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ON PRINCIPLES BETWEEN Σ_1- AND Σ_2-INDUCTION, AND MONOTONE ENUMERATIONS

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ABSTRACT. We show that many principles of first-order arithmetic, previously only known to lie strictly between Σ_1-induction and Σ_2-induction, are equivalent to the well-foundedness of ω^ω. Among these principles are the iteration of partial functions ($P\Sigma_1$) of Hájek and Paris, the bounded monotone enumerations principle (non-iterated, BME_1) by Chong, Slaman, and Yang, the relativized Paris-Harrington principle for pairs, and the totality of the relativized Ackermann-Péter function. With this we show that the well-foundedness of ω^ω is a far more widespread than usually suspected.

Further, we investigate the k-iterated version of the bounded monotone iterations principle (BME_k), and show that it is equivalent to the well-foundedness of the $k+1$-height ω-tower $\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}$.

In this paper we will investigate principles between Σ_1-induction ($I\Sigma_1$) and Σ_2-induction ($I\Sigma_2$). The following principles will be considered.

1) Iteration of partial functions, as introduced by Hájek, Paris in [7].
2) The bounded monotone enumeration principle (non-iterated), as introduced by Chong, Slaman, Yang in their proof of the fact that Ramsey's theorem for pairs and two colors (RT^2_2) does not imply Σ_2-induction in [5, 4].
3) The relativized Paris-Harrington principle for pairs and arbitrarily many colors.
4) The totality of the Ackermann-Péter function relativized to a total function.
5) The well-foundedness of ω^ω ($\mathsf{WF}(\omega^\omega)$).

Of all of these principles it is well known that they lie strictly between $I\Sigma_1$ and $I\Sigma_2$. However, their relations were mostly unknown. To the knowledge of the authors it was only known that the well-foundedness of ω^ω implies the totality of the Ackermann-Péter function, and that this is equivalent to the (non-relativized) Paris-Harrington principle for pairs.

We will show that all of the above-enumerated principles are equivalent over $I\Sigma_1$. This is surprising since these principles usually have been investigated separately,

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and the connection was apparently not expected. For instance in [8] the Pairs-Harrington principle, iteration of partial functions, and $\mathsf{WF}(\omega^\omega)$ are considered but in separate sections. The system $\mathsf{WF}(\omega^\omega)$ has shown up in even more places before. In [13] Simpson showed that it is equivalent to Hilbert's basis theorem. Recently, Hatzikiriakou and Simpson that also a related result by Formanek and Lawrence on group algebras is equivalent to, see [9].

Given the many equivalent forms of $\mathsf{WF}(\omega^\omega)$ of which many are natural statements, we believe that $\mathsf{WF}(\omega^\omega)$ must be considered as a natural and robust system just like $B\Sigma_2$, which for instance occurs in the natural description as the infinite pigeonhole principle or as a certain partition principle, see [6].

In addition to this we also investigate k-iterated bounded monotone enumeration principle as used in [5], and characterize its strength. We will show that the k-iterated version BME_j is equivalent to the well-foundedness of $k+1$-high ω-tower $\omega^{\cdot^{\cdot^{\cdot^{\omega}}}} = \omega^\omega_k$. In particular, the Π^0_3-consequence of $\mathsf{BME} = \bigcup_{k\in\mathbb{N}} \mathsf{BME_k}$ are all Π^0_3-sentences of PA.

The paper is structured as follows. The first chapter will introduce the principles mentioned above. In the following chapter the equivalences between them are proven. The third chapter deals with the iterated bounded monotone enumeration. The last chapter consists of concluding remarks.

1. Introduction

We will work over $I\Sigma_1$, that is Peano Arithmetic where the induction axiom is restricted to Σ_1-formulas. We will make use of stronger forms of induction (i.e., $I\Sigma_n$ with $n \geq 2$) and the bounded collection principle (i.e., $B\Sigma_n$). If the reader is not familiar with these systems and principles, we refer him to [8].

1.1. **Iteration of functions.**

A formula $\phi(x,y)$ represents a total function if $\forall x\, \exists! y\, \phi(x,y)$, it represents a partial function if for all x there is at most one y satisfying $\phi(x,y)$. We shall denote these statements by $\mathsf{TFUN}(\phi)$, respectivly $\mathsf{PFUN}(\phi)$. We shall say that s is an approximation to the iteration of such a function, if s is a finite sequence such that

$$\forall i < \mathrm{lth}(s){-}1\ \forall x,y\ ((x \leq (s)_i \wedge \phi(x,y)) \rightarrow y \leq (s)_{i+1})\,.$$

We will denote this statement by $\mathsf{Approx}_\phi(s)$. The statement that all finite approximations of the iterations of a total resp. partial function is given by ϕ is then given by the following.

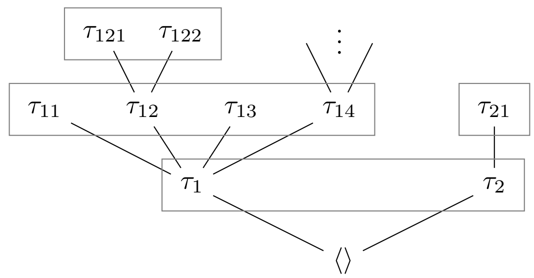
$$(T\phi): \qquad \mathsf{TFUN}(\phi) \rightarrow \forall z\, \exists s\, \mathsf{Approx}_\phi(s) \wedge \mathrm{lth}(s) = z$$
$$(P\phi): \qquad \mathsf{PFUN}(\phi) \rightarrow \forall z\, \exists s\, \mathsf{Approx}_\phi(s) \wedge \mathrm{lth}(s) = z$$

These definitions are made relative to $I\Sigma_1$. For a class of formulas $\mathcal{K}$, the sets $\{T\phi \mid \phi \in \mathcal{K}\} \cup I\Sigma_1$, $\{P\phi \mid \phi \in \mathcal{K}\} \cup I\Sigma_1$ will be denoted by $T\mathcal{K}$ resp. $P\mathcal{K}$.

The following theorem collects the known facts about T, P.

Theorem 1 ([7], [8, Chap. I.2.(b)])**.**

1) *$T\Sigma_{n+1} \leftrightarrow T\Pi_n$, $P\Sigma_{n+1} \leftrightarrow P\Pi_n$, $P\Sigma_0 \leftrightarrow P\Sigma_1$.*
2) *$T\Sigma_{n+1} \leftrightarrow I\Sigma_{n+1}$.*
3) *$I\Sigma_{n+2} \rightarrow P\Sigma_{n+1} \rightarrow I\Sigma_{n+1}$. Here all implications are strict.*
4) *$P\Sigma_{n+1}$ is incomparable with $B\Sigma_{n+2}$.*
5) *$P\Sigma_{n+1} + B\Sigma_{n+2}$ is strictly weaker than $I\Sigma_{n+2}$.*



Strings τ_i in a box are enumerated at the same stage. The stage-by-stage enumeration of say τ_{121} is $\langle\rangle, \tau_1, \tau_{12}, \tau_{121}$. Not visible in the diagram is that notes enumerate at the same stage, say τ_1, τ_2, might be of different length.

FIGURE 1. Tree enumerated by a monotone enumeration.

1.2. **Bounded monotone iterations.** Bounded monotone iterations will deal with enumerations of trees of natural numbers ($\mathbb{N}^{<\mathbb{N}}$).

Let E be a function given by a quantifier-free formula. We will regard $E[s]$ via a suitable coding as a finite subset of $\mathbb{N}^{<\mathbb{N}}$ and assume that $E[s] \subseteq E[s+1]$. We will refer to the parameter s of E as the stage of the enumeration and use E also to refer to the tree enumerated by E, i.e.,

$$\left\{ \tau \in \mathbb{N}^{<\mathbb{N}} \;\middle|\; \exists s \, \exists \sigma \in E[s] \; (\tau \prec \sigma) \right\}.$$

Definition 2 ([5])**.** E is a *monotone enumeration* if the following holds.

