A Survey On the Logical Notions of Negation

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1 Introduction

When one wishes to consider the notion of negation in logic, a good strategy would be to start from a logic with a relatively weaker negation. Thus arises as a candidate the system of *intuitionistic logic*, a logic which identifies truth with provability and first formalised by Heyting(1930) [6]. Intuitionistic logic has a weaker negation than the more common *classical logic*, in that it invalidates the inference of *double negation elimination* $\neg\neg A \to A$. But one can in fact go weaker, and if he does so soon he aquaints himself with another system called *minimal logic*.

Minimal logic is first formulated by Johansson(1937) [7]. It is born out of his dissatisfaction with Heyting's acceptance of the law of *ex falso quodlibet* in his formulation of intuitionistic logic [8]. Informally, this law states that every proposition is inferable from a contradiction. Formally, we can express this as $(A \wedge \neg A) \to B$.

The validity of *ex falso* has been questioned by various people, and many alternative logics have been proposed to overcome this. Minimal logic is one such logic. It is obtained simply by dropping *ex falso* from intuitionistic logic. This gives minimal logic a flavour very similar to intuitionistic logic, but there are a few significant differences. The most notable amongst them is the loss of *disjunctive syllogism*: $[(A \wedge B) \wedge \neg A] \to B$. On the other hand, a fragment of *ex falso* still remain in minimal logic; the *negation* of any proposition is inferable from a contradiction, viz. $(A \wedge \neg A) \to \neg B$.

This last feature is controversial still, as it makes negations trivial once a contradiction is obtained. Odintsov(2008) [9] attempted to avoid this negative *ex falso*, by restricting one of the formulas used to derive it, $(A \to B) \wedge (A \to \neg B) \to \neg A$. In order to achieve this, He weakenend the negation by splitting up the contradiction $\bot$ into *contradiction operators* $C(A)$ for each formula A. The negation $\neg A$ is then defined as $A \to C(A)$.

In minimal logic, there are two ways to define negation. One is to take $\neg$ as primitive. The other is to take $\bot$ as primitive, and define $\neg A$ as in $A \to \bot$. In the former case, it turns out that the formula $(A \to B) \wedge (A \to \neg B) \to \neg A$ above defines the negation of minimal logic, whereas in the latter case, no axiom for $\bot$ exists. It thus becomes apparent, that by weakening this axiom we can also obtain logics weaker than minimal logic. This is exactly the direction taken in A. Colacito(2016) [2] and A. Colacito et al.(2017) [3]. There, such subsystems are named *subminimal logics*, and the weaker negations *subminimal negations*. They took as the basic system $\mathbf{IPC}^+ + \mathrm{N}$, where $\mathbf{IPC}^+$ is the positive fragment of intuitionistic logic and N is the subminimal negation axiom $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$. On this basis proof systems and semantics for some subminimal systems are given, as well as metalogical results like completeness and cut-elimination.

Subminimal logic is a relatively unstudied area of mathematical logic. It however has a huge potential in facilitating our understanding of negation. By not altering the behaviour of other connectives, it allows us to investigate negation in isolation. This enables us to see more finely the relationship among various inferences concerning negation. It is therefore of interest to mathematicians and philosophers alike. In addition, it is also conceivable that the logics shed light on the nature of negative expressions in natural language. Furthermore, it can contribute to the enhancement of our understanding of logical paradoxes, as negation plays a significant role in many paradoxes, including the liar paradox. Paradoxes are one of the sources for the popular appreciation of logic, so this is potentially an important application.

In this paper, we shall begin with providing some preliminary information about propositional minimal logic and its fragment, positive logic. Then we move on to investigate properties of subminimal logics. Our investigation is threefold. In the first part, the relationship among subminimal axioms are studied, using proof-theoretic approach. In the second part, the correspondence between subminimal axioms and kripke frames is studied. In the third part, the relationship between subminimal logic and Odintsov's logics with multiple contradictions is studied. Finally, we conclude with remarks on possible future directions.

2 Positive and Minimal Logic

In this section, we shall provide some basic information about the syntax and semantics of a formalisation of propositional minimal logic, $\mathbf{MPC}_\neg$ (**PC** stands for *Propositional Calculus*), formulated in [2] and [3]. Along with it, we shall also introduce a simpler system called **PPC**, which is a formalisation of the negation-less fragment of minimal logic(*positive logic*).

2.1 Proof theory for Positive and Minimal Logic

Let us start with specifying the language. We specify the symbols (*vocabulary*) to use, and declare what concatenations of them (*formulas*) we shall deem well-formed.

Definition 2.1.1 (vocabulary of $\mathbf{PPC}/\mathbf{MPC}_\neg$)**.**
The vocabulary of **PPC** consists of the following symbols.

· *Countable number of propositional variables* $p_0, p_1, p_2, \ldots$.
· *Connectives* $\wedge, \vee, \rightarrow$.
· *Parentheses* (,).

$\mathbf{MPC}_\neg$ in addition contains an additional connective $\neg$.

Definition 2.1.2 (formulas of $\mathbf{PPC}/\mathbf{MPC}_\neg$)**.**
We inductively define the formulas of $\mathbf{PPC}/\mathbf{MPC}_\neg$ as follows.

· Each propositional variable is a formula.
· If A, B are formulas, then $(A \wedge B)$, $(A \vee B)$. $(A \rightarrow B)$ are formulas.
· If A is a formula, then $\neg A$ is a formula. [$\mathbf{MPC}_\neg$ only]

We shall use the abbreviation $A \leftrightarrow B$ for $(A \rightarrow B) \wedge (B \rightarrow A)$.

As for proof system, we shall employ Hilbert-type systems. A proof of a formula A from a set of formulas(*assumptions*) Γ is a finite sequence $A_1, \ldots A_n$ of formulas, where $A_n \equiv A$. Each A_i is either an axiom, an assumption or obtained from previous terms by a deduction rule. We denote $\Gamma \vdash A$ if such a sequence exists.

Definition 2.1.3 (Hilbert-type system for $\mathbf{PPC}/\mathbf{MPC}_\neg$)**.**
The proof system for positive logic contains the following axioms and a deduction rule.

<u>Axioms</u>
$A \rightarrow (B \rightarrow A)$; $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;
$A \rightarrow (A \vee B)$; $B \rightarrow (A \vee B)$; $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$;
$A \wedge B \rightarrow A$; $A \wedge B \rightarrow B$; $A \rightarrow (B \rightarrow (A \wedge B))$.

<u>Rule</u>
MP: If $\Gamma \vdash A$ and $\Gamma \vdash A \rightarrow B$, deduce $\Gamma \vdash B$.

The system for $\mathbf{MPC}_\neg$ is identical, except that we have an additional axiom called M: $(A \to B) \wedge (A \to \neg B) \to \neg A$.

Henceforth, we shall denote these systems as $\mathbf{hPPC}$ and $\mathbf{hMPC}_\neg$. When it needs differentiating, we shall denote $\vdash_{\mathbf{P}}$, $\vdash_{\mathbf{M}_\neg}$ etc.. The same convention applies to other consequence relations as well.

It is also worth mentioning a few things about classical/intuitionistic logic. They have the same language as $\mathbf{MPC}_\neg$ (when $\neg$ is primitive). In Hilbert-type proof system, intuitionistic logic has an additional axiom $(A \wedge \neg A) \to B$, and classical logic further has $\neg\neg A \to A$.

2.2 Semantics for Positive and Minimal Logic

Let us now turn our attention to the semantics for these logics.

Definition 2.2.1 (Kripke frame for **PPC**)**.**
Let W be an inhabited set, and $\leq$ be a partially ordered set on W. We shall call $(W, \leq)$ a *Kripke frame* for **PPC**.

Definition 2.2.2 (Kripke model for **PPC**)**.**
Let $\mathcal{F} = (W, \leq)$ be a kripke frame, and let $\mathcal{V}$ be a mapping(*valuation*) from the propositional variables of **PPC** to the subsets of W, satisfying what is called the *persistency* of valuation:

- for all $w, w' \in W$, if $w \in \mathcal{V}(p)$ and $w \leq w'$ then $w' \in \mathcal{V}(p)$.

We shall write $w \in \mathcal{V}(p)$ also as $w \Vdash_{\mathbf{P}} p$ (or more explicitly, $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} p$). We now extend $\mathcal{V}$ to non-atomic formulas by setting:

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A \wedge B \Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A \text{ and } (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} B.$$
$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A \vee B \Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A \text{ or } (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} B.$$
$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A \to B \Leftrightarrow \text{ for all } w' \geq w, \text{ if } (\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{P}} A \text{ then } (\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{P}} B.$$

We call the pair $(\mathcal{F}, \mathcal{V})$ a *Kripke model* for **PPC**.

Definition 2.2.3 (validity for **PPC**)**.**
We write $(\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{P}} A$ if $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{P}} A$ for all $w \in W$.
We write $\mathcal{F} \vDash_{\mathbf{P}} A$ if $(\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{P}} A$ for all $\mathcal{V}$.
We write $\Gamma \vDash_{\mathbf{P}} A$ if $(\mathcal{F}, \mathcal{V})$ is such that $(\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{P}} B$ for all $B \in \Gamma$, then $(\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{P}} A$. In particular, if $\Gamma = \emptyset$ we write $\vDash A$.

Definition 2.2.4 (Kripke frame for $\mathbf{MPC}_\neg$)**.**
A *Kripke frame* for $\mathbf{MPC}_\neg$ is a triple $(W, \leq, F)$, where $(W, \leq)$ are as in **PPC**, and F is an upward closed subset of W.

Definition 2.2.5 (Kripke model for $\mathbf{MPC}_\neg$)**.**
A *Kripke model* for $\mathbf{MPC}_\neg$ is a pair $(\mathcal{F}, \mathcal{V})$, where:

- $\mathcal{F}$ is a Kripke frame for $\mathbf{MPC}_\neg$.
- $\mathcal{V}$ is defined like **PPC**, but with the following valuation for negation:

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{M}_\neg} \neg A \Leftrightarrow \text{ for all } w' \geq w, \text{ if } (\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{M}_\neg} A \text{ then } w' \in F.$$

The validity for $\mathbf{MPC}_\neg$ is defined analogously to that of **PPC**. To understand how this semantics works, let us see an example.

Example 2.2.1 (validity of M)**.**
$\vDash_{\mathbf{M}_\neg} (A \to B) \wedge (A \to \neg B) \to \neg A$

Proof.
Let $(\mathcal{F}, \mathcal{V})$ be a model and $w \in W$. Suppose $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{M}_\neg} (A \to B) \wedge (A \to \neg B)$ for $w' \geq w$. Then if $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{M}_\neg} A$, $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{M}_\neg} B$ and $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{M}_\neg} \neg B$. So $w'' \in F$. Thus $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{M}_\neg} \neg A$, and consequently $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{M}_\neg} (A \to B) \wedge (A \to \neg B) \to \neg A$. As $(\mathcal{F}, \mathcal{V})$ and w are arbitrary, we conclude $\vDash_{\mathbf{M}_\neg} (A \to B) \wedge (A \to \neg B) \to \neg A$ □

The following theorem provides the necessary connection between the semantics and the proof theory.

Theorem 2.2.1 (Soundness and Completeness of **PPC**/$\mathbf{MPC}_\neg$ [3])**.**
(i) $\Gamma \vdash_{\mathbf{P}} A \Leftrightarrow \Gamma \vDash_{\mathbf{P}} A$.
(ii) $\Gamma \vdash_{\mathbf{M}_\neg} A \Leftrightarrow \Gamma \vDash_{\mathbf{M}_\neg} A$.

Proof.
The left-to-right directions (soundness) are demonstrable by induction on the depth of deduction of $\Gamma \vdash A$. The right-to-left directions (completeness) are slight variations of the proof for **AnPC** presented in Chapter 5. We postpone the details until then. The proof there is smoothly transferable to the present cases once we tweak the canonical models: For (i) simply omit Φ from the model, and for (ii) use $F := \{\Delta | A \in \Delta \text{ and } \neg A \in \Delta \text{ for some A}\}$ instead of Φ. □

The semantics for intuitionistic logic is obtained from the one for $\mathbf{MPC}_\neg$ by imposing $F = \emptyset$. The one for classical logic is then obtained by restricting W to singletons.

3 Proof Theory for Subminimal Logics

In this section, we consider some properties of subminimal axioms, and subminimal logics employing (combinations of) such axioms. We take **PPC** as the basic system when we consider the inter-deducibility of subminimal axioms. For the sake of simplicity, whenever we add subminimal axioms to **PPC**, we assume that the language is implicitly expanded by the negation symbol. For example, Given an axiom containing negation symbol, Ax, the language of the system **PPC**+Ax is the language of **PPC** plus $\neg$, and $\neg A \to \neg A$ is among the derivable formulas of the system, while not in **PPC**.

The structure of this section is as follows. In 3.1, we introduce subminimal axioms from the previous research [2] and [3], and then introduce consider some combinations of subminimal axioms which are equivalent to the negation axiom of $\mathbf{MPC}_\bot$, $[(A \to B) \wedge (A \to \neg B)] \to \neg A$. In 3.2, We concentrate on the axiom NeF (negative *ex falso*), and see how this axiom relates to other axioms. In 3.3., We turn our eyes on negation axioms for classical/intuitionistic logic (which we shall call *superminimal* negation axioms), and see their connection with subminimal axioms. In 3.4, we change our proof system to sequent calculus, and give some separation results among others.

Because proofs in Hilbert-type systems are rather tedious, the proofs hereafter are given in sketches. They are given in tree-form, and the basic unit of these sketch-proofs are consequences of the form $\Gamma \vdash A$. Whenever clarification is needed in the deduction, we show the main axiom/rule used in square brackets. We use double lines to indicate abbreviations of deduction.

3.1 Subminimal and Minimal Axioms

Definition 3.1.1 (known subminimal axioms [3]).
The following subminimal axioms are considered in [3].

An: $(A \to \neg A) \to \neg A$ Co: $(A \to B) \to (\neg B \to \neg A)$
NeF: $(A \land \neg A) \to \neg B$ N: $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$
DN: $(A \to \neg\neg A)$

Among these axioms, the following relations are known to hold.

Proposition 3.1.1 (Known relations among subminimal axioms [2], [3]).
(i) Co$\Rightarrow$NeF, Co$\Rightarrow$N, Co$\Rightarrow$ $\neg\neg\neg A \to \neg A$
(ii) N+An$\Leftrightarrow$ M

Proof.
(i)
'Co$\Rightarrow$NeF'

$$\cfrac{\cfrac{A \land \neg A \vdash_{\mathbf{P}} (B \to \ A) \land \neg A}{A \land \neg A \vdash_{\mathbf{P}} \neg B}\,[\mathrm{Co}]}{\vdash_{\mathbf{P}} (A \land \neg A) \to \neg B}$$

'Co$\Rightarrow$N'

$$\cfrac{\cfrac{\cfrac{A \leftrightarrow B \vdash_{\mathbf{P}} A \to B}{A \leftrightarrow B \vdash_{\mathbf{P}} \neg B \to \neg A}\,[\mathrm{Co}] \qquad \cfrac{A \leftrightarrow B \vdash_{\mathbf{P}} B \to A}{A \leftrightarrow B \vdash_{\mathbf{P}} \neg A \to \neg B}\,[\mathrm{Co}]}{A \leftrightarrow B \vdash_{\mathbf{P}} \neg A \leftrightarrow \neg B}}{\vdash_{\mathbf{P}} (A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)}$$

'Co$\Rightarrow$ $\neg\neg\neg A \to \neg A$'

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A, \neg A \vdash_{\mathbf{P}} \neg\neg A}\,[\text{NeF, deducible from Co by above}]}{A \vdash_{\mathbf{P}} \neg A \to \neg\neg A}}{A \vdash_{\mathbf{P}} \neg\neg\neg A \to \neg\neg A}\,[\mathrm{Co}]}{\neg\neg\neg A \vdash_{\mathbf{P}} A \to \neg\neg A}}{\neg\neg\neg A \vdash_{\mathbf{P}} \neg\neg\neg A \to \neg A}\,[\mathrm{Co}]}{\vdash_{\mathbf{P}} \neg\neg\neg A \to \neg A}$$

(ii) We show the right implication.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\overline{A, (A \to B) \land (A \to \neg B) \vdash_{\mathbf{P}} A \leftrightarrow B}}}{A, (A \to B) \land (A \to \neg B) \vdash_{\mathbf{P}} \neg B \to \neg A}\,[\mathrm{N}]}{A, (A \to B) \land (A \to \neg B) \vdash_{\mathbf{P}} \neg A}}{(A \to B) \land (A \to \neg B) \vdash_{\mathbf{P}} A \to \neg A}}{(A \to B) \land (A \to \neg B) \vdash_{\mathbf{P}} \neg A}\,[\mathrm{An}]}{\vdash_{\mathbf{P}} (A \to B) \land (A \to \neg B) \to \neg A}$$

$\square$

Let us observing some other axioms and combinations, which are equivalent to M. But before that, let us see that the axiom DN can be strengthened while remaining subminimal.

Definition 3.1.2 (axioms wM, IDN)**.**
wM: $(\neg A \to B) \wedge (\neg A \to \neg B) \to \neg\neg A$
IDN: $(\neg A \to A) \to \neg\neg A$

Proposition 3.1.2 (wM and IDN are subminimal)**.**
(i) M $\Rightarrow$ wM, M $\Rightarrow$ IDN.
(ii) IDN$\Rightarrow$ DN

Proof.
(i) wM is an instance of M, and IDN is obtained from wM by taking $B := A$.
(ii) If A, then $\neg A \to A$. So by IDN, $\neg\neg A$. hence $A \to \neg\neg A$. $\square$

Proposition 3.1.3 (axioms equivalent to M)**.**
(i) $(A \to \neg B) \to (B \to \neg A) \Leftrightarrow$ M.
(ii) $(A \to B) \wedge (B \to \neg B) \to \neg A \Leftrightarrow$ M.
(iii) NeF + An $\Leftrightarrow$ M.
(iv) DN + Co $\Leftrightarrow$ M.

