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Japan Advanced Institute of Science and Technology

# A study of left residuated lattices and logics without contraction and exchange rules

By Shin-ichiro Kaneko

A thesis submitted to
School of Information Science,
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Master of Information Science
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Written under the direction of
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March, 2002

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Associate Professor Hajime Ishihara
Professor Atsushi Ohori

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# Contents

# Chapter 1

# Introduction

Recently a lot of studies have been done in the fields of substructural logics. Both classic logic (**Cl**) and intuitionistic logic (**Int**) have three structural rules which are exchange, contraction and weakening rules. However, substructural logics does not have some or all structural rules. For example, Lambek calculus does not have all structural rules, linear logic has neither contraction nor weakening rules, and BCK logic does not have contraction rule. Studying properties of substructural logic in contrast to properties of **Cl** and **Int** can clarify the role the structural rules play. Also by using algebraic semantics we are able to consider substructural logics from more universal points. From that points of view, we can investigate relations between substructural logics.

In this paper, we will deal with the class of logics without contraction and exchange rules called $\mathbf{FL_w}$. The notation **FL** means Full Lambek calculus which is extension of Lambek calculus by adding binary connectives $\vee$ and $\wedge$, and rules pertaining to these connectives. (also we say that **FL** is intuitionistic logics without structural rules). The subscripts $\mathbf{w}, \mathbf{c}, \mathbf{e}$ denote weakening rule, contraction rule, exchange rule, respectively. Thus, $\mathbf{FL_w}$ means Full Lambek calculus which is added by weakening rule. In this paper, by comparing to $\mathbf{FL_w}$, we also deal with $\mathbf{FL_{ew}}$ which is the class of logics without contraction rule.

Residuated lattices have been studied since 1930. But recently it is noticed that residuated lattices are algebraic semantics of substractual logics. We have already had many results of algebraic semantics of $\mathbf{FL_{ew}}$ (see [6], [9]). However, there are not so many studies of algebraic semantics of $\mathbf{FL_w}$. Many problems still remain in it.

Exchange rule in sutractural rules corresponds to commutativity in algebraic semantics. Thus, an algebra for $\mathbf{FL_w}$ does not always have commutativity. We have known that noncommutative algebras sometimes have different properties from those of commutative algebras. Thus, study of noncomutative algebraic semantics will be interesting topic.

In this paper, we will introduce left residuated lattices which are the algebraic counterparts of $\mathbf{FL'_w}$, which in turn is a reduct of $\mathbf{FL_w}$. We will study basic properties of left residuated lattices, comparing them with commutative residuated lattices. Next, we will introduce the identity $C_n$. And we will study a classification of left residuated lattices by $C_n$.

Next, we will give a summary of this paper.

In Chapter 2, we will describe the systems $\mathbf{FL_{ew}}$ and $\mathbf{FL_w}$ by using sequent calculi. We will also introduce $\mathbf{FL'_w}$ which is a fragment of $\mathbf{FL_w}$.

In Chapter 3, we will introduce left residuated lattices which is algebras for $\mathbf{FL'_w}$. We will show characterizations of some basic properties which are filters for left residuated lattices, subdirectly irreducible left residuated lattices and simple left residuated lattices. Lastly we will show the existence of a lattice isomorphism between the set of all filters and the set of all congruences.

In Chapter 4, we will introduce a condition $(C_n)$ on left residuated lattices, for each $n$. When a left residuated lattice satisfies the condition $C_n$, we will demonstrate that filters of left residuated lattices coincide those of commutative residuated lattices. Next, we will introduce a classification of left residuated lattices by $C_n$.

# Chapter 2

# Preliminaries

The sequent system **LJ** which Gentzen introduced for intuitionistic logic has three kind of structural rules (exchange, contraction and weakening). Roughly speaking, the sequent calculus $\mathbf{FL_w}$ is obtained from intuitionistic logic **Int** by eliminating contraction and exchange rules. Also, sequent calculus $\mathbf{FL_{ew}}$ is obtained from intuitionistic logic by eliminating only contraction rule. Since $\mathbf{FL_{ew}}$ is closely related to $\mathbf{FL_w}$, we compare $\mathbf{FL_w}$ and $\mathbf{FL_{ew}}$. In this chapter, first we introduce the notations in this thesis and preparation for lattices. Next, we will introduce a sequent calculus $\mathbf{FL_{ew}}$. Lastly we will introduce sequent calculi $\mathbf{FL_w}$, and $\mathbf{FL'_w}$. The latter has single implication.