1) The empty sequence $\langle\rangle$ is enumerated at the first stage.
2) At each stage only finitely many sequences are enumerated by E. (This is by our coding automatically the case.)
3) If τ is enumerated by E at stage s and τ_0 is the longest initial segment enumerated by E at a prior stage. Then
   (a) no extension of τ_0 has been enumerated by E before the stage s and
   (b) all sequences enumerated at stage s are extensions of τ_0.

Let E be a monotone enumeration. For an element τ enumerated by E at stage s we call the maximal initial segments (τ_i) of τ enumerated at stages prior to s the stage-by-stage sequence of τ. We say that *a monotone enumeration E is bounded by b* if for each τ in E the length of its stage-by-stage sequence is bounded by b.

Definition 3. BME_* is the statement that a tree enumerated by a bounded monotone enumeration is finite.

The following is known about the first-order strength of BME_*.

Theorem 4 ([5, Propositions 3.5, 3.6])**.**

1) $I\Sigma_2 \vdash \mathsf{BME}_*$
2) $B\Sigma_2 \nvdash \mathsf{BME}_*$

Note that BME_* is equivalent to BME_1 as defined by Chong, Slaman an Yang. This follows for instance from Theorems 5 and 16.

1.3. **Paris-Harrington theorem.** The Paris-Harrington theorem (PH) is a strengthening of the finite Ramsey's theorem. It is one of the classical examples of a natural first-order theorem which is not provable from Peano Arithmetic. In this paper we will be only concerned with a (variant of a) fragment of PH.

As usual in this context, we will write $X \to (q)^u_z$ for the statement that each coloring of unordered u-tuples of X with z colors has a homogenous set of cardinality q. In this notation finite Ramsey's theorem is simply the statement

$$\forall q \geq 1\, \forall u \geq 1\, \forall z\, \exists y \left([0,y] \to (q)^u_z\right).$$

To state the Paris-Harrington variant of Ramsey's theorem we will need the following. A finite set X is called *relatively large* if $\min X < |X|$.

We will write $X \underset{*}{\to} (q)^u_z$ if each coloring of unordered u-tuples of X with z colors has a *relatively large* homogenous set of cardinality at least q. The Paris-Harrington theorem is then the following statement.

$$(\mathsf{PH})\colon \forall x\, \forall q \geq 1\, \forall u \geq 1\, \forall z\, \exists y \left([x,y] \underset{*}{\to} (q)^u_z\right).$$

(Note that we need to vary the starting point x of the interval since the property of being relatively large is not translation invariant.)

We will write $\mathsf{PH}(u,z)$ for the restriction of PH to u-tuples and z many colors. We will write $\mathsf{PH}(u)$ for $\forall z\, \mathsf{PH}(u,z)$.

We will also need the relativization $\mathsf{PH}^*(u,z)$ of $\mathsf{PH}(u,z)$ given by the following. Let $\phi(n)$ be a Σ_1-formula describing an infinite set. Then $\mathsf{PH}^*(u,z)$ states that $\mathsf{PH}(u,z)$ holds relativized to $[x,y] \cap \{n \mid \phi(n)\}$. In other words,

$$\mathsf{PH}^*(u,z) : \forall k\, \exists n > k\, \phi(n) \to \forall x\, \forall q \geq 1\, \exists y \left(\left([x,y] \cap \{n \mid \phi(n)\}\right) \underset{*}{\to} (q)^u_z\right).$$

$\mathsf{PH}^*(u)$ is defined as above.

We will be mainly concerned with $\mathsf{PH}(2)$, $\mathsf{PH}^*(2)$.

1.4. **Ackermann function.** The Ackermann-Péter function is given by the following defining equations.

$$A(m,n) := \begin{cases} n+1 & \text{if } m = 0, \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0, \\ A(m-1, A(m,n-1)) & \text{if } m,n > 0, \end{cases} \tag{1}$$

It is known that $I\Sigma_2$ or even the statement that ω^ω is well-order implies the totality of Ackermann-Péter function. Let f be a strictly monotonic function. The relativized Ackermann-Péter function A_f is defined as A but with the base case set to f, i.e.,

$$A_f(0,n) := f(n). \tag{2}$$

We will write A^* for the statement that for each function f (given by a quantifier-free formula) the Ackermann-Péter function relative to f is total.

1.5. **Ordinals.** We will use ordinals $< \epsilon_0$. For this we will fix a suitable ordinal notation. See e.g. [8, Section II.3] for details. We shall write $\mathsf{WF}(\alpha)$ for the statement that α if *well-ordered* or *well-founded*, that is there is no infinite descending sequence of ordinals α_i starting from α. In the context of fragments of first-order arithmetic the sequence α_i is understood to be primitive recursive in the theory, which is equivalent to saying that α_i is given as a Σ_1-function, as defined in Section 1.1.

Since our work is motivated by results in second-order arithmetic/reverse mathematics, we would note that in that context well-foundedness is defined differently, see [13]. There the descending sequence α_i is given by a second-order object X

coding the function $f\colon i \mapsto \alpha_i$. Since that Σ_1-function in the sense of Section 1.1 are exactly the functions from which a theory proves to be recursive, recursive comprehension gives that the Σ_1-functions and the second-order functions coincide. This immediately shows that the first- and second-order definitions of well-foundedness are equivalent.

The main result of this paper is the following.

Theorem 5. *Over $I\Sigma_1$ the following are equivalent:*

(i) $P\Sigma_1$,
(ii) BME_*,
(iii) $\mathsf{PH}^*(2)$,
(iv) A^*,
(v) $\mathsf{WF}(\omega^\omega)$.

The proof will proceed as follows.

- $(i) \Leftrightarrow (ii)$ (Proposition 6),
- $(i) \Rightarrow (iv)$ (Proposition 7),
- $(iv) \Rightarrow (iii)$ (Proposition 8),
- $(iii) \Rightarrow (v)$ (Proposition 12)
- $(v) \Rightarrow (ii)$ (Proposition 13)
- $(v) \Rightarrow (iv)$ is a classical result.

2. The proof of Theorem 5

Proposition 6. $I\Sigma_1 \vdash P\Sigma_1 \leftrightarrow \mathsf{BME}_*$

Proof. "$\rightarrow$": Let E be a monotone enumeration. Assume that E is bounded by b. Define the partial functions

$$F'(\tau) := [\text{first stage } s \text{ such that extensions of } \tau \text{ are enumerated in } E]$$

and

$$F(\tau) := E[F'(\tau)] \setminus E[F'(\tau) - 1].$$

The partial function $F(\tau)$ yields the set of all extensions of τ that are newly enumerated at the first stage where extensions of τ enter into E. Since E is a monotone enumeration these are all direct extensions of τ.

The graph of F' can be defined by the following Σ_0-formula

$$\phi'(\tau, s) :\equiv \exists \tau' \in E[s] \setminus E[s-1] \ \ (\tau \prec \tau') \wedge \forall \tau' \in E[s-1] \ \ (\tau \not\prec \tau')\,.$$

The partial function F can then be defined by the Σ_1-formula

$$\phi(\tau, x) :\equiv \exists s \ \left(\phi'(\tau, s) \wedge x = \left[E[s] \setminus E[s-1]\right]\right).$$

We make the assumption that for each code of a finite set x we have that $y \in x$ implies $y \le x$. (This is for instance the case for the usual coding based on Cantor pairing.)

Then we have for each stage-by-stage enumeration (τ_i) that $\tau_{i+1} \le F(\tau_i)$. Hence $\tau_{i+1} \le F^{i+1}(\tau_0) = F^{i+1}(\langle\rangle)$. As a consequence each element in any b-bounded stage-by-stage enumeration is bounded by $\max_{i \le b}\{F^i(\langle\rangle)\}$. Now by $P\Sigma_1$ we can bound this value and obtain that E is finite.

"$\leftarrow$": Let $\phi(x, y)$ be a quantifier-free formula and assume $\mathsf{PFUN}(\phi)$. (Quantifier-free is sufficient by Theorem 1.(1).) Let b be given. We will construct a b-bounded

monotone enumeration E which will give an approximation s of length b to the iteration of ϕ.
At stage 0 we will enumerate $\langle 0\rangle$ into the tree.
At stage $s+1$ we search for the smallest $\sigma = \langle x_0, \dots, x_k\rangle \in E[s]$ such that $|\sigma| < b$ and $\exists y < s+1\, \phi(x_k, y)$. If such a σ exists then enumerate $\sigma * \langle 0\rangle, \sigma * \langle 1\rangle, \dots, \sigma * \langle y\rangle$. Otherwise do nothing.