Proof. We shall give proof sketches.
(i) '$\Leftarrow$':

$$\dfrac{\dfrac{\dfrac{A \to \neg B \vdash_{\mathbf{M}_\neg} A \to \neg B \qquad \dfrac{B \vdash_{\mathbf{M}_\neg} B}{B \vdash_{\mathbf{M}_\neg} A \to B}}{A \to \neg B, B \vdash_{\mathbf{M}_\neg} (A \to \neg B) \wedge (A \to B)}}{A \to \neg B, B \vdash_{\mathbf{M}_\neg} \neg A}\,[\mathrm{M}]}{\vdash_{\mathbf{M}_\neg} (A \to \neg B) \to (B \to \neg A)}$$

'$\Rightarrow$': Note that $(A \to \neg B) \to (B \to \neg A) \Rightarrow$ NeF. Thus:

$$\dfrac{\dfrac{\dfrac{\overline{\overline{A \to B, A \to \neg B \vdash_{\mathbf{P}} A \to (B \wedge \neg B)}}}{A \to B, A \to \neg B \vdash_{\mathbf{P}} A \to \neg(A \to B)}\,[\mathrm{NeF}]}{A \to B, A \to \neg B \vdash_{\mathbf{P}} (A \to \neg(A \to B)) \wedge (A \to B)} \qquad \dfrac{\vdash_{\mathbf{P}} (A \to \neg(A \to B)) \to ((A \to B) \to \neg A)}{\vdash_{\mathbf{P}} [(A \to \neg(A \to B)) \wedge (A \to B)] \to \neg A}}{\dfrac{A \to B, A \to \neg B \vdash_{\mathbf{P}} \neg A}{\vdash_{\mathbf{P}} (A \to B) \wedge (A \to \neg B) \to \neg A}}\,[\mathrm{MP}]$$

(ii)
'$\Leftarrow$': Immediate since from $A \to B$ and $B \to \neg B$ we obtain $A \to \neg B$.
'$\Rightarrow$': It suffices to show $(A \to B) \to (B \to \neg B) \to \neg A \Rightarrow$ Co + An.
-<u>Co</u>

$$\dfrac{\dfrac{\dfrac{(A \to B) \wedge \neg B \vdash_{\mathbf{P}} (A \to B) \wedge \neg B}{(A \to B) \wedge \neg B \vdash_{\mathbf{P}} (A \to B) \wedge (B \to \neg B)} \qquad \vdash_{\mathbf{P}} (A \to B) \wedge (B \to \neg B) \to \neg A}{(A \to B) \wedge \neg B \vdash_{\mathbf{P}} \neg A}\,[\mathrm{MP}]}{\vdash_{\mathbf{P}} (A \to B) \wedge \neg B \to \neg A}$$

-<u>An</u>

$$\dfrac{\dfrac{}{\vdash_{\mathbf{P}} (A \to A) \wedge (A \to \neg A) \to \neg A}\;[\text{Take } B := A]}{\vdash_{\mathbf{P}} (A \to \neg A) \to \neg A}$$

(iii) It suffices to show the right direction.
'$\Rightarrow$':

$$\dfrac{\dfrac{\dfrac{}{(A \to B) \wedge (A \to \neg B) \vdash_{\mathbf{P}} A \to (B \wedge \neg B)} \qquad \dfrac{}{\vdash_{\mathbf{P}} (B \wedge \neg B) \to \neg A}\;[\text{NeF}]}{(A \to B) \wedge (A \to \neg B) \vdash_{\mathbf{P}} A \to \neg A} \qquad \dfrac{}{\vdash_{\mathbf{P}} (A \to \neg A) \to \neg A}\;[\text{An}]}{\dfrac{(A \to B) \wedge (A \to \neg B) \vdash_{\mathbf{P}} \neg A}{\vdash_{\mathbf{P}} (A \to B) \wedge (A \to \neg B) \to \neg A}}\;[\text{MP}]$$

(iv) It suffices to show the right direction.
'$\Rightarrow$': It suffices to show DN + Co $\Rightarrow$ An

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to \neg A, A \vdash_{\mathbf{P}} (A \to \neg A) \to \neg A}{A \to \neg A, A \vdash_{\mathbf{P}} \neg\neg A \to \neg(A \to \neg A)}\;[\text{Co}]}{A \to \neg A \vdash_{\mathbf{P}} A \to \neg(A \to \neg A)}\;[\text{DN}]}{A \to \neg A \vdash_{\mathbf{P}} \neg\neg(A \to \neg A) \to \neg A}\;[\text{Co}]}{A \to \neg A \vdash_{\mathbf{P}} \neg A}\;[\text{DN}]}{\vdash_{\mathbf{P}} (A \to \neg A) \to \neg A}$$

□

3.2 The Axiom NeF

As we have mentioned in the introduction, NeF stands out among the subminimal axioms introduced in [3] for its counter-intuitivity. We can partially illuminate the nature of this axiom by observing some related axioms.

Definition 3.2.1 (axioms related to NeF)**.**
We define the following axioms:
EC: $(A \leftrightarrow \neg A) \to \neg B$
D: $(A \wedge \neg A) \to (B \to \neg B)$

Note that we can think of EC as an instance of the liar paradox, if we regard the equivalence of A as giving the definition of its meaning. Then EC says that if any A is defined by its negation $\neg A$ (i.e. *this sentence is false*), we can deduce any $\neg B$, which immediately implies contradiction, and in particular $A \wedge \neg A$.

Proposition 3.2.1 (deducibility for EC and D)**.**
(i) NeF $\Rightarrow$ D, N $\Rightarrow$ D, D + An $\Rightarrow$ M.
(ii) M $\Rightarrow$ EC, EC $\Rightarrow$ NeF.

Proof.
(i)
<u>NeF $\Rightarrow$ D</u>

$$\dfrac{\dfrac{}{\vdash_{\mathbf{P}} (A \wedge \neg A) \to \neg B}\,[\mathrm{NeF}]}{\vdash_{\mathbf{P}} (A \wedge \neg A) \to (B \to \neg B)}$$

<u>N $\Rightarrow$ D</u>

$$\dfrac{\dfrac{\dfrac{A \wedge B \vdash_{\mathbf{P}} A \leftrightarrow B \qquad \dfrac{}{\vdash_{\mathbf{P}} (A \leftrightarrow B) \to (\neg A \to \neg B)}\,[\mathrm{N}]}{A \wedge B \vdash_{\mathbf{P}} \neg A \to \neg B}\,[\mathrm{MP}]}{\overline{A \wedge \neg A \vdash_{\mathbf{P}} B \to \neg B}}}{\vdash_{\mathbf{P}} (A \wedge \neg A) \to (B \to \neg B)}$$

<u>D + An $\Rightarrow$ M</u>: It suffices to show D + An $\Rightarrow$ NeF, for NeF + An $\Rightarrow$ M.

$$\dfrac{\dfrac{\dfrac{}{A \wedge \neg A \vdash_{\mathbf{P}} B \to \neg B}\,[\mathrm{D}]}{A \wedge \neg A \vdash_{\mathbf{P}} \neg B}\,[\mathrm{An}]}{\vdash_{\mathbf{P}} (A \wedge \neg A) \to \neg B}$$

(ii)
<u>M $\Rightarrow$ EC</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \to A \vdash_{\mathbf{M}_\neg} \neg\neg A}\,[\mathrm{IDN}] \qquad \dfrac{}{A \to \neg A \vdash_{\mathbf{M}_\neg} \neg A}\,[\mathrm{An}]}{A \leftrightarrow \neg A \vdash_{\mathbf{M}_\neg} \neg A \wedge \neg\neg A}}{A \leftrightarrow \neg A \vdash_{\mathbf{M}_\neg} \neg B}\,[\mathrm{NeF}]}{\vdash_{\mathbf{M}_\neg} (A \leftrightarrow \neg A) \to \neg B}$$

<u>EC $\Rightarrow$ NeF</u>

$$\dfrac{\dfrac{A \wedge \neg A \vdash_{\mathbf{P}} A \leftrightarrow \neg A \qquad \dfrac{}{\vdash_{\mathbf{P}} (A \leftrightarrow \neg A) \to \neg B}\,[\mathrm{EC}]}{A \wedge \neg A \vdash_{\mathbf{P}} \neg B}\,[\mathrm{MP}]}{\vdash_{\mathbf{P}} (A \wedge \neg A) \to \neg B}$$

$\square$

This result shows that among negative inferences, NeF is situated below the liar paradox EC. It also shows that D is derivable from N, which does not express a particularly negative meaning; it merely states that the unary operator is extensional. D, however, appears as problematic an axiom as NeF; if we allow ourselves a natural reading, the principle says that a single contradictory proposition turns all true propositions into false ones. This suggests that the root of counter-intuitivity of NeF does not specifically lie in the negative meaning we assign to '$\neg$'. Further, we cannot eliminate all counterintuitive inferences in minimal logic by merely removing NeF.

It has been pointed out by authors like Došen [5], that it is possible to think $\neg$ as a modal operator. If we couple this reading with the above, we informally observe the following.

<u>Observation</u>
Let P be a property for propositions that is faithfully expressible by means of an extensional modal operator $\Box$. Then If there exists a true proposition satisfying the property P, (i.e.. $A \wedge \Box A$), then all true proposition have that property, (i.e. $B \to \Box B$).

It would be a matter of debate how logically acceptable this principle is. However it is *prima facie* unclear, at least, why this must hold for any such P.

3.3 Superminimal Axioms

Let us now move attention to negation axioms that are classically/intuitionistically valid, but not derivable in minimal logic. Let us call them *superminimal* axioms, and investigate their relationship with minimal and subminimal axioms.

Definition 3.3.1 (superminimal axioms)**.**
We introduce the following axiom.
DNE: $\neg\neg A \to A$ LEM: $A \vee \neg A$
CM: $(\neg A \to A) \to A$ EFQ: $(A \wedge \neg A) \to B$

As already mentioned, M+EFQ defines the negation of intuitionistic logic, and M+EFQ+DNE (or M+EFQ+LEM) defines the negation for classical logic.

It is known that DNE $\Rightarrow$ LEM, EFQ; LEM+EFQ $\Rightarrow$ DNE; and LEM$\Leftrightarrow$ CM; all over minimal logic [4]. But it is not guaranteed these relations still hold without minimal negation (i.e. axiom M). In what follows, we shall investigate deducibility over **PPC**, so without assuming M.

Lemma 3.3.1 (deducibility for CM+EFQ)**.**
(i) CM+EFQ$\Rightarrow$DNE
(ii) CM+EFQ$\Rightarrow$LEM

Proof.
(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash_{\mathbf{P}} \neg A \qquad \neg\neg A \vdash_{\mathbf{P}} \neg\neg A}{\neg A, \neg\neg A \vdash_{\mathbf{P}} \neg A \wedge \neg\neg A} \qquad \dfrac{}{\vdash_{\mathbf{P}} (\neg A \wedge \neg\neg A) \to A}\,[\text{EFQ}]}{\neg A, \neg\neg A \vdash_{\mathbf{P}} A}\,[\text{MP}]}{\neg\neg A \vdash_{\mathbf{P}} \neg A \to A} \qquad \dfrac{}{\vdash_{\mathbf{P}} (\neg A \to A) \to A}\,[\text{CM}]}{\neg\neg A \vdash_{\mathbf{P}} A}\,[\text{MP}]}{\vdash_{\mathbf{P}} \neg\neg A \to A}$$

(ii)

$$\dfrac{\dfrac{\dfrac{\neg(A \vee \neg A) \vdash_{\mathbf{P}} \neg(A \vee \neg A) \qquad \dfrac{\dfrac{}{\neg\neg A \vdash_{\mathbf{P}} A}\,[\text{DNE (by (i))}]}{\neg\neg A \vdash_{\mathbf{P}} A \vee \neg A}}{\neg\neg A, \neg(A \vee \neg A) \vdash_{\mathbf{P}} (A \vee \neg A) \wedge \neg(A \vee \neg A)} \qquad \dfrac{}{\vdash_{\mathbf{P}} [(A \vee \neg A) \wedge \neg(A \vee \neg A)] \to \neg A}\,[\text{EFQ}]}{\neg\neg A, \neg(A \vee \neg A) \vdash_{\mathbf{P}} \neg A}\,[\text{MP}]}{\neg(A \vee \neg A) \vdash_{\mathbf{P}} \neg\neg A \to \neg A \cdots \mathit{(a)}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A\lor\neg A)\vdash_{\mathbf{P}}\neg\neg A\to\neg A}\,\mathit{(a)} \qquad \dfrac{}{\vdash_{\mathbf{P}}(\neg\neg A\to\neg A)\to\neg A}\,[\mathrm{CM}]}{\neg(A\lor\neg A)\vdash_{\mathbf{P}}\neg A}\,[\mathrm{MP}]}{\neg(A\lor\neg A)\vdash_{\mathbf{P}} A\lor\neg A}}{\vdash_{\mathbf{P}}\neg(A\lor\neg A)\to A\lor\neg A\cdots\mathit{(b)}}$$

$$\dfrac{\dfrac{}{\vdash_{\mathbf{P}}\neg(A\lor\neg A)\to A\lor\neg A}\,\mathit{(b)} \qquad \dfrac{}{\vdash_{\mathbf{P}}[\neg(A\lor\neg A)\to(A\lor\neg A)]\to(A\lor\neg A)}\,[\mathrm{CM}]}{\vdash_{\mathbf{P}} A\lor\neg A}\,[\mathrm{MP}]$$

□

Lemma 3.3.2 (deducibility of subminimal axioms from LEM, CM+EFQ)**.**
(i) LEM⇒An
(ii) CM+EFQ⇒M

Proof.
(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{A\vdash_{\mathbf{P}}A \qquad A\to\neg A\vdash_{\mathbf{P}}A\to\neg A}{A, A\to\neg A\vdash_{\mathbf{P}}\neg A}\,[\mathrm{MP}]}{A\vdash_{\mathbf{P}}(A\to\neg A)\to\neg A} \qquad \dfrac{\dfrac{\neg A\vdash_{\mathbf{P}}\neg A}{\neg A, A\to\neg A\vdash_{\mathbf{P}}\neg A}}{\neg A\vdash_{\mathbf{P}}(A\to\neg A)\to\neg A}}{\dfrac{A\lor\neg A\vdash_{\mathbf{P}}(A\to\neg A)\to\neg A}{\vdash_{\mathbf{P}}A\lor\neg A\to[(A\to\neg A)\to\neg A]}} \qquad \dfrac{}{\vdash_{\mathbf{P}}A\lor\neg A}\,[\mathrm{LEM}]}{\vdash_{\mathbf{P}}(A\to\neg A)\to\neg A}\,\mathrm{MP}$$

(ii) This follows from (i), CM+EFQ ⇒ LEM, EFQ ⇒ NeF and An+NeF ⇒ M.
□

Using these lemmas, we obtain the following theorem about the relationship among super/subminimal axioms.

Theorem 3.3.1 (hierarchy of super/subminimal axioms)**.**
(i) EFQ+An defines the intuitionistic negation.
(ii) CM+EFQ defines the classical negation.
(iii) C: $(\neg A\to B)\land(\neg A\to\neg B)\to A$ also defines the classical negation

Proof.
(i) We have seen that NeF+An⇔ M. NeF is clearly and instance of EFQ.
(ii) This follows from (i) and the above lemma.
(iii) '⇒' We show C⇔CM+EFQ.
<u>CM</u>

$$\dfrac{\dfrac{}{\vdash_{\mathbf{P}}(\neg A\to A)\land(\neg A\to\neg A)\to A}\,[\mathrm{C}]}{\vdash_{\mathbf{P}}(\neg A\to A)\to A}$$

<u>EFQ</u>

$$\dfrac{\dfrac{B\land\neg B\vdash_{\mathbf{P}}(\neg A\to B)\land(\neg A\to\neg B)}{B\land\neg B\vdash_{\mathbf{P}}A}\,[\mathrm{C}]}{\vdash_{\mathbf{P}}(B\land\neg B)\to A}$$

'⇐'

$$\dfrac{\dfrac{\dfrac{(\neg A \to B) \wedge (\neg A \to \neg B) \vdash_{\mathbf{P}} \neg A \to (B \wedge \neg B)}{(\neg A \to B) \wedge (\neg A \to \neg B) \vdash_{\mathbf{P}} \neg A \to A}\,[\mathrm{EFQ}]}{(\neg A \to B) \wedge (\neg A \to \neg B) \vdash_{\mathbf{P}} A}\,[\mathrm{CM}]}{\vdash_{\mathbf{P}} (\neg A \to B) \wedge (\neg A \to \neg B) \to A}$$

□

This result hints that CM, EFQ(NeF) and An play a substantial role in defining negation. We see that NeF is a weaker version of EFQ, and paired with An, which seems to constitute a couple with CM (though the conclusion is weaker), to define the minimal negation. Thus it appears that the pairs of axioms allow us to naturally descend to weaker negations. But it is not clear if we can consider the minimal negation the bottom of this descent. After all, EFQ can be weakened still by adding more negations to the conclusion. It is a natural question then, to ask if we can climb down this ladder of classical-intuitionistic-minimal negation any further. The below result shows that this is possible: we can generalise CM, EFQ and An so that we can indefinitely obtain weaker negations.

Definition 3.3.2 (generalisation of CM, EFQ and An)**.**
Let $(\mathrm{CM}_i)_{i\in\mathbb{N}}$, $(\mathrm{EFQ}_i)_{i\in\mathbb{N}}$, $(\mathrm{An}_i)_{i\in\mathbb{N}}$ be as follows.

CM_i: $(\neg^{(2i+1)}A \to \neg^{(2i)}A) \to \neg^{(2i)}A$
EFQ_i: $(A \wedge \neg A) \to \neg^{(i)}B$
An_i: $(\neg^{(2i)}A \to \neg^{(2i+1)}A) \to \neg^{(2i+1)}A$

For instance, CM_0 =CM, EFQ_0=EFQ, EFQ_1=NeF and An_0=An.

Lemma 3.3.3 (generalised double negation elimination)**.**
(i) $\mathrm{CM}_i + \mathrm{EFQ}_{2i} \Rightarrow \neg^{(2i+2)}A \to \neg^{(2i)}A$
(ii) $\mathrm{An}_i + \mathrm{EFQ}_{2i+1} \Rightarrow \neg^{(2i)}A \to \neg^{(2i+2)}A$
(iii) $\mathrm{An}_i + \mathrm{EFQ}_{2i+1} \Rightarrow \neg^{(2i+3)}A \to \neg^{(2i+1)}A$

Proof.
(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\mathbf{P}} \neg^{(2i+1)}A \wedge \neg^{(2i+2)}A \to \neg^{(2i)}A}\,[\mathrm{EFQ}_{2i}]}{\neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+1)}A \to \neg^{(2i)}A} \qquad \dfrac{}{\vdash_{\mathbf{P}} (\neg^{(2i+1)}A \to \neg^{(2i)}A) \to \neg^{(2i)}A}\,[\mathrm{CM}_i]}{\neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i)}A}\,[\mathrm{MP}]}{\vdash_{\mathbf{P}} \neg^{(2i+2)}A \to \neg^{(2i)}A}$$

(ii)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\mathbf{P}} (\neg^{(2i)}A \wedge \neg^{(2i+1)}A) \to \neg^{(2i+2)}A}\,[\mathrm{EFQ}_{2i+1}]}{\neg^{(2i)}A \vdash_{\mathbf{P}} \neg^{(2i+1)}A \to \neg^{(2i+2)}A} \qquad \dfrac{}{\vdash_{\mathbf{P}} (\neg^{(2i+1)}A \to \neg^{(2i+2)}A) \to \neg^{(2i+2)}A}\,[\mathrm{An}_i]}{\neg^{(2i)}A \vdash_{\mathbf{P}} \neg^{(2i+2)}A}\,[\mathrm{MP}]}{\vdash_{\mathbf{P}} \neg^{(2i)}A \to \neg^{(2i+2)}A}$$

(iii)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg^{(2i+3)}A, \neg^{(2i)}A \vdash_{\mathbf{P}} \neg^{(2i+2)}A \wedge \neg^{(2i+3)}A}\,[(\mathrm{ii})]}{\neg^{(2i+3)}A, \neg^{(2i)}A \vdash_{\mathbf{P}} \neg^{(2i+1)}A}\,[\mathrm{EFQ}_{2i+1}]}{\neg^{(2i+3)}A \vdash_{\mathbf{P}} \neg^{(2i)}A \to \neg^{(2i+1)}A}}{\neg^{(2i+3)}A \vdash_{\mathbf{P}} \neg^{(2i+1)}A}\,[\mathrm{An}_i]}{\vdash_{\mathbf{P}} \neg^{(2i+3)}A \to \neg^{(2i+1)}A}$$

□

Proposition 3.3.1 (deducibility of CM_i, An_i via EFQ_i)**.**
(i) $\mathrm{CM}_i + \mathrm{EFQ}_{2i} \Rightarrow \mathrm{An}_i$
(ii) $\mathrm{An}_i + \mathrm{EFQ}_{2i+1} \Rightarrow \mathrm{CM}_{i+1}$