## 2.1 Preparations for lattices

We will explain notions in this paper. We will assume a familiarity with the most basic notions of sets. A class of sets is frequently called a *family* of sets. Define $I$ as $I = \{0, 1, 2, 3, \cdots\}$. The notations $A_i$, $i \in I$, and $(A_i)_{i \in I}$ refer to a *family of sets indexed by a set I*. We assume readers are familiar with membership ($\in$), subset ($\subseteq$), union ($\cup$), intersection($\cap$) and ordered n-tuples($\langle x_1, \cdots, x_n \rangle$).

**Definition 1 (Partial order sets (posets))** *A binary relation $\leq$ defined on a set $A$ is a* **partial order** *on the set $A$ if it satisfies, for any $x, y, z \in A$*

(O1) $x \leq x$ *(reflexivity),*

(O2) $x \leq y$ *and* $y \leq x$ *imply* $a = b$ *(antisymmetry),*

(O3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ *(transitivity).*

*A nonempty set with a partial order on it is called a* **partial order set***, or briefly a* **poset***.*

we call (O1)-(O3) axioms of posets. We define *lattices*, as follows.

**Definition 2 (Lattices)** *Let $\langle M, \leq \rangle$ be a poset. When for any $x, y \in M$, there exists supremum $x \vee y$ and infimum $x \wedge y$ of the set $\{x, y\}$ on $M$, $\mathbf{M} = \langle M, \vee, \wedge \rangle$ is said to be a* **lattice**.

Now, we call that $\vee$ is **join** and $\wedge$ is **meet**. The following proposition gives an alternative definition of lattices.

**Proposition 1** *An algebra $\mathbf{M} = \langle M, \vee, \wedge \rangle$ is a lattice, if it satisfies, for $x, y, z \in M$,*

(L1) $x \vee x = x,\ x \wedge x = x,$ *(idempotent laws),*

(L2) $x \vee y = y \vee x,\ x \wedge y = y \wedge x,$ *(commutative laws),*

(L3) $x \vee (y \vee z) = (x \vee y) \vee z,\ x \wedge (y \wedge z) = (x \wedge y) \wedge z,$ *(associative laws),*

(L4) $x \vee (x \wedge y) = x,\ x \wedge (x \vee y) = x.$ *(absorption laws).*

We call (L1)-(L4) axioms of lattices. Next, we define *bounded lattices* as follows.

**Definition 3 (Bounded lattices)** *An algebra $\mathbf{M} = \langle M, \vee, \wedge, 0, 1 \rangle$ with two binary and two nullary operations is a* **bounded lattice** *if it satisfies,*

1. $\mathbf{M} = \langle M, \vee, \wedge, 0, 1 \rangle$ *is a lattice,*
2. $x \wedge 0 = 0,\ x \vee 1 = 1$ *for any $x \in M$.*

For brevity's sake, we suppose that any lattice under consideration in this paper is non-trivial, i.e. it has at least two elements. In this paper, we do not consider a trivial lattice which has only one element. Lastly we define *congruences*.

**Definition 4 (Congruences)** *Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$ and $\theta$ be an equivalence relation (that is, a reflexive, symmetric, and transitive binary relation) on $\mathbf{A}$. Then, $\theta$ is a* **congruence** *on $\mathbf{A}$ if $\theta$ satisfies the following;*

*for each $n$-ary function symbol $f \in \mathcal{F}$ and elements $a_i, b_i \in A$, if $\langle a_i, b_i \rangle \in \theta$ holds for $1 \leq i \leq n$ then*

$$\langle f^{\mathbf{A}}(a_1, \ldots, a_n), f^{\mathbf{A}}(b_1, \ldots, b_n) \rangle \in \theta$$