By BME_* this tree is finite. Let m_i be the maximum of the elements in the $\le i$ levels of E. We claim that $s = \langle m_0, m_1, \dots, m_b\rangle$ satisfies $\mathsf{Approx}_\phi(s)$. We prove this by induction on the length of s. For $\langle m_0\rangle = \langle 0\rangle$ this is clear. Assume that the statement is true for $\langle m_0, \dots, m_i\rangle$. First we consider the case that the maximum m_{i+1} is attained at a level $< i$, i.e., $m_i = m_{i+1}$ and by the construction of E we have that $\forall x \le m_i\, \forall y\, \phi(x, y) \to y \le m_i$. From this it follows immediately that also $\langle m_0, \dots, m_i, m_i\rangle$ satisfies Approx_ϕ. Now consider the case that m_{i+1} is attained at the $(i+1)$-th level and no prior level. By construction of E there must be the elements $[0; m_i]$ on the i-th level, and we have $\forall x < m_i\, \forall y\, \phi(x, y) \to y \le m_{i+1}$, which yields that $\langle m_0, \dots, m_i, m_{i+1}\rangle$ satisfies Approx_ϕ. ☐

Proposition 7. $I\Sigma_1 \vdash P\Sigma_1 \to \mathsf{A}^*$.

Proof. For notational ease we will only show that $A(m, n)$ is total. The relativization to $A_f(m, n)$ is straightforward.

Let $\phi_A(m, n, k)$ be the Σ_1-formula describing the graph of the (relativized) Ackermann-Péter function A as in (1) and $\psi_A(m, n) \equiv \exists k\, \phi_A(m, n, k)$ be the Σ_1-formula which states that $A(m, n)$ is defined. Clearly,

$$\forall n\ \psi_A(0, n). \tag{3}$$

We claim that $I\Sigma_1$ proves

$$\forall m, n\ (\neg\psi_A(m, n) \to \exists n'\, \neg\psi_A(m-1, n'))\,. \tag{4}$$

Indeed, suppose $\neg\psi_A(m, n)$ and in particular that $m > 0$. Then by $I\Sigma_1$ we can find a k which is minimal with $\neg\psi_A(m, k)$. If $k = 0$ then by definition of A we have $\neg\psi_A(m-1, 1)$. If $k > 0$ then by minimality $A(m, k-1)$ is defined, thus $A(m-1, A(m, k-1))$ cannot be defined and therefore $\neg\psi_A(m-1, A(m, k-1))$.

Σ_2-induction applied to (4) would now immediately give that $\neg\psi_A(m, n)$ implies $\exists n'\, \neg\psi_A(0, n')$. ($\Sigma_2$-induction is required since $\exists n'\, \neg\psi_A(m, n')$ is Σ_2.) Together with (3) this would yield the totality of A.

We will show how to use $P\Sigma_1$ to bound n' occurring in (4). With this, $I\Sigma_1$ suffices to carry out this induction.

Let $\langle m, n\rangle$ denote the Cantor pairing function and $(x)_0, (x)_1$ the unpairing functions. Recall that $m, n < \langle m, n\rangle$. To cover both parameters of $A(m, n)$ we will use the following modification

$$A'(x) := \big\langle A\big((x)_0, (x)_1\big), A\big((x)_0, (x)_1\big)\big\rangle$$

Let $\phi_{A'}(x, k)$ be the Σ_1-formula describing the graph of A'.

Suppose that $A(m, n)$ is not defined or in other words $\neg\psi_A(m, n)$. Let $c := \max(m, n)$.

Now by $P\Sigma_1$ arbitrary long approximations to A' exists. Since $A(0, n) = n+1$, and assuming that $\langle 0, 0\rangle = 0$, which is the case for Cantor pairing, we have for any approximation s of A'

$$(s)_j \ge \langle j+1, j+1\rangle, \qquad \text{for } j < \mathrm{lth}(s).$$

Therefore, if $A(m,n)$ with $m,n < c$ is defined then $A(m,n) \leq (s)_c$ for any approximation s to A' of length $> c$.

Now as in the argument above, assume that $A(m,n)$ is not defined. Then we know that there is a $k < m$ such that $A(m,k-1)$ is defined and $A(m-1, A(m,k-1))$ is not defined or $A(m-1,1)$ is not defined. In particular, for a long enough approximation s of A' we have

$$\exists n' < (s)_c \, \neg\phi_A(m-1, n').$$

Since m, n' are bound by $(s)_c$ one obtains by the same argument that

$$\exists n'' < (s)_{c+1} \, \neg\phi_A(m-2, n'').$$

Iterating this argument gives then

$$\exists n^* < (s)_{c+m-1} \, \neg\phi_A(0, n^*)$$

and with this the desired contradiction to (3). This argument can be carried out in $P\Sigma_1$ since this iteration is—after building the approximation s of sufficient ($= 2c$) length—provable in $I\Sigma_1$ which is a consequence of $P\Sigma_1$. □

It is known that the totality of the Ackermann function implies PH, see Theorem II.3.36 and Fact II.3.34 of [8]. We show here how to relativize this proof to obtain the following theorem.

Proposition 8. *$I\Sigma_1 \vdash \mathsf{A}^* \to \mathsf{PH}^*(2)$.*

Before we can prove this theorem we will need some notation and lemmata. In a canonical way we can define a fundamental sequence $\{\alpha\}(n)$ for each $\alpha < \epsilon_0$. That is a sequence such that $\{\alpha\}(n)$ converges monotonically from below to α if α is a limit and the predecessor otherwise. For instance $\{\omega\}(n) = n$. This sequence will be Δ_1. See [8, II.3.a)] for details.

We say that a finite set $X = \{x_0 < x_1 < x_2 < x_3 < \cdots < x_n\}$ is α-large if the sequence

$$\{\alpha\}(x_0), \big\{\{\alpha(x_0)\}\big\}(x_1), \big\{\big\{\{\alpha(x_0)\}\big\}(x_1)\big\}(x_2), \ldots$$

reaches 0. It is easy to see that ω-large is the same as relatively large (by using the fact $\{\omega\}(n) = n$ and $\{n\}(m) = n-1$).

Lemma 9 ([12, Section 6.2])**.** *Let $z \geq 2$, $\theta := \omega^{z+3} + \omega^3 + z + 4$. Further, let X be an θ-large set. Assume that the pairs of X are colored with z many colors. There exists a subset Y of X that is homogenous and relatively large.*

In other words, for X we have that the conclusion of $\mathsf{PH}^*(2,z)$ holds.

Lemma 10 ([8, Lemma II.3.21.(3)])**.** *Suppose $\alpha \gg \beta > 0$ (that means, looking at the Cantor-normals forms of $\alpha = \sum_{i=0}^{x} \omega^{\mu_i} a_i$, $\beta = \sum_{i=0}^{y} \omega^{\nu_i} b_i$ we have that $\mu_0 \geq \nu_y$). Then X is $(\alpha+\beta)$-large iff there are X_α, X_β such that $X = X_\beta \cup X_\alpha$, $\max(X_\beta) < \min(X_\alpha)$, and X_α is α-large and X_β is β-large.*

Lemma 11 (cf. [8, Lemma II.3.30.(3)])**.** *Let g be the strictly increasing enumeration of an infinite set X. Let f_α be the fast growing hierarchy relativized to g as follows.*

$$\begin{aligned} & f_0(n) := g(n) \\ & f_{\beta+1}(n) := f_\beta^n(g(n+1)), \quad \text{where } f^n \text{ is the } n\text{-fold iteration} \\ & f_\lambda(n) := f_{\{\lambda\}(n)}(g(n+1)). \end{aligned} \tag{5}$$

If $x \in X$, the set $[x, f_\alpha(x)] \cap X$ is ω^α-large.