Proof.
(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\mathbf{P}} \neg^{(2i+2)}A \to \neg^{(2i)}A}\,[(\mathrm{i}),\ \mathrm{lemma}] \qquad \neg^{(2i)}A \to \neg^{(2i+1)}A \vdash_{\mathbf{P}} \neg^{(2i)}A \to \neg^{(2i+1)}A}{\neg^{(2i)}A \to \neg^{(2i+1)}A \vdash_{\mathbf{P}} \neg^{(2i+2)}A \to \neg^{(2i+1)}A}}{\neg^{(2i)}A \to \neg^{(2i+1)}A \vdash_{\mathbf{P}} \neg^{(2i+1)}A}\,[\mathrm{CM}_i]}{\vdash_{\mathbf{P}} (\neg^{(2i)}A \to \neg^{(2i+1)}A) \to \neg^{(2i+1)}A}$$

(ii)

$$\dfrac{\dfrac{\neg^{(2i+3)}A \to \neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+3)} \to \neg^{(2i+2)}A \qquad \dfrac{}{\vdash_{\mathbf{P}} \neg^{(2i+2)}A \to \neg^{(2i+4)}A}\,[(\mathrm{ii}),\ \mathrm{lemma}]}{\neg^{(2i+3)}A \to \neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+3)}A \to \neg^{(2i+4)}A}}{\neg^{(2i+3)}A \to \neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+4)}A \cdots \mathit{(a)}}\,[\mathrm{An}_i]$$

$$\dfrac{\dfrac{\overline{\overline{\neg^{(2i+3)}A \to \neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+4)}A}}\,\mathit{(a)} \qquad \dfrac{}{\vdash_{\mathbf{P}} \neg^{(2i+4)}A \to \neg^{(2i+2)}A}\,[(\mathrm{iii}),\ \mathrm{lemma}]}{\neg^{(2i+3)}A \to \neg^{(2i+2)}A \vdash_{\mathbf{P}} \neg^{(2i+2)}A}}{\vdash_{\mathbf{P}} (\neg^{(2i+3)}A \to \neg^{(2i+2)}A) \to \neg^{(2i+2)}A}$$

□

Hence $\mathrm{CM}_i + \mathrm{EFQ}_{2i}$ and $\mathrm{An}_i + \mathrm{EFQ}_{2i+1}$ defines weaker versions of classical and minimal negation, with $\mathrm{An}_i + \mathrm{EFQ}_{2i}$ in between defining weaker intuitionistic negations. If we write them as $\mathbf{C}_i$, $\mathbf{I}_i$ and $\mathbf{M}_i$, then the negations are weakening in the order:

$$\mathbf{C}_0 \Rightarrow \mathbf{I}_0 \Rightarrow \mathbf{M}_0 \Rightarrow \mathbf{C}_1 \Rightarrow \mathbf{I}_1 \Rightarrow \mathbf{M}_1 \Rightarrow \mathbf{C}_2 \ldots$$

3.4 Subminimal Axioms and Sequnt Calculus

In what follows, we shall investigate some axioms with the tool of sequent calculus. Sequent calculus is a proof system wherein a logical consequence (a *sequent*) is taken to be the basic unit of deduction. This is different from Hilbert-type system, which takes a formula as the basic unit of deduction. We begin with an observation on the provable sequents in classical logic. (with $\bot$ taken as primitive: $\mathbf{CPC}_\bot$).

Definition 3.4.1 ($\mathbf{sCPC}_\bot$, sequent calculus for $\mathbf{CPC}_\bot$ (**G1cp** in [11])). A proof in $\mathbf{sCPC}_\bot$ is a labeled finite tree with a single node, such that the top nodes are instances of axioms, and other nodes are suitable instances of one of the rules.

Axioms:

$$\text{Ax } A \Rightarrow A \qquad\qquad \text{L}\bot\ \bot \Rightarrow$$

Structural rules:

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad\qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad\qquad \text{RC}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Rules for connectives:

$$\text{L}\wedge\ \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \wedge A_1 \Rightarrow \Delta}\ (i \in \{0,1\}) \qquad\qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad\qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_0 \vee A_1 \Delta}\ (i \in \{0,1\})$$

$$\text{L}\rightarrow\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad\qquad \text{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

(Γ, Δ finite multisets of formulas)

We now divide the formulas into two classes, the class of *positive* formulas and *negative* formulas.

Definition 3.4.2 (positive/negative formulas). We define the class of positive/negative formulas F^+ and F^- as follows

$F^+ ::= p|P_1 \wedge P_2|P \vee A|A \vee P|A \rightarrow P|N \rightarrow A$
$F^- ::= \bot|N \wedge A|A \wedge N|N_1 \vee N_2|P \rightarrow N$
$(P \in F^+, N \in F^-)$

Note that $F^+ \cap F^- = \emptyset$, $F^+ \cup F^- = FORM(\mathbf{CPC}_\bot)$.

The proposition below shows that there exists a constraint for deducing negative formulas. This means that we cannot allow a negation axiom that defies this constraint.

Proposition 3.4.1 (derivability of negative formulas in $\mathbf{sCPC}_\perp$**).**
If $\vdash_{\mathbf{C}} \Gamma \Rightarrow \Delta$, and all the formulas in Δ are negative, then there exists a negative formula in Γ.

Proof.
We prove by induction on the depth of deduction.

Basis:
<u>Ax</u> Suppose the deduction ends with an instance of Ax:

$$A \Rightarrow A$$

Then if A is negative in the consequent, so it is in the antecedent.

<u>L$\perp$</u> The statement vacuously holds.

Inductive step:
<u>LW</u> Suppose the deduction ends with an instance of LW:

$$\text{LW}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

If all formulas in Δ are negative, then by I.H. Γ contains a negative formula. So $\{A\} \cup \Gamma$ contains a negative formula.

<u>RW</u> Suppose the deduction ends with an instance of RW:

$$\text{RW}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Then if all formulas in $\Delta \cup \{A\}$ are negative, all formulas in Δ are negative. So by I.H., Γ contains a negative formula.

<u>LC</u> Suppose the deduction ends with an instance of LC:

$$\text{LC}\,\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

Then if all formulas in Δ are negative, by I.H. $\Gamma \cup \{A, A\}$ contains a negative formula. So $\Gamma \cup \{A\}$ contains a negative formula.

<u>RC</u> Suppose the deduction ends with an instance of RC:

$$\text{RC}\,\frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Then if all formulas in $\Delta \cup \{A\}$ are negative, $\Delta \cup \{A, A\}$ contains a negative formula. So by I.H., Γ contains a negative formula.

<u>L$\wedge$</u> Suppose the deduction ends with an instance of L$\wedge$:

$$\text{L}\wedge\,\frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \wedge A_1 \Rightarrow \Delta}\ (i \in \{0, 1\})$$

Then if all formulas in Δ are negative, by I.H., either A_i is negative, or Γ contains a negative formula. If the former, $A_0 \wedge A_1$ is also negative.

R$\wedge$ Suppose the deduction ends with an instance of R$\wedge$:

$$\mathrm{R}\wedge \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

Then if all formulas in $\Delta \cup \{A \wedge B\}$ are negative, $A \wedge B$ is negative; so either A is negative or B is negative. Either way, by I.H. Γ contains a negative formula.

L$\vee$ Suppose the deduction ends with an instance of L$\vee$:

$$\mathrm{L}\vee \frac{A, \Gamma \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

Then if all formulas in Δ are negative, by I.H. either Γ contains a negative formula, or A, B are both negative. If the latter, $A \vee B$ is also negative.

R$\vee$ Suppose the deduction ends with an instance of R$\vee$:

$$\mathrm{R}\vee \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_0 \vee A_1, \Delta} \ (i \in \{0, 1\})$$

Then if all formulas in $\Delta \cup \{A_0 \vee A_1\}$ are negative, $A_0 \vee A_1$ is negative. So both A_0 and A_1 are negative. Hence by I.H., Γ contains a negative formula.

L$\to$ Suppose the deduction ends with an instance of L$\to$:

$$\mathrm{L}\to \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

Then if all formulas in Δ are negative, by I.H. either B is negative, or Γ contains a negative formula. If the former, then if A is positive, $A \to B$ is negative. On the other hand, if A is negative, $\Delta \cup \{A\}$ are all negative. So Γ contains a negative formula by I.H..

R$\to$ Suppose the deduction ends with an instance of R$\to$:

$$\mathrm{R}\to \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

Then if all formulas in $\Delta \cup \{A \to B\}$ are negative, A is positive and B is negative. So by I.H., Γ contains a negative formula. $\square$

To see that this proposition holds for $\mathbf{MPC}_\neg$, we must check that the property holds for the rules N and An, devised in [2]. Also note that the result holds for the Cut rule

$$\mathrm{Cut} \frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$$

even in subminimal systems where it is not admissible; if it does not, we can derive a sequent (in $\mathbf{CPC}_\perp$) whose consequents only contain negative formulas, while the antecedent containing only positive formulas, contradicting the last proposition.

Definition 3.4.3 (**sPPC**, a sequent calculus for **PPC**).
Axioms:

Ax: $A \Rightarrow A$

Structural rules:

$$\text{LW: } \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad \text{LC: } \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

Rules for connectives:

$$\text{L}\wedge\text{: } \frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_0 \wedge A_1 \Rightarrow C}\ (i \in \{0,1\}) \qquad \text{R}\wedge\text{: } \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee\text{: } \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \text{R}\vee\text{: } \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ (i \in \{0,1\})$$

$$\text{L}\to\text{: } \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \qquad \text{R}\to\text{: } \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

(Γ a finite multiset of formulas)

In [2], there is an additional axiom R$\top$: $\Rightarrow \top$. $\top$ is only required in proving the base case for the interpolation theorem, in place of $\bot \to \bot$. For our purpose the presence of this axiom is negligible, so it is dropped. Also, the formulation of L$\wedge$ is slightly different, but each formulation is easily shown to be admissible under the presence of the other.

It is easily shown by induction on the depth of deduction, that:

$$\vdash_{\mathbf{sP}} \Gamma \Rightarrow A \text{ if and only if } \Gamma \vdash_{\mathbf{hP}} A$$

and so the two systems are equivalent.

The Sequent calculus **sMPC**$_\neg$ for **MPC**$_\neg$ is **sPPC** plus rules N and An below.

Definition 3.4.4 (N, An).
N, An refer to the following rules.

$$\text{N } \frac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B} \qquad \text{An } \frac{\Gamma, A \Rightarrow \neg A}{\Gamma \Rightarrow \neg A}$$

The equivalence of these rules to the corresponding axioms (over the systems for **PPC**) has been shown in [2]. In general, establishing the correspondence between a Hilbert-type axiom Ax and a sequent calcural rule Ru is straightforward: From Ax to Ru, we show $\vdash\ \Rightarrow$ Ax is deriveble with the presence of Ru; from Ru to Ax, we show that given the premises of Ru, the conclusion of Ru is derivable with the presence of Ax. (Strictly speaking, we need to prove $\Gamma \vdash_{Hilbert} A$ if and only if $\vdash_{Seq.Calc.} \Gamma \Rightarrow A$ by induction, and the above constitutes a part of this).

Now as claimed earlier, we shall show the above result about negative formulas hold for **sMPC**$_\neg$ as well.

Definition 3.4.5 (positive/negative formulas for **MPC**$_\neg$).
We define the class of positive/negative formulas F^+ and F^- in **MPC**$_\neg$ as:

$F^+ ::= p|\neg N|P_1 \wedge P_2|P \vee A|A \vee P|A \to P|N \to A$
$F^- ::= \neg P|N \wedge A|A \wedge N|N_1 \vee N_2|P \to N$
$(P \in F^+, N \in F^-)$

Proposition 3.4.2 (derivability of negative formulas in $\mathbf{sMPC}_\neg$)**.**
If $\vdash \Gamma \Rightarrow A$, and A is negative, then there exists a negative formula in Γ.

Proof.
It suffices to check the cases for N and An.

<u>N</u> Suppose the deduction ends with an instance of N:

$$\mathrm{N}\ \frac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B}$$

Then if $\neg B$ is negative, B is positive, so by I.H. either Γ contains a negative formula, or A is negative. If the latter, by I.H. Γ has to contain a negative formula, as B is positive.

<u>An</u> Suppose the deduction ends with an instance of *An*:

$$\mathrm{An}\ \frac{\Gamma, A \Rightarrow \neg A}{\Gamma \Rightarrow \neg A}$$

Then if $\neg A$ is negative, A is positive, so by I.H. Γ has to contain a negative formula.

□

As explained earlier, any subminimal axiom has to observe this condition. Note however that this condition is not sufficient, as $\neg p \Rightarrow \neg q$ is not valid.

We can also obtain some separation results for subminimal axioms, by slightly adjusting the notion of positive/negative formulas.

Definition 3.4.6 (formally positive/negative formulas)**.**
We define the class of formally positive/negative formulas F^+ and F^- in $\mathbf{MPC}_\neg$ as follows

$F^+ ::= p|P_1 \wedge P_2|P \vee A|A \vee P|A \to P|N \to A$
$F^- ::= \neg A|N \wedge A|A \wedge N|N_1 \vee N_2|P \to N$
$(P \in F^+, N \in F^-)$

Definition 3.4.7 (An^-)**.**
We define the following axiom:

$$\mathrm{An}^-\ \frac{\Gamma, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}$$

Let us call $\mathrm{An}^- + \mathrm{N} + \mathbf{sPPC}$ as $\mathbf{sAn^-PC}$. In the corresponding Hilbert-type system, An^- can be expressed as $(A \to \neg A) \to (\neg B \to \neg A)$. As usual, it is straightforward to show the equivalence between the two versions.

Proposition 3.4.3 (formal negativity and $\mathbf{An^{-}PC}$)**.**
If $\mathbf{An^{-}PC} \vdash \Gamma \Rightarrow A$, where A is formally negative, then Γ contains a formally negative formula.

Proof. We prove by induction on the depth of deduction. Most of the cases proceed as in the previous proposition. N is treated slightly differently, and An^{-} needs to be checked upon.

<u>N</u> Suppose the deduction ends with an instance of N:

$$\text{N}\ \dfrac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B}$$

Then $\neg B$ is formally negative, but so is $\neg A$.

<u>An^{-}</u> Suppose the deduction ends with an instance of An^{-}:

$$\text{An}^{-}\ \dfrac{\Gamma, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}$$

Then $\neg A$ is formally negative, but so is $\neg B$. □

Corollary 3.4.1 (relationship between An and An^{-})**.**
(i) $\text{An} + \mathbf{PPC}$ derives An^{-}.
(ii) $\mathbf{An^{-}PC}$ does not derive An.

Proof.

(i) $$\dfrac{\dfrac{\Gamma, A \Rightarrow \neg A}{\Gamma \Rightarrow \neg A}\,[\text{An}]}{\Gamma, \neg B \Rightarrow \neg A}\,[\text{LW}]$$

(ii) If $\mathbf{An^{-}PC}$ derives An, then $\mathbf{An^{-}PC}$ is equivalent to $\mathbf{MPC}_{\neg}$. But then $\mathbf{An^{-}PC} \vdash \quad \Rightarrow \neg(p \wedge \neg p)$, where the consequent is formally negative but without formally negative antecedent. This contradicts the previous proposition. □

Let us now observe how An^{-} relates to other subminimal axioms.
We start with axioms Co and NeF. In sequent calculus, they are each realised as follows [2].

$$\text{Co}\ \dfrac{\Gamma, A \Rightarrow B}{\Gamma, \neg B \Rightarrow \neg A} \qquad\qquad \text{NeF}\ \dfrac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B}$$

Proposition 3.4.4 (An^{-} and Co)**.**
(i) $\mathbf{An^{-}PC}$ derives Co.
(ii) $\text{An}^{-} + \text{NeF} + \mathbf{PPC}$ derives Co.

Proof.
(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A \Rightarrow B}{\Gamma, A.A \Rightarrow B}\,[\text{LW}] \qquad \dfrac{A \Rightarrow A}{\Gamma, A, B \Rightarrow A}\,[\text{LW}]}{\Gamma, A, \neg B \Rightarrow \neg A}\,[\text{N}]}{\Gamma, \neg B, \neg B \Rightarrow \neg A}\,[\text{An}^{-}]}{\Gamma, \neg B \Rightarrow \neg A}\,[\text{LC}]$$

(ii)

$$\cfrac{\cfrac{\cfrac{\Gamma, B \Rightarrow A}{\Gamma, B, \neg A \Rightarrow \neg B}\,[\mathrm{NeF}]}{\Gamma, \neg A, \neg A \Rightarrow \neg B}\,[\mathrm{An}^-]}{\Gamma, \neg A \Rightarrow \neg B}\,[\mathrm{LC}]$$

□

Hence we see that $\mathbf{An^-PC} = \mathrm{An}^- + \mathrm{NeF} + \mathbf{PPC}$, as Co derives NeF and N.

Next we consider the axiom DN. It is easy to see that the axiom DN is realised in sequent calculus as:

$$\mathrm{DN}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \neg\neg A}$$

Proposition 3.4.5 (An$^-$ and DN)**.**
$\mathbf{An^-PC}$ does not derive DN.

Proof.
If $\mathbf{An^-PC}$ derives DN, then:

$$\cfrac{\cfrac{\Rightarrow p \to p}{\Rightarrow \neg\neg(p \to p)}\,[\mathrm{DN}] \qquad \cfrac{\Gamma, A \Rightarrow \neg A}{\Gamma, \neg\neg(p \to p) \Rightarrow \neg A}\,[\mathrm{An}^-]}{\Gamma \Rightarrow \neg A}\,[\mathrm{Cut}]$$

And so An becomes admissible in $\mathbf{An^-PC}$, a contradiction. □

Thus in particular, we see that Co does not derive DN, as we saw $\mathbf{An^-PC}$ derives Co.

We now move on to another topic. In [9], an operator $C(\phi)$ satisfying the axiom $C(\phi) \to \phi$ over positive logic is considered. There it is shown that if negation is defined as $\neg\phi := \phi \to C(\phi)$, then

$$C(p) \leftrightarrow p \wedge \neg p$$

is provable, and

$$(p \to q) \to ((p \to \neg q) \to \neg p)$$

is provable if and only if

$$C(p \wedge q) \leftrightarrow C(p) \wedge q,\ C(p \wedge q) \leftrightarrow p \wedge C(q)$$

are provable.

Using the first equivalence, we can translate the last two formulas into:

$$((p \wedge q) \wedge \neg(p \wedge q)) \leftrightarrow ((p \wedge \neg p) \wedge q),\ ((p \wedge q) \wedge \neg(p \wedge q)) \leftrightarrow (p \wedge (q \wedge \neg q))$$

These are then (over positive logic) equivalent to:

$$((p \wedge q) \to (\neg p \leftrightarrow \neg(p \wedge q)),\ ((p \wedge q) \to (\neg q \leftrightarrow \neg(p \wedge q))$$

When expressed as sequent-calcual rules, these axioms become:

$$\text{(RCN):} \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma, \neg A \Rightarrow \neg(A \wedge B)} \; ; \; \frac{\Gamma \Rightarrow A \wedge B}{\Gamma, \neg B \Rightarrow \neg(A \wedge B)}$$

$$\text{(LCN):} \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma, \neg(A \wedge B) \Rightarrow \neg A} \; ; \; \frac{\Gamma \Rightarrow A \wedge B}{\Gamma, \neg(A \wedge B) \Rightarrow \neg B}$$

Let us call these rules collectively as CN. The question to ask now is where these rules lie compared with other subminimal axioms.