*holds.*

## 2.2 Sequent calculus $\mathbf{FL_{ew}}$

The language of $\mathbf{FL_{ew}}$ consists of a logical constance $\bot$, logical connectives $\supset, \wedge_L, \vee_L$ and $*$ (called *multiplicative conjunction* or *fusion*). The negation $\neg A$ is defined as an abbreviation of $A \supset \bot$. Sometimes we abbreviate the formula $(A \supset B) \wedge_L (B \supset A)$ to $A \equiv B$. A sequent is of the form $A_1, A_2, \ldots, A_m \to B$ for $m \geq 0$ where $A_1, A_2, \ldots, A_m, B$ are formulas. The system $\mathbf{FL_{ew}}$ consist of the initial sequents and rules of inference given below. (Here, A, B are formulas, C is either a formula or empty, and $\Gamma, \Delta, \Sigma$ are a (possibly empty) sequence of formulas.)

1. Initial sequents:

   (a) $A \to A$

   (b) $\bot \to$

2. Structural rules:
   (Cut rule)

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}$$

   (Weakening rule)

$$\frac{\Gamma \to C}{A, \Gamma \to C}(weakening\ left) \qquad \frac{\Gamma \to}{\Gamma \to A}(weakening\ right)$$

   (Exchange rule)

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C}(ex)$$

3. Logical rules:

$$\frac{A, \Gamma \to C}{A \wedge_L B, \Gamma \to C}(\wedge_{L1} \to) \qquad \frac{B, \Gamma \to C}{A \wedge_L B, \Gamma \to C}(\wedge_{L2} \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge_L B}(\to \wedge_L)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee_L B}(\to \vee_{L1}) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}(\to \vee_{L2})$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee_L B, \Gamma \to C}(\vee_L \to)$$

$$\frac{\Gamma \to A \quad B, \Delta \to C}{A \supset B, \Gamma, \Delta \to C}(\supset\to) \quad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}(\to\supset)$$

$$\frac{\Gamma \to A}{\neg A, \Gamma \to}(\neg\ Left) \quad \frac{A, \Gamma \to}{\Gamma \to \neg A}(\neg Right)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B}(\to *) \quad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A * B, \Delta \to C}(* \to)$$

The provability of a given sequent is defined in the usual way. We say that in $\mathbf{FL_{ew}}$ a formula A is provable, when the sequent $\to A$ is provable in it. For more information on syntactic and semantic properties of $\mathbf{FL_{ew}}$, see [8]. The cut elimination, the decidability and Craig's interpolation theorem of $\mathbf{FL_{ew}}$ hold. Moreover, a Hilbert-style formulation of $\mathbf{FL_{ew}}$ is given, a Kripke-type semantics for $\mathbf{FL_{ew}}$ and related systems is introduced and their completeness with respect to the semantics is proved.

## 2.3 Sequent calculus $\mathbf{FL_w}$

It is natural to introduce two implications on $\mathbf{FL_w}$ since it does not have exchange rule. The negation $\neg_1 A$ and $\neg_2 A$ are defined as an abbreviations of $A \supset_1 \bot$ and $A \supset_2 \bot$, respectively. The system $\mathbf{FL_w}$ consists of the initial sequents and rules of inference given below.

1. Initial sequents:

   (a) $A \to A$

   (b) $\bot \to$

2. Structural rules:
   (Cut rule)

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta, \Gamma, \Sigma \to C}$$

   (Weakening rule)

$$\frac{\Gamma, \Delta \to C}{\Gamma, A, \Delta \to C}(weakening\ left) \quad \frac{\Gamma \to}{\Gamma \to A}(weakening\ right)$$

