Linear Algebraic Semantics for Multi-agent Communication

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Keywords: Logic of Belief, Belief Revision, Dynamic Epistemic Logic, Propositional Dynamic Logic, Linear Algebra.

Abstract: When we study multi-agent communication system, it forces us to manage an existence of communication channels between agents, such as phone numbers or e-mail addresses, while ordinary modal logic for multi-agent system does not consider the notion of channel. This paper proposes a decidable and semantically complete logic of belief with communication channels, and then expands the logic with informing action operators to change agents' beliefs via communication channels. Moreover, for a better formalism for handling these semantics efficiently, we propose a linear algebraic representation of these. That is, with the help of Fitting (2003) and van Benthem and Liu (2007), we reformulate our proposed semantics of the doxastic static logic and its dynamic extensions in terms of boolean matrices. We also implement and publicize a calculation system of our matrix reformulations as an open system on the web.

1 INTRODUCTION

One of the most important aspects of multi-agent communication is changes of an agent's knowledge or belief (Gärdenfors, 2003). Nowadays, such changes are well-discussed in terms of modal logic, as dynamic epistemic logic (van Ditmarsch et al., 2007). For example, public announcement logic, proposed by Plaza (Plaza, 1989), can capture how an agent's knowledge change after a piece of information is *publicly* announced to all the agents, while we do not assume any structure among agents. On the other hand, in communication of multiple agents, we can naturally consider the existence of *channels* between them (Barwise and Seligman, 1997), e.g., phone numbers or e-mail addresses. Then, communicability in those agents can be represented in a directed graph, where a vertex is an agent and an edge a channel.

There are several studies integrating the notion of structure among agents into dynamic epistemic logic. (Seligman et al., 2011) proposes a two-dimensional modal logic which can handle both agents' knowledge and a friendship relation between agents. Based on the two-dimensional framework, (Sano and Tojo, 2013) implemented the idea of communication channel in terms of a modal operator and studies belief changes of agents, where they raised the following requirements:

(R1) An effect of an informing action is restricted to some specified agents determined by communication channels.

(R2) An existence of communication channel between agents depends on a given situation, i.e., it is not constant or rigid for all situations.

One of the deficiencies of the two dimensional framework is that it is still unknown whether the resulting logics in (Seligman et al., 2011; Sano and Tojo, 2013) are decidable, i.e., we can effectively test if a given formula is a theorem of a given logic. One of the purposes of this paper is to propose a *decidable* multi-agent doxastic logic which satisfies the two requirements above and can talk about communication channels among agents. Instead of communication channel as a modal operator, we implement the notion of channel as a constant symbol c_{ab} whose reading is 'there is a channel from agent a to agent b'. Moreover, instead of public announcement operators, this paper proposes two dynamic operators satisfying the requirement (R1), called *semi-private announcement* and *introspective announcement* operators.

When we study logic of multi-agent system, it forces us to manage many indices, such as agent IDs and names of the worlds in our syntax and its semantics. What seems to be lacking is an introduction of a better formalism or notation for handling such many indices. Thus far, (Fitting, 2003) proposed a linear algebraic reformulation of Kripke semantics of

Hatano R., Sano K. and Tojo S..
Linear Algebraic Semantics for Multi-agent Communication.
DOI: 10.5220/0005219001740181
In *Proceedings of the International Conference on Agents and Artificial Intelligence* (ICAART-2015), pages 174-181
ISBN: 978-989-758-073-4

modal logic. (Tojo, 2013) has employed the notion of boolean matrix and tried to integrate the notion of communication channel with dynamic logic of multiple agents' beliefs in term of linear algebra. In this research, we give a more rigorous logical formalisms to (Tojo, 2013). That is, we reformulate our proposed doxastic logic and its dynamic extensions in terms of boolean matrices.

To sum up, this paper first proposes a decidable multi-agent doxastic logic and its dynamic extensions with two informing action operators, and then reformulate our Kripke semantics in terms of boolean matrices.

This paper is organized as follows. Section 2 introduces a static logic of agents' belief equipped with the notion of channel between agents and establish that all the valid formulas on all the *finite* Kripke models for our syntax is completely axiomatizable (Theorem 1). Moreover, our proposed axiomatization is decidable (Theorem 2). In order to deal with changes of agents' belief via communication channel, Section 3 provides two dynamic operators to our syntax of static logic with sets of reduction axioms. Following the idea by (Fitting, 2003), Section 4 reformulates our Kripke semantics in terms of boolean matrix. With the help of (Van Benthem and Liu, 2007), Section 5 reveals that we can regard our two dynamic operators as program terms in propositional dynamic logic and also reformulates the semantics of two operators in terms of boolean matrix. Section 6 use our boolean matrix reformulation to present an algorithm for checking agent's belief at a given world and an algorithm for rewriting a given Kripke model by one of our dynamic operators. Finally, Section 7 concludes this paper.

Related Works. Here we comment on linear algebraic approach to multi-agent belief revision. (Fitting, 2003) proposed a linear algebraic approach to Kripke semantics, but he did not consider any dynamic operators. On the other hand, we reformulate (Van Benthem and Liu, 2007)'s idea of *relation changer* over propositional dynamic logic in terms of matrices and provide a linear algebraic treatment with our dynamic operators. In this sense, this paper can be regarded as a generalization of (Fitting, 2003) to dynamic extensions. While (Liau, 2004) also used boolean matrices to represent an accessibility relation of an agent and (Fusaoka et al., 2007) used real-valued matrices to represent qualitative belief change in multi-agent setting, both of them did not provide any concrete axiomatization of logics they study.

2 STATIC LOGIC FOR AGENTS' BELIEF

2.1 Syntax and Semantics

This section introduces a modal epistemic language which enables us to formalize agents' beliefs and communication channels.