Proof of Lemma 11. First observe that for all α, n we have $f_\alpha(n) \in X$. We will use the following claim.
Claim: Assume that the statement of the lemma holds for α and that $x \in X$. Then the set $[x, f^y_\alpha(x)] \cap X$ is $\omega^\alpha \cdot y$-large.
Proof of claim: The statement is shown by induction in y. Suppose $[x, f^y_\alpha(x)] \cap X$ is $\omega^\alpha \cdot y$ large. By the assumption we have that $[x, f_\alpha(x)] \cap X$ is ω^α-large, and by induction hypothesis that $[f_\alpha(x), f^y_\alpha(f_\alpha(x))] \cap X$ is $\omega^\alpha \cdot y$-large. Now Lemma 10 gives the claim.

We prove the lemma by quantifier-free transfinite induction. (We will use it only for $\alpha < \omega$ in the proof of Proposition 8.) Consider $\alpha + 1$ and $x = g(n) \in X$. Now $[x, z] \cap X$ is $\omega^{\alpha+1}$-large iff $[g(n+1), z] \cap X$ is $\omega^\alpha \cdot x$-large, i.e., if $z \geq f^x_\alpha(g(n+1))$. Since $f^x_\alpha(g(n+1)) \leq f^x_\alpha(g(x+1)) = f_{\alpha+1}(x)$, the claim follows. For the limit case consider λ and again $x = g(n) \in X$. Then $[x, z] \cap X$ is ω^λ-large iff $[g(n+1), z] \cap X$ is $\omega^{\{\lambda\}(x)}$-large, i.e., $z \geq f_{\{\lambda\}(x)}(g(n+1))$. Thus it suffices if $z \geq f_{\{\lambda\}(x)}(g(x+1)) = f_\lambda(x)$. ☐

Proof of Proposition 8. Let $\phi(n)$ be a Σ_1-formula describing an infinite set. Assume that a number of colors z is given. By Lemma 9 (we check that it formalizes in $I\Sigma_1$) it is sufficient to find a θ-large subset of $X := \{n \mid \phi(n)\}$. We can apply Lemma 11 to X (a suitable g exists by $I\Sigma_1$) and reduce the problem to showing that $f_{z+4}(x)$ as in (5) is total. This follows from the totality of the relativized Ackermann-Péter function. (We have for instance that $A_g(2k, n)$ majorizes $f_k(n)$.) ☐

Proposition 12. $I\Sigma_1 \vdash \mathsf{PH}^*(2) \rightarrow \mathsf{WF}(\omega^\omega)$.

Proof. It is well known that the order of ω^ω is isomorphic to the lexicographic order $<^*$ of $\mathbb{N}^{<\mathbb{N}}$. (To see this consider the order-isomorphism $n_0 n_1 n_2 \cdots n_k \mapsto \omega^k \cdot (n_k + 1) + \cdots + \omega^2 \cdot n_2 + \omega^1 \cdot n_1 + n_0$.)

Assume that ω^ω is not well-ordered. Then there is a function $f\colon \mathbb{N} \longrightarrow \mathbb{N}^{<\mathbb{N}}$ such that $f(n) \mathrel{{}^*{>}} f(n+1)$. We will show that this contradicts $\mathsf{PH}^*(2)$. Let $b := \mathrm{lth}(f(0))$. By definition of the lexicographic order we know that $\mathrm{lth}(f(n)) \leq b$ for all n. We define a Δ_1-set X and a strictly increasing Δ_1-function $h\colon X \longrightarrow \mathbb{N}$, such that $\max_i \left(f(h(n))\right)_i < n$ and $\min(X) > b$. Such X, h can be build by primitive recursion by

$$h(0) := 0,$$
$$h(n+1) := \begin{cases} h(n) + 1 & \text{if } \max_i \left(f(h(n)+1)\right)_i < n+1, \\ h(n) & \text{otherwise.} \end{cases}$$
$$X := \{n > \max_i \left(f(0)\right)_i, b \mid h(n) \neq h(n-1)\}.$$

It is clear that X is infinite.

Define the coloring $c\colon [X]^2 \longrightarrow b \cup \{-1\}$ by the following

$$c(\{n,m\}) := \begin{cases} \max\left(\left\{ i < b \;\middle|\; \begin{array}{r} \left(f(h(n))\right)_i \neq \left(f(h(m))\right)_i \\ \wedge\, i < \mathrm{lth}(f(h(m))) \end{array} \right\}\right) & \text{if such an } i \text{ exists,} \\ -1 & \text{otherwise.} \end{cases}$$

By $\mathsf{PH}^*(2)$ there exists a c-homogenous, relatively large set $Y \subseteq X$. First assume that $c([Y]^2) = -1$. This implies that for $n, m \in Y$ we have

$$n < m \rightarrow f(h(n)) \supsetneq f(h(m)).$$

Therefore, $\mathrm{lth}(f(h(n)) > \mathrm{lth}(f(h(m))$. Since the length of $f(n)$ is bounded by b, there must be a strictly decreasing sequence of natural numbers $\leq b$ of length $|Y| > \min Y > b$, which is a contradiction.

Now assume $c([Y]^2) = i \neq -1$. Then for $n, m \in Y$ we have

$$n < m \rightarrow \big(f(h(n))\big)_i > \big(f(h(m))\big)_i.$$

Since $\big(f(h(\min Y))\big)_i < \min Y$, we have decreasing sequence of length $|Y| > \min Y$ of natural numbers $< \min Y$, which is again a contradiction. □

Proposition 13. *$I\Sigma_1 \vdash \mathsf{WF}(\omega^\omega) \rightarrow \mathsf{BME}_*$*

Proof. Let $E[s]$ be a b-bounded monotone enumeration. We will assign to the trees E and $E[s]$ an ordinal in the following way.

For $\tau \in E$ let $|\tau|_E$ be length of the stage-by-stage enumeration of τ. We say a τ is maximal in its stage if there is no extension $\tau' \in E$ of τ with $|\tau|_E = |\tau'|_E$. For maximal $\tau, \tau' \in E$ define $\tau \sqsubset_E \tau'$ if $\tau \sqsubset \tau'$ and $|\tau|_E = |\tau'|_E - 1$. To a maximal $\tau \in E$ we assign the following ordinal.

$$\zeta_E(\tau) := \begin{cases} 0 & \text{if } |\tau|_E = b, \\ \omega^{b-|\tau|_E} & \text{if } \tau \text{ is a leaf in } E \text{ and } |\tau|_E < b, \\ \sum_{\tau' \sqsupset_E \tau} \zeta_E(\tau') & \text{if } \tau \text{ is not a leaf.} \end{cases} \tag{6}$$

Same for $E[s]$ instead of E. We define the ordinal ζ_E, $\zeta_{E[s]}$ for E respectively $E[s]$ to be $\zeta_E(\langle\rangle)$, $\zeta_{E[s]}(\langle\rangle)$.

By definition it is clear that $\zeta_E, \zeta_{E[s]} \leq \omega^b$. Moreover, we claim that if new elements are enumerated into $E[s+1]$ then $\zeta_{E[s+1]} < \zeta_{E[s]}$. Indeed, if there are new elements enumerated at stage $s+1$ there must be a leaf $\tau \in E[s]$ such that all elements are successors of τ. Then by definition we have $\zeta_{E[s+1]}(\tau) < \zeta_{E[s]}(\tau)$. Induction on $|\tau|_E$, gives that $\zeta_{E[s+1]}(\tau') < \zeta_{E[s]}(\tau')$ for all maximal $\tau' \sqsubset \tau$. In particular $\zeta_{E[s+1]} < \zeta_{E[s]}$.

Now the stages s_i where new elements are enumerated into E gives a decreasing sequence of ordinals $\zeta_{E[s_i]} < \omega^b$. Since $\omega^b < \omega^\omega$ and ω^ω is well-founded by assumption, there can be only finitely many stages where new elements are enumerated and thus E is finite. □

Note that Theorem 5 can be relativizable with set parameters. In the second-order setting with the recursive comprehension, we can replace primitive recursive sequences / Σ_1-definable infinite sets / functions defined by quantifier-free or Σ_1-formulas by sets. Thus, we have the following.