3. Logical rules:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge_L B, \Delta \to C}(\wedge_{L1} \to) \quad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge_L B, \Delta \to C}(\wedge_{L2} \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge_L B}(\to \wedge_L)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee_L B}(\to \vee_{L1}) \quad \frac{\Gamma \to B}{\Gamma \to A \vee_L B}(\to \vee_{L2})$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee_L B, \Delta \to C}(\vee_L \to)$$

$$\frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, A \supset_1 B, \Gamma, \Sigma \to C}(\supset_1\to) \quad \frac{\Gamma, A \to B}{\Gamma \to A \supset_1 B}(\to\supset_1)$$

$$\frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, \Gamma, A \supset_2 B, \Sigma \to C}(\supset_2\to) \quad \frac{A, \Gamma \to B}{\Gamma \to A \supset_2 B}(\to\supset_2)$$

$$\frac{\Gamma \to A}{\neg_1 A, \Gamma \to}(\neg_1\ Left) \quad \frac{\Gamma, A \to}{\Gamma \to \neg_1 A}(\neg_1 Right)$$

$$\frac{\Gamma \to A}{\Gamma, \neg_2 A \to}(\neg_2\ Left) \quad \frac{A, \Gamma \to}{\Gamma \to \neg_2 A}(\neg_2 Right)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B}(\to *) \quad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A * B, \Delta \to C}(* \to)$$

The provability of a given sequent is defined similarly to $\mathbf{FL_{ew}}$. We say that in $\mathbf{FL_{ew}}$ a formula A is provable, when the sequent $\to A$ is provable in it. In this paper, for simplifying the discussion, we consider only one implication ($\supset_1$) and one negation ($\neg_1$). We call this restricted system $\mathbf{FL'_w}$. Clearly, $\mathbf{FL'_w} \subsetneq \mathbf{FL_w}$ holds.

The class of logics over $\mathbf{FL'_w}$ is ordered by the set inclusion $\subseteq$. Of course, $\mathbf{FL'_w}$ is the smallest logic among them and the set of all formulas is the greatest one. The latter is called the inconsistent logic. Here we are concerned only with consistent logics, among which the classical logic **Cl** is the greatest.

Suppose that $\{L_i\}_{i \in I}$ is a set of logics, where $I$ is a (possibly infinite) nonempty set of indices. Then, the set intersection $\cap_{i \in I} L_i$ of them becomes also a logic. It is obvious that their set union is not always a logic, so we define,

$$\vee_{i \in I} L_i = \{A : there\ exist\ j_1, \ldots, j_k \in I\ and\ formulas\ B_{jt} \in L_{jt}\ for\ 1 \leq t \leq k\ such\ that\ the\ formula\ (B_{j1} * \cdots * B_{jk}) \supset A\ is\ provable\ in\ \mathbf{FL'_w}\}$$

Then, we can show that $\vee_{i \in I} L_i$ is the smallest logic which includes all logics $L_i$s.

Let $L_0$ and $L$ be logics such that $L_0 \subseteq L$. Then, $L$ is said to be *finitely axiomatized over* $L_0$ *by the axioms* $A_1, \ldots, A_m$, if $L$ is the smallest logic contains both $L_0$ and the set $\{A_1, \ldots, A_m\}$. Then, for any formula $C$, $C$ is in $L$ if and only if there exist formulas $B_1, \ldots, B_n (for\ some\ n \geq 0)$, each of substitution instance of some $A_k$, such that the formula $(B_1 * \cdots * B_n) \supset C$ belongs to $L_0$. The logic $L$ is denoted by $L_0[A_1, \ldots, A_m]$ in this case. A logic $L$ is said to be *finitely axiomatizable* over $L_0$ when there exist some axioms by which $L$ is finitely axiomatized over $L_0$. We will omit the word "over $L_0$" when $L_0$ is $\mathbf{FL'_w}$. It is easy to see that $L[A_1, \ldots, A_m] = L[A_1 * \cdots * A_n]$, by the help of the *weakening rule* of $FL_w$, i.e. by using the fact that $(C * D) \supset C$ is provable in $\mathbf{FL'_w}$. It is easy to see the following.