Let G be a fixed *finite* set of agents. Our syntax $\mathcal{L}$ consists of the following vocabulary: a finite set $\mathsf{Prop} = \{p, q, r, \ldots\}$ of propositional letters; boolean connectives $\neg$, $\vee$; belief operators B_a ($a \in G$); channel constants c_{ab} ($a, b \in G$). A set of formulas of $\mathcal{L}$ is inductively defined as:

$$\varphi ::= p \mid \mathsf{c}_{ab} \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{B}_a \varphi$$

where $p \in \mathsf{Prop}$, $a, b \in G$. We define $\widehat{\mathsf{B}_a}\varphi := \neg \mathsf{B}_a \neg\varphi$ whose reading is 'agent a considers it possible that φ'. We also introduce the boolean connectives $\wedge, \rightarrow, \leftrightarrow$ as ordinary abbreviations. $\mathsf{B}_a\, p$ stands for 'agent a believes that p' and c_{ab} is to read 'there is a communication channel from a to b'. Then, let us provide Kripke semantics with our syntax. A *model* $\mathfrak{M}$ is a tuple $(W, (R_a)_{a\in G}, (C_{ab})_{a,b\in G}, V)$ where W is a non-empty set of worlds, called *domain*, $R_a \subseteq W \times W$, $C_{ab} \subseteq W$ is a *channel relation* such that $C_{aa} = W$ for all $a \in G$, and $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ is a *valuation function*. Note that we require $C_{aa} = W$ for all $a \in G$ in order to capture our notion of communication channel. A *frame* (denoted by $\mathfrak{F}$, etc.) is the result of dropping a valuation function from a model.

Given any model $\mathfrak{M}$, any world $w \in W$, and any formula φ, we define the *satisfaction relation* $\mathfrak{M}, w \models \varphi$ inductively as follows:

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \mathsf{c}_{ab}$	iff	$w \in C_{ab}$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \mathsf{B}_a\varphi$	iff	$\mathfrak{M}, v \models \varphi$ for all v with wR_av.

We define the *truth set* $[\![\varphi]\!]_{\mathfrak{M}}$ of φ in $\mathfrak{M}$ by $[\![\varphi]\!]_{\mathfrak{M}} = \{ w \in W \mid \mathfrak{M}, w \models \varphi \}$. φ is *valid* on $\mathfrak{M}$ if $\mathfrak{M}, w \models \varphi$ for all worlds $w \in W$. We say that φ is *valid* in a class of Kripke models if φ is valid on $\mathfrak{M}$ belongs to the class. It is clear that c_{aa} is always valid in any Kripke model $\mathfrak{M}$. Moreover, given any Kripke model $\mathfrak{M}$, it is easy to see that all the axioms in Table 1 are valid in $\mathfrak{M}$ and all the rules of Table 1 preserve validity on $\mathfrak{M}$.

Example 1 (Running Example)**.** Let $G = \{a, b\}$. Define $\mathfrak{M}$ (see Figure 1) by: $W = \{w_1, w_2, w_3\}$, $R_a = \{(w_1,w_1),(w_1,w_2),(w_1,w_3),(w_2,w_2),(w_3,w_3)\}$, $R_b = W \times W$, $V(p) = \{w_2\}$, $C_{ab} = \{w_1, w_2\}$, $C_{ba} = \emptyset$,

Table 1: Hilbert-style Axiomatization $\mathbf{K}_{\mathsf{c}}$ of Static Logic.

(**Taut**)	φ, φ is a tautology
($\mathbf{K}_{\mathsf{B}}$)	$\mathsf{B}_a(\varphi \to \psi) \to (\mathsf{B}_a \varphi \to \mathsf{B}_a \psi) \quad (a \in G)$
(**Selfchn**)	$\mathsf{c}_{aa} \quad (a \in G)$
(**MP**)	From φ and $\varphi \to \psi$, infer ψ
($\mathbf{Nec}_{\mathsf{B}}$)	From φ, infer $\mathsf{B}_a \varphi \quad (a \in G)$

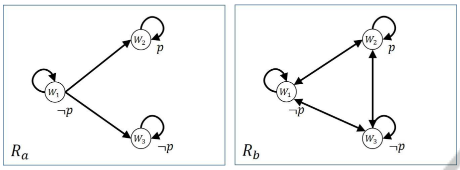

Figure 1: Accessibility relations of agents a and b.

$C_{aa} = C_{bb} = W$. Agent a believes p in w_2 and $\neg p$ in w_3, but he/she is not sure of p or $\neg p$ in w_1. On the other hand, agent b does not believe p nor $\neg p$ at all the worlds. There are channels from a to b in w_1 and w_2, but there is no channel between them in w_3.

2.2 Hilbert-style Axiomatization

The following theorem implies that we can axiomatize all the valid formlas on the class of all *finite* Kripke models. The restriction to the finite models is important for us, since our matrix representation of Kripke model is always in terms of *finite matrix*.

Theorem 1. For all formulas φ in $\mathcal{L}$, φ is a theorem in $\mathbf{K}_{\mathsf{c}}$ of Table 1 iff φ is valid on the class of all *finite* Kripke models.

Proof. (Outline) Since the soundness is easy to establish, we focus on the completeness with respect to the class of all finite Kripke models. We show that any unprovable formula φ in $\mathbf{K}_{\mathsf{c}}$ is falsified in a finite Kripke model. Let φ be an unprovable formula in $\mathbf{K}_{\mathsf{c}}$. First, we define the canonical model $\mathfrak{M}$ where φ is falsified at some point of $\mathfrak{M}$. Second, since the domain of the canonical model is infinite, we employ the technique of *filtration* to boil the model down to a finite model where φ is still falsified at some point. For both steps, we basically follow the standard techniques, e.g. found in (Blackburn et al., 2002).

We say that a set Γ of formulas is $\mathbf{K}_{\mathsf{c}}$-*consistent* (for short, *consistent*) if $\bigwedge \Gamma'$ is unprovable in $\mathbf{K}_{\mathsf{c}}$, for all finite subsets Γ' of Γ, and that Γ is *maximally consistent* if Γ is consistent and $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for all formulas φ. Note that ψ is unprovable in $\mathbf{K}_{\mathsf{c}}$ iff $\neg\psi$ is $\mathbf{K}_{\mathsf{c}}$-consistent, for any formula ψ. We define the canonical model $(W, (R_a)_{a \in G}, (C_{ab})_{a,b \in G}, V)$ by:

- W is the set of all maximal consistent sets;
- $\Gamma R_a \Delta$ iff ($\mathsf{B}_a \psi \in \Gamma$ implies $\psi \in \Delta$) for all ψ;
- $C_{ab} := \{ \Gamma \in W \mid \mathsf{c}_{ab} \in \Gamma \}$;
- $\Gamma \in V(p)$ iff $p \in \Gamma$.

