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# Single-Player and Two-Player Buttons & Scissors Games

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## Abstract

The Buttons & Scissors puzzle was recently shown to be NP-hard. In this paper we continue studying the complexity of various versions of the puzzle. For example, we show that it is NP-hard when the puzzle consists of  $C = 2$  colors, and polytime solvable for  $C = 1$ . Similarly, it is NP-hard when each color is used by at most  $F = 4$  buttons, and polytime solvable for  $F = 3$ . We also consider restrictions on the board size, cut directions, and cut lengths. Finally, we introduce new two-player games and show that they are PSPACE-complete.

## 1 Introduction

Buttons & Scissors is a single-player puzzle by KyWorks that was recently studied by Gregg et al. [2]. A *level* is an  $n \times n$  grid of buttons of different colors; each position is either empty or has a button of a single color sewn to it. The goal is to remove all buttons using horizontal, vertical and diagonal scissor cuts. A cut is *feasible* if it removes at least two buttons of the same color and no buttons of any other color. Figures 1(a)–(b) show a sample level and solution. See [2] for further clarification of the rules and terminology.

Deciding whether a level is solvable is NP-complete [2]. We show that several restricted versions of the puzzle remain hard, and provide polytime algorithms for a number of easier versions. We also introduce two-player Buttons & Scissors games and show that they are PSPACE-complete. Due to space restrictions, most proofs are sketched or omitted.

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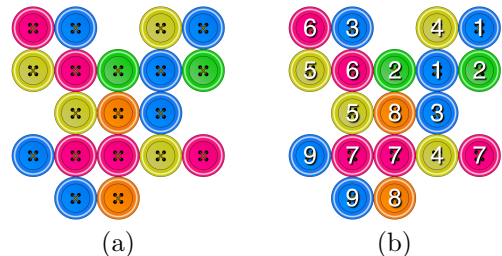


Figure 1: (a) Level 6 in the Buttons & Scissors app is on a  $5 \times 5$  grid with  $C = 5$  colors, each used at most  $F = 6$  times; (b) a solution using nine cuts of minimum length  $\ell = 2$  and all four directions ( $d = *$ ).

## 2 Notation

Each Buttons & Scissors level can be parameterized as follows (see Figure 1 for an example):

1. The *board size*  $m \times n$ .
2. The *number of colors*  $C$ .
3. The *maximum frequency*  $F$  of an individual color.
4. The *cut directions*  $d$  can be limited from the original four directions, which we denote by  $*$ . We only consider  $d \in \{*, *, +, -, -\}$  because an  $n \times m$  board can be rotated  $90^\circ$  to an equivalent  $m \times n$  board, or  $45^\circ$  to an equivalent  $k \times k$  board for  $k = n + m - 1$  by adding blank squares.
5. The *cut length*  $\ell$  is the minimum number of buttons required to be removed by a feasible cut.

These parameters give the following decision problem (the original problem is  $B\&S[n \times n, \infty, \infty, *, 2](B)$ ):

**Decision Problem:**  $B\&S[m \times n, C, F, d, \ell](B)$ .

**Input:** Given an  $m \times n$  board  $B$  with buttons of  $C$  colors, where each color is used at most  $F$  times.

**Output:** True if  $B$  has a solution with minimum cut length  $\ell$  using  $d$  directions. Otherwise, False.

## 3 Single-Player Puzzle

We now present our results on the single-player puzzle.

### 3.1 Board Size

**Remark 1** *If a Buttons & Scissors board can be solved, then it can be solved using only cuts of 2 or 3 buttons.*

**Theorem 1** *Buttons & Scissors is polytime solvable on  $1 \times m$  boards. That is,  $B\&S[1 \times n, \infty, \infty, -, 2](B) \in P$ .*

**Proof.** Consider the following context-free grammar,

$$\begin{array}{llll} S \rightarrow \epsilon & S \rightarrow S\Box & S \rightarrow xSx & S \rightarrow xSxx \\ S \rightarrow SS & S \rightarrow \Box S & & S \rightarrow xxSx \end{array}$$

where  $\Box$  is an empty square and  $x \in \{1, 2, \dots, C\}$ . By Remark 1, the solvable  $1 \times m$  boards are in one-to-one correspondence with the strings in this language.  $\square$