Proposition 3.4.6 (deducibility of CN)**.**
(i) CN + **PPC** derives D
(ii) NeF + **PPC** derives CN

Proof.
(i) Note that the axiom D is equivalent to $(A \wedge B) \rightarrow (\neg A \rightarrow \neg B)$. So as a rule, we can express D as:

$$[\mathrm{D}] \; \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma, \neg A \Rightarrow \neg B}$$

Then:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}[R\wedge]}{\Gamma, \neg A \Rightarrow \neg(A \wedge B)}[RCN] \qquad \dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}[R\wedge]}{\Gamma, \neg(A \wedge B) \Rightarrow \neg B}[LCN]}{\Gamma, \neg A \Rightarrow \neg B}[\mathrm{Cut}, LC]$$

(ii)

$$\text{(RCN)} \quad \dfrac{\dfrac{\Gamma \Rightarrow A \wedge B}{\Gamma \Rightarrow A}}{\Gamma, \neg A \Rightarrow \neg(A \wedge B)}[\mathrm{NeF}] \quad \text{(LCN)} \quad \dfrac{\Gamma \Rightarrow A \wedge B}{\Gamma, \neg(A \wedge B) \Rightarrow \neg A}[\mathrm{NeF}]$$

and analogous for the other cases.

□

Thus it is revealed that CN lies between NeF and D. As D+An ⇔ M, we expect that $C(\phi)$ version of D,

$$C(p) \rightarrow (q \rightarrow C(q))$$

also proves equivalent to

$$(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$$

Proposition 3.4.7 (D and Odintsov's system)**.**
$C(p) \rightarrow (q \rightarrow C(q))$ holds if and only if $(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$ holds.

Proof.
'⇒'

$$
\dfrac{\dfrac{\dfrac{p \to q, p \to \neg q \vdash_{\mathbf{P}} p \to [p \wedge (q \wedge \neg q)]}{p \to q, p \to \neg q \vdash_{\mathbf{P}} p \to [p \wedge C(q)]}\;[q \wedge \neg q \leftrightarrow C(q)]}{p \to q, p \to \neg q \vdash_{\mathbf{P}} p \to C(p)}\;[C(q) \to (p \to C(p))]}{\vdash_{\mathbf{P}} (p \to q) \wedge (p \to \neg q) \to \neg p}\;[\neg p := p \to C(p)]
$$

'$\Leftarrow$'

$$
\dfrac{\dfrac{\dfrac{C(p), q \vdash_{\mathbf{P}} p \wedge \neg p}{C(p), q \vdash_{\mathbf{P}} (q \to p) \wedge (q \to \neg p)}}{C(p), q \vdash_{\mathbf{P}} \neg q}\;[(q \to p) \wedge (q \to \neg p) \to \neg q]}{\vdash_{\mathbf{P}} C(p) \to (q \to \neg q)}
$$

$\square$

4 Subminimal Correspondence Theory

In this section, we shall turn our attention to the semantic side of subminimal logics. In terms of semantics, [2] and [3] take the basic subminimal system to be N+**PPC** (hereafter called **NPC**). A key element in the semantics of **NPC** and its extensions is a mapping between the set of upward closed sets of worlds, N. Given the set of worlds in which a proposition is true, N returns the set of worlds in which the *negation* of that proposition is true. Moreover, by giving adequate restrictions on N, we can validate corresponding subminimal axioms. This means we can ask the question of what characterisations for Kripke frames correspond to the validity of subminimal axioms. We shall call this enquiry *subminimal correspondence theory*, after similar frameworks for modal&intermediate logics.

In what follows, we first describe the semantics for **NPC**. After that, we investigate subminimal corresponding theory, restricting our attention to axioms containing a single type of propositional variables. We abbreviate this and call them as *stpv* formulas. So for instance, $p \to (\neg p \wedge p)$ is a stpv formulas because it contains only one type (albeit three tokens) of propositional variable, p. On the other hand, $p \vee \neg q$ is not a stpv formula, because it contains two types (albeit one token each) of propositional variables, p and q.

4.1 Subminimal Semantics

We start with describing the Kripke semantics for **NPC**.

Definition 4.1.1 (Kripke frame for **NPC**).
A Kripke frame for **NPC** is a triple $(W, \leq, N)$, where$(W, \leq)$ is as in **PPC**, and N is a mapping from the set $\mathcal{U}(W)$ of all upward closed subsets of W to itself. Additionally, N satisfies the restriction:

- $w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))$ for all $w \in W$
(where $\mathcal{R}(w) := \{w' \in W | w' \geq w\}$ is the upward closed subset generated by w.)

Kripke models for **NPC** is a slight modification from that of **MPC**. The valuation for negation is now defined as:

- $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg A \Leftrightarrow w \in N(\mathcal{V}(A))$.
(where $\mathcal{V}(A) := \{w \in W | (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A\}$)

The soundness and completeness of **NPC** is verified in [3]. It is appropriate here to give an example of how this semantics works.

Example 4.1.1 (Validity of D)**.**
$\vDash_{\mathbf{N}} (A \wedge \neg A) \to (B \to \neg B)$

Proof.
Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be given. Suppose $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} A \wedge \neg A$ for $w' \geq w$. We wish to show $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} B \to \neg B$, so assume $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{N}} B$ for $w'' \geq w'$. Note that $w'' \in \mathcal{V}(\neg A) = N(\mathcal{V}(A))$ and $\mathcal{V}(B) \cap \mathcal{R}(w'') = \mathcal{V}(A) \cap \mathcal{R}(w'')$.
We then observe:

$$\begin{aligned}
(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{N}} \neg B &\Leftrightarrow w'' \in N(\mathcal{V}(B)) && [\text{dfn}] \\
&\Leftrightarrow w'' \in N(\mathcal{V}(B) \cap \mathcal{R}(w'')) && [\text{rest. on } N] \\
&\Leftrightarrow w'' \in N(\mathcal{V}(A) \cap \mathcal{R}(w'')) && [\mathcal{V}(B) \cap \mathcal{R}(w'') = \mathcal{V}(A) \cap \mathcal{R}(w'')] \\
&\Leftrightarrow w'' \in N(\mathcal{V}(A)) && [\text{rest. on } N]
\end{aligned}$$

and so $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{N}} \neg B$. Therefore $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} B \to \neg B$ and so $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} (A \wedge \neg A) \to (B \to \neg B)$. Since $(\mathcal{F}, \mathcal{V})$ and w are arbitrary, we conclude $\vDash_{\mathbf{N}} (A \wedge \neg A) \to (B \to \neg B)$. □

4.2 Subminimal Correspondence Theory for stpv Formulas

Now we are going to investigate the correspondence between axioms and restrictions on frames. As discussed earlier, given such a restriction for a formula, a frame validates the formula if and only if the frame satisfies the condition specified by the formula. We shall restrict ourselves to formulas containing a single type of variable, p.

Before moving on to our analysis, we mention known correspondences verified in [2].

Proposition 4.2.1 (known correspondences [2])**.**
The following conditions correspond to the axioms NeF and Co[1].

(i) $\mathcal{F} \vDash_{\mathbf{N}} (p \wedge \neg p) \to \neg q \Leftrightarrow \forall U, U' \in \mathcal{R}(W)[U \cap N(U) \subseteq N(U')]$
(ii) $\mathcal{F} \vDash_{\mathbf{N}} (p \to q) \to (\neg q \to \neg p) \Leftrightarrow \forall U, U' \in \mathcal{R}(W)[U \subseteq U' \Rightarrow N(U') \subseteq N(U)]$

Proof.
(i) '$\Rightarrow$'
Let $U, U' \in \mathcal{R}(W)$ and suppose $w \in U \cap N(U)$. Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = U$ and $\mathcal{V}(q) = U'$. Then $\mathcal{V}(\neg p) = N(U)$, and $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} p \wedge \neg p$. So by assumption, $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg q$. Hence $w \in N(U')$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Suppose $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} p \wedge \neg p$ for $w' \geq w$. Then

[1] Here given in terms of propositional variables.

$w' \in \mathcal{V}(p) \wedge N(\mathcal{V}(p))$. Hence by assumption $w' \in N(\mathcal{V}(q))$, i.e. $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} \neg q$. So $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} (p \wedge \neg p) \rightarrow \neg q$. Since $\mathcal{V}$, w are arbitrary, we conclude $\mathcal{F} \vDash_{\mathbf{N}} (p \wedge \neg p) \rightarrow \neg q$.

(ii) '$\Rightarrow$'
Let $U, U' \in \mathcal{R}(W)$ and suppose $U \subseteq U'$. Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = U$ and $\mathcal{V}(q) = U'$. Then if $w \in \mathcal{N}(U')$, we see $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} p \rightarrow q$ and $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg q$. So by assumption, $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg p$. Thus $w \in N(U)$.

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Suppose $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} p \rightarrow q$ for $w' \geq w$. Then $\mathcal{V}(p) \cap \mathcal{R}(w') \subseteq \mathcal{V}(q)$. So by assumption $N(\mathcal{V}(q)) \subseteq N(\mathcal{V}(p) \cap \mathcal{R}(w'))$ Now if $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{N}} \neg q$ for $w'' \geq w'$, $w'' \in N(\mathcal{V}(q)) \subseteq N(\mathcal{V}(p) \cap \mathcal{R}(w'))$. By restriction on N,

$$\begin{aligned} w'' \in N(\mathcal{V}(p) \cap \mathcal{R}(w')) &\Leftrightarrow w'' \in N([\mathcal{V}(p) \cap \mathcal{R}(w')] \cap \mathcal{R}(w'')) \\ &\Leftrightarrow w'' \in N(\mathcal{V}(p) \cap [\mathcal{R}(w') \cap \mathcal{R}(w'')]) \\ &\Leftrightarrow w'' \in N(\mathcal{V}(p) \cap \mathcal{R}(w'')) \\ &\Leftrightarrow w'' \in N(\mathcal{V}(p)) \end{aligned}$$

hence $(\mathcal{F}, \mathcal{V}), w'' \Vdash_{\mathbf{N}} \neg p$. So $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} \neg q \rightarrow \neg p$ and $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$. Since $\mathcal{V}$, w are arbitrary, we conclude $\mathcal{F} \vDash_{\mathbf{N}} (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$. □

We shall try now to define frame conditions for stpv formulas inductively. To achieve this, we shall begin with defining some classes for the formulas.

Definition 4.2.1 ($\mathcal{B}$,$\mathcal{E}$,$\mathcal{I}$,$\mathcal{L}$)**.**
We define the following four classes of formulas.

$$\begin{aligned} \mathcal{B} &::= p|B_1 \wedge B_2|B \wedge E|E \wedge B|B \wedge I|I \wedge B|B \wedge L|L \wedge B|B_1 \vee B_2 \\ \mathcal{E} &::= \neg B|\neg E|\neg I|\neg L|E_1 \wedge E_2|E \wedge I|I \wedge E|I_1 \wedge I_2|E \wedge L|L \wedge E \\ &\quad |I \wedge L|L \wedge I|L_1 \wedge L_2|B \vee E|E \vee B|E_1 \vee E_2|B \vee I|I \vee B|E_1 \vee E_2 \\ &\quad |I \vee E|I_1 \vee I_2|B \vee L|L \vee B|E \vee L|L \vee E|I \vee L|L \vee I|L_1 \vee L_2 \\ \mathcal{I} &::= B_1 \rightarrow B_2|B \rightarrow E|E \rightarrow B|E_1 \rightarrow E_2 \\ \mathcal{L} &::= B \rightarrow I|I \rightarrow B|E \rightarrow I|I \rightarrow E|I_1 \rightarrow I_2|B \rightarrow L|L \rightarrow B|E \rightarrow L \\ &\quad |L \rightarrow E|I \rightarrow L|L \rightarrow I|L_1 \rightarrow L_2 \end{aligned}$$

$(B \in \mathcal{B}, E \in \mathcal{E}, I \in \mathcal{I}, L \in \mathcal{L})$

Each stpv formula belongs to one of the classes. We shall establish the correspondence for the last three classes of formula, $\mathcal{E}$,$\mathcal{I}$,$\mathcal{L}$. (The validity of formulas in the class $\mathcal{B}$ essentially depends on the valuation, so no meaningful correspondences can be established.)

In the remainder of this section, we aim to is to prove the following theorem giving the general correspondence for stpv formulas.

Theorem 4.2.1 (correspondence for stpv formulas)**.**
Let $\mathcal{F} = (W, \leq, N)$ be a frame.

Let A be a formula with a single type of propositional variable, p.

(i) If $A \in \mathcal{B} \cup \mathcal{E}$, then for each $U \in \mathcal{U}(W)$, there exists $\Sigma_A^U \in \mathcal{U}(W)$, such that:
- (a) If $\mathcal{V}$ is a valuation s.t. $\mathcal{V}(p) = U$, then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A \Leftrightarrow w \in \Sigma_A^U$.
- (b) Further, if $A \in \mathcal{E}$, $\mathcal{F} \vDash_{\mathbf{N}} A \Leftrightarrow \forall U[\Sigma_A^U = W]$.

(ii) If $A \equiv A_1 \to A_2 \in \mathcal{I}$ (where $A_i \in \mathcal{B} \cup \mathcal{E}$), then:
- (a) If $\mathcal{V}$ is a valuation s.t. $\mathcal{V}(p) = U$, then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A \Leftrightarrow \mathcal{R}(w) \cap \Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U$
- (b) $\mathcal{F} \vDash_{\mathbf{N}} A \Leftrightarrow \forall U[\Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U]$.

(iii) If $A \in \mathcal{L}$, then there exists a proposition $\lambda_A^{(U,w)}$ such that:
- (a) If $\mathcal{V}$ is a valuation s.t. $\mathcal{V}(p) = U$, then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A \Leftrightarrow \forall w' \geq w[\lambda_A^{(U,w')}]$.
- (b) $\mathcal{F} \vDash_{\mathbf{N}} A \Leftrightarrow \forall U \forall w[\lambda_A^{(U,w)}]$.

The proof of this theorem is done by induction on the complexity of A. We shall split it into several lemmas.

Firstly, we shall consider the case for the class $\mathcal{B}$. Note that throughout the following lemmas, we are assuming the inductive hypothesis of the main theorem.

Lemma 4.2.1 ($\mathcal{B}$).
Let $\mathcal{F} = (W, \leq, N)$ be a frame and let $B \in \mathcal{B}$.
Then there exists $\Sigma_B^U \in \mathcal{U}(W)$ such that $\mathcal{V}(p) = U \Rightarrow \mathcal{V}(B) = \Sigma_B^U$.

Proof. We consider by cases.

$\underline{B \equiv p}$
Let $\Sigma_B^U := U$. Then $\mathcal{V}(B) = \mathcal{V}(p) = U = \Sigma_B^U$.

$\underline{B \equiv B_1 \wedge B_2}$
Let $\Sigma_{B_1 \wedge B_2}^U := \Sigma_{B_1}^U \cap \Sigma_{B_2}^U$.
(Recall that upward closed sets are closed under union and intersection.)
Then if $\mathcal{V}(p) = U$,
$\mathcal{V}(B_1 \wedge B_2) = \mathcal{V}(B_1) \cap \mathcal{V}(B_2) = \Sigma_{B_1}^U \cap \Sigma_{B_2}^U$ [I.H.].

$\underline{B \equiv B' \wedge E, E \wedge B'}$
Let $\Sigma_{B' \wedge E}^U, \Sigma_{E \wedge B'}^U := \Sigma_{B'}^U \cap \Sigma_E^U$.
Then if $\mathcal{V}(p) = U$,
$\mathcal{V}(B' \wedge E) = \mathcal{V}(E \wedge B') = \mathcal{V}(B') \cap \mathcal{V}(E) = \Sigma_{B'}^U \cap \Sigma_E^U$ [I.H.].

$\underline{B \equiv B' \wedge I, I \wedge B'}$ (where $I \equiv A_1 \to A_2$ for $A_1, A_2 \in \mathcal{B} \cup \mathcal{E}$)
Let $\Sigma_{B' \wedge I}^U, \Sigma_{I \wedge B'}^U := \Sigma_{B'}^U \cap \{w | \mathcal{R}(w) \cap \Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U\}$.
(Notice that $\{w | \mathcal{R}(w) \cap \Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U\}$ is an upward closed set.)
Then if $\mathcal{V}(p) = U$,
$\mathcal{V}(B' \wedge I) = \mathcal{V}(I \wedge B') = \mathcal{V}(B') \cap \mathcal{V}(I) = \Sigma_{B'}^U \cap \{w | \mathcal{R}(w) \cap \Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U\}$ [I.H.].

$\underline{B \equiv B' \wedge L, L \wedge B'}$
Let $\Sigma_{B' \wedge L}^U, \Sigma_{L \wedge B'}^U := \Sigma_{B'}^U \cap \{w | \forall w' \geq w(\lambda_L^{(U,w')})\}$.
(Notice that $\{w | \forall w' \geq w(\lambda_L^{(U,w')})\}$ is an upward closed set.)

Then if $\mathcal{V}(p) = U$,
$\mathcal{V}(B' \wedge L) = \mathcal{V}(L \wedge B') = \mathcal{V}(B') \cap \mathcal{V}(L) = \Sigma^U_{B'} \cap \{w | \forall w' \geq w(\lambda_L^{(U,w')})\}$ [I.H.].

$\underline{B \equiv B_1 \vee B_2}$
Let $\Sigma^U_{B_1 \vee B_2} := \Sigma^U_{B_1} \cup \Sigma^U_{B_2}$.
Then if $\mathcal{V}(p) = U$,
$\mathcal{V}(B_1 \vee B_2) = \mathcal{V}(B_1) \cup \mathcal{V}(B_2) = \Sigma^U_{B_1} \cup \Sigma^U_{B_2}$ [I.H.].

□

Next, we consider the cases for $\mathcal{E}$.

Lemma 4.2.2 ($\mathcal{E}$)**.**
Let $\mathcal{F} = (W, \leq, N)$ be a frame and let $E \in \mathcal{E}$.
Then there exists $\Sigma^U_E \in \mathcal{U}(W)$, such that:
- (a) If $\mathcal{V}$ is a valuation such that $\mathcal{V}(p) = U$, then $\mathcal{V}(E) = \Sigma^U_E$.
- (b) $\mathcal{F} \vDash_{\mathbf{N}} E \Leftrightarrow \forall U[\Sigma^U_E = W]$.

Proof. We consider by cases.

$\underline{E \equiv \neg B}$
Let $\Sigma^U_{\neg B} := N(\Sigma^U_B)$.

(a) If $\mathcal{V}(p) = U$, then by I.H, $\mathcal{V}(B) = \Sigma^U_B$. So $\mathcal{V}(\neg B) = N(\mathcal{V}(B)) = N(\Sigma^U_B)$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg B$ by assumption. Also by I.H, $\mathcal{V}(B) = \Sigma^U_B$.
To show $\forall U[N(\Sigma^U_B) = W]$, it suffices to show $w \in N(\Sigma^U_B)$.

$$\begin{aligned}(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg B &\Leftrightarrow w \in N(\mathcal{V}(B)) \\ &\Leftrightarrow w \in N(\Sigma^U_B) \quad [I.H.]\end{aligned}$$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Then $\Sigma^{\mathcal{V}(p)}_{\neg B} = W$ by assumption.
So $w \in \Sigma^{\mathcal{V}(p)}_{\neg B}$. As $\Sigma^{\mathcal{V}(p)}_{\neg B} := N(\Sigma^{\mathcal{V}(p)}_B)$, we see:

$$\begin{aligned}w \in N(\Sigma^{\mathcal{V}(p)}_B) &= N(\mathcal{V}(B)) \; [I.H.] \\ &= \mathcal{V}(\neg B)\end{aligned}$$

Hence $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg B$. As $\mathcal{V}, w$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} \neg B$.