Then, we can show the following equivalence (Truth Lemma (Blackburn et al., 2002, Lemma 4.21)): $\mathfrak{M}, \Gamma \models \psi$ iff $\psi \in \Gamma$ for all formulas ψ and $\Gamma \in W$, where we note that we need to use the axiom ($\mathbf{K}_{\mathsf{B}}$) and the rule ($\mathbf{Nec}_{\mathsf{B}}$) for the case where ψ is of the form of $\mathsf{B}_a \gamma$.) Given any unprovable formula φ in $\mathbf{K}_{\mathsf{c}}$, we can find a maximal consistent set Δ such that $\neg\varphi \in \Gamma$ (where we need to use (**Taut**) and (**MP**)). Then, by the equivalence above, φ is falsified at Δ of the canonical model $\mathfrak{M}$, where we can assure that $C_{aa} = W$ for all $a \in G$ by the axiom (**Selfch**). This finishes the first step of our proof.

Let us move to the second step. Let $\mathfrak{N} = (W, (R_a)_{a \in G}, (C_{ab})_{a,b \in G}, V)$ be a Kripke model and Γ a finite set of formulas that is closed under taking *subformulas*. Without loss of generality, we can assume that Γ contains c_{aa} for all agents a occurring in Γ (otherwise, we can just add c_{aa}s to Γ for all as occurring in Γ where note that the number of such as is finite). Let us define an equivalence relation $\sim_\Gamma$ by $w \sim_\Gamma w'$ iff ($\mathfrak{N}, w \models \psi$ iff $\mathfrak{N}, w' \models \psi$) for all $\psi \in \Gamma$. Then, we define a finite model $\mathfrak{N}_\Gamma$ as follows:

- $W^\Gamma := \{ [w] \mid w \in W \}$, where $[w]$ is the equivalence class of w with respect to $\sim_\Gamma$.
- $[w] R_a^\Gamma [w']$ iff $v R_a v'$ for some $v \in [w]$ and $v' \in [w']$.
- $C_{ab}^\Gamma := \{ [w] \mid w \in C_{ab} \}$ for $\mathsf{c}_{ab} \in \Gamma$.
- $[w] \in V^\Gamma(p)$ iff $w \in V(p)$ for $p \in \Gamma$.

Remark that C_{aa}^Γ always holds, since we assumed that $\mathsf{c}_{aa} \in \Gamma$ for all as occurring in Γ. Remark also that the size of W^Γ is less than or equal to $2^{\#\Gamma}$, hence finite. By induction on $\psi \in \Gamma$, we can show that $\mathfrak{N}, w \models \psi$ iff $\mathfrak{N}, [w] \models \psi$ for all $w \in W$ (the proof can be found in (Blackburn et al., 2002, Theorem 2.39)). Recall that any unprovable formula φ in $\mathbf{K}_{\mathsf{c}}$ is falsified at Γ of the canonical model $\mathfrak{M}$. Now we can apply the filtration technique to obtain a finite model $\mathfrak{M}^\Gamma$ where φ is falsified at $[\Delta]$ and Γ is the union of $\{ \mathsf{c}_{aa} \mid a \text{ occurs in } \varphi \}$ and the finite set $\mathrm{Sub}(\varphi)$ of all subformulas of φ and this finishes the second (and last) step of our proof. □

Theorem 2. $\mathbf{K}_{\mathsf{c}}$ is decidable.

Proof. When φ is unprovable in $\mathbf{K}_{\mathsf{c}}$, Theorem 2 tells us that φ has a finite countermodel. Since we can recursively check if a given finite model satisfies the condition $C_{aa} = W$ for all agents $a \in G$ (note G is finite), we can construct an effective procedure generating all the finite Kripke models and checking if φ is falsified at some point of a finite model. Together with an effective procedure of enumerating all the theorems of $\mathbf{K}_{\mathsf{c}}$, we obtain the decision procedure of Theoremhood of $\mathbf{K}_{\mathsf{c}}$. □

3 DYNAMIC OPERATORS FOR CHANNEL COMMUNICATION

This section introduces two dynamic operators which allows us to talk about agents' belief changes in terms of informing action. The first dynamic operator (semi-private announcement) specifies both the sender and the receiver, but the second operator (introspective announcement via channel) just specified the sender agents and we need to calculate the receivers of the information via communication channels.

3.1 Semi-private Announcement

One of the most well-known dynamic operators is public announcement operator (Plaza, 1989), but our operator of this section differs from it by the following requirement:

(R3) Our introducing operators are *semi-private* or *non-public* announcements to some specific agents. We assume that an agent a can send a message to an agent b only when there is a channel from a to b.

When an agent informs one of the other agents of something, our basic assumption is that we need a (context-dependent) channel between those agents. The notion of channel was formalized as channel propositions c_{ab}.

Let us denote our intended dynamic operator by $[\varphi\downarrow_b^a]$, whose reading is 'after the agent a informs the agent b of the message φ via channel'. Our intended reading of $[\varphi\downarrow_b^a]\psi$ is 'after the agent a informs the agent b to φ, ψ'. We provide the semantic clause for $[\varphi\downarrow_b^a]\psi$ on a model $\mathfrak{M} = (W, (R_a)_{a\in G}, (C_{ab})_{a,b\in G}, V)$ is given as follows:

$$\mathfrak{M}, w \models [\varphi\downarrow_b^a]\psi \quad \text{iff} \quad \mathfrak{M}^{\varphi\downarrow_b^a}, w \models \psi$$

where $\mathfrak{M}^{\varphi\downarrow_b^a} = (W, (R'_a)_{a\in G}, (C_{ab})_{a,b\in G}, V)$ and $(R'_c)_{c\in G}$ is defined as: if $c = b$, for all $x \in W$, we set

$$R'_b(x) := \begin{cases} R_b(x) \cap [\![\varphi]\!]_{\mathfrak{M}} & \text{if } \mathfrak{M}, x \models \mathsf{B}_a\varphi \wedge \mathsf{c}_{ab} \\ R_b(x) & \text{otherwise.} \end{cases}$$

If $c \neq b$, $R'_c := R_c$. Semantically speaking, $[\varphi\downarrow_b^a]$ restricts b's attention to the φ's worlds if there is a channel from the agent a to b and agent a believes φ. Otherwise, the action $[\varphi\downarrow_b^a]$ will not change b's belief.

