(G_0)) \sqsubseteq S' \cup G_{1'} = G_3$.

Let $G_4 = \langle V_4, \rightarrow_4 \rangle$ be given by $G_4 := G_3|_D > M_C(\overline{M_C}^{k'}(G))$. We show $G_0 \sqsubseteq G_4$ by showing $G_3 \sqsubseteq G_4$. Since $G_{1'} \sqsubseteq G_{2_{k'}}$ by ii) where $G_{2_{k'}}$ contains no top edges, we have $V_{1'}|_C \subseteq V_{2_{k'}}|_C$ and $\rightarrow_{1'}|_C \subseteq (\leftrightarrow_{2_{k'}}|_C)^*$. Since $\rightarrow_{2_{k'}}|_C =$ $\rightarrow_{M_C(\overline{M_C}^{k'}(G))}$, we have $G_{1'}|_C \sqsubseteq \langle V_{1'}, \emptyset \rangle \cup M_C(\overline{M_C}^{k'}(G))$. Thus, $G_{1'} = G_{1'}|_D \cup G_{1'}|_C \sqcup G_{1'}|_D \cup M_C(\overline{M_C}^{k'}(G))$. By $S' = S'|_D$, we have $G_3 = S' \cup G_{1'} \sqsubseteq S'|_D \cup G_{1'}|_D \cup M_C(\overline{M_C}^{k'}(G)) = G_4$.

Now, our goal is to show that G_4 is convergent and subterm-closed. The convergence of $G_4 = G_3|_D > M_C(\overline{M_C}^{k'}(G))$ is reduced to that of $G_3 = G_3|_D >$

 $\langle V_3|_C, \rightarrow_3|_C \rangle$ by Proposition 4 and Lemma 5. Their requirements are satisfied from $\langle V_3|_C, \rightarrow_3|_C \rangle = \langle V_{1'}|_C, \rightarrow_{1'}|_C \rangle \sqsubseteq M_C(\overline{M_C}^{k'}(G))$ by ii) and the convergence of $M_C(\overline{M_C}^{k'}(G))$ by Corollary 19 (3) and Proposition 14 (1). We will prove that G_4 is subterm-closed by showing $\operatorname{sub}(G_4^{\neq\varepsilon}) \sqsubseteq \overline{M_C}^{k'}(G)$

We will prove that G_4 is subterm-closed by showing $\operatorname{sub}(G_4^{\neq\varepsilon}) \sqsubseteq \overline{M_C}^{\kappa}(G)$ and $\overline{M_C}^{k'}(G) \sqsubseteq G_4$. Note that $\operatorname{sub}(G_4^{\neq\varepsilon}) = \operatorname{sub}((S'|_D)^{\neq\varepsilon} \cup (G_{1'}|_D)^{\neq\varepsilon} \cup (M_C(\overline{M_C}^{k'}(G)))^{\neq\varepsilon}) \subseteq \operatorname{sub}(S'^{\neq\varepsilon}) \cup \operatorname{sub}(G_{1'}|_D) \cup \overline{M_C}^{k'}(G)$. We have $\operatorname{sub}(S'^{\neq\varepsilon}) = \langle \operatorname{sub}(V_{1'}), \emptyset \rangle$. Since $G_{2_{k'}}$ has no top edges and $G_{1'} \sqsubseteq G_{2_{k'}}$ by ii), $\operatorname{sub}(G_{1'}) \sqsubseteq \operatorname{sub}(G_{2_{k'}}) = \operatorname{sub}(M_F(\overline{M_C}^{k'}(G))) \subseteq \overline{M_C}^{k'}(G)$. Thus, $\operatorname{sub}(G_4^{\neq\varepsilon}) \sqsubseteq \overline{M_C}^{k'}(G)$.

It remains to show $\overline{M_C}^{k'}(G) \sqsubseteq G_4$, which is reduced to $G|_D \sqsubseteq G_4$ from $\overline{M_C}^{k'}(G) = G|_D \cup M_C(\overline{M_C}^{k'-1}(G)), M_C(\overline{M_C}^{k'}(G)) \subseteq G_4$, and Proposition 21. Since $G|_D \subseteq G \sqsubseteq G^{\varepsilon} \cup M_F(G)$ by Lemma 17 (2), it is sufficient to show that $G^{\varepsilon} \sqsubseteq G_4$ and $M_F(G) \sqsubseteq G_4$.

Obviously, $M_F(G) \sqsubseteq G_{1'} \subseteq G_3 \sqsubseteq G_4$ holds, since $M_F(G) \sqsubseteq G_{1'}$ by iv). We show $G^{\varepsilon} \sqsubseteq G_4$. Since $V_G \subseteq V_{M_F(G)}$ by Proposition 14 (2), we have $V_{G^{\varepsilon}} = V_G \subseteq V_{M_F(G)} \subseteq V_{1'} \subseteq V_3 \subseteq V_4$. By Lemmas 27 (1) and 28 (2), $\rightarrow_{G^{\varepsilon}} \subseteq (\leftrightarrow_{G_{1'}} \cup \leftrightarrow_{S'})^*$ holds, and by ii) we have $\rightarrow_{G_{1'}|_C} \subseteq \leftrightarrow^*_{M_C(\overline{M_C}^{k'}(G))}$. Hence, $\rightarrow_{G^{\varepsilon}} \subseteq (\leftrightarrow_{G_{1'}|_D} \cup \bigoplus_{S'} \cup \bigoplus_{M_C(\overline{M_C}^{k'}(G))})^* = \leftrightarrow^*_{G_4}$. Therefore G_4 is subterm-closed.

Example 30. Let us consider applying Lemma 29 on G_1 and T in Example 26, where k = 1. The edge $c \to g(c)$ in T is simply moved to S. For the edge $f(c,g(c)) \to g^3(c)$ in T, \vdash_{ℓ} adds $f(g^2(c),g^2(c)) \to f(g^2(c),g^3(c))$ to G_1 . \vdash_r adds $g^3(c) \to g^4(c) \to g^5(c)$ to G_1 and increases k to 3. \vdash_e adds $f(g^2(c),g^3(c)) \to g^5(c)$ to S. Since $M_C(\overline{M_C}^3(G))$ is $\{g(c) \to g^2(c) \to \cdots \to g^4(c) \to g^5(c), g^6(c)\}, G_4 = (S \cup G_1|_D) \gg M_C(\overline{M_C}^3(G))$ is as in Fig. 3.

Theorem 31. Non-E-overlapping, weakly shallow, and non-collapsing TRSs are confluent.

Proof. Let $u \leftarrow_R^* s \rightarrow_R^* t$. We obtain G_4 by applying Lemma 29 to an *R*-reduction graph G_0 consisting of the sequence. By $G_0 \sqsubseteq G_4$ and the convergence of G_4 , $u \downarrow_{G_4} = t \downarrow_{G_4}$. Thus we have $u \rightarrow_R^* s' \leftarrow_R^* t$ for some s'.

Corollary 32. Strongly non-overlapping, weakly shallow, and non-collapsing TRSs are confluent.

7 Conclusion

This paper extends the reduction graph technique [SO10] and has shown that non-E-overlapping, weakly shallow, and non-collapsing TRSs are confluent.

We think that the *non-collapsing* condition can be dropped by refining the reduction graph techniques. A further step will be to relax the *weakly shallow* to the *almost weakly shallow* condition, which allows at most one occurrence of a defined function symbol in each path from the root to a variable.

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