$\underline{E \equiv \neg E'}$
Let $\Sigma^U_{\neg E'} := N(\Sigma^U_{E'})$.

(a) If $\mathcal{V}(p) = U$, then by I.H. $\mathcal{V}(E') = \Sigma^U_{E'}$. So $\mathcal{V}(\neg E') = N(\mathcal{V}(E')) = N(\Sigma^U_{E'})$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg E'$ by assumption. Also by I.H, $\mathcal{V}(E') = \Sigma^U_{E'}$.
To show $\forall U[N(\Sigma^U_{E'}) = W]$, it suffices to show $w \in N(\Sigma^U_{E'})$.

$$\begin{aligned}(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg E' &\Leftrightarrow w \in N(\mathcal{V}(E')) \\ &\Leftrightarrow w \in N(\Sigma^U_{E'}) \quad [I.H.]\end{aligned}$$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Then $\Sigma^{\mathcal{V}(p)}_{\neg E'} = W$ by assumption.
So $w \in \Sigma^{\mathcal{V}(p)}_{\neg E'}$. As $\Sigma^{\mathcal{V}(p)}_{\neg E'} := N(\Sigma^{\mathcal{V}(p)}_{E'})$, we see:

$$\begin{aligned} w \in N(\Sigma^{\mathcal{V}(p)}_{E'}) &= N(\mathcal{V}(E'))[I.H.] \\ &= \mathcal{V}(\neg E') \end{aligned}$$

Hence $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} \neg E'$. As $\mathcal{V}, w$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} \neg E'$.

$\underline{E \equiv \neg I}$(where $I \equiv A_1 \to A_2$ for $A_1, A_2 \in \mathcal{B} \cup \mathcal{E}$)
Let $\Sigma^U_{\neg I} := N(\{w | \mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}\})$.

(a) If $\mathcal{V}(p) = U$, then by I.H, $\mathcal{V}(A_i) = \Sigma^U_{A_i}$ for $i \in \{1, 2\}$.
So $\mathcal{V}(\neg I) = N(\mathcal{V}(I)) = N(\mathcal{V}(A_1 \to A_2)) = N(\{w | \mathcal{R}(w) \cup \mathcal{V}(A_1) \subseteq \mathcal{V}(A_2)\}) = N(\{w | \mathcal{R}(w) \cup \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}\})$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $u \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg I$ by assumption. Also by I.H, $\mathcal{V}(A_i) = \Sigma^U_{A_i}$ for $i \in \{1, 2\}$.
To show $N(\{w | \mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}\}) = W$, it suffices to show $u \in N(\{w | \mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}\})$.

$$\begin{aligned} (\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg I &\Leftrightarrow u \in \mathcal{V}(\neg(A_1 \to A_2)) \\ &\Leftrightarrow u \in N(\mathcal{V}(A_1 \to A_2)) \\ &\Leftrightarrow u \in N(\{w | \mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}\})[I.H.] \end{aligned}$$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $u \in W$.
Then $\Sigma^{\mathcal{V}(p)}_{\neg I} = W$ by assumption. So $u \in \Sigma^{\mathcal{V}(p)}_{\neg I}$.
As $\Sigma^{\mathcal{V}(p)}_{\neg I} := N(\{w | \mathcal{R}(w) \cap \Sigma^{\mathcal{V}(p)}_{A_1} \subseteq \Sigma^{\mathcal{V}(p)}_{A_2}\})$ we see:

$$\begin{aligned} u \in N(\{w | \mathcal{R}(w) \cap \Sigma^{\mathcal{V}(p)}_{A_1} \subseteq \Sigma^{\mathcal{V}(p)}_{A_2}\}) &= N(\mathcal{V}(A_1 \to A_2)) \\ &= N(\mathcal{V}(I)) \qquad [I.H.] \\ &= \mathcal{V}(\neg I) \end{aligned}$$

Hence $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg I$. As $\mathcal{V}, u$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} \neg I$.

$\underline{E \equiv \neg L}$
Let $\Sigma^U_{\neg E'} := N(\{w | \forall w' \geq w (\lambda^{(U, w')}_L)\})$.

(a) If $\mathcal{V}(p) = U$, then by I.H, $\mathcal{V}(L) = \{w | \forall w' \geq w (\lambda^{(U, w')}_L)\}$.
So $\mathcal{V}(\neg L) = N(\mathcal{V}(L)) = N(\{w | \forall w' \geq w (\lambda^{(U, w')}_L)\})$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $u \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg L$ by assumption. Also by I.H, $\mathcal{V}(L) = \{w | \forall w' \geq w (\lambda^{(U, w')}_L)\}$.
To show $\forall U [\mathcal{N}(\{w | \forall w' \geq w (\lambda^{(U, w')}_L)\}) = W$, it suffices to show $u \in N(\{w | \forall w' \geq w (\lambda^{(U, w')}_L)\})$.

$$\begin{aligned} (\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg L &\Leftrightarrow u \in N(\mathcal{V}(L)) \\ &\Leftrightarrow u \in N(\{w | \forall w' \geq w (\lambda^{(U, w')}_L)\})[I.H.] \end{aligned}$$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $u \in W$.
Then $\Sigma_{\neg L}^{\mathcal{V}(p)} = W$ by assumption. So $u \in \Sigma_{\neg L}^{\mathcal{V}(p)}$.
As $\Sigma_{\neg L}^{\mathcal{V}(p)} := N(\{w | \forall w' \geq w(\lambda_L^{(\mathcal{V}(p),w')})\})$, we see:

$$\begin{aligned} u \in N(\{w | \forall w' \geq w(\lambda_L^{(\mathcal{V}(p),w')})\}) &= N(\mathcal{V}(L))[I.H.] \\ &= \mathcal{V}(\neg L) \end{aligned}$$

Hence $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} \neg L$. As $\mathcal{V}, u$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} \neg L$.

$\underline{E \equiv A_1 \wedge A_2}$(where $A_i \in \mathcal{E} \cup \mathcal{I} \cup \mathcal{L}$).
Let $\Sigma_{A_1 \wedge A_2}^U := \Pi_{A_1}^U \cap \Pi_{A_2}^U$, where:

$$\Pi_C^U = \begin{cases} \Sigma_C^U & \text{if } C \in \mathcal{B} \cup \mathcal{E}, \\ \{w | \mathcal{R}(w) \cap \Sigma_{C_1}^U \subseteq \Sigma_{C_2}^U\} & \text{if } C \equiv C_1 \to C_2 \in \mathcal{I}, \\ \{w | \forall w' \geq w(\lambda_C^{(U,w')})\} & \text{if } C \in \mathcal{L}. \end{cases}$$

By inductive hypothesis, $\mathcal{V}(p) = U \Rightarrow \mathcal{V}(A_i) = \Pi_{A_i}^U$.

(a) Suppose $\mathcal{V}(p) = U$. Then $\mathcal{V}(A_1 \wedge A_2) = \mathcal{V}(A_1) \cap \mathcal{V}(A_2) = \Pi_{A_1}^U \cap \Pi_{A_2}^U$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \wedge A_2$ by assumption.
So $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1$ and $(\mathcal{F}, \mathcal{V}), w \Vdash A_2$.
$\Leftrightarrow w \in \mathcal{V}(A_1)$ and $w \in \mathcal{V}(A_2)$.
$\Leftrightarrow w \in \Pi_{A_1}^U$ and $w \in \Pi_{A_2}^U$. [I.H.]
$\Leftrightarrow w \in \Pi_{A_1}^U \cap \Pi_{A_2}^U$.
Hence $\Pi_{A_1}^U \cap \Pi_{A_2}^U = W$. Since U, w are arbitrary, we conclude $\forall U[\Pi_{A_1}^U \cap \Pi_{A_2}^U = W]$.

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation, and $w \in W$.
Then $\Pi_{A_1}^{\mathcal{V}(p)} \cap \Pi_{A_2}^{\mathcal{V}(p)} = W$ by assumption.
So $\mathcal{V}(A_1) \cap \mathcal{V}(A_2) = W$ by I.H..
$\Rightarrow w \in \mathcal{V}(A_1) \cap \mathcal{V}(A_2)$
$\Leftrightarrow w \in \mathcal{V}(A_1 \wedge A_2)$
$\Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \wedge A_2$
Since $\mathcal{V}, w$ are arbitrary, we conclude $\mathcal{F} \vDash_{\mathbf{N}} A_1 \wedge A_2$.

$\underline{E \equiv A_1 \vee A_2}$(where not both of A_1 and A_2 are in $\mathcal{B}$).
Let $\Sigma_{A_1 \vee A_2}^U := \Pi_{A_1}^U \cup \Pi_{A_2}^U$.

(a) Suppose $\mathcal{V}(p) = U$. Then $\mathcal{V}(A_1 \vee A_2) = \mathcal{V}(A_1) \cup \mathcal{V}(A_2) = \Pi_{A_1}^U \cup \Pi_{A_2}^U$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$.
Then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \vee A_2$ by assumption.
So $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1$ or $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_2$.

$\Leftrightarrow w \in \mathcal{V}(A_1)$ or $w \in \mathcal{V}(A_2)$.
$\Leftrightarrow w \in \Pi^U_{A_1}$ or $w \in \Pi^U_{A_2}$. [I.H.]
$\Leftrightarrow w \in \Pi^U_{A_1} \cup \Pi^U_{A_2}$.
Hence $\Pi^U_{A_1} \cup \Pi^U_{A_2} = W$. Since U, w are arbitrary, we conclude $\forall U[\Pi^U_{A_1} \cup \Pi^U_{A_2} = W]$.

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation, and $w \in W$.
Then $\Pi^{\mathcal{V}(p)}_{A_1} \cup \Pi^{\mathcal{V}(p)}_{A_2} = W$ by assumption.
So $\mathcal{V}(A_1) \cup \mathcal{V}(A_2) = W$ by I.H..
$\Rightarrow w \in \mathcal{V}(A_1) \cup \mathcal{V}(A_2)$
$\Leftrightarrow w \in \mathcal{V}(A_1 \vee A_2)$
$\Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \vee A_2$
Since $\mathcal{V}, w$ are arbitrary, we conclude $\mathcal{F} \vDash_{\mathbf{N}} A_1 \vee A_2$.

$\square$

These lemmas establish (i) of the theorem. We shall now move on to (ii), namely the cases for $\mathcal{I}$.

Lemma 4.2.3 ($\mathcal{I}$)**.**
Let $\mathcal{F} = (W, \leq, N)$ be a frame. Let $A \equiv A_1 \to A_2$, where $A_i \in \mathcal{B} \cup \mathcal{E}$. Then:
(a) If $\mathcal{V}(p) = U$, then $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \to A_2 \Leftrightarrow \mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}$
(b) $\mathcal{F} \vDash_{\mathbf{N}} A_1 \to A_2 \Leftrightarrow \forall U[\Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}]$

Proof.
(a) '$\Rightarrow$'
Let $u \in \mathcal{R}(w) \cap \Sigma^U_{A_1}$. Then $u \in \mathcal{R}(w) \cap \mathcal{V}(A_1)$ by I.H.. So $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} A_1$ and therefore $(\mathcal{F}, \mathcal{V}), u \Vdash_{\mathbf{N}} A_2$ by assumption. Thus $u \in \mathcal{V}(A_2) = \Sigma^U_{A_2}$.

'$\Leftarrow$'
Let $w' \geq w$ be s.t. $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} A_1$. Then $w' \in \mathcal{R}(w) \cap \mathcal{V}(A_1) = \mathcal{R}(w) \cap \Sigma^U_{A_1}$ So by assumption, $w' \in \Sigma^U_{A_2} = \mathcal{V}(A_2)$. Hence $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} A_2$, so $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} I$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$. Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = U$.
Now for any $w \in W$, $w \in \Sigma^U_{A_1}$
$\Leftrightarrow w \in \mathcal{V}(A_1)$ [I.H.]
$\Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1$
$\Rightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_2$ [by assumption, $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} A_1 \to A_2$]
$\Leftrightarrow w \in \mathcal{V}(A_2)$
$\Leftrightarrow w \in \Sigma^U_{A_2}$ [I.H.]
Hence $\Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}$. As U is arbitrary, we conclude $\forall U[\Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}]$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation, and $w \in W$. Let $w' \geq w$.
Then $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} A_1$
$\Leftrightarrow w' \in \mathcal{V}(A_1)$
$\Leftrightarrow w' \in \Sigma^{\mathcal{V}(p)}_{A_1}$ [I.H.]

$\Rightarrow w' \in \Sigma^{\mathcal{V}(p)}_{A_2}$ [Assumption]
$\Leftrightarrow w' \in \mathcal{V}(A_2)$
$(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} A_2$
Hence $(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} A_1 \to A_2$.
As $\mathcal{V}, w$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} A_1 \to A_2$. □

This lemmas establishes (ii) of the theorem. Now we move on to the last part of the theorem.

Lemma 4.2.4 ($\mathcal{L}$)**.** If $L \in \mathcal{L}$, then there exists a proposition $\lambda^{(U,w)}_L$ such that:
- (a) If $\mathcal{V}$ is a valuation s.t. $\mathcal{V}(p) = U$, then $(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{N}} L \Leftrightarrow \forall w' \geq w[\lambda^{(U,w')}_L]$.
- (b) $\mathcal{F} \vDash_{\mathbf{N}} L \Leftrightarrow \forall U \forall w[\lambda^{(U,w)}_L]$.

Proof.
We define(where $I, I_1 \equiv A_1 \to A_2, I_2 \equiv A_3 \to A_4$):
$\lambda^{(U,w)}_{B\to I} = [\mathcal{R}(w) \subseteq \Sigma^U_B] \Rightarrow [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}]$
$\lambda^{(U,w)}_{I\to B} = [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}] \Rightarrow [\mathcal{R}(w) \subseteq \Sigma^U_B]$
$\lambda^{(U,w)}_{E\to I} = [\mathcal{R}(w) \subseteq \Sigma^U_E] \Rightarrow [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}]$
$\lambda^{(U,w)}_{I\to E} = [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}] \Rightarrow [\mathcal{R}(w) \subseteq \Sigma^U_E]$
$\lambda^{(U,w)}_{I_1\to I_2} = [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}] \Rightarrow [\mathcal{R}(w) \cap \Sigma^U_{A_3} \subseteq \Sigma^U_{A_4}]$
$\lambda^{(U,w)}_{B\to L} = [\mathcal{R}(w) \subseteq \Sigma^U_B] \Rightarrow [\forall w' \geq w(\lambda^{(U,w')}_L)]$
$\lambda^{(U,w)}_{L\to B} = [\forall w' \geq w(\lambda^{(U,w')}_L)] \Rightarrow [\mathcal{R}(w) \subseteq \Sigma^U_B]$
$\lambda^{(U,w)}_{E\to L} = [\mathcal{R}(w) \subseteq \Sigma^U_E] \Rightarrow [\forall w' \geq w(\lambda^{(U,w')}_L)]$
$\lambda^{(U,w)}_{L\to E} = [\forall w' \geq w(\lambda^{(U,w')}_L)] \Rightarrow [\mathcal{R}(w) \subseteq \Sigma^U_E]$
$\lambda^{(U,w)}_{I\to L} = [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}] \Rightarrow [\forall w' \geq w(\lambda^{(U,w')}_L)]$
$\lambda^{(U,w)}_{L\to I} = [\forall w' \geq w(\lambda^{(U,w')}_L)] \Rightarrow [\mathcal{R}(w) \cap \Sigma^U_{A_1} \subseteq \Sigma^U_{A_2}]$
$\lambda^{(U,w)}_{L_1\to L_2} = [\forall w' \geq w(\lambda^{(U,w')}_{L_1})] \Rightarrow [\forall w' \geq w(\lambda^{(U,w')}_{L_2})]$

We shall treat the cases $E \to I$ and $L_1 \to L_2$. the arguments for the other cases are analogous.

<u>$L \equiv E \to I$</u>(where $I \equiv A_1 \to A_2$)
(a) '$\Rightarrow$'
Let $w' \geq w$ and suppose $\mathcal{R}(w') \subseteq \Sigma^U_E$. Let $u \in \mathcal{R}(w') \cap \mathcal{V}(A_1)$. By I.H., $\mathcal{V}(p) = U$ implies $\Sigma^U_E = \mathcal{V}(E)$, so $(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} E$. Thus $(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} I$ by assumption, and consequently $(\mathcal{F},\mathcal{V}), u \Vdash_{\mathbf{N}} I$. As $(\mathcal{F},\mathcal{V}), u \Vdash_{\mathbf{N}} A_1$, $(\mathcal{F},\mathcal{V}), u \Vdash_{\mathbf{N}} A_2$. So $u \in \mathcal{V}(A_2) = \Sigma^U_{A_2}$.

'$\Leftarrow$'
Let $w' \geq w$ be s.t. $(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} E$. Then $\mathcal{R}(w') \subseteq \mathcal{V}(E)$. So $\mathcal{R}(w') \cap \mathcal{V}(A_1) \subseteq \mathcal{V}(A_2)$ by assumption. This is equivalent to $(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{N}} A_1 \to A_2$. Hence $(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{N}} E \to I$.

(b) '$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Suppose $\mathcal{R}(w) \subseteq \Sigma^U_E$. Let $u \in \mathcal{R}(w) \cap \mathcal{V}(A_1)$. Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = U$. Then by assumption, $(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{N}} E \to I$. Also,

by I.H. $\Sigma_E^U = \mathcal{V}(E)$. So $u \in \mathcal{V}(E)$. Hence $u \in \mathcal{V}(I)$, and consequently $u \in \mathcal{V}(A_2)$. By I.H. again, $u \in \Sigma_{A_2}^U$. As U, w are arbitrary, we conclude $\forall U \forall x[\mathcal{R}(w) \subseteq \Sigma_E^U \Rightarrow \mathcal{R}(w) \cap \Sigma_{A_1}^U \subseteq \Sigma_{A_2}^U]$

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Let $w' \geq w$ be s.t. $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} E$. Then $\mathcal{R}(w') \subseteq \mathcal{V}(E) = \Sigma_E^{\mathcal{V}(p)}$ by I.H.. So by assumption, $\mathcal{R}(w') \cap \Sigma_{A_1}^{\mathcal{V}(p)} \subseteq \Sigma_{A_2}^{\mathcal{V}(p)}$. Thus by I.H., $\mathcal{R}(w') \cap \mathcal{V}(A_1) \subseteq \mathcal{V}(A_2)$. So $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} I$. Therefore $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} E \to I$. As $\mathcal{V}, w$ are arbitrary, $\mathcal{F} \vDash_{\mathbf{N}} E \to I$.

$\underline{L \equiv L_1 \to L_2}$
(a) We need to show: If $\mathcal{V}(p) = U$,
$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_1 \to L_2 \Leftrightarrow \forall w' \geq w[\forall w'' \geq w'(\lambda_{L_1}^{(U,w'')}) \Rightarrow \forall w'' \geq w'(\lambda_{L_2.}^{(U,w'')})]$

'$\Rightarrow$'
Let $w' \geq w$ and suppose $\forall w'' \geq w'(\lambda_{L_1}^{(U,w'')})$. Then by I.H., $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} L_1$. So $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_2$ by assumption. Hence by I.H. again, $\forall w'' \geq w'(\lambda_{L_2}^{(U,w'')})$.