Table 2: Hilbert-style Axiomatization $\mathbf{K}_{\mathsf{c}[\cdot\downarrow_b^a]}$.

In addition to all the axioms and rules of $\mathbf{K}_{\mathsf{c}}$, we add:

$[\varphi\downarrow_b^a]p$	$\leftrightarrow$	p,
$[\varphi\downarrow_b^a]\mathsf{c}_{cd}$	$\leftrightarrow$	c_{cd},
$[\varphi\downarrow_b^a]\neg\psi$	$\leftrightarrow$	$\neg[\varphi\downarrow_b^a]\psi$,
$[\varphi\downarrow_b^a](\psi\vee\chi)$	$\leftrightarrow$	$[\varphi\downarrow_b^a]\psi \vee [\varphi\downarrow_b^a]\chi$,
$[\varphi\downarrow_b^a]\mathsf{B}_c\psi$	$\leftrightarrow$	$\mathsf{B}_c[\varphi\downarrow_b^a]\psi \quad (c \neq b)$
$[\varphi\downarrow_b^a]\mathsf{B}_b\psi$	$\leftrightarrow$	$((\mathsf{c}_{ab} \wedge \mathsf{B}_a\varphi) \to \mathsf{B}_b(\varphi \to [\varphi\downarrow_b^a]\psi)) \wedge$ $(\neg(\mathsf{c}_{ab} \wedge \mathsf{B}_a\varphi) \to \mathsf{B}_b[\varphi\downarrow_b^a]\psi)$

($\mathbf{Nec}_{[\varphi\downarrow_b^a]}$) From ψ, infer $[\varphi\downarrow_b^a]\psi$

Theorem 3. For all formulas φ in the expanded syntax $\mathcal{L}$ with $[\psi\downarrow_b^a]$, φ is a theorem in $\mathbf{K}_{\mathsf{c}[\cdot\downarrow_b^a]}$ of Table 2 iff φ is valid on the class of all finite Kripke models.

Proof. By $\vdash\psi$ (or $\vdash^+\psi$), we mean that ψ is a theorem of the axiomatization $\mathbf{K}_{\mathsf{c}}$ (or, $\mathbf{K}_{\mathsf{c}[\cdot\downarrow_b^a]}$, respectively.) The soundness of the axioms is easy. One can also check that the necessitation rule ($\mathbf{Nec}_{[\varphi\downarrow_b^a]}$) preserves the validity on the class of all finite models. As for the completeness part, we can reduce the completeness of our dynamic extension to the static counterpart (i.e., Theorem 1) as follows. With the help of the axioms of Table 2, we can define a mapping t sending a formula ψ of the expanded syntax (we denote this by $\mathcal{L}^+$ below) with the dynamic operators $[\varphi\downarrow_b^a]$ to a formula $t(\psi)$ of the original syntax $\mathcal{L}$, where we start rewriting the *innermost occurrences* of $[\varphi\downarrow_b^a]$. For example, $t([\varphi\downarrow_b^a]\mathsf{B}_c(p\vee\mathsf{c}_{ac})) := \mathsf{B}_c(p\vee\mathsf{c}_{ac})$. For this mapping t, we can show that $\psi\leftrightarrow t(\psi)$ is valid on all finite models and $\vdash^+\psi\leftrightarrow t(\psi)$. Then, we can proceed as follows. Fix any formula ψ of $\mathcal{L}^+$ such that ψ is valid on all finite models. By the validity of $\psi\leftrightarrow t(\psi)$ on all finite models, we obtain that $t(\psi)$ is valid on all finite models. By Theorem 1, $\vdash t(\psi)$, which implies $\vdash^+ t(\psi)$. Finally, it follows from $\vdash^+\psi\leftrightarrow t(\psi)$ that $\vdash^+\psi$, as desired. □

Example 2. In Example 1, we obtain the truth of $[p\downarrow_b^a]\mathsf{B}_b p$ at w_2, i.e., 'after agent a informs agent b of the message φ via channel, agent b comes to believe p' in w_2. Figure 2 is the updated model of $\mathfrak{M}$ by $[p\downarrow_b^a]$. On the other hand, agent a does not have any channel to b in w_3, and so, the accessible worlds from w_3 will be unchanged even after the update of $\mathfrak{M}$ by $[p\downarrow_b^a]$. Therefore, $[p\downarrow_b^a]\mathsf{B}_b p$ is false at w_3. Similarly, agent a does not believe $\neg p$ in w_1, i.e., $B_a\neg p$ fails in w_1, and so, the informing action $[p\downarrow_b^a]$ will not change the accessible worlds from w_1.

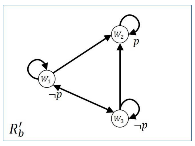

Figure 2: Updated accessibility relation of agent b.

Table 3: Hilbert-style Axiomatization $\mathbf{K}_{\mathsf{c}[\cdot\downarrow^H]}$.