**Proposition 2** *Suppose that logics $L$ and $L'$ are finitely axiomatized over $L_0$ by the axioms $A$ and $B$, respectively. Then, logics $L \cap L'$ and $L \vee L'$ are finitely axiomatized over $L_0$ by axioms $A \vee B$ and $A * B$, respectively. (In the latter case, it is necessary to assume moreover that $A$ and $B$ have no propositional variable in common)*

# Chapter 3

# Left residuated lattices

In this section, we will introduce left residuated lattices. As shown later, the class of left residuated lattices gives algebraic semantics for $\mathbf{FL'_w}$, as the class of *Boolean algebras* (and of *Heyting algebras*) does for classical logic (and intuitionistic logic, respectively). For a left residuated lattice, the commutativity of the monoidal operator is not always assumed, since commutativity corresponds to exchange rule. The class of commutative residuated lattices is algebraic semantics for $\mathbf{FL_{ew}}$. We will discuss properties of left residuated lattices, comparing with commutative residuated lattices.

## 3.1 Definition of left residuated lattices

**Definition 5 (Left residuated lattices)** *An algebra* $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ *is a* **left residuated lattice** *if it satisfies*

1. $\langle M, \wedge, \vee, 0, 1 \rangle$ *is a bounded lattice with the greatest element 1 and the least 0,*
2. $\langle M, \cdot, 1 \rangle$ *is a monoid,*
3. $c \cdot a \leq b \Leftrightarrow c \leq a \rightarrow b$ *(left-residuation),*
4. $w \cdot (x \vee y) \cdot z = (w \cdot x \cdot z) \vee (w \cdot y \cdot z)$.

The relation between the monoidal operator $\cdot$ and its *left residual* $\rightarrow$ shown in 3 of Definition 1 is called *the law of left residuation*. In the following, we assume that left residuated lattices are always *non-degenerate* ones, i.e. left residuated lattices satisfying $0 \neq 1$. We define $\sim x$ by $\sim x = x \rightarrow 0$. The term $\underbrace{x \cdots x}_{\text{n}}$ is denoted by $x^n$.

It is easy to see following properties from the above definition.

**Lemma 1** *Let* $\boldsymbol{M}$ *be a left residuated lattice. For all* $x, y, z \in M$

1. *If* $x \leq y$*, then* $z \cdot x \leq z \cdot y$ *and* $x \cdot z \leq y \cdot z$*,*
2. *If* $x \leq y$*, then* $z \rightarrow x \leq z \rightarrow y$*,*

3. *If* $x \leq y$, *then* $y \to z \leq x \to z$,

4. $(x \to y) \cdot x \leq y$,

5. $z \cdot (x \to y) \leq x \to z \cdot y$,

6. $(z \to x)(y \to z) \leq y \to x$,

7. $z \to x \leq (y \to z) \to (y \to x)$,

8. $z \to (y \to x) = (z \cdot y) \to x$,

9. $(z \to x) \to \{(y \to z) \to (y \to x)\} = 1$,

10. $1 \to x = x$,

11. $x \to 1 = 1$,

12. $x \to x = 1$,

13. $x \to y = max\,\{z | z \cdot x \leq y\}$.

Here, we can show the following, similarly to Idziak [3] for commutative residuated lattices.

**Proposition 3** *The class all left residuated lattices forms a variety.*

A variety $\mathcal{K}$ is *congruence-distributive* when the lattice of all congruence relations of any algebra in $\mathcal{K}$ is distributive, and is *congruence permutable* when every two congruences of any algebra in $\mathcal{K}$ is permute. Moreover, if $\mathcal{K}$ is both congruence-distributive and congruence permutable, it is said to be *arithmetical*. Then the following holds.

**Proposition 4** *The variety of left residuated lattices is arithmetical.*

### 3.1.1 Comparison with commutative residuated lattices

**Definition 6 (Commutative residuated lattice)** *An algebra* $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \to, 0, 1\rangle$ *is a* **commutative residuated lattice** *if it satisfies*

1. $\langle M, \wedge, \vee, 0, 1\rangle$ *is bounded lattice with the greatest element 1 and least 0,*

2. $\langle M, \cdot, 1\rangle$ *is a commutative monoid,*

3. $c \cdot a \leq b \Leftrightarrow c \leq a \to b$.