'$\Leftarrow$'
Let $w' \geq w$ be s.t. $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} L_1$. Then by I.H., $\forall w'' \geq w'(\lambda_{L_1}^{(U,w'')})$. So $\forall w'' \geq w'(\lambda_{L_2}^{(U,w'')})$ by assumption. Hence by I.H. again, $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} L_2$. Therefore $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_1 \to L_2$.

(b) We need to show:
$\mathcal{F} \vDash_{\mathbf{N}} L_1 \to L_2 \Leftrightarrow \forall U \forall w[\forall w' \geq w(\lambda_{L_1}^{(U,w')}) \Rightarrow \forall w' \geq w(\lambda_{L_2.}^{(U,w')})]$

'$\Rightarrow$'
Let $U \in \mathcal{U}(W)$ and $w \in W$. Let $\mathcal{V}$ be s.t. $\mathcal{V}(p) = U$. Suppose $\forall w' \geq w(\lambda_{L_1}^{(U,w')})$. Then by I.H., $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_1$ So by assumption, $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_2$. By I.H. again, $\forall w' \geq w(\lambda_{L_2}^{(U,w')})$. Since U, w are arbitrary, we conclude $\forall U \forall w[\forall w' \geq w(\lambda_{L_1}^{(U,w')}) \Rightarrow \forall w' \geq w(\lambda_{L_2.}^{(U,w')})]$.

'$\Leftarrow$'
Let $\mathcal{V}$ be a valuation and $w \in W$. Suppose for $w' \geq w$, $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} L_1$. then by I.H., $\forall w'' \geq w'(\lambda_{L_1}^{(\mathcal{V}(p),w'')})$. so by assumption, $\forall w'' \geq w'(\lambda_{L_2}^{(\mathcal{V}(p),w'')})$. By I.H. again, $(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{N}} L_2$. Hence $(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{N}} L_1 \to L_2$. Since $\mathcal{V}, w$ are arbitrary, $\mathcal{F} \vDash L_1 \to L_2$. $\square$

This completes the whole proof. It is now possible to calculate the frame properties corresponding to stpv formulas.

For instance to the axiom $(p \to \neg p) \to \neg p \in \mathcal{L}$ corresponds the property $\forall U \forall w[\lambda_{((p \to \neg p) \to \neg p)}^{(U,w)}]$, where:

$$\begin{aligned}\lambda_{((p \to \neg p) \to \neg p)}^{(U,w)} &= [\mathcal{R}(w) \cap \Sigma_p^U \subseteq \Sigma_{\neg p}^U] \Rightarrow [\mathcal{R}(w) \subseteq \Sigma_{\neg p}^U] \\ &= [\mathcal{R}(w) \cap U \subseteq N(U)] \Rightarrow [\mathcal{R}(w) \subseteq N(U)]\end{aligned}$$

5 Multi-Absurdity Logic

In this section, we shall consider logical systems where each formula A has its own absurdity/contradiction operator $\perp_A$. As we briefly mentioned in section 3.4, Odintsov [9] has already treated such a logic. He in fact had two operators $C(\phi)$ and $A(\phi)$, where $C(\phi)$ satisfies the formula $C(\phi) \to \phi$, while $A(\phi)$ does not. His motive in splitting contradiction was to identify the condition it has to satisfy in order to validate the negative *ex falso*. Along this enquiry he discovered the results discussed in 3.4, and as a corollary

$$\phi, \neg\phi \vdash_L \neg\psi \textit{ if and only if } C(p \wedge q) \leftrightarrow C(p) \wedge q, C(p \wedge q) \leftrightarrow C(p) \wedge q \in L.$$

for such logic L and $\neg\phi := \phi \to C(\phi)$.

The aim of this section is to investigate how logics with multiple contradictions correspond to subminimal logics. Let us call them *multi-absurdity* logics. We begin with considering multi-absurdity version of $\mathbf{MPC}_\neg$. After that, We look for the multi-absurdity version of the logic **AnPC** defined as An+**PPC**.

5.1 $\mathbf{MPC}_\neg$ and Multi-Absurdity

We shall now explain how to split $\perp$ into $\perp_A$ for each A. This is achieved by expanding the language of **PPC** with the clause:

-If A is a formula, then $\perp_A$ is a formula.

$\neg A$ is now defined as $A \to \perp_A$. If we add the axiom $\perp_A \to A$, this definition of $\perp_A$ becomes identical to Odintsov's $C(\phi)$ we discussed earlier.

Odintsov mentions that minimal negation can be defined by setting $\neg\phi := \phi \to C(\phi)$ (p.108, [9]), although no proof is given. He however proves that if M holds with this definition of negation, then C is extensional (p.110, *ibid.*), i.e.

$$\frac{\phi \leftrightarrow \psi}{C(\phi) \leftrightarrow C(\psi)}$$

Let us begin with checking the converse, namely that extensional $\perp_A$ defines minimal negation. For this, we need to see that the logic with multiple absurdities is equivalent to the minimal logic. So let us introduce a logical system with multiple contradictions.

Definition 5.1.1 ($\mathbf{MPC}_{\perp_*}$)**.**
We introduce the following rules:

$$\mathrm{N}_\perp \ \frac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \perp_A \Rightarrow \perp_B} \qquad\qquad \mathrm{L}\perp_* \ \ \perp_A \Rightarrow A$$

We call $\mathrm{N}_\perp + \mathrm{L}_{\perp_*}$+**PPC** as $\mathbf{MPC}_{\perp_*}$.

The next proposition shows that if we read $A \wedge \neg A$ as $\perp_A$ in $\mathbf{MPC}_\neg$ and $A \to \perp_A$ as $\neg A$ in $\mathbf{MPC}_{\perp_*}$, then we can interpret $\neg A$ and $\perp_A$ as equivalent to $A \to \perp_A$ and $A \wedge \neg A$, respectively.

Proposition 5.1.1 (equivalence for $\neg A$ and $\perp_A$)**.**
(i) $\vdash_{\mathbf{M}_\neg} \quad \Rightarrow \neg A \leftrightarrow [A \to (A \wedge \neg A)]$
(ii) $\vdash_{\mathbf{M}_{\perp_*}} \quad \Rightarrow \perp_A \leftrightarrow [A \wedge (A \to \perp_A)]$

Proof.
(i)
'→'

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, \neg A \Rightarrow A}\,[\mathrm{LW}] \qquad \dfrac{\neg A \to \neg A}{A, \neg A \Rightarrow \neg A}\,[\mathrm{LW}]}{A, \neg A \Rightarrow A \wedge \neg A}\,[\mathrm{R}\wedge]}{\neg A \Rightarrow A \to (A \wedge \neg A)}\,[\mathrm{R}\to]}{\Rightarrow \neg A \to [A \to (A \wedge \neg A)]}\,[\mathrm{R}\to]
$$

'←'

$$
\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{\dfrac{\neg A \Rightarrow \neg A}{A \wedge \neg A \Rightarrow \neg A}\,[\mathrm{L}\wedge]}{A, A \wedge \neg A \Rightarrow \neg A}\,[\mathrm{LW}]}{A \to (A \wedge \neg A), A \Rightarrow \neg A}\,[\mathrm{L}\to]}{A \to (A \wedge \neg A) \Rightarrow \neg A}\,[\mathrm{An}]}{\Rightarrow [A \to (A \wedge \neg A)] \to \neg A}\,[\mathrm{R}\to]
$$

(ii)
'→'

$$
\dfrac{\dfrac{\dfrac{\dfrac{\bot_A \Rightarrow \bot_A}{A, \bot_A \Rightarrow \bot_A}\,[\mathrm{LW}]}{\bot_A \Rightarrow A \to \bot_A}\,[\mathrm{R}\to] \qquad \dfrac{}{\bot_A \Rightarrow A}\,[\mathrm{L}\bot_*]}{\bot_A \Rightarrow A \wedge (A \to \bot_A)}\,[\mathrm{R}\wedge]}{\Rightarrow \bot_A \to [A \wedge (A \to \bot_A)]}\,[\mathrm{R}\to]
$$

'←'

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{\bot_A \Rightarrow \bot_A}{A, \bot_A \Rightarrow \bot_A}\,[\mathrm{LW}]}{A, A \to \bot_A \Rightarrow \bot_A}\,[\mathrm{L}\to]}{A, A \wedge (A \to \bot_A) \Rightarrow \bot_A}\,[\mathrm{L}\wedge]}{A \wedge (A \to \bot_A), A \wedge (A \to \bot_A) \Rightarrow \bot_A}\,[\mathrm{L}\wedge]}{A \wedge (A \to \bot_A) \Rightarrow \bot_A}\,[\mathrm{LC}]}{\Rightarrow [A \wedge (A \to \bot_A)] \to \bot_A}\,[\mathrm{R}\to]
$$

□

From here we study the relationship between $\mathbf{MPC}_\neg$ and $\mathbf{MPC}_{\bot_*}$. To prepare fore this, we must introduce some definitions.

Definition 5.1.2 (faithful embedding)**.**
Let L_1 and L_2 be two logical systems in languages $\mathcal{L}_1$ and $\mathcal{L}_1$. We say L_1 is *faithfully embeddable* into L_2, if there exists a mapping(*translation*) $t : \mathcal{L}_1 \to \mathcal{L}_2$ such that for all $\mathcal{L}_1$ formula A,

$$\vdash_{L_1} \Gamma \Rightarrow A \text{ if and only if } \vdash_{L_2} \Gamma^t \Rightarrow A^t$$

(where $\Gamma^t = \{G^t | G \in \Gamma\}$)

Definition 5.1.3 (definition equivalence)**.**
Let L_1 and L_2 be two logical systems in languages $\mathcal{L}_1$ and $\mathcal{L}_1$. We say L_1 is *definition equivalent* to L_2, if there exist translations $t : \mathcal{L}_1 \to \mathcal{L}_2$ and $s : \mathcal{L}_2 \to \mathcal{L}_1$ such that:

(i) L_1 is faithfully embeddable into L_2 via t.
(ii) L_2 is faithfully embeddable into L_1 via s.
(iii) $\vdash_{L_1} \Rightarrow A \leftrightarrow A^{t^s}$ and $\vdash_{L_2} \Rightarrow B \leftrightarrow B^{s^t}$ for all $\mathcal{L}_1$ formula A and $\mathcal{L}_2$ formula B.

We are going to prove the definition equivalence between $\mathbf{MPC}_\neg$ and $\mathbf{MPC}_{\perp_*}$, via translations $\dagger$ and $\star$ described below. We first give the embeddability of each logic to the other.

Lemma 5.1.1 (embedding of $\mathbf{MPC}_{\perp_*}$ into $\mathbf{MPC}_\neg$)**.**
$\vdash_{\mathbf{M}_{\perp_*}} \Gamma \Rightarrow A$ implies $\vdash_{\mathbf{M}_\neg} \Gamma^\dagger \Rightarrow A^\dagger$, where:

$p^\dagger \equiv p$
$(\perp_A)^\dagger \equiv A^\dagger \wedge \neg A^\dagger$
$(A \circ B)^\dagger \equiv A^\dagger \circ B^\dagger$, $\circ \in \{\wedge, \vee \to\}$

Proof. We prove by induction on the depth of deduction.

Base
Ax: Immediate.

L$\perp_*$: Assume the deduction ends with an instance of L$\perp_*$.

$$\perp_A \Rightarrow A$$

We need to show $\vdash_{\mathbf{M}_\neg} A^\dagger \wedge \neg A^\dagger \Rightarrow A^\dagger$.

$$\dfrac{A^\dagger \Rightarrow A^\dagger}{A^\dagger \wedge \neg A^\dagger \Rightarrow A^\dagger}\,[\mathrm{L}\wedge]$$

Inductive step
LW, LC, LR$\wedge$, LR$\vee$, LR$\to$: Immediate.

N$_\perp$: Assume the deduction ends with an instance of N$_\perp$.

$$\mathrm{N}_\perp\,\dfrac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \perp_A \Rightarrow \perp_B}$$

We need to show $\vdash_{\mathbf{M}_\neg} \Gamma^\dagger, A^\dagger \wedge \neg A^\dagger \Rightarrow B^\dagger \wedge \neg B^\dagger$. By I.H., we know $\vdash_{\mathbf{M}_\neg} \Gamma^\dagger, A^\dagger \Rightarrow B^\dagger$, $\vdash_{\mathbf{M}_\neg} \Gamma^\dagger, B^\dagger \Rightarrow A^\dagger$.

$$\dfrac{\dfrac{\Gamma^\dagger, A^\dagger \Rightarrow B^\dagger}{\Gamma^\dagger, A^\dagger \wedge \neg A^\dagger \Rightarrow B^\dagger}\,[\mathrm{L}\wedge] \qquad \dfrac{\dfrac{\Gamma^\dagger, A^\dagger \Rightarrow B^\dagger \qquad \Gamma^\dagger, B^\dagger \Rightarrow A^\dagger}{\Gamma^\dagger, \neg A^\dagger \Rightarrow \neg B^\dagger}\,[\mathrm{N}]}{\Gamma^\dagger, A^\dagger \wedge \neg A^\dagger \Rightarrow \neg B^\dagger}\,[\mathrm{L}\wedge]}{\Gamma^\dagger, A^\dagger \wedge \neg A^\dagger \Rightarrow B^\dagger \wedge \neg B^\dagger}\,[\mathrm{R}\wedge]$$

□

Lemma 5.1.2 (embedding of $\mathbf{MPC}_\neg$ into $\mathbf{MPC}_{\bot_*}$)**.**
$\vdash_{\mathbf{M}_\neg} \Gamma \Rightarrow A$ implies $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star \Rightarrow A^\star$, where:

$p^\star \equiv p$
$(\neg A)^\star \equiv A^\star \to \bot_{A^\star}$
$(A \circ B)^\star \equiv A^\star \circ B^\star$, $\circ \in \{\wedge, \vee \to\}$

Proof.
We prove by induction on the depth of deduction.

Base
Ax: Immediate.

Inductive step
LW, LC, LR$\wedge$, LR$\vee$, LR$\to$: Immediate.

N: Assume that the deduction ends with an instance of N.

$$\frac{\Gamma, A \Rightarrow B \qquad \Gamma, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B}$$

We need to show $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star, A^\star \to \bot_{A^\star} \Rightarrow B^\star \to \bot_{B^\star}$. By I.H., we know $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star, A^\star \Rightarrow B^\star$, $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star, B^\star \Rightarrow A^\star$

$$\dfrac{\dfrac{\Gamma^\star, B^\star \Rightarrow A^\star \qquad \dfrac{\dfrac{\Gamma^\star, A^\star \Rightarrow B^\star \qquad \Gamma^\star, B^\star \Rightarrow A^\star}{\Gamma^\star, \bot_{A^\star} \Rightarrow \bot_{B^\star}}\,[\mathrm{N}_\bot]}{\Gamma^\star, B^\star, \bot_{A^\star} \Rightarrow \bot_{B^\star}}\,[\mathrm{LW}]}{\Gamma^\star, B^\star, A^\star \to \bot_{A^\star} \Rightarrow \bot_{B^\star}}\,[\mathrm{L}{\to}]}{\Gamma^\star, A^\star \to \bot_{A^\star} \Rightarrow B^\star \to \bot_{B^\star}}\,[\mathrm{R}{\to}]$$

An: Assume that the deduction ends with an instance of An.

$$\frac{\Gamma, A \Rightarrow \neg A}{\Gamma \Rightarrow \neg A}$$

We need to show $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star \Rightarrow A^\star \to \bot_{A^\star}$. By I.H., we know $\vdash_{\mathbf{M}_{\bot_*}} \Gamma^\star, A^\star \Rightarrow A^\star \to \bot_{A^\star}$

$$\dfrac{\dfrac{\dfrac{\Gamma^\star, A^\star \Rightarrow A^\star \to \bot_{A^\star} \qquad \dfrac{A^\star \Rightarrow A^\star \qquad \dfrac{\bot_{A^\star} \Rightarrow \bot_{A^\star}}{A^\star, \bot_{A^\star} \Rightarrow \bot_{A^\star}}\,[\mathrm{LW}]}{A^\star, A^\star \to \bot_{A^\star} \Rightarrow \bot_{A^\star}}\,[\mathrm{L}{\to}]}{\Gamma^\star, A^\star, A^\star \Rightarrow \bot_{A^\star}}\,[\mathrm{Cut}]}{\Gamma^\star, A^\star \Rightarrow \bot_{A^\star}}\,[\mathrm{LC}]}{\Gamma^\star \Rightarrow A^\star \to \bot_{A^\star}}\,[\mathrm{R}{\to}]$$

$\square$

We have to prove the other directions to establish faithful embeddings; these are obtainable once we show that the last condition of definition equivalence hold with respect to $\dagger$ and $\star$.

Lemma 5.1.3 ($\dagger$ and $\star$)**.**
(i) $\vdash_{\mathbf{M}_{\bot_*}} A \Leftrightarrow (A^\dagger)^\star$
(ii) $\vdash_{\mathbf{M}_\neg} A \Leftrightarrow (A^\star)^\dagger$

Proof.
We prove by induction on the complexity of formula.
(i) If $A \equiv p$, then $(A^\dagger)^\star \equiv A$. If $A \equiv B \circ C$, $\circ \in \{\wedge \vee \to\}$, then by I.H. $\vdash B \Leftrightarrow (B^\dagger)^\star$ and $\vdash C \Leftrightarrow (C^\dagger)^\star$. Also, $(A^\dagger)^\star \equiv (B^\dagger)^\star \circ (C^\dagger)^\star$. We consider by cases.

$\underline{\wedge}$

$$\dfrac{\dfrac{B \Rightarrow B^{\dagger^\star}}{B \wedge C \Rightarrow B^{\dagger^\star}}[\mathrm{L}\wedge] \qquad \dfrac{C \Rightarrow C^{\dagger^\star}}{B \wedge C \Rightarrow C^{\dagger^\star}}[\mathrm{L}\wedge]}{B \wedge C \Rightarrow B^{\dagger^\star} \wedge C^{\dagger^\star}}[\mathrm{R}\wedge]$$

The other direction is similar.
$\underline{\vee}$

$$\dfrac{\dfrac{B \Rightarrow B^{\dagger^\star}}{B \Rightarrow B^{\dagger^\star} \vee C^{\dagger^\star}}[\mathrm{R}\vee] \qquad \dfrac{C \Rightarrow C^{\dagger^\star}}{C \Rightarrow B^{\dagger^\star} \vee C^{\dagger^\star}}[\mathrm{R}\vee]}{B \vee C \Rightarrow B^{\dagger^\star} \vee C^{\dagger^\star}}[\mathrm{L}\vee]$$

The other direction is similar.
$\underline{\to}$

$$\dfrac{\dfrac{B^{\dagger^\star} \Rightarrow B \qquad \dfrac{C \Rightarrow C^{\dagger^\star}}{B^{\dagger^\star}, C \Rightarrow C^{\dagger^\star}}[\mathrm{LW}]}{B \to C, B^{\dagger^\star} \Rightarrow C^{\dagger^\star}}[\mathrm{L}\to]}{B \to C \Rightarrow B^{\dagger^\star} \to C^{\dagger^\star}}[\mathrm{R}\to]$$

The other direction is similar.