In addition to all the axioms and rules of $\mathbf{K}_{\mathsf{c}}$, we add:

$[\varphi\downarrow^H]p$	$\leftrightarrow$	p,
$[\varphi\downarrow^H]\mathsf{c}_{ab}$	$\leftrightarrow$	c_{ab},
$[\varphi\downarrow^H]\neg\psi$	$\leftrightarrow$	$\neg[\varphi\downarrow^H]\psi$,
$[\varphi\downarrow^H](\psi\vee\chi)$	$\leftrightarrow$	$[\varphi\downarrow^H]\psi\vee[\varphi\downarrow^H]\chi$,
$[\varphi\downarrow^H]\mathsf{B}_a\psi$	$\leftrightarrow$	$(\bigvee_{b\in H}(\mathsf{c}_{ba}\wedge\mathsf{B}_b\varphi)\to\mathsf{B}_b(\varphi\to[\varphi\downarrow^H]\psi))$
		$\wedge(\neg(\bigvee_{b\in H}(\mathsf{c}_{ba}\wedge\mathsf{B}_b\varphi))\to\mathsf{B}_b[\varphi\downarrow^H]\psi)$

($\mathbf{Nec}_{[\varphi\downarrow^H]}$) From ψ, infer $[\varphi\downarrow^H]\psi$

3.2 Introspective Announcement Via Communication Channels

In the dynamic operator $[\psi\downarrow^a_b]$, we specified a and b as the sender and the receiver of the information φ, respectively. Even so, we may consider the situation where more than one agents, say a and b, send a piece of information to the other agents, and who will receive the information may change, depending on communication channels between agents. In this sense, we do not specify the receivers in advance here. Rather, we calculate the receivers of the information from the senders and the communication channels. We may expand our static syntax $\mathcal{L}$ with a dynamic operator $[\varphi\downarrow^H]$ ($H\subseteq G$) whose reading is 'after a group H of agents sends a piece φ of information via communication channels'. Given a Kripke model $\mathfrak{M}$ = $(W,(R_a)_{a\in G},(C_{ab})_{a,b\in G},V)$ and a world $w\in W$, we define the semantics of $[\varphi\downarrow^H]\psi$ by:

$$\mathfrak{M},w\models[\varphi\downarrow^H]\psi \quad \text{iff} \quad \mathfrak{M}^{\varphi\downarrow^H},w\models\psi,$$

where $\mathfrak{M}^{\varphi\downarrow^H}$ = $(W,(R'_a)_{a\in G},(C_{ab})_{a,b\in G},V)$ and R'_a is defined as follows: for all $w\in W$, if there is some $b\in H$ such that $w\in C_{ba}$ and $\mathfrak{M},w\models\mathsf{B}_b\varphi$, we put

$$R'_a(w):=R_a(w)\cap[\![\varphi]\!]_{\mathfrak{M}}.$$

Otherwise, we put $R'_a(w):=R_a(w)$.

By the similar argument to Theorem 3, we can prove the completeness theorem for $\mathbf{K}_{\mathsf{c}[\cdot\downarrow^a_b]}$ over the class of all the finite Kripke models.

Theorem 4. For all formulas φ in the expanded syntax $\mathcal{L}$ with $[\psi\downarrow^H]$, φ is a theorem in $\mathbf{K}_{\mathsf{c}[\cdot\downarrow^H]}$ of Table 3 iff φ is valid on the class of all finite Kripke models.

Example 3. In Example 1, let $H=\{a\}$ be a group of senders. Then, when we focus on the world w_2, we can calculate the receivers by the calculation just before this example and specify the receivers as $\{a,b\}$, since there is a channel from a to b in w_2 and a believes p in w_2. So, we obtain the truth of $[p\downarrow^H]\mathsf{B}_b p$ at w_2, i.e., 'after the group of agent H sends a piece p of information via communication channel, agent b comes to believe p' in w_2. Moreover, the updated model of $\mathfrak{M}$ by $[p\downarrow^H]$ is the same as Figure 2.

However, when we change the group of senders to $H'=\{b\}$, agent b does not believe p in w_2 (i.e., $\mathsf{B}_b p$ is false in w_2), and so, the accessible worlds from w_2 will be unchanged even after the update of $\mathfrak{M}$ by $[p\downarrow^{H'}]$. Therefore, $[p\downarrow^{H'}]\mathsf{B}_b p$ is still false at w_2.

4 MATRIX REPRESENTATION OF KRIPKE SEMANTICS

A usual Kripke frame (W,R) (for a single agent) can be regarded as a directed graph, i.e., a set W of possible worlds corresponds to a set of nodes, and a set R of accessibility relation corresponds to a set of edges. Generally speaking, such set of edges can be written as a boolean matrix. Therefore, the accessibility relation (= a belief state of an agent) can be represented in a matrix. In this case, the accessibility from possible world i to j can be mapped to the (i,j)-element of the matrix. In what follows, we use $M(m\times n)$ to mean the set of all $m\times n$-boolean matrix.

Let us provide a matrix representation of our notions of frame and model. First, we start with frames. Given any Kripke frame $\mathfrak{F}=(W,(R_a)_{a\in G},(C_{ab})_{a,b\in G})$ with $\#W$ = n, we write W = $\{w_1,w_2,\dots,w_n\}$ and define matrix representations of C_{ab} and R_a as follows.

In accordance with $C_{ab}\subseteq W$ ($a,b\in G$), C^M_{ab} is a matrix in $M(n\times 1)$, i.e., a column vector where the k's component is 1 if $w_k\in C_{ab}$, otherwise 0. In general, given any relation $R\subseteq W\times W$, R^M is a matrix in $M(n\times n)$ such that

$$R^M(i,j)=\begin{cases}1 & \text{if } (w_i,w_j)\in R\\ 0 & \text{otherwise}\end{cases}$$

Now we move to define a matrix representation of a model $\mathfrak{M}=(W,(R_a)_{a\in G},(C_{ab})_{a,b\in G},V)$. Here we assume that the number $\#\mathsf{Prop}$ of propositional letters is m and $\#W$ of possible worlds is n. Our matrix representation of $V(p)$ is similar to a channel relation C_{ab}. That is, $V(p)^M$ is a matrix in $M(n\times 1)$ (= a column vector) where the k's component is 1 if $w_k\in V(p)$, otherwise 0.