Theorem 14. *Over RCA_0 the following are equivalent:*

(i) $P\Sigma^0_1$*: $P\phi$ for any Σ^0_1-formulas (Σ_1-formulas with set parametes),*
(ii) BME_**: $\forall E\,(E$ is a monotone enumeration bounded by $b \rightarrow E$ is finite$)$,*
(iii) $\mathsf{PH}^*(2)$*: $\forall X\,\forall z\,\Big((\forall k\,\exists n > k\; n \in X) \rightarrow \forall x\,\forall q \geq 1\,\exists y\,\Big(([x,y] \cap X) \underset{*}{\rightarrow} (q)^2_z\Big)\Big)$,*
(iv) A^**: $\forall f\,($the Ackermann-Péter function relative to f is total$)$,*
(v) $\mathsf{WF}(\omega^\omega)$*: $\neg\exists f\,(f$ is an infinite descending sequence of ordinals α_i starting from $\omega^\omega)$.*

3. Full BME

Chong, Slaman, Yang actually used certain iterations of the principle BME_* in [5] called BME_k and $\mathsf{BME} := \bigcup_k \mathsf{BME}_k$ for the union of all these. In these principles, bounded monotone enumerations will be enumerated relative to a real (in a continuous way). We will write $E(\sigma)$, with $\sigma \in \mathbb{N}^{<\mathbb{N}}$, for such an enumeration and understand that the stage s will be implicitly given by $s = |\sigma|$. Further, we will compute a bounded tree in a similar fashion, i.e., by a function $V(\tau)$ where $\tau \in \mathbb{N}^{<\mathbb{N}}$. Here, we again consider functions E and V defined by Σ_1-formulas to work within $I\Sigma_1$, but one can easily lift-up the following discussion into the second-order setting as same as Theorem 14.

Definition 15.

1) Let $E(\sigma)$ be a monotone enumeration as above. For a tree enumerated by V a $\sigma \in V$ is called *E-expansionary* if in $E(\sigma)$ a new element is enumerated a stage $|\sigma|$.
2) A level ℓ in a tree V is E-expansionary if there is an n such that ℓ is minimal with for all $\sigma \in V$ with $|\sigma| = \ell$ and there are at least n E-expansionary initial segments of σ.
3) A *k-iterated monotone enumeration* is a sequence $(V_i, E_i)_{1 \leq i \leq k}$ such that
   (a) each V_i is a relativized recursively bounded tree as above,
   (b) each E_i is a relativized monotone enumeration procedure as above,
   (c) for each $1 \leq j < k$, if $\sigma \in V_j$ is E_j-expansionary, then for each new element τ enumerated in $E_j(\sigma)$, $V_{j+1}(\tau)$ is a proper E_{j+1}-expansionary extension of $V_{j+1}(\tau_0)$, where τ_0 is the longest initial segment of τ that had been enumerated into $E_j(\sigma)$ before.
4) A *k-path* for a k-iterated monotone enumeration (as above) is a sequence $(\sigma_i, \tau_i)_{1 \leq i \leq k}$ such that $\sigma_1 \in V_1$, τ_1 is a maximal sequence in $E_1(\sigma_1)$, and for each $1 < j \leq k$ we have that σ_j is a maximal sequence in $V_j(\tau_{j-1})$ and τ_j is a maximal sequence in $E_j(\sigma_j)$.
5) A k-iterated monotone enumeration is b-bounded if $E_k(\sigma)$ is b-bounded for each σ.
6) BME_k is the statement that each bounded k-iterated monotone enumeration procedure contains only finitely many E_1-expansionary levels in V_1.

Let $\omega_0^\delta := \delta$ and $\omega_{k+1}^\delta := \omega^{\omega_k^\delta}$. In particular $\omega_k^\omega = \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{k+1 \text{ many } \omega}$. We will show the following theorem.

Theorem 16. *For all k*

$$I\Sigma_1 \vdash \mathsf{BME}_k \leftrightarrow \mathsf{WF}(\omega_k^\omega).$$

Corollary 17. $I\Sigma_1 \vdash \forall k\, \mathsf{BME}_k \leftrightarrow \mathsf{WF}(\epsilon_0)$.

The proof of Theorem 16 proceeds by exhibiting a one-to-one correspondence between k-iterated monotone enumerations and ordinals $< \omega_k^\omega$.

For the backward direction of the proof we will consider bounded monotone enumerations of $\mathbb{N}$ together with a special termination symbol $\perp$. This will not cause any problems since $\mathbb{N} \cup \{\perp\}$ can of course be code into $\mathbb{N}$. We will extend the assignment of ordinals to bounded monotone enumerations as in (6) to include a case for $\bot$.

$$\zeta_E(\tau) := \begin{cases} 0 & \text{if } \tau(|\tau|-1) = \bot, \\ 0 & \text{if } |\tau|_E = b, \\ \omega^{b-|\tau|_E} & \text{if } \tau \text{ is a leaf in } E \text{ and } |\tau|_E < b, \\ \sum_{\tau' \sqsupset_E \tau} \zeta_E(\tau') & \text{if } \tau \text{ is not a leaf.} \end{cases}$$

Now let a k-iterated monotone enumeration $(V_i, E_i)_{1\leq i\leq k}$ be given. Further assume that $\langle \sigma_1, \tau_1, \dots, \sigma_k, \tau_k\rangle$ is a k-path in $(V_i, E_i)_{1\leq i\leq k}$. We assign the following ordinals.

$$\begin{aligned} \zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_k\rangle}(\tau) &:= \zeta_{E_k(\sigma_k)}(\tau), \\ \zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j\rangle} &:= \max_{\substack{\sigma\in V_{j+1}(\tau_j)\\ |\sigma|=\ell}} \zeta_{\langle \sigma_1,\tau_1,\dots,\tau_j,\sigma\rangle}(\langle\rangle), \end{aligned}$$

where ℓ is the maximal E_{j+1}-expansionary level in $V_{j+1}(\tau_j)$,

$$\zeta_{\langle \sigma_1,\tau_1\dots,\sigma_j\rangle}(\tau) := \begin{cases} 0 & \text{if } \tau \text{ is a leaf in } E_j(\sigma_j) \\ & \quad\text{and } \tau(|\tau|-1) = \bot, \\ \omega^{\zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau\rangle}} & \text{if } \tau \text{ is a leaf in } E_j(\sigma_j) \\ & \quad\text{and } \tau(|\tau|-1) \neq \bot, \\ \sum_{\tau'\sqsupset_{E_j(\sigma_j)}\tau} \zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j\rangle}(\tau') & \text{if } \tau \text{ is not a leaf in } E_j(\sigma_j). \end{cases}$$

To the full k-iterated monotone enumeration we assign the following ordinal.

$$\zeta_{(V_i,E_i)_{1\leq i\leq k}} := \zeta_{\langle\rangle}.$$

Note that $\zeta_{(V_i,E_i)_{1\leq i\leq k}} \leq \omega_k^b < \omega_k^\omega$.

Lemma 18. *Let $(V_i, E_i)_{1\leq i\leq k}$ be a k-iterated monotone enumeration and a tree V_1' be given, such that V_1' properly extends V_1. If V_1' contains strictly more E_1-expansionary levels thant V_1, then*

$$\zeta_{(V_i,E_i)_{1\leq i\leq k}} > \zeta_{(V_i',E_i)_{1\leq i\leq k}},$$

where for $i \geq 2$ we set $V_i' := V_i$.

Proof. We prove my induction that

(a) $\zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j\rangle} > \zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j'\rangle}$, if $\tau_j' \sqsupseteq \tau_j$ enumerates a new E_{j+1}-expansionary level in V_{j+1},

(b) $\zeta_{\langle \sigma_1,\tau_1\dots,\sigma_j\rangle} > \zeta_{\langle \sigma_1,\tau_1\dots,\sigma_j'\rangle}$, if $\sigma_j' \sqsupseteq \sigma_j$ enumerates a new element into E_j.

This directly implies then the lemma.

To prove the induction we start with (b) for $j = k$. This case follows as in Proposition 13.

