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Summary of the Research

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1. Background

As an alternative to classical logic, **intuitionistic logic** has been quite influential since its foundation. Its regarding of truth as provability enabled the formalisation of constructive mathematics, and also helped connecting logic and computer science via so-called Curry-Howard correspondence between proofs and computations[?].

Although perhaps less known than the classical&intuituitionistic siblings, there exists a subsystem of intuitionistic logic which is worthy of attention: **minimal logic**. The first formulation of the system of minimal logic is due to Ingebrigt Johansson in [3]. It shares the same set of positive axioms as intuitionistic logic, and contains the axiom for negation $(A \to A) \land (A \to \neg A) \to \neg A$. There is another formulation of minimal logic, whose language employs the symbol $\bot$ for contradiction, instead of the negation symbol $\neg$. In this formulation a negation $\neg$A is then defined as $A \to \bot$, and no special axiom for $\bot$ is required. In other words, the symbol behaves in the same way as propositional variables.

Nowadays, this second formulation is more common, and its lack of contradiction axiom enforces the impression that it certainly is the 'minimal' logic. But if we remember the first formulation, then we see no reason that the negation axiom cannot be further weakened, and obtain subsystems of minimal logic containing weaker negation. This is the approach taken in Colacito et al. [2] and Colacito [1]. There they have studied the following axioms:

Co: $(A \to B) \to (\neg B \to \neg A)$ An: $(A \to \neg A) \to \neg A$

NeF: $(A \land \neg A) \to \neg B$ N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$

These axioms all define negations weaker than that of minimal logic. For this reason they are called **subminimal negation** axioms, and logics employing them **subminimal logics**. It has been shown that the combinations of axioms N+An and NeF+N+An both define the minimal negation. In addition, it is known that Co implies NeF and N. After formulating these axioms, they defined propositional logical systems CoPC, NeFPC and NPC, respectively by adding Co, NeF+N, N to the positive axioms of minimal logic. Then proof systems (Hilbert-type, sequent calculus) and semantics (Kripke, algebraic) were given for these systems, and formal characteristics like the possibility of cut-elimination and disjunction property were investigated.

The aim of this research is to obtain a more general understanding of subminimal negations and prepare for a further investigation later on, when the author studies for doctorate. The significance of subminimal negation primarily lies in its enabling us to classify various inferences of negation in terms of their inter-derivability (with the positive fragment of minimal logic as the common basis), which is lost at the level of minimal logic. Secondly, it help us diagnose why some counter-intuitive inferences concerning negation, such as NeF, hold in minimal logic. This would in turn makes it possible to create a subminimal system whose negation behaves less counter-intuitively. Finally, the study of subminimal negation may help us analyse and classify negative expressions in natural language.

2. Structure

The report is divided into six sections. The first section provides the information on the previous research, as well as providing the motivation and the overview of the research.

The second section introduces some preliminary information on minimal logic and other related systems, such as positive logic, the negation-less fragment of minimal logic. Subsection 2.1 introduces language and the proof theory for these logics, where $\neg$ is taken to be primitive (for minimal logic) and the proof systems are Hilbert-type. In the section 2.2, Kripke semantics for positive and minimal logic are introduced.

The third section concerns proof theoretic analysis of subminimal systems. In 3.1, some additional subminimal axioms are introduced, and their relationships are observed. The axioms introduced here are as follows:

$$\text{DN: } A \to \neg\neg A,$$

$$\text{IDN: } (\neg A \to A) \to \neg\neg A$$

$$\text{wM: } (\neg A \to B) \wedge (\neg A \to \neg B) \to \neg\neg A.$$

It is shown that wM implies IDN, which in turn implies DN. Also, among others, we see that NeF+An, DN+Co each defines the minimal negation. Subsection 3.2 deals with the counter-intuitive axiom NeF. There it is seen that there exists a weaker version of it, D: $(A \wedge \neg A) \to (B \to \neg B)$. D is also derivable from N, and thus it is noted that the counter-intuitivity lies deeper than NeF. Subsection 3.3. studies the relationship between subminimal axioms on the one hand and negation axioms defining intuitionistic&classical negation on the other hand. The latter axioms are named **superminimal axioms** in this report. We mainly focus on two superminimal axioms:

$$\text{CM: } (\neg A \to A) \to A, \qquad \text{EFQ: } (A \wedge \neg A) \to B$$

It is observed that the combination EFQ+An defines the intuitionistic negation, and the combination CM+EFQ defines the classical negation. This result is followed by a generalisation of the relation between CM, EFQ and An. Subsection 3.4changes the proof system to sequent calculus. The main result in this subsection is obtained through defining the classes of formally positive/negative formulas. Using these classes, we see that several separation results can be established, such as Co not deriving DN. This subsection also treats subminimal axioms hinted in [4].

The fourth section adopts a semantic approach. In 4.1, Kripke semantics for NPC introduced in [1], [2] are explained. briefly, in this semantics the valuation of negation is given by a mapping *N*, which when given an upward closed set of worlds gives another such set. Then a negative formula $\neg A$ is forced in a world w if $w \in N(V(A))$. With this semantics, in 4.2 we set out to investigate the correspondence between subminimal axioms and the properties of Kripke frames validating them: **subminimal correspondence theory**. We concentrate on formulas with just one type of propositional variables. Such formulas are divided into different classes, and the correspondence is established for each of the classes.

The fifth section considers another set of logical systems which are weaker than minimal logic. These systems take contradiction as primitive, but each formula A has its own contradiction $\bot_A$, such that $\neg A$ is defined to be $\mathrm{A} \to \bot_A$. This sort of **multi-absurdity** has already been considered in [4]. the task in this section is to observe how such logics related to minimal and subminimal logics. In 5.1 we establish that a multi-absurdity system with extensional $\bot_A$ and axiom $\bot_A \to \mathrm{A}$ is so-called definition equivalent to minimal logic. In 5.2, we see that the same logic but without extensionality turns out to be closely related to the subminimal logic defined by An+positive logic (though not definition equivalent to it).

The last section provides some concluding remarks, including the future tasks.

3. Reference

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[4] S. Odintsov (2008), Constructible Negations and Paraconsistensy. Trends in Logic, Volume 26, Springer Netherlands.