If $A \equiv \bot_B$, then $A^{\dagger^\star} \equiv (B^\dagger \wedge \neg B^\dagger)^\star \equiv B^{\dagger^\star} \wedge (B^{\dagger^\star} \to \bot_{B^{\dagger\star}})$. By I.H., $\vdash B \Leftrightarrow B^{\dagger^\star}$. We have:

$$\dfrac{\dfrac{\dfrac{}{\bot_B \Rightarrow B}[\mathrm{L}\bot_*] \qquad B \Rightarrow B^{\dagger^\star}}{\bot_B \Rightarrow B^{\dagger^\star}}[\mathrm{Cut}] \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B^{\dagger^\star} \qquad B^{\dagger^\star} \Rightarrow B}{\bot_B \Rightarrow \bot_{B^{\dagger\star}}}[\mathrm{N}_\bot]}{\bot_B, B^{\dagger^\star} \Rightarrow \bot_{B^{\dagger\star}}}[\mathrm{LW}]}{\bot_B \Rightarrow B^{\dagger^\star} \to \bot_{B^{\dagger\star}}}[\mathrm{R}\to]}{\bot_B \Rightarrow B^{\dagger^\star} \wedge (B^{\dagger^\star} \to \bot_{B^{\dagger\star}})}[\mathrm{R}\wedge]$$

$$\dfrac{\dfrac{B^{\dagger^\star} \Rightarrow B^{\dagger^\star} \qquad \dfrac{\dfrac{B \Rightarrow B^{\dagger^\star} \qquad B^{\dagger^\star} \Rightarrow B}{\bot_{B^{\dagger\star}} \Rightarrow \bot_B}[\mathrm{N}_\bot]}{B^{\dagger^\star}, \bot_{B^{\dagger\star}} \Rightarrow \bot_B}[\mathrm{LW}]}{B^{\dagger^\star}, B^{\dagger^\star} \to \bot_{B^{\dagger\star}} \Rightarrow \bot_B}[\mathrm{L}\to]}{B^{\dagger^\star} \wedge (B^{\dagger^\star} \to \bot_{B^{\dagger\star}}) \Rightarrow \bot_B}[\mathrm{L}\wedge,\mathrm{LC}]$$

(ii) If $A \equiv p$, $A^{\star^\dagger} \equiv A$. If $A \equiv B \circ C$: $\circ \in \{\wedge, \vee, \to\}$, then $A^{\star^\dagger} \equiv B^{\star^\dagger} \circ C^{\star^\dagger}$. The statement can be shown by the same argument as (i).

If $A \equiv \neg B$, then $A^{\star^\dagger} \equiv (B^\star \to \bot_{B^\star})^\dagger \equiv B^{\star^\dagger} \to (B^{\star^\dagger} \wedge \neg B^{\star^\dagger})$. By I.H. $\vdash B^{\star^\dagger} \Leftrightarrow B$. We have:

$$\dfrac{\dfrac{\dfrac{B^{\star\dagger} \Rightarrow B^{\star\dagger}}{\neg B, B^{\star\dagger} \Rightarrow B^{\star\dagger}}\,[\mathrm{LW}] \qquad \dfrac{\dfrac{B \Rightarrow B^{\star\dagger} \qquad B^{\star\dagger} \Rightarrow B}{\neg B \Rightarrow \neg B^{\star\dagger}}\,[\mathrm{N}]}{\neg B, B^{\star\dagger} \Rightarrow \neg B^{\star\dagger}}\,[\mathrm{LW}]}{\neg B, B^{\star\dagger} \Rightarrow B^{\star\dagger} \wedge \neg B^{\star\dagger}}\,[\mathrm{R}\wedge]}{\neg B \Rightarrow B^{\star\dagger} \to (B^{\star\dagger} \wedge \neg B^{\star\dagger})}\,[\mathrm{R}\to]$$

$$\dfrac{\dfrac{B \Rightarrow B^{\star\dagger} \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B^{\star\dagger} \qquad B^{\star\dagger} \Rightarrow B}{\neg B^{\star\dagger} \Rightarrow \neg B}\,[\mathrm{N}]}{B, \neg B^{\star\dagger} \Rightarrow \neg B}\,[\mathrm{LW}]}{B, B^{\star\dagger} \wedge \neg B^{\star\dagger} \Rightarrow \neg B}\,[\mathrm{L}\wedge]}{B, B^{\star\dagger} \to B^{\star\dagger} \wedge \neg B^{\star\dagger} \Rightarrow \neg B}\,[\mathrm{L}\to]}{B^{\star\dagger} \to B^{\star\dagger} \wedge \neg B^{\star\dagger} \Rightarrow \neg B}\,[\mathrm{An}]$$

□

Theorem 5.1.1 (faithful embeddings for $\mathbf{MPC}_{\neg}$ and $\mathbf{MPC}_{\perp_*}$)**.**
(i) $\vdash_{\mathbf{M}_{\perp_*}} \Gamma \Rightarrow A$ iff $\vdash_{\mathbf{M}_{\neg}} \Gamma^{\dagger} \Rightarrow A^{\dagger}$
(ii)$\vdash_{\mathbf{M}_{\neg}} \Gamma \Rightarrow A$ iff $\vdash_{\mathbf{M}_{\perp_*}} \Gamma^{\star} \Rightarrow A^{\star}$

Proof. The left-to-right directions are already established. For the other direction, in (i), $\vdash_{\mathbf{M}_{\neg}} \Gamma^{\dagger} \Rightarrow A^{\dagger}$ then $\vdash_{\mathbf{M}_{\perp*}} \Gamma^{\dagger^{\star}} \Rightarrow A^{\dagger^{\star}}$, so $\vdash_{\mathbf{M}_{\perp*}} \Gamma \Rightarrow A$ by the preceding lemma. Similarly for (ii). □

This establishes faithful embeddings, and so the definition equivalence between $\mathbf{MPC}_{\neg}$ and $\mathbf{MPC}_{\perp_*}$.

5.2 AnPC and Multi-Absurdity

We shall now consider the system obtained from $\mathbf{MPC}_{\perp_*}$ by dropping $\mathrm{N}_{\perp}$. We claim that this system is equivalent to **AnPC**:=An+**PPC**, albeit in a weaker sense than the equivalence between $\mathbf{MPC}_{\perp_*}$ and $\mathbf{MPC}_{\neg}$. They are faithfully embeddable to each other, but not definition equivalent. We shall call this non-extensional system as $\mathbf{AnPC}_{\perp}$. To establish the equivalence, we have to make use of semantic tools.

We start by outlining the semantics for each logic. In addition, we go back to Hilbert-type proof systems for the sake of convenience in proving soundness and completeness.

The following gives the Kripke semantics for **AnPC**.

Definition 5.2.1 (Kripke frame for **AnPC**)**.**
A Kripke frame for **AnPC** is a triple $(W, \leq, \Phi)$, where $(W, \leq)$ is a partially ordered set and Φ is a mapping $\Phi : FORM \to \mathcal{U}(W)$. (recall that $\mathcal{U}(W)$ is the set of all upward closed sets of W)

Definition 5.2.2 (Kripke model for **AnPC**)**.**
Let $\mathcal{F}$ be a frame. Then a Kripke model is a pair $(\mathcal{V}, \mathcal{F})$, where $\mathcal{V}$ is a persistent valuation for the propositional variables. The valuation for negation is defined as:

$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}} \neg A$ if and only if $\forall w' \geq w[w' \Vdash_{\mathbf{An}} A$ implies $w' \in \Phi(A)]$.

We check that some desired properties hold with this semantics.

Proposition 5.2.1 (persistence)**.**
$w \Vdash_{\mathbf{An}} A$ and $w \leq w'$ implies $w' \Vdash_{\mathbf{An}} A$

Proof.
We prove by induction on the complexity of formula. We only need to check the case for negation.

$w \Vdash_{\mathbf{An}} \neg A$
$\Leftrightarrow \forall w'' \geq w[w'' \Vdash_{\mathbf{An}} A$ implies $w'' \in \Phi(A)]$
$\Rightarrow \forall w'' \geq w'[w'' \Vdash_{\mathbf{An}} A$ implies $w'' \in \Phi(A)]$
$\Leftrightarrow w' \Vdash_{\mathbf{An}} \neg A$ □

Lemma 5.2.1 (validity of An)**.**
$\vDash_{\mathbf{An}} (A \to \neg A) \to \neg A$

Proof. Let $(\mathcal{F}, \mathcal{V})$ be given, and $w \in W$. Suppose for $w' \geq w$, $w' \Vdash_{\mathbf{An}} A \to \neg A$.Then for any $w'' \geq w'$, $w'' \Vdash_{\mathbf{An}} A$ implies $w'' \Vdash_{\mathbf{An}} \neg A$. So for any $w''' \geq w''$, $w''' \Vdash A$ implies $w''' \in \Phi(A)$. Thus in particular, $w'' \Vdash_{\mathbf{An}} A$ implies that $w'' \Vdash_{\mathbf{An}} A \Rightarrow w'' \in \Phi(A)$. So $w'' \Vdash_{\mathbf{An}} A$ implies $w'' \in \Phi(A)$. Therefore $w' \Vdash_{\mathbf{An}} \neg A$, and so $w \Vdash_{\mathbf{An}} (A \to \neg A) \to \neg A$. □

Now we are going to check the soundness and completeness.

Proposition 5.2.2 (soundness)**.**
$\vdash_{\mathbf{An}} A \Rightarrow \vDash_{\mathbf{An}} A$

Proof.
We need to prove that:
(i) the axioms of **AnPC** are valid.
(ii) MP preserves validity.

(i) That the positive axioms are valid is readily checked. By the previous lemma, An is valid.

(ii) Suppose $\Gamma \vDash_{\mathbf{An}} A$ and $\Gamma \vDash_{\mathbf{An}} A \to B$. Let $\mathcal{M}$ be s.t. $\mathcal{M} \vDash_{\mathbf{An}} G$ for all $G \in \Gamma$, and $w \in W$ be arbitrary. Then $\mathcal{M}, w \Vdash_{\mathbf{An}} A$ and $\mathcal{M}, w \Vdash_{\mathbf{An}} A \to B$. So $\mathcal{M}, w \Vdash_{\mathbf{An}} B$. Thus $\Gamma \vDash_{\mathbf{An}} B$. □

Definition 5.2.3 (saturation)**.**
Let Γ be a set of formulas. Γ is called *saturated*, if:
(i) $\Gamma \vdash A \Rightarrow A \in \Gamma$
(ii) $\Gamma \vdash A \vee B \Rightarrow \Gamma \vdash A$ or $\Gamma \vdash B$

Lemma 5.2.2 (Lindenbaum lemma)**.**
Suppose $\Gamma \nvdash_{\mathbf{An}} A$. Then there is a saturated $\Gamma^\omega \supseteq \Gamma$ s.t. $\Gamma^\omega \nvdash_{\mathbf{An}} A$.

Proof.
We define $(\Gamma^n)_{n \in \omega}$, and set $\Gamma^\omega := \bigcup_{n \in \omega} \Gamma_n$.

Let $(B_i \vee C_i)_{i\in\omega}$ be an enumeration of all disjunctions with infinite repetitions. Then:

$$\Gamma_0 := \Gamma$$
$$\Gamma^{i+1} := \begin{cases} \Gamma^i \cup \{B_i\} & \text{if } \Gamma^i \vdash_{\mathbf{An}} B_i \vee C_i \text{ and } \Gamma^i \cup \{B_i\} \nvdash_{\mathbf{An}} A. \\ \Gamma^i \cup \{C_i\} & \text{if } \Gamma^i \vdash_{\mathbf{An}} B_i \vee C_i \text{ and } \Gamma^i \cup \{C_i\} \nvdash_{\mathbf{An}} A. \\ \Gamma^i & \text{otherwise.} \end{cases}$$

We confirm that Γ^ω satisfies (i), (ii) and $\Gamma^\omega \nvdash_{\mathbf{An}} A$.
(i) Suppose $\Gamma^\omega \vdash_{\mathbf{An}} B$. Then (as proofs are finite objects), there is $k \in \omega$ s.t. $\Gamma^k \vdash_{\mathbf{An}} B$. Since disjunctions are infinitely repeated, there is $k' \geq k$ s.t. $B_{k'} \vee C_{k'} = B \vee B$. As $\Gamma^{k'} \vdash_{\mathbf{An}} B \vee B$, we have $B \in \Gamma^{k'+1} \subseteq \Gamma^\omega$.

(ii) If $\Gamma^\omega \vdash_{\mathbf{An}} B \vee C$, then for some k, $\Gamma^k \vdash_{\mathbf{An}} B \vee C$. So there must be $k' \geq k$ s.t. $B \in \Gamma^{k'+1}$ or $C \in \Gamma^{k'+1}$. Hence $\Gamma^\omega \vdash_{\mathbf{An}} B$ or $\Gamma^\omega \vdash_{\mathbf{An}} C$.

$\Gamma^\omega \nvdash_{\mathbf{An}} A$
Assume $\Gamma^\omega \vdash_{\mathbf{An}} A$. Then $\Gamma^k \vdash_{\mathbf{An}} A$ for some k, which is impossible by our choice of $(\Gamma^n)_{n\in\omega}$. □

Definition 5.2.4 (canonical model for **AnPC**).
A canonical model for **AnPC** is $(W, \leq, \Phi.\mathcal{V})$, where:

$W := \{\Delta | \Delta \text{ is saturated}\}$
$\leq := \subseteq$
$\Phi := \{(A, \{\Delta | A \in \Delta \text{ and } \neg A \in \Delta\})\}$
$\mathcal{V} := \{(p, \{\Delta | p \in \Delta\})\}$

Theorem 5.2.1. (completeness for **AnPC**)
$\Gamma \vDash_{\mathbf{An}} A \Rightarrow \Gamma \vdash_{\mathbf{An}} A$

Proof.
We prove the contraposition. Assume $\Gamma \nvdash_{\mathbf{An}} A$. Then by Lindenbaum lemma, there exists a saturated $\Gamma_0 \supseteq \Gamma$ s.t. $\Gamma_0 \nvdash_{\mathbf{An}} A$.

We construct a canonical model w.r.t. Γ, as $\mathcal{M}_n = (W, \leq, \Phi, \mathcal{V})$, where $W := \{\Delta \supseteq \Gamma | \Delta \text{ is saturated}\}$. Then for all $B \in \Gamma$ and $\Delta \in W$, $B \in \Delta$. Assuming $B \in \Delta$ iff $\Delta \Vdash_{\mathbf{An}} B$ (CLAIM), it follows $\mathcal{M}_n \vDash_{\mathbf{An}} B$. Now, $\Gamma_0 \in W$, but $A \notin \Gamma_0$. So $\Gamma_0 \nVdash_{\mathbf{An}} A$ by CLAIM. Hence $\mathcal{M}_n \nvDash_{\mathbf{An}} A$. Therefore $\Gamma \nvDash_{\mathbf{An}} A$.

proof of CLAIM
We prove by induction on the complexity of B.

'p': When $B \equiv p$,

$$\begin{aligned} p \in \Delta &\Leftrightarrow \Delta \in \mathcal{V}(p) \text{ [definition of } \mathcal{V}] \\ &\Leftrightarrow \Delta \Vdash_{\mathbf{An}} p \end{aligned}$$

'$\wedge$': When $B \equiv B_1 \wedge B_2$,

$$\begin{aligned} B_1 \wedge B_2 \in \Delta &\Leftrightarrow \Delta \vdash_{\mathbf{An}} B_1 \wedge B_2 && [\Delta \text{ is saturated}] \\ &\Leftrightarrow \Delta \vdash_{\mathbf{An}} B_1 \text{ and } \Delta \vdash B_2 \\ &\Leftrightarrow B_1 \in \Delta \text{ and } B_2 \in \Delta && [\Delta \text{ is saturated}] \\ &\Leftrightarrow \Delta \Vdash_{\mathbf{An}} B_1 \text{ and } \Delta \Vdash_{\mathbf{An}} B_2 && [\text{I.H.}] \\ &\Leftrightarrow \Delta \Vdash_{\mathbf{An}} B_1 \wedge B_2 \end{aligned}$$

'$\vee$': When $B \equiv B_1 \vee B_2$,

$$\begin{aligned} B_1 \vee B_2 \in \Delta &\Leftrightarrow \Delta \vdash_{\mathbf{An}} B_1 \vee B_2 && [\Delta \text{ is saturated}] \\ &\Leftrightarrow \Delta \vdash_{\mathbf{An}} B_1 \text{ or } \Delta \vdash_{\mathbf{An}} B_2 && [\Delta \text{ is saturated}] \\ &\Leftrightarrow B_1 \in \Delta \text{ or } B_2 \in \Delta && [\Delta \text{ is saturated}] \\ &\Leftrightarrow \Delta \Vdash_{\mathbf{An}} B_1 \text{ or } \Delta \Vdash_{\mathbf{An}} B_2 && [\text{I.H.}] \\ &\Leftrightarrow \Delta \Vdash_{\mathbf{An}} B_1 \vee B_2 \end{aligned}$$

'$\rightarrow$':
'$\Rightarrow$' Suppose $B \equiv B_1 \rightarrow B_2 \in \Delta$. Then $\Delta \vdash_{\mathbf{An}} B_1 \rightarrow B_2$. Now, if $\Delta' \Vdash_{\mathbf{An}} B_1$ for $\Delta' \geq \Delta$, then $B_1 \in \Delta'$ by I.H.. Thus $\Delta' \vdash_{\mathbf{An}} B_1$ and so $\Delta' \vdash_{\mathbf{An}} B_2$ by MP. This means $B_2 \in \Delta'$, so by I.H. $\Delta' \Vdash_{\mathbf{An}} B_2$. Hence $\Delta \Vdash_{\mathbf{An}} B_1 \rightarrow B_2$.

'$\Leftarrow$'
Suppose $\Delta \Vdash_{\mathbf{An}} B_1 \rightarrow B_2$ and assume $B_1 \rightarrow B_2 \notin \Delta$. Since Δ is saturated, $\Delta \nvdash_{\mathbf{An}} B_1 \rightarrow B_2$. Then $\Delta \cup \{B_1\} \nvdash_{\mathbf{An}} B_2$, so by Lindenbaum lemma there exists a saturated $\Delta_0 \supseteq \Delta \cup \{B_1\}$ s.t. $\Delta_0 \nvdash_{\mathbf{An}} B_2$. As Δ_0 is saturated, this means $B_1 \in \Delta_0$ but $B_2 \notin \Delta_0$. by I.H., $\Delta_0 \Vdash_{\mathbf{An}} B_1$ but $\Delta_0 \nVdash_{\mathbf{An}} B_2$, contradicting $\Delta \Vdash_{\mathbf{An}} B_1 \rightarrow B_2$. Therefore $B_1 \rightarrow B_2 \in \Delta$.

'$\neg$':
'$\Rightarrow$' Suppose $B \equiv \neg B_1$ and assume $\neg B_1 \in \Delta$. Then $\Delta \vdash_{\mathbf{An}} \neg B_1$. Now if $\Delta' \Vdash_{\mathbf{An}} B_1$ for $\Delta' \geq \Delta$, by I.H. $B_1 \in \Delta'$. Also, $\Delta \vdash_{\mathbf{An}} \neg B_1$ means $\neg B_1 \in \Delta \subseteq \Delta'$. Adding the two, we see $\Delta' \in \Phi(B_1)$ if $\Delta' \Vdash_{\mathbf{An}} B_1$. Thus $\Delta \Vdash_{\mathbf{An}} \neg B_1$

'$\Leftarrow$' Suppose $\Delta \Vdash_{\mathbf{An}} \neg B_1$ and assume $\neg B_1 \notin \Delta$. Then $\Delta \nvdash_{\mathbf{An}} \neg B_1$ as Δ is saturated. Since $\Delta \vdash_{\mathbf{An}} (B_1 \rightarrow \neg B_1) \rightarrow \neg B_1$, we have $\Delta \nvdash_{\mathbf{An}} B_1 \rightarrow \neg B_1$ and so $\Delta \cup \{B_1\} \nvdash_{\mathbf{An}} \neg B_1$. By Lindenbaum lemma, there exists a saturated $\Delta_0 \supseteq \Delta \cup \{B_1\}$ s.t. $\Delta_0 \nvdash_{\mathbf{An}} \neg B_1$. Because $B_1 \in \Delta_0$, by I.H. $\Delta_0 \Vdash_{\mathbf{An}} B_1$. Also, $\Delta_0 \nvdash_{\mathbf{An}} \neg B_1$ implies $\neg B_1 \notin \Delta_0$. Hence by definition of Φ, $\Delta_0 \notin \Phi(B_1)$. Thus $\Delta \nVdash_{\mathbf{An}} \neg B_1$, a contradiction. Hence $\neg B_1 \in \Delta$. □

We shall now turn our attention to $\mathbf{AnPC}_\perp$.