**Theorem 2** *Given a full  $2 \times m$  Buttons & Scissors board with  $C = 2$  and a constant  $s$ , there exists a polynomial time algorithm that removes all but  $s$  buttons from the board with feasible cuts.*

### 3.2 Number of Colors

**Theorem 3** *Buttons & Scissors is polytime solvable for 1-color, and is NP-complete for 2-colors. That is,  $B\&S[n \times n, 1, \infty, *, 2](B) \in P$  and  $B\&S[n \times n, 2, \infty, *, 2](B)$  is NP-complete.*

**Proof Sketch:** In a graph  $G$ , the maximum number of vertices that can be covered by edge-disjoint  $K_2$  and  $K_3$  subgraphs is polytime computable in the size of  $G$  (see Cornuéjols et al. [1]). We convert each 1-color board  $B$  into a graph whose vertices can be perfectly covered if and only if  $B$  is solvable. The transformation uses Remark 1 and is non-trivial since the order of cuts is important. The 2-color reduction is in the full paper.  $\square$

### 3.3 Frequency of Colors

**Remark 2** *If board  $B'$  is obtained from board  $B$  by removing every button of a single color, then  $B\&S[m \times n, C, F, d, \ell](B) \implies B\&S[m \times n, C, F, d, \ell](B')$  (i.e., it is impossible that Buttons & Scissors is solvable on  $B$ , but not solvable on  $B'$  with the same parameters).*

**Theorem 4** *Buttons & Scissors is polytime solvable for maximum color frequency  $F = 3$ , and is NP-complete for  $F = 4$ . That is,  $B\&S[n \times n, \infty, 3, *, 2](B) \in P$  and  $B\&S[n \times n, \infty, 4, *, 2](B)$  is NP-complete.*

**Proof Sketch:** If  $F = 3$ , then each color is removed by a single cut in any solution. By Remark 2, these cuts cannot make a solvable board unsolvable. Thus, there is a simple greedy algorithm for deciding solvability.

It was proven that  $B\&S[n \times n, \infty, 7, *, 2](B)$  is NP-complete via 3-SAT [2]. We use the same reduction for  $F = 4$ , with Figure 2 replacing each OR gadget.  $\square$

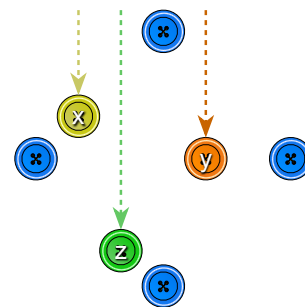


Figure 2: The left blue button can be removed if and only if “ $x$  is removed  $\vee$   $y$  is removed  $\vee$   $z$  is removed”.

### 3.4 Cut Directions

NP-completeness for horizontal and vertical cuts and  $F = 7$  was proven in [2]. We improve this to  $F = 6$ .

**Theorem 5**  *$B\&S[n \times n, \infty, 6, +, 2](B)$  is NP-complete.*

### 3.5 Cut Lengths

In Section 3.2 we saw that the 1-color version has a polytime algorithm. However, if the minimum cut length is set to  $\ell = 3$  (instead of  $\ell = 2$ ) then it is NP-complete.

**Theorem 6**  *$B\&S[n \times n, 1, \infty, *, 3](B)$  is NP-complete.*

## 4 Two-Player Games

In our two-player games, the players take turns making feasible cuts on a common board  $B$ . In a *partisan game*, some cuts are available to one player, but not the other. In an *impartial game* all feasible cuts can be made by both players. We consider two losing conditions:

1. The player cannot execute a feasible cut (LAST).
2. The player removed fewer buttons (MAX).

**Theorem 7** *The partisan LAST two-player game is PSPACE-complete, where player 1 cuts blue buttons, player 2 cuts red, and both cut green (if any).*

**Theorem 8** *The impartial MAX and LAST two-player Buttons & Scissors games are PSPACE-complete.*

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