For (a) and j we assume that (b) already holds for j. By the induction hypothesis each of the terms in the maximum in the definition $\zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j\rangle}$ decreases. Therefore, $\zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j'\rangle} < \zeta_{\langle \sigma_1,\tau_1,\dots,\sigma_j,\tau_j\rangle}$.

For (b) and $j < k$ we assume that (a) already holds for $j+1$. This case follows by a similar proof as in Proposition 13 together with the induction hypothesis. ☐

For the backward direction we will only consider simplified iterated monotone enumerations where the trees $V_k(\tau)$ are trivial, i.e., they contain only branches of the form $\langle 0, \dots, 0, 1\rangle$, where the length codes τ. Thus, we can omit the V_i and assume that E_{j+1} is of the form $E_{j+1}(\tau_j)$ with $\tau_j \in E_j$. With this the bound on the E_1-expansionary levels in V_1 then becomes a bound cardinality of E_1.

Further we make the assumption that each tree contains $\langle\perp\rangle$ and that $E_j(\langle\perp\rangle) = \{\perp\}$. For ease of notation we will omit the V_j.

Lemma 19. *For any $\alpha < \omega^b_{k+1}$, one can effectively find an $k+1$-iterated bounded enumeration $\langle E_1, \dots, E_{k+1}\rangle$ where E_1 is bounded by b and such that $\zeta_{\langle E_1,\dots,E_{k+1}\rangle} = \alpha$.*

Proof. We will prove this lemma by induction on k.

For the case $k = 0$ and $\alpha = 0$, set $E_1 := \{\langle\perp\rangle\}$. For $k = 0$ and $\alpha > 0$, write $\alpha = \sum_{1\le j\le l} \omega^{e_j}$ such that $b > e_1 \ge e_2 \ge \cdots \ge e_l$. In this case one easily checks that, the enumeration the constant sequences $\langle j, \dots, j\rangle$ of length $b - e_j$ in $b - e_j$ steps for $j \in [1; l]$, i.e.,

$$E_1 := \{\langle j\rangle^{*m} \mid 1 \le j \le l \wedge m \le b - e_j\},$$

such that

$$|\tau|_{E_1} = |\tau| \quad \text{for any } \tau \in E_1$$

is the desired tree. (We write $\langle j\rangle^{*m}$ for the m-fold repetition of j.)

For the case $k > 0$ and $\alpha = 0$, we again set $E_1 := \{\langle\perp\rangle\}$, and $E_i(\langle\perp\rangle) = \{\langle\perp\rangle\}$ for any $j \in [1; k{+}1]$. If $\alpha > 0$, write $\alpha = \sum_{1\le j\le l} \omega^{\alpha_j}$ such that $\omega^b_k > \alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_l$. By induction hypothesis, one can find effectively k-iterated bounded enumerations $(E^j_i)_{1\le i\le k}$ such that $\zeta_{(E^j_i)_{1\le i\le k}} = \alpha_j$.

Let

$$E_1 := \{\langle j\rangle \mid 1 \le j \le l\},$$

$$E_{i+1}(\langle j\rangle * \tau) := \langle j\rangle * E^j_i(\tau) = \{\langle j\rangle * \sigma \mid \sigma \in E^j_i(\tau)\},$$

for $i \in [1; k]$. We can easily check that $\zeta_{(E_i)_{1\le i\le l}} = \alpha$. □

We say that a bounded enumeration E is *separating* if

$$E(\tau_1) \cap E(\tau_2) = E(\tau) \quad \text{where } \tau \text{ longest common initial substring of } \tau_1, \tau_2.$$

In other words, E is separating if different paths enumerate separate sets of strings, or each σ is enumerated at most once into E. We say that a k-iterated bounded enumeration $(E_i)_{1\le i\le k}$ is separating if each E_i is separating.

We can make any enumeration separating by just coding into each string where it has been enumerated without changing the ordinal.

Lemma 20. *For any separating $k+1$-iterated bounded enumeration $(E_i)_{1\le i\le k+1}$ bounded by $b{+}1$ with $\zeta_{(E_i)_{1\le i\le k+1}} =: \alpha < \omega^b_{k+1}$ and for any $\beta < \alpha$, one can effectively find a separating proper monotone extension $(E'_i)_{1\le i\le k+1}$ also bounded by b, such that $\zeta_{(E'_i)_{1\le i\le k+1}} \ge \beta$.*

Proper extension means hear that only leafs of E_i are extended in E'_i and $E_1 \subsetneq E'_1$.

Proof. We will prove this lemma by induction on k.

For the case $k = 0$, write $\alpha = \sum_{1\le i\le l} \omega^{e_j}$ and $\beta = \sum_{1\le j\le l'} \omega^{f_j}$ such that $b > e_0 \ge e_1 \ge \cdots \ge e_l$ and $b > f_0 \ge \cdots \ge f_{l'}$.
If $l' < l$ and $e_j = f_j$ for all $j \le l'$, find a leaf $\tau \in E_1$ such that $|\tau|_{E_1} = b - e_{l'+1}$, and put $E'_1 = E_1 \cup \{\tau * \langle\perp\rangle\}$.
Otherwise, there exists $j^* < l, l'$ such that $e_{j^*} > f_{j^*}$. Find a leaf $\tau \in E_1$ such that $|\tau|_{E_1} = b - e_{j^*}$. Let $E'_1 := E_1 \cup \{\tau * \langle j\rangle^{*m} \mid j^* \le j \le l' \wedge m \le e_{j^*} - f_{j^*}\}$ where $\langle j\rangle^{*m}$ is enumerated step by step, i.e., $|\sigma * \langle j\rangle^{*(e_{j^*}-f_{j^*})}|_{E_1} = b - f_{j^*}$.

For the case $k > 0$, write $\alpha = \sum_{1\leq j\leq l}\omega^{\alpha_j}$ and $\beta = \sum_{1\leq j\leq l'}\omega^{\beta_j}$ such that $\omega_k^b > \alpha_1 \geq \cdots \geq \alpha_l$ and $\omega_k^b > \beta_1 \geq \cdots \geq \beta_{l'}$. If $l' < l$ and $\alpha_j = \beta_j$ for all $j \leq l'$. Find a leaf $\tau_1 \in E_1$ such that $\zeta_{\langle\rangle}(\tau_1) = \omega^{\alpha_{l'+1}}$. Set $E_1' := E_1 \cup \{\tau * \langle\bot\rangle\}$, and set

$$\tau_{i+1} := \min\{\tau \mid \tau \text{ is leaf in } E_{i+1}(\tau_i) \text{ and } \zeta_{\langle\tau_1,\ldots,\tau_i\rangle}(\tau) > 0\},$$

$$E_{i+1}'(\tau) := \begin{cases} E_{i+1}(\tau_i) \cup \{\tau_{i+1} * \langle\bot\rangle\} & \text{if } \tau = \tau_i * \langle\bot\rangle, \\ E_{i+1}(\tau) & \text{otherwise.}\end{cases}$$

Otherwise, there exists $j^* < l, l'$ such that $\alpha_{j^*} > \beta_{j^*}$. Find a leaf $\tau_1 \in E_1$ such that $\zeta_{\langle\rangle}(\tau_1) = \omega^{\alpha_{j^*}}$. By induction hypothesis, there exist proper a extensions $(E_i')_{1\leq i\leq k}$ of $(E_2(\tau_1), E_3, \ldots, E_k)$ such that $\zeta_{(E_i^*)_{1\leq i\leq k}} \geq \beta_j$ for $j \in [j^*; l']$. (One can effectively find these extensions.) We may further assume that the new elements enumerated into $E_i'^j$ for different j are different.