Definition 5.2.5 (Kripke frame/model for $\mathbf{AnPC}_\perp$).
A Kripke frame for $\mathbf{AnPC}_\perp$ is identical to that of $\mathbf{AnPC}$, and a Kripke model has a valuation $\mathcal{V}$, where the valuation for contradiction is defined as:

$$w \Vdash_{\mathbf{An}_\perp} \perp_A \text{ if and only if } w \in \Phi(A) \text{ and } w \Vdash_{\mathbf{An}_\perp} A$$

As with $\mathbf{AnPC}$, we need to check that various properties hold.

Proposition 5.2.3 (persistence).
$w \Vdash_{\mathbf{An}_\perp} A$ and $w \le w'$ implies $w' \Vdash_{\mathbf{An}_\perp} A$.

Proof.
We shall only check the case for contradiction.

$$\begin{aligned} w \Vdash \perp_A &\Leftrightarrow w \in \Phi(A) \text{ and } w \Vdash_{\mathbf{An}_\perp} A & \text{[dfn]} \\ &\Leftrightarrow \forall w'' \ge w[w'' \in \Phi(A) \text{ and } w'' \Vdash_{\mathbf{An}_\perp} A] & \text{[I.H.]} \\ &\Rightarrow w' \Vdash_{\mathbf{An}_\perp} \perp_A \end{aligned}$$

□

Lemma 5.2.3 (validity of $\mathrm{L}_{\perp_*}$).
$\vDash_{\mathbf{An}_\perp} \perp_A \to A$

Proof.
Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be given. Suppose for $w' \ge w$, $w' \Vdash_{\mathbf{An}_\perp} \perp_A$. Then by definition, $w' \Vdash_{\mathbf{An}_\perp} A$. So $w \Vdash_{\mathbf{An}_\perp} \perp_A \to A$. □

The proofs of soundness and completeness are largely analogous to those of **AnPC**.

Proposition 5.2.4 (soundness).
$\vdash_{\mathbf{An}_\perp} A \Rightarrow \vDash_{\mathbf{An}_\perp} A$

Proof.
That the axioms of $\mathbf{AnPC}_\perp$ are valid follows from the previous lemma. That MP preserves validity is checkable as in **AnPC**. □

Definition 5.2.6 (canonical model).
The canonical model for $\mathbf{AnPC}_\perp$ is the same as that of **AnPC**, except that Φ is defined as:
$\Phi := \{(A, \{\Delta | \perp_A \in \Delta\})\}$

Theorem 5.2.2 (completeness for $\mathbf{AnPC}_\perp$).
$\Gamma \vDash_{\mathbf{An}_\perp} A \Rightarrow \Gamma \vdash_{\mathbf{An}_\perp} A$

Proof.
The proof proceeds as in **AnPC**. The only difference is to establish $A \in \Delta \Leftrightarrow \Delta \Vdash_{\mathbf{An}_\perp} A$ for $A \equiv \perp_B$.

'$\Rightarrow$' Suppose $\perp_B \in \Delta$. Then $\Delta \vdash_{\mathbf{An}_\perp} \perp_B$. As $\Delta \vdash_{\mathbf{An}_\perp} \perp_B \to B$, $\Delta \vdash_{\mathbf{An}_\perp} B$. As Δ is saturated, $B \in \Delta$ and thus by I.H. $\Delta \Vdash_{\mathbf{An}_\perp} B$. Also, $\perp_B \in \Delta$ implies $\Delta \in \Phi(B)$. Hence $\Delta \in \Phi(B)$ and $\Delta \Vdash_{\mathbf{An}_\perp} B$. So $\Delta \Vdash_{\mathbf{An}_\perp} \perp_B$.

'$\Leftarrow$' Suppose $\Delta \Vdash_{\mathbf{An}_\perp} \perp_B$. Then $\Delta \in \Phi(B)$ and $\Delta \Vdash_{\mathbf{An}_\perp} B$. So in particular $\perp_B \in \Delta$ from the definition of Φ. □

Now we are ready to check the faithful embedding. We mainly rely on semantic argument, and so the statements take the following form.

Theorem 5.2.3 (faithful embedding between $\mathbf{AnPC}_\perp$ and **AnPC**).
(i) $\Gamma \vDash_{\mathbf{An}_\perp} A \Leftrightarrow \Gamma^\dagger \vDash_{\mathbf{An}} A^\dagger$
(ii) $\Gamma \vDash_{\mathbf{An}} A \Leftrightarrow \Gamma^\star \vDash_{\mathbf{An}_\perp} A^\star$
($\dagger, \star$ as before.)

Proof.
(i) '$\Leftarrow$'
Given an $\mathbf{AnPC}_\perp$ model $(\mathcal{F}, \mathcal{V})$, we define an $\mathbf{AnPC}$ model $(\mathcal{F}', \mathcal{V}')$ s.t.:

$$\begin{cases} (W', \leq') & := (W, \leq) \\ \Phi'(A^\dagger) & := \Phi(A) \\ \mathcal{V}'(p) & := \mathcal{V}(p) \end{cases}$$

Assume $\Gamma^\dagger \vDash_{\mathbf{An}} A^\dagger$. Let $(\mathcal{F}, \mathcal{V})$ be an $\mathbf{AnPC}_\perp$ frame s.t. $(\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{An}_\perp} G$ for all $G \in \Gamma$. We shall prove by induction on the complexity of A, that

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A \Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A^\dagger.\text{(CLAIM)}$$

Then,

$$\begin{aligned} (\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{An}_\perp} G \text{ for all } G \in \Gamma &\Leftrightarrow (\mathcal{F}', \mathcal{V}') \vDash_{\mathbf{An}} G^\dagger \text{ for all } G^\dagger \in \Gamma^\dagger \\ &\Rightarrow (\mathcal{F}', \mathcal{V}') \vDash_{\mathbf{An}} A^\dagger \\ &\Leftrightarrow (\mathcal{F}, \mathcal{V}) \vDash_{\mathbf{An}_\perp} A \end{aligned}$$

and so $\Gamma \vDash_{\mathbf{An}_\perp} A$, as required.

proof of CLAIM
- When $A \equiv p$, $A^\dagger \equiv p$. As $\mathcal{V}$ and $\mathcal{V}'$ coincide in the valuation of propositional variables,

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} p \Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} p$$

- When $A \equiv A_1 \wedge A_2$, $A^\dagger \equiv A_1^\dagger \wedge A_2^\dagger$.

$$\begin{aligned} (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_1 \wedge A_2 &\Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_1 \text{ and } (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_2 \\ &\Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_1^\dagger \text{ and } (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_2^\dagger \text{ [I.H.]} \\ &\Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_1^\dagger \wedge A_2^\dagger \end{aligned}$$

- When $A \equiv A_1 \vee A_2$, $A^\dagger \equiv A_1^\dagger \vee A_2^\dagger$.

$$\begin{aligned} (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_1 \vee A_2 &\Leftrightarrow (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_1 \text{ or } (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_2 \\ &\Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_1^\dagger \text{ or } (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_2^\dagger \text{ [I.H.]} \\ &\Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_1^\dagger \vee A_2^\dagger \end{aligned}$$

- When $A \equiv A_1 \to A_2$, $A^\dagger \equiv A_1^\dagger \to A_2^\dagger$.

$$\begin{aligned} (\mathcal{F}, \mathcal{V}), w \Vdash_{\mathbf{An}_\perp} A_1 \to A_2 &\Leftrightarrow \forall w' \geq w[(\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{An}_\perp} A_1 \text{ implies } (\mathcal{F}, \mathcal{V}), w' \Vdash_{\mathbf{An}_\perp} A_2] \\ &\Leftrightarrow \forall w' \geq w[(\mathcal{F}', \mathcal{V}'), w' \Vdash_{\mathbf{An}} A_1^\dagger \text{ implies } (\mathcal{F}', \mathcal{V}'), w' \Vdash_{\mathbf{An}} A_2^\dagger] \text{ [I.H.]} \\ &\Leftrightarrow (\mathcal{F}', \mathcal{V}'), w \Vdash_{\mathbf{An}} A_1^\dagger \to A_2^\dagger \end{aligned}$$

- When $A \equiv \bot_B$, $A^\dagger \equiv B^\dagger \wedge \neg B^\dagger$.

$$
\begin{aligned}
(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{An}_\bot} \bot_B &\Leftrightarrow (\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{An}_\bot} B \text{ and } w \in \Phi(B) && \text{[dfn]}\\
&\Leftrightarrow (\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{An}_\bot} B \text{ and } \forall w' \geq w[(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{An}_\bot} B \text{ implies } w' \in \Phi(B)]\\
&\Leftrightarrow (\mathcal{F}',\mathcal{V}'), w \Vdash_{\mathbf{An}} B^\dagger \text{ and } \forall w' \geq w[(\mathcal{F}',\mathcal{V}'), w' \Vdash_{\mathbf{An}} B^\dagger \text{ implies } w' \in \Phi'(B^\dagger)] && \text{[I.H.]}\\
&\Leftrightarrow (\mathcal{F}',\mathcal{V}'), w \Vdash_{\mathbf{An}} B^\dagger \wedge \neg B^\dagger
\end{aligned}
$$

'$\Rightarrow$'
Given soundness and completeness of the systems, we can give a proof-theoretic proof. Then it turns out that this direction is materially contained in the proof of embeddability of $\mathbf{MPC}_{\bot_*}$ to $\mathbf{MPC}_\neg$. (Notice that there in the cases other than $\mathrm{N}_\bot$, we did not appeal to the now omitted rule N; so the same derivations are available in establishing embeddability for the present case.)

(ii) '$\Leftarrow$'
For any **AnPC** model $(\mathcal{F},\mathcal{V})$, define an $\mathbf{AnPC}_\bot$ model $(\mathcal{F}',\mathcal{V}')$ s.t.

$$
\begin{cases}
(W', \leq') & := (W, \leq)\\
\Phi'(A^\star) & := \Phi(A)\\
\mathcal{V}'(p) & := \mathcal{V}(p)
\end{cases}
$$

Assume $\Gamma^\star \Vdash_{\mathbf{An}_\bot} A^\star$ and let $(\mathcal{F},\mathcal{V})$ be an **AnPC** model s.t. $(\mathcal{F},\mathcal{V}) \vDash_{\mathbf{An}} G$ for all $G \in \Gamma$. We shall show $(\mathcal{F},\mathcal{V}) \vDash_{\mathbf{An}} A$.

As in (i), it suffices to prove

$$(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{An}} A \Leftrightarrow (\mathcal{F}',\mathcal{V}'), w \Vdash_{\mathbf{An}_\bot} A^\star \text{ for all } w \in W$$

The cases for $A \equiv p$ and $A_1 \circ A_2, \circ \in \{\wedge, \vee, \rightarrow\}$ are similar to (i).

- When $A \equiv \neg B$, $A^\star \equiv B^\star \rightarrow \bot_{B^\star}$.

$$
\begin{aligned}
(\mathcal{F},\mathcal{V}), w \Vdash_{\mathbf{An}} \neg B &\Leftrightarrow \forall w' \geq w[(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{An}} B \text{ implies } w' \in \Phi(B)] && \text{[dfn]}\\
&\Leftrightarrow \forall w' \geq w[(\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{An}} B \text{ implies } (\mathcal{F},\mathcal{V}), w' \Vdash_{\mathbf{An}} B \text{ and } w' \in \Phi(B)]\\
&\Leftrightarrow \forall w' \geq w[(\mathcal{F}',\mathcal{V}'), w' \Vdash_{\mathbf{An}_\bot} B^\star \text{ implies } (\mathcal{F}',\mathcal{V}'), w' \Vdash_{\mathbf{An}_\bot} B^\star \text{ and } w' \in \Phi'(B^\star)] && \text{[I.H.]}\\
&\Leftrightarrow (\mathcal{F}',\mathcal{V}'), w \Vdash_{\mathbf{An}_\bot} B^\star \rightarrow \bot_{B^\star} && \text{[dfn]}
\end{aligned}
$$

'$\Rightarrow$'
This direction is materially contained in the proof of embbedability of $\mathbf{MPC}_\neg$ to $\mathbf{MPC}_{\bot_*}$. (there in the cases other than N, we did not appeal to the now omitted rule of $\mathrm{N}_\bot$, so again the same derivations are available for establishing embeddability for the present case).

□

Lastly we show that † and ⋆ do not define definitional equivalence between the two systems.

Proposition 5.2.5 (**AnPC**$_\perp$ and **AnPC** are not definition equivalent via †, ⋆)**.**
(i)$\nvDash_{\mathbf{An}} A \leftrightarrow (A^\star)^\dagger$
(ii)$\nvDash_{\mathbf{An}_\perp} A \leftrightarrow (A^\dagger)^\star$

Proof.
(i) We shall show $\nvDash_{\mathbf{An}} \neg\neg p \to (\neg\neg p)^{\star^\dagger}$.

$$(\neg\neg p)^{\star^\dagger} \equiv p \to (p \wedge \neg p) \to [p \to (p \wedge \neg p) \wedge \neg(p \to (p \wedge \neg p))]$$

Let $\mathcal{M}$ be a model s.t.

$$\begin{cases} W & := \{w, w'\} \\ \leq & := \{(w,w), (w,w'), (w',w')\} \\ \Phi(p) & := \{w'\} \\ \Phi(\neg p) & := \{w, w'\} \\ \Phi(p \to (p \wedge \neg p)) & := \emptyset \\ \mathcal{V}(p) & := \{w'\} \end{cases}$$

Then $w' \Vdash_{\mathbf{An}} p$ and $w' \in \Phi(p)$. So $w \Vdash_{\mathbf{An}} \neg p$. As $w \in \Phi(\neg p)$, this implies $w \Vdash_{\mathbf{An}} \neg\neg p$. Now, as $w' \Vdash_{\mathbf{An}} p$ and $w' \in \Phi(\neg p)$, $w \Vdash_{\mathbf{An}} p \to (p \wedge \neg p)$. But $w \notin \Phi(p \to (p \wedge \neg p))$. Hence $w \nVdash_{\mathbf{An}} p \to (p \wedge \neg p) \to [p \to (p \wedge \neg p) \wedge \neg(p \to (p \wedge \neg p))]$. Therefore $w \nVdash_{\mathbf{An}} \neg\neg p \to (\neg\neg p)^{\star^\dagger}$.

(ii) we shall show $\nVdash_{\mathbf{An}_\perp} \perp_{\perp_p} \to (\perp_{\perp_p})^{\dagger^\star}$.

$$(\perp_{\perp_p})^{\dagger^\star} \equiv p \wedge (p \to \perp_p) \wedge [p \wedge (p \to \perp_p) \to \perp_{p \wedge (p \to \perp_p)}]$$

Let $\mathcal{M}$ be a model s.t.

$$\begin{cases} W & := \{w\} \\ \Phi(p) & := \{w\} \\ \Phi(\perp_p) & := \{w\} \\ \Phi(p \wedge (p \to \perp_p)) & := \emptyset \\ \mathcal{V}(p) & := \{w\} \end{cases}$$

Then $w \Vdash_{\mathbf{An}_\perp} p$ and $w \in \Phi(p)$, so $w \Vdash_{\mathbf{An}_\perp} \perp_p$. In addition, $w \in \Phi(\perp_p)$, so $w \Vdash_{\mathbf{An}_\perp} \perp_{\perp_p}$. Further, it is easily seen than $w \Vdash_{\mathbf{An}_\perp} p \wedge (p \to \perp_p)$. But $w \notin \Phi(p \wedge (p \to \perp_p))$. So $w \nVdash_{\mathbf{An}_\perp} \perp_{p \wedge (p \to \perp_p)}$. Therefore $w \nVdash_{\mathbf{An}_\perp} \perp_{\perp_p} \to (\perp_{\perp_p})^{\dagger^\star}$. □

6 Concluding Remarks

Let us briefly look back what we have discussed in the last three sections.

In section 3, we mainly considered various subminimal axioms. As a result,

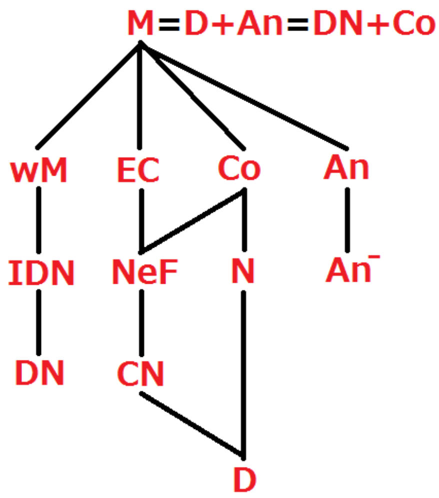

Figure 1: deducibility of subminimal axioms

we have obtained the above map of the deducibility relations among them (including the known ones). We also dealt with superminimal axioms, and found that An+EFQ, CM+EFQ each defines intuitionistic/classical negation.

In section 4, we considered subminimal correspondence theory. This was the enquiry of the correspondence between subminimal axioms and Kripke frames. We obtained a general method to give frame property for formulas with single type of propositional variables.

In section 5, We looked at another way of getting logic weaker tha minimal logic, by splitting up contradictions. We studied the relationship between subminimal logics and multi-absurdity logics. We established the definition equivalence between $\mathbf{MPC}_{\neg}$ and $\mathbf{MPC}_{\perp_*}$. We also showed the mutual faithful embedding between **AnPC** and $\mathbf{An}_{\perp}\mathbf{PC}$.

Given these results, we can raise several candidates for possible future research directions.

One is to investigate the inter-derivability of subminimal axioms further. An important task in this is to give separation results for the axioms. In 3.4 we gave some such results by defining certain classes of formulas. It is worth considering whether this approach can be generalised. We can also take a more standard, semantic approach for giving separability. This makes it more reasonable to study logics above **NPC**, because the current semantics assume the presence of N. Alternatively, we can try to generalise the current semantics, so that more axioms can be treated.

As to subminimal correspondence theory, we did not touch upon axioms with more than two types of propositional variables. These axioms are obviously in need of examination. In addition, we can turn our attention to the condition N imposes, $w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))$. This condition is the very thing that makes the investigation of the current subminimal semantics difficult (and interesting). Hence a closer study of it has potentially huge implication for our understanding of the semantics. For example it is examined in[3] how this condition is affected inside a linear frame. It is of interest, among others, to see how in general the condition is influenced by the shape of the frame (corresponding to a certain subminimal axiom).

With regards to multi-absurdity logics, it is desirable to general the achieved result, and clarify to what extent we can establish the correspondence between subminimal logics, in terms of definition equivalence/faithful embedding and other related conditions. Another, but related task would be to formulate a general method for giving semantics to multi-absurdity logics, so as to enable the semantic approach to the first task easier.

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