Now we can rewrite Kripke semantics to our syntax in terms of matrix. We inductively associate each formula φ of $\mathcal{L}$ with a column vector $\|\varphi\|_{\mathfrak{M}} \in M(n \times 1)$ as follows:[1]

$$\begin{array}{lllllll} \|p\|_{\mathfrak{M}} & := & V(p)^M & \|\mathsf{c}_{ab}\|_{\mathfrak{M}} & := & C^M_{ab} \\ \|\neg\varphi\|_{\mathfrak{M}} & := & \overline{\|\varphi\|_{\mathfrak{M}}} & \|\mathsf{B}_a\varphi\|_{\mathfrak{M}} & := & \overline{R^M_a\overline{\|\varphi\|_{\mathfrak{M}}}}. \end{array}$$
$$\|\varphi\vee\psi\|_{\mathfrak{M}} := \|\varphi\|_{\mathfrak{M}} + \|\psi\|_{\mathfrak{M}}$$

where, for $X \in M(n \times n)$, $\overline{X}$ means the boolean complementation of X. For the dual $\widehat{\mathsf{B}_a}$ of B_a, it is easy to see that $\|\widehat{\mathsf{B}_a}\varphi\|_{\mathfrak{M}} = R^M_a\|\varphi\|_{\mathfrak{M}}$. If the underlying model is clear from the context, we drop the subscript '$\mathfrak{M}$' from $\|\varphi\|_{\mathfrak{M}}$. We use $\|\varphi\|_{w_i}$ to means the i-th component $\|\varphi\|(i)$ of the column vector $\|\varphi\|_{\mathfrak{M}}$, i.e., the truth value of the formula φ at w_i of $\mathfrak{M}$.

Example 4. $\|\mathsf{B}_a\, p\|_{\mathfrak{M}}$ in Example 1 is calculated as:

$$\overline{R^M_a\overline{\|p\|_{\mathfrak{M}}}} = \overline{\begin{pmatrix}1\,1\,1\\0\,1\,0\\0\,0\,1\end{pmatrix}\overline{\begin{pmatrix}1\\0\\1\end{pmatrix}}} = \overline{\begin{pmatrix}1\\0\\1\end{pmatrix}} = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

This result coincides with our explanation in Example 1 (recall also Figure 1).

This match up can be captured by the following proposition.

Proposition 5. Given any finite model $\mathfrak{M}$ and any formula φ of $\mathcal{L}$, we can show that $([\![\varphi]\!]_{\mathfrak{M}})^M = \|\varphi\|_{\mathfrak{M}}$.

5 MATRIX REPRESENTATION OF DYNAMIC OPERATORS

Given a Kripke model $\mathfrak{M}$ with a domain $W = \{w_1,\ldots,w_n\}$, we may easily rewrite semantic clauses of $[\varphi\downarrow^a_b]$ and $[H\downarrow^\varphi]$ in terms of matrix such as: $\|[\varphi\downarrow^a_b]\psi\|_{\mathfrak{M}} := \|\psi\|_{\mathfrak{M}^{\varphi\downarrow^a_b}}$ and $\|[H\downarrow^\varphi]\psi\|_{\mathfrak{M}} := \|\psi\|_{\mathfrak{M}^{\varphi\downarrow^H}}$ where $\|[\varphi\downarrow^a_b]\psi\|_{\mathfrak{M}}$ and $\|[H\downarrow^\varphi]\psi\|_{\mathfrak{M}}$ are matrices in $M(n\times 1)$. However, it is not so clear if we can capture processes of updating $\mathfrak{M}$ to $\mathfrak{M}^{\varphi\downarrow^a_b}$ and $\mathfrak{M}^{\varphi\downarrow^H}$ in terms of operations over matrices. (Van Benthem and Liu, 2007) propose a general framework of updating agents' accessibility relations in terms of program term of propositional dynamic logic. With the help of their ideas, this section provides matrix representations of our two dynamic operators $[\varphi\downarrow^a_b]$ and $[H\downarrow^\varphi]$. First, we expand our syntax of static logic of agents' belief with terms of (iteration free) propositional dynamic logic, and then we explain the main idea of (Van Benthem and Liu, 2007) in Section 5.1. Finally, we rewrite their semantic idea in terms of matrix in Section 5.2.

[1] In order to handle multiple agents G, (Fitting, 2003) employed the notion of $\mathcal{P}(G)$-valued matrix. However, we keep ourselves to the boolean matrices in this paper.

5.1 Propositional Dynamic Logic of Relation Changers

The syntax of PDL-extension of $\mathcal{L}$ is defined by simultaneous induction on a program term π and a formula φ:

$$\pi ::= \mathsf{R}_a \mid (\pi\cup\pi) \mid (\pi;\pi) \mid \varphi? \quad (a\in G)$$
$$\varphi ::= p \mid \mathsf{c}_{ab} \mid \neg\varphi \mid \varphi\vee\varphi \mid [\pi]\varphi \quad (p\in\mathsf{Prop},\, a,b\in G)$$

Here we regard R_a as an *atomic program* (for agent a). $[\mathsf{R}_a]$ corresponds to the previous belief operator B_a. So, in what follows, we also write B_a for $[\mathsf{R}_a]$, if no confusion arises from the context. Then, we may read the program terms as follows: $(\pi\cup\pi')$ is to read 'do π or π', non-deterministically"; $(\pi;\pi')$ is to read "do π followed by π'"; $\varphi?$ is to read "proceed if φ true, else fail". As is well-known, we can introduce some standard programming constructs by definitional abbreviation. For example,

if φ **then** π **else** $\pi' := (\varphi?;\pi)\cup((\neg\varphi)?;\pi')$.