When we assume the commutativity of the monoidal operator $\cdot$, clearly $\langle M, \cdot, 1\rangle$ is a commutative monoid. The following condition (4) becomes redundant in commutative residuated lattices, as shown below.

$$w \cdot (x \vee y) \cdot z = (w \cdot x \cdot z) \vee (w \cdot y \cdot z) \tag{4}$$

Before proving (4), we show the following i and ii.

i $\quad x \leq y \quad implies \quad x \cdot z \leq y \cdot z$

ii $\quad (x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$

Proof of i.
Suppose that $x \leq y$.

$$\begin{aligned} y \cdot z \leq y \cdot z &\Leftrightarrow y \leq z \to y \cdot z \\ &\Rightarrow x \leq y \leq z \to y \cdot z \quad (\because \textit{By the assumption}) \\ &\Rightarrow x \cdot z \leq y \cdot z \end{aligned}$$

Proof of ii.
The inequality $(x \cup y) \cdot z \leq (x \cdot z) \cup (y \cdot z)$ can be shown as follows.

$$\begin{aligned} \begin{cases} x \cdot z \leq (x \cdot z) \vee (y \cdot z) \textit{ and} \\ y \cdot z \leq (x \cdot z) \vee (y \cdot z) \end{cases} &\Leftrightarrow \begin{cases} x \leq z \to (x \cdot z) \vee (y \cdot z) \textit{ and} \\ y \leq z \to (x \cdot z) \vee (y \cdot z) \end{cases} \\ &\Leftrightarrow x \vee y \leq z \to (x \cdot z) \vee (y \cdot z) \\ &\Leftrightarrow (x \vee y) \cdot z \leq (x \cdot z) \vee (y \cdot z) \end{aligned}$$

To show $(x \cdot z) \vee (y \cdot z) \leq (x \vee y) \cdot z$, by using i.

$$\begin{aligned} \begin{cases} x \leq x \vee y \textit{ and} \\ y \leq x \vee y \end{cases} &\Rightarrow \begin{cases} x \cdot z \leq (x \vee y) \cdot z \textit{ and} \\ y \cdot z \leq (x \vee y) \cdot z \end{cases} \\ &\Leftrightarrow (x \cdot z) \vee (y \cdot z) \leq (x \vee y) \cdot z \end{aligned}$$

Hence, $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$.
Now, by using the commutativity, we can show (4) easily.

## 3.2 Connections between the class of left residuated lattices and $\mathbf{FL'_w}$

We will focus on connections between the class of left residuated lattices and $\mathbf{FL'_w}$. At first, we define the *validity* of a formula (of $\mathbf{FL'_w}$) in a given left residuated lattice.

Let $\mathbf{M}$ be a left residuated lattice. Any mapping $v$ from the set of all propositional variables to the set $M$ is called a *valuation* on $\mathbf{M}$. A given valuation $v$ can be extended to a mapping from the set of all formulas to $M$, inductively as follow.

1. $v(\bot) = 0$,
2. $v(A \wedge_L B) = v(A) \wedge v(B)$,
3. $v(A \vee_L B) = v(A) \vee v(B)$,

4. $v(A * B) = v(A) \cdot v(B)$,

5. $v(A \supset B) = v(A) \to v(B)$.

A formula $A$ is valid in $\mathbf{M}$ if $v(A) = 1$ holds for any valuation on $\mathbf{M}$. The set of formulas which are valid in $\mathbf{M}$ is denoted by $L(\mathbf{M})$. Next, a given sequent $A_1, \cdots, A_m \to B$ is said to be valid in $M$ if the formula $(A_1 * \cdots * A_m) \supset B$ is valid in it. Then, the following completeness theorem of $\mathbf{FL'_w}$ can be shown.

**Theorem 1** *A sequent $S$ is provable in $\mathbf{FL'_w}$ if and only if it is valid in all left residuated lattices.*

It is easy to see that $L(\mathbf{M})$ is a logic over $\mathbf{FL'_w}$ for any left residuated lattices $\mathbf{M}$, which is called the logic determined by $\mathbf{M}$. Conversely, we can show that for any logic $L$ over $\mathbf{FL'_w}$ there exists a left residuated lattice $\mathbf{M}$ such that $L = L(\mathbf{M})$. The latter can be proved by taking the *Lindenbaum algebra* of $L$ for $\mathbf{M}$.