Set

$$E_1' := E_1 \cup \{\tau_1 * \langle j\rangle^{*m} \mid j^* \leq j \leq l'\},$$

m is minimal with $\zeta_{(E_i'^j[|\tau_1|+m])_{1\leq i\leq k}} < \alpha_j$,

$$E_2'(\tau) := \begin{cases} E_i'^j[|\tau|] & \text{if } \tau = \tau_1 * \langle j\rangle, \\ E_2(\tau) & \text{otherwise,}\end{cases}$$

$$E_{i+2}'(\tau) := \begin{cases} E_{i+1}'^j(\tau) & \text{for } \tau \text{ being enumerated below a } \tau_1 * \langle j\rangle. \\ E_j(\tau) & \text{otherwise.}\end{cases}$$

The last case distinction is possible by separability. We can easily check that $(E_i')_{1\leq i\leq k+1}$ is again separable and $\alpha > \zeta_{(E_i')_{1\leq i\leq k+1}} \geq \beta$. □

Proof of Theorem 16. The forward direction follows directly from Lemma 18 and the fact that $\zeta_{(V_i,E_i)_{1\leq i\leq k}} \leq \omega_k^\omega$ for any k-iterated monotone enumeration. For the backward direction assume that there exists an infinite descending sequence of ordinals $(\alpha_n)_n$ with $\alpha_0 = \omega_k^\omega$. Take b large enough that $\alpha_1 \leq \omega_k^b$. By Lemma 19 and the comments below it, there exists a separating k-iterated $b+1$-bounded monotone enumeration $(E_i^1)_{1\leq i\leq k}$ with $\zeta_{(E_i^1)_{1\leq i\leq k}} = \alpha_1$. Lemma 20 gives a sequence $\big((E_i^n)_{1\leq i\leq k}\big)_n$ of separating k-iterated b-bounded monotone enumerations with $\zeta_{(E_i^n)_{1\leq i\leq k}} \geq \alpha_n$. Now set $E_i' := \bigcup_n E_i^n$, then $(E_i')_{1\leq i\leq k}$ is again k-iterated $b+1$-bounded monotone enumeration. However by construction E_0 is infinite and thus we get $\neg\mathsf{BME}_k$. □

We close this section with showing that weak König's lemma, a formulation of the Baire Category theorem, and the cohesive principle are Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ for each primitive recursive linear order $\mathcal{O}$. Here $\mathsf{WF}(\mathcal{O})$ stands for the statement that $\mathcal{O}$ is well-founded. (In particular one can take for $\mathcal{O}$ any ordinal $\alpha < \epsilon_0$.) This shows that BME is stable with those axioms.

Theorem 21 (Folklore)**.** *For each primitive recursive linear order $\mathcal{O}$, the system $\mathsf{WKL}_0 + \mathsf{WF}(\mathcal{O})$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$.*

Proof. The proof proceeds as the classical proof of the Π^1_1-conservativity of WKL_0 over RCA_0, see [14, IX.2]. By a standard argument it is sufficient to show that each countable model of $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ can be extended to an ω-submodel of $\mathsf{WKL}_0 + \mathsf{WF}(\mathcal{O})$. This follows, again by a standard argument, from the fact that for each model $M = (|M|, \mathcal{S}_M)$ of $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ and each tree infinite tree $T \in \mathcal{S}_M$

one can find an ω-submodel $M' \models \mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ containing an infinite branch of T. To establish this, let $M = (|M|, \mathcal{S}_M)$ be a model of $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$. The model will be extended by forcing along the set $\mathcal{T}_M$ of infinite 0/1-trees in M ordered by inclusion, i.e.,

$$\mathcal{T}_M := \left\{ T \in \mathcal{S}_M \;\middle|\; M \models T \text{ is an infinite subtree of } 2^{\mathbb{N}} \right\}.$$

For $T_1, T_2 \in \mathcal{T}_M$ we set $T_1 \geq T_2$ iff $T_1 \supseteq T_2$. A set $\mathcal{D} \subseteq \mathcal{S}_M$ is called dense if for every $T \in \mathcal{T}_M$ there is an $T' \in \mathcal{D}$ with $T \geq T'$. A set G is called $\mathcal{T}_M$-generic iff it meets every definable, dense subset of $\mathcal{T}_M$.

One can show that any infinite tree in M has a $\mathcal{T}_M$-generic path and that for each $\mathcal{T}_M$-generic G we have that $M[G] \models I\Sigma^0_1$, where $M[G] := \big(|M|, \{X \subseteq |M| \mid X \text{ is recursive in } G \text{ and sets from } \mathcal{S}_M\}\big)$. See Lemmas X.2.3–5 of [14].

To prove this theorem it is thus sufficient to show the following lemma.

Lemma 22. *For each $M \models \mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ and each $\mathcal{T}_M$-generic G, we have that $M[G] \models \mathsf{WF}(\mathcal{O})$.*

Proof of Lemma 22. To show this lemma it is sufficient to show that the e-th Turing functional Φ^G_e relative to G for any $(e \in |M|)$ does not give an infinite descending chain in $\mathcal{O}$.

For a $\sigma \in |M|$ viewed as a finite binary sequence in M, and $T \in \mathcal{T}_M$ we will write $\sigma \prec T$ iff $M \models$ "any $\tau \in T$ is compatible with σ". For $e, m \in |M|$, put

$$\mathcal{D}^1_e := \left\{ T \in \mathcal{T}_M \;\middle|\; \exists n\, \exists \sigma \left(\begin{aligned} &\sigma \prec T \wedge \forall i \leq n\, (\Phi^\sigma_{e,|\sigma|}[i]{\downarrow}) \\ &\quad \wedge \left(\Phi^\sigma_{e,|\sigma|}[i]\right)^n_{i=0} \text{ is not strictly decreasing in } \mathcal{O} \end{aligned} \right) \right\},$$

$$\mathcal{D}^2_{e,m} := \left\{ T \in \mathcal{T}_M \;\middle|\; \forall \tau \in T \left(\Phi^\tau_{e,|\tau|}[m]{\uparrow} \right) \right\},$$

$$\mathcal{D}_e := \mathcal{D}^1_e \cup \bigcup_{m \in |M|} \mathcal{D}^2_{e,m}.$$

Clearly, if $T \in \mathcal{D}_e$ and $G \in [T]$, then, Φ^G_e is not an infinite descending sequence of $\mathcal{O}$.

Now, we want to show that $\mathcal{D}_e$ is dense. Assume not then there exists an infinite tree $T \in \mathcal{T}_M$ such that any infinite subtree is not in $\mathcal{D}_e$. Put $l_0 := 0$ and $l_{m+1} := \min \left\{ l > l_m \;\middle|\; \forall \tau \in T \cap 2^l\, (\Phi^\tau_{e,|\tau|}[m+1]{\downarrow}) \right\}$. Such an l always exists since there are only finitely many $\tau \in T$ such that $\Phi^\tau_{e,|\tau|}[m]{\uparrow}$. Otherwise they would form an infinite subtree of T belonging to $\mathcal{D}^2_{e,m}$. Note that the sequence l_m is computable in M.

For each $m \in |M|$ and each $\tau \in T \cap 2^{l_m}$ the finite sequence $\left(\Phi^\tau_{e,|\tau|}[i]\right)^m_{i=0}$ is strictly decreasing in $\mathcal{O}$, since otherwise the subtree below τ would lie in $\mathcal{D}^1_e$. Therefore, the function $f(m) := \min_{\mathcal{O}} \left\{ \phi^\tau_{e,|\tau|} \;\middle|\; \tau \in T \cap 2^{l_m} \right\}$ is computable in M and one easily checks that it gives an infinite strictly decreasing sequence in $\mathcal{O}$. This contradicts the fact $M \models \mathsf{WF}(\mathcal{O})$ and hence $\mathcal{D}_e$ must be dense. ∎

The Baire Category theorem for Cantor space can be formulated in the following way. For a $\sigma \in 2^{<\mathbb{N}}$ and $X \in 2^{\mathbb{N}}$ we will write $\sigma \subseteq X$ if X extends σ. A set D is called dense if for each $\sigma \in 2^{<\mathbb{N}}$ there is a $\tau \in D$ with $\tau \supseteq \sigma$. We say that X meets D if $\exists \sigma \in D$ $(\sigma \subseteq D)$. The Baire category theorem (BCT) is then the statement that every sequence of dense sets $D_i \subseteq 2^{<\mathbb{N}}$ there exists a set G that meets every D_i.

Theorem 23. *For each primitive recursive linear order $\mathcal{O}$, the system* $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O}) + \mathsf{BCT}$ *is Π^1_1-conservative over* $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$.

Proof. As in the proof of Theorem 21 it is sufficient to show that each countable model $M = (|M|, \mathcal{S}_M)$ of $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ can be extended to an ω-submodel of BCT.