Given a model $\mathfrak{M} = (W,(R_a)_{a\in G},(C_{ab})_{a,b\in G},V)$, we define the semantics of our PDL-extension by:

$$\begin{array}{lll}
[\![\mathsf{R}_a]\!]_{\mathfrak{M}} & := & R_a \\
[\![\pi\cup\pi']\!]_{\mathfrak{M}} & := & [\![\pi]\!]_{\mathfrak{M}}\cup[\![\pi']\!]_{\mathfrak{M}} \\
[\![\pi;\pi']\!]_{\mathfrak{M}} & := & [\![\pi]\!]_{\mathfrak{M}}\circ[\![\pi']\!]_{\mathfrak{M}} \\
[\![\varphi?]\!]_{\mathfrak{M}} & := & \{(w,v)\mid w=v \text{ and } w\in[\![\varphi]\!]_{\mathfrak{M}}\} \\
[\![p]\!]_{\mathfrak{M}} & := & V(p) \\
[\![\mathsf{c}_{ab}]\!]_{\mathfrak{M}} & := & C_{ab} \\
[\![\neg\varphi]\!]_{\mathfrak{M}} & := & W\setminus[\![\varphi]\!]_{\mathfrak{M}} \\
[\![\varphi\vee\psi]\!]_{\mathfrak{M}} & := & [\![\varphi]\!]_{\mathfrak{M}}\cup[\![\psi]\!]_{\mathfrak{M}} \\
[\![[\pi]\varphi]\!]_{\mathfrak{M}} & := & \{w\in W\mid [\![\pi]\!]_{\mathfrak{M}}(w)\subseteq[\![\varphi]\!]_{\mathfrak{M}}\},
\end{array}$$

where $R\circ S$ is the relational composition of R with S, i.e., $(w,v)\in R\circ S$ iff $(w,u)\in R$ and $(u,v)\in S$ for some $u\in W$, and $[\![\pi]\!]_{\mathfrak{M}}(w) := \{v\in W\mid (w,v)\in[\![\pi]\!]_{\mathfrak{M}}\}$. Note that $[\![[\mathsf{R}_a]\varphi]\!]_{\mathfrak{M}}$ is the same meaning as the truth set $\{w\in W\mid \mathfrak{M},w\models\varphi\}$ of the previous Kripke semantics.

Recall that, in the semantics of $[\varphi\downarrow^a_b]$ and $[\varphi\downarrow^H]$ ($H\subseteq G$), we keep the domain of a model, channel relations, and a valuation for proposition letters but *redefine* the accessibility relation $(R_a)_{a\in G}$. In this sense, we may say that those operations are *relation changers*. (Van Benthem and Liu, 2007) observed that, if relation changing operations are written in terms of program terms generated from atomic programs by the composition ;, the union $\cup$ and the test φ?, then we can automatically generate the set of reduction axioms (as in Tables 2 and 3) to assure semantic completeness of propositional dynamic logic with relation changing operations. Let us suppose that our relation changer for a relation $R_a = [\![\mathsf{R}_a]\!]_{\mathfrak{M}}$ is written in terms of a program term π_a ($a\in G$). Then, we may denote by $[(\mathsf{R}_a := \pi_a)_{a\in G}]$ our dynamic operator which changes an original relation R_a into a new relation R'_a via π_a for all agents $a \in G$. Then, our key equivalence for generating the reduction axioms takes the following form:

$$[(\mathsf{R}_a := \pi_a)_{a\in G}][\mathsf{R}_b]\varphi \leftrightarrow [\pi_b][(\mathsf{R}_a := \pi_a)_{a\in G}]\varphi.$$

where we generalize van Benthem and Liu's equivalence for a single agent to multi-agents.

Example 6. 1. Semi-private Announcement: In the semantics of $[\varphi{\downarrow}^a_b]$, we have rewritten the accessibility relations $(R_a)_{a\in G}$ into the new ones $(R'_a)_{a\in G}$. We may reformulate the semantics in terms of binary relations.

- Let $c = b$. Then, $R'_c := (R_c \cap ([\![\mathsf{c}_{ac} \wedge \mathsf{B}_a\,\varphi]\!] \times [\![\varphi]\!])) \cup (R_c \cap ([\![\neg(\mathsf{c}_{ac} \wedge \mathsf{B}_a\,\varphi)]\!] \times W))$.
- Let $c \neq b$. Then, $R'_c := R_c$.

Then, the corresponding relation changer agent b to $[\varphi{\downarrow}^a_b]$ is the following. When $c = b$,

$$\pi_b := ((\mathsf{c}_{ab} \wedge \mathsf{B}_a\,\varphi)?;\mathsf{R}_b;\varphi?) \cup (\neg(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,\varphi)?;\mathsf{R}_b).$$

If we employ the previous definitional abbreviation, we may write π_b as:

$$\pi_b := \textbf{if } \mathsf{c}_{ab} \wedge \mathsf{B}_a\,\varphi \textbf{ then } \mathsf{R}_b;\varphi? \textbf{ else } \mathsf{R}_b.$$

When $c \neq b$, the relation changer for agent c for $[\varphi{\downarrow}^a_b]$ is: $\pi_c := \mathsf{R}_c$. Then, we may regard $[\varphi{\downarrow}^a_b]$ as $[(\mathsf{R}_a := \pi_a)_{a\in G}]$.

2. Introspective Announcement via Communication Channel: Let a be any agent. The corresponding relation changer to $[\varphi{\downarrow}^H]$ is the following program term $\pi'_b := (\psi?;\mathsf{R}_b;\varphi?) \cup (\neg\psi?;\mathsf{R}_b)$, where $\psi := \bigvee_{a\in H}(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,\varphi)$. By the previous definitional abbreviation, we may write π'_b as:

$$\pi'_b := \textbf{if } \left(\bigvee_{a\in H}(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,\varphi)\right) \textbf{ then } \mathsf{R}_b;\varphi? \textbf{ else } \mathsf{R}_b.$$

Then, we may regard $[\varphi{\downarrow}^H]$ as $[(\mathsf{R}_a := \pi'_a)_{a\in G}]$.

5.2 Relation Changers in Matrix Form

Given two relations $R_1, R_2 \subseteq W \times W$. Relational union and composition fit well with matrix addition and multiplication as follows:

$$(R_1 \cup R_2)^M = R_1^M + R_2^M, \quad (R_1 \circ R_2)^M = R_1^M R_2^M$$

Let φ be a formula of static logic of agents' belief. Since $[\![\varphi?]\!]_{\mathfrak{M}} = \{\,(w,v) \mid w = v \text{ and } \mathfrak{M}, w \models \varphi\,\}$ is also a relation on W, we may provide a matrix representation $[\![\varphi?]\!]_{\mathfrak{M}}$. By definition of R^M, we obtain:

$$[\![\varphi?]\!]^M_{\mathfrak{M}}(i,j) = \begin{cases} 1 & \text{if } i = j \text{ and } \mathfrak{M}, w_i \models \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $[\![\varphi?]\!]^M_{\mathfrak{M}}$ is the matrix from which diagonal components we may read off the information of truth set of $[\![\varphi]\!]_{\mathfrak{M}}$ of the formula φ. For test program, we note the following proposition.