## 3.3 Definition of filters

**Definition 7 (filters)** *A nonempty subset $F$ of a left residuated lattice $\boldsymbol{M}$ is an* **implicative filter** *(or, simply a* **filter***) if for $a, b \in M$ it satisfies*

*1.* $1 \in F$,

*2.* $a, a \to b \in F \quad \textit{imply} \quad b \in F$,

*3.* $a \in F \quad \textit{implies} \quad (a \to b) \to b \in F$.

The following proposition is equal to the above definition.

**Proposition 5** *A nonempty subset $F$ of a left residuated lattice $\boldsymbol{M}$ is a filter, if for $a, b \in M$ it satisfies,*

*1.* $a \leq b$ *and* $a \in F \quad \textit{imply} \quad b \in F$,

*2.* $a, b \in F \quad \textit{implies} \quad a \cdot b \in F$,

*3.* $a \in F \quad \textit{implies} \quad (a \to b) \to b \in F$.

**Proof.**

1. Conditions of filters implies conditions in Proposition 5.

(a) $a \leq b$ and $a \in F$ imply $b \in F$:

$$\begin{aligned} a \leq b &\Leftrightarrow 1 \cdot a \leq b \\ &\Leftrightarrow 1 \leq a \to b \\ &\Leftrightarrow a \to b = 1 \in F \\ &\quad (\because \textit{The greatest element on } M \textit{ is } 1) \\ &\Rightarrow b \in F \\ &\quad (\because \textit{By the assumption and the condition } 2 \textit{ of filters}) \end{aligned}$$

(b) $a, b \in F$ implies $a \cdot b \in F$:

$$\begin{aligned} a \cdot b \leq a \cdot b &\Leftrightarrow a \leq b \to (a \cdot b) \\ &\Leftrightarrow 1 \leq a \to \{b \to (a \cdot b)\} \\ &\Leftrightarrow a \to \{b \to (a \cdot b)\} = 1 \\ &\quad (\because \textit{The greatest element on } M \textit{ is } 1) \\ &\Rightarrow b \to (a \cdot b) \in F \\ &\quad (\because \textit{By the assumption and the condition } 2 \textit{ of filters}) \\ &\Rightarrow a \cdot b \in F \\ &\quad (\because \textit{By the assumption and the condition } 2 \textit{ of filters}) \end{aligned}$$

(c) $a \in F$ implies $(a \to b) \to b \in F$:
It is the same as the condition 3 of filters.

2. Conditions of Proposition 5 imply conditions of filters.

(a) $1 \in F$:
Since F is not empty, there exists at least one element in F. Let us call it $x$ holds. Clearly, $x \leq 1$. By the first condition of Proposition 5, $1 \in F$ holds.

(b) $a, a \to b \in F$ implies $b \in F$:
we have $(a \to b) \cdot a \leq b$ by $a \to b \leq a \to b$. Here, $(a \to b) \cdot a \in F$ since using the condition 2 of Proposition 5. Thus, by the condition 1 of Proposition 5 and $(a \to b) \cdot a \leq b$, $b \in F$ holds.

(c) $a \in F$ implies $(a \to b) \to b \in F$:
It is the same as the condition 3 of Proposition 5. ∎

The following proposition is also equal to the definition of filters.

**Proposition 6** *A nonempty subset $F$ of a left residuated lattice $\boldsymbol{M}$ is a filter, if for $a, b, c \in M$ it satisfies,*

1. $1 \in F$,

2. $a, b \to (a \to c) \in F$ *implies* $b \to c \in F$.