In [2, Lemma 6.2] it is shown that one can find a G such that $M[G] \models \mathsf{RCA}_0 + \mathsf{BCT}$ and G intersects all dense M-definable sets. Such a set G will be called M-generic. The theorem follows by showing the following lemma.

Lemma 24. *For each $M \models \mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ and each M-generic $M[G] \models \mathsf{WF}(\mathcal{O})$.*

Proof of Lemma 24. As in Lemma 22 we construct for each Turing-functional Φ^X_e a dense set D_e. Hence put,

$$\begin{aligned} D^1_e &:= \left\{ \sigma \in 2^{<|M|} \;\middle|\; \exists n \in |M| \begin{pmatrix} \forall i \leq n\, (\Phi^\sigma_{e,|\sigma|}[i]{\downarrow}) \wedge \left(\Phi^\sigma_{e,|\sigma|}[i]\right)^n_{i=0} \\ \text{is not strictly decreasing in } \mathcal{O} \end{pmatrix} \right\}, \\ D^2_{e,m} &:= \left\{ \sigma \in 2^{<|M|} \;\middle|\; \forall \tau \supseteq \sigma \left(\Phi^\tau_{e,|\tau|}[m]{\uparrow}\right) \right\}, \\ D &:= D^1_e \cup \bigcup_{m \in M} D^2_{e,m}. \end{aligned}$$

Clearly, if a generic G meets D_e then Φ^G_e is not an infinite descending sequence of $\mathcal{O}$. Now, we want to show that D_e is dense. Assume not, then there exists a σ_0 such that for any $\sigma \supseteq \sigma_0$ we have $\forall i \leq n \left(\Phi^\sigma_{e,|\sigma|}[i]{\downarrow}\right)$ implies that $\left(\Phi^\sigma_{e,|\sigma|}[i]\right)^n_{i=0}$ is strictly decreasing in $\mathcal{O}$, and for any $m \in \langle M \rangle$ the set $\left\{\tau \;\middle|\; \Phi^\tau_{e,|\tau|}[m]{\downarrow}\right\}$ is dense below σ_0. By the latter condition one can easy construct a computable in M set $X \supseteq \sigma_0$ such that $\Phi^X_e(m){\downarrow}$ for any $m \in |M|$. By the former Φ^X_e outputs a strictly decreasing sequence of $\mathcal{O}$ in M which is a contradiction. □

A sentence of the form

$$\forall X\ (\phi(X) \to \exists Y\, \eta(X,Y))$$

where ϕ is arithmetical and $\eta \in \Sigma^0_3$, is called *restricted* Π^1_2*-sentence* (r-Π^1_2). Hirschfeld and Shore showed that the cohesive principle (COH) is r-Π^1_2-conservative over RCA_0, see [10, Theorem 7.18] and [11]. Assume that $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$ does prove a r-Π^1_2-sentence, i.e.,

$$\mathsf{RCA}_0 + \mathsf{COH} + \mathsf{WF}(\mathcal{O}) \vdash \forall X\ (\phi(X) \to \exists Y\, \eta(X,Y)).$$

By the deduction theorem this is equivalent to

$$\mathsf{RCA}_0 + \mathsf{COH} \vdash \forall Z\, \mathsf{WF}(\mathcal{O})[Z] \to \forall X\ (\phi(X) \to \exists Y\, \eta(X,Y)),$$

where $\mathsf{WF}(\mathcal{O})[Z]$ denotes each Z-computable sequence is well-founded. Note that this can be written as a $\Sigma^0_2[Z]$-formula. By logical transformation this is equivalent to

$$\mathsf{RCA}_0 + \mathsf{COH} \vdash \forall X\ (\phi(X) \to \exists Y\, \exists Z\ (\eta(X,Y) \vee \neg \mathsf{WF}[Z])).$$

Since this is again a r-Π^1_2-sentence we can apply the above-mentioned result. This proves the following theorem.

Theorem 25. *For each primitive recursive linear order $\mathcal{O}$, the system* $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O}) + \mathsf{COH}$ *is r-Π^1_2-conservative over* $\mathsf{RCA}_0 + \mathsf{WF}(\mathcal{O})$.

$$I\Sigma_1 \Longrightarrow B\Sigma_2 \Longrightarrow I\Sigma_2 \Longrightarrow B\Sigma_3 \Longrightarrow \cdots$$

FIGURE 2. Pairs-Kirby hierarchy

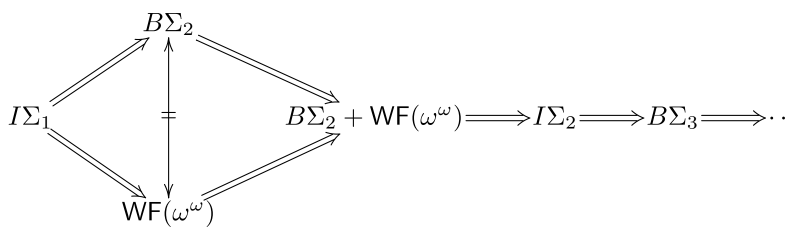


FIGURE 3. Extended Paris-Kirby hierarchy

4. Conclusion

We have shown in Theorem 5 that $\mathsf{WF}(\omega^\omega)$ has many equivalent formulations and occurs far more often than expected. It has been rediscovered in different contexts, see for instance [7] and [5] as already mentioned above. This shows that there are only a few natural first-order principles between $I\Sigma_1$ and $I\Sigma_2$, and $\mathsf{WF}(\omega^\omega)$ has to be considered one of them, besides induction and bounded collection principles. For this reason we believe that the usually in reverse mathematics considered Kirby-Paris hierarchy as shown in Figure 2 has to be extended to give a comprehensive picture. Figure 3 displays such an extension by $\mathsf{WF}(\omega^\omega)$. This hierarchy has been defined in [8]. There the considered formulation of $\mathsf{WF}(\omega^\omega)$ was $P\Sigma_1$, and more generally $P\Sigma_{n+1}$ for all n was considered. As mentioned above $P\Sigma_{n+1}$ lies between $I\Sigma_{n+1}$ and $I\Sigma_{n+2}$. However, a similar equivalence as in Theorem 5 for $P\Sigma_{n+1}$ with $n > 0$ cannot hold for quantifier reasons. In detail, $P\Sigma_2$ is Π_4 while $\mathsf{WF}(\omega_2^\omega)$ is still Π_3. Thus, it is unlikely to find similar extensions between $I\Sigma_{n+1}$ and $I\Sigma_{n+2}$ that are equally natural.

We furthermore characterize the principles BME and BME_n in terms of well-foundedness of ordinals. This allows us to answer the question whether Ramsey's theorem for pairs and two colors (RT^2_2) implies BME, as ask by Chong, Slaman, and Yang in [4, Question 5.2], negatively. This cannot be the case since it is known that RT^2_2 is Π^1_1-conservative over $I\Sigma_2$, see [3], where ω_3^ω cannot be seen to be well-founded in $I\Sigma_2$. Thus $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{BME}_3$.

Let ME be the *monotone enumeration principle* which states that each unbounded monotone enumeration has an infinite branch. This principle is formalized in RCA_0. For ME we have the following well-known result.

Theorem 26 (Folklore, RCA_0)**.** ACA_0 *and* ME *are equivalent.*

BME can be seen as a miniaturization of ME as certain iterations of the Paris-Harrington principle are for Ramsey's theorem for pairs, see [1, 16, 15], or has been done for $P\Sigma_1$ in [7]. For the Paris-Harrington principle equivalences between these miniaturizations and the provably recursive functions (in some cases even provable Π^0_3 or Π^0_4 statements) of RT^2_2 over different systems (in detail WKL_0, WKL_0^*) have been established. For $P\Sigma_1$ this has not been done yet. Our characterization in Theorem 5 together with Theorem 3 of [7] shows that the miniaturization of [7] is faithful, in the sense that they prove the same Π^0_2-sentences. Since Theorem 16 shows that the Π^0_3-sentences of BME are exactly the same as of PA, BME is a faithful miniaturization of ME in the same way.

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