Proposition 7. Let φ and ψ be formulas. Then, $[\![(\varphi \wedge \psi)?]\!] = [\![\varphi?]\!] \circ [\![\psi?]\!]$. Therefore, $[\![(\varphi \wedge \psi)?]\!]^M = [\![\varphi?]\!]^M [\![\psi?]\!]^M$.

Example 8. Let us see whether our matrix representation of model update for semi-private announcement works on our running example (Example 1). As is the same as in Example 2, we consider the update by $[p{\downarrow}^a_b]$. There are channel between agent a and b, and agent a believes that p at w_2. By Proposition 7, the first part of a matrix calculation of R_b becomes:

$$[\![(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,p)?]\!]^M R_b^M [\![p?]\!]^M = [\![\mathsf{c}_{ab}?]\!]^M [\![\mathsf{B}_a\,p?]\!]^M R_b^M [\![p?]\!]^M$$
$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Then calculate also the remaining part of R'_b, i.e.,$[\![\neg(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,p)?]\!]^M R_b^M$, we combine both results to obtain updated relation R'_b of agent b as:

$$R'_b = [\![(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,p)?]\!]^M R_b^M [\![p?]\!]^M + [\![\neg(\mathsf{c}_{ab} \wedge \mathsf{B}_a\,p)?]\!]^M R_b^M$$
$$= \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}1&1&1\\0&0&0\\1&1&1\end{pmatrix} = \begin{pmatrix}1&1&1\\0&1&0\\1&1&1\end{pmatrix}$$

This coincides with the result of Example 2 (see Figure 2)

6 IMPLEMENTATION

This section introduces two algorithms. One of them calculates the truth value of a formula $\mathsf{B}_a\,p$ and the other one calculates the relation updates by $[p{\downarrow}^a_b]$. For both algorithms, we assume that an input model $\mathfrak{M} = (W, (R_a)_{a\in G}, (C_{ab})_{a,b\in G}, V)$ is represented in terms of boolean matrix.

Algorithm 1: Calculation of $\|B_a p\|_w$.

procedure BELIEF-OF
  input $\mathfrak{M}$, $w_i \in W$, $a \in G$, $p \in \mathsf{Prop}$
  $\|\mathsf{B}_a\,p\| := R_a^M \overline{V(p)^M}$
  return True **if** $\|\mathsf{B}_a\,p\|(i) > 0$; False **otherwise**
end procedure

Here we comment just on Algorithm 2. In order to update an accessibility relation of agent b, the algorithm loops to find agent b. If the algorithm finds agent b, a model updating procedure (for a single agent) will be started, otherwise it just put $R'_c = R_c$. At the beginning of the updating procedure, the algorithm generates test matrices through `Test` function where an input of this function is a column vector, and it enumerates the elements of the input vector

Algorithm 2: Calculation of $[p{\downarrow}_b^a]$.

procedure SEMI-PRIVATE-ANNOUNCEMENT
  input $\mathfrak{M}$, $a, b \in G$, $p \in \mathsf{Prop}$
  for $c \in G$ **do**
    if $c = b$ **then**
      $X := \mathrm{Test}({C_{ab}}^M)$
      $Y := \mathrm{Test}(\|\mathsf{B}_a\, p\|)$
      $Z := \mathrm{Test}(V(p)^M)$
      ${R'_b}^M := XYR_b^M Z + \overline{XY}R_b^M$
    else
      ${R'_c}^M := {R_c}^M$
    end if
  end for
  return $\mathfrak{M}' = (W, (R'_a)_{a\in G}, (C)_{a,b\in G}, V)$
end procedure

in the diagonal components of an output matrix, and fills 0 in the non-diagonal components of the matrix. Then, it calculates the updated accessibility relation of agent b in terms of boolean matrix. Note that $\|\neg\varphi?\|$ can be calculated as $\overline{\|\varphi?\|}$. Finally, the algorithm returns the updated model $\mathfrak{M}'$.

Implemented Program. We have implemented the preceding algorithms in a single calculator with GUI by Java^{TM} 7. It is now available on our web site[2]. The main features of the calculator are summarized as follows. First, we may edit the numbers of both agents and worlds, and also accessibility relations for agents in terms of boolean matrix. Second, it also implemented an algorithm checking if a given accessibility relation satisfies frame properties such as reflexivity, transitivity, etc. Third, the calculator can visualize both an accessibility relation of an agent and a channel relation (communication channels) between agents at a world, with the help of Graphviz. [3]

7 CONCLUSION

The main contribution of this paper can be summarized as follows. First, we introduced the static doxastic logic with communication channels (where we always assume self-channel on all agents) with the complete axiomatization $\mathbf{K}_c$ that is also decidable (Theorems 1 and 2). We also extended such static logic with two dynamic operators $[\varphi{\downarrow}_b^a]$ (semi-private announcement) and $[\varphi{\downarrow}^H]$ (introspective announcement) with reduction axioms (so extensions of both of them enjoy completeness results, Theorems 3 and 4). A key feature of our dynamic operators are *non-public*, i.e., effects of announcements are restricted to some specified agents determined by communication channels. Second, we followed the idea by (Fitting, 2003) to reformulate Kripke semantics to our doxastic logic in linear algebraic form, and employ the idea of PDL-format by (Van Benthem and Liu, 2007) to provide matrix representations to our two dynamic operators. Finally, based on this linear algebraic reformulation, we implemented the calculation system of agents' beliefs and updates of Kripke models by $[\varphi{\downarrow}_b^a]$. An implementation of $[\varphi{\downarrow}^H]$ is a direction of further work.

[2] http://cirrus.jaist.ac.jp:8080/soft/bc
[3] http://www.graphviz.org/

ACKNOWLEDGEMENTS

We would like to thank anonymous reviewers for their helpful comments. The works of the second and third authors were partially supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B) No. 24700146 and Scientific Research (C) No. 25330434, respectively.

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