**Proof.**

1. Conditions of filters imply conditions of Proposition 6
   The condition 1 of Proposition 6 is the same as the condition 1 of filters. It is enough to show that $a, b \to (a \to c) \in F \quad \text{implies} \quad b \to c \in F$.
   Since $a \in F$, $(a \to c) \to c \in F$ holds for any $c \in M$. By the condition 9 of Lemma 1,

$$\{(a \to c) \to c\} \to [\{b \to (a \to c)\} \to (b \to c)] = 1 \in F.$$

   By the second condition of the definition of filters,

$$\{b \to (a \to c)\} \to (b \to c) \in F$$

   By the assumption, $b \to (a \to c) \in F$ holds. Thus, by the second condition of the definition of filters,

$$b \to c \in F.$$

2. Conditions of Proposition 6 imply conditions of filters.
   (a) $1 \in F$:
   It is the same as the condition 1 of Proposition 6.
   (b) $a, a \to b \in F \quad \textit{implies} \quad b \in F$:

$$\begin{aligned} 1 &= (a \to b) \to (a \to b) = 1 \to \{(a \to b) \to (a \to b)\} \in F \\ &\Rightarrow 1 \to (a \to b) \in F \\ &\quad (\because \textit{By the assumption and the condition } 2 \textit{ of Propsition} 5) \\ &\Rightarrow 1 \to b = b \in F \\ &\quad (\because \textit{By the assumption and the condition } 2 \textit{ of Propsition} 5) \end{aligned}$$

   (c) $a \in F \quad \textit{implies} \quad (a \to b) \to b \in F$:

$$\begin{aligned} &(a \to b) \to (a \to b) = 1 \in F \\ \Rightarrow\ &(a \to b) \to b \in F \\ &(\because \textit{By the assumption and the condition } 2 \textit{ of Propsition} 5) \end{aligned}$$

∎

Thus, we can define filters by using either conditions in Proposition 5 or those in Proposition 6.

Let $S$ be a nonempty subset of a left residuated lattice $\mathbf{M}$. We can define the minimum filter including $S$, as follows.

**Lemma 2** *Let $S$ be a nonempty subset of a left residuated lattice* $\mathbf{M}$. *Define* $A_m^S$ *by induction on* $\mathbf{M}$, *as follows.*

$$\begin{array}{rcl} D_0^S & = & S \\ A_0^S & = & \{w_1 \cdots w_k | w_i \in D_0^s, k \geq 1\} \\ D_1^S & = & \big\{(x \to y) \to y | x \in A_0^S, y \in M\big\} \\ A_1^s & = & \{w_1 \cdots w_k | w_i \in D_1^s, k \geq 1\} \\ \vdots & & \\ D_{n+1}^S & = & \{(x \to y) \to y | x \in A_n^s, y \in M\} \\ A_{n+1}^S & = & \big\{w_1 \cdots w_k | w_i \in D_{n+1}^S, k \geq 1\big\} \end{array}$$

*Then,* $H = \big\{x | m \geq 0,\ z \in A_m^S,\ z \leq x\big\}$ *is the minimum filter including $S$, called the filter generated by $S$.*

**Proof.** At first, we confirm $D_0^S \subseteq D_1^S \subseteq D_2^S \subseteq \cdots$. For any $z \in D_l^S$, $(z \to z) \to z = z \in D_{l+1}^S$ holds by $z \in A_l^S$. Thus, $D_i^S \subseteq A_i^S \subseteq D_{i+1}^S$ holds.

Next, we will show that $H = \big\{x | m \geq 0,\ z \in A_m^S,\ z \leq x\big\}$ is the minimum filter including S. First, we show that $H$ is a filter including $S$.

1. $H$ is a filter including $S$.
   We will demonstrate this by using Proposition 5.

   (a) $x \in H$ and $x \leq y$ for $y \in M$ imply $y \in H$:
   By the assumption, $z \leq x$ for some $z \in A_m^S$ and some $m(\geq 1)$. Since $x \leq y$, $z \leq y$ holds. Therefore, $y \in H$.

   (b) $x, y \in H$ implies $x \cdot y \in H$:
   By the assumption, there exist $z_0, z_1$ which satisfy $z_0 \in A_m^S$, $z_0 \leq x$ and $z_1 \in A_n^S$, $z_1 \leq y$, respectively. We have $z_0 \cdot z_1 \leq x \cdot y$ by the condition 1 of Lemma 1. Since $z_0 \cdot z_1 \in A_{max\{m,n\}+1}^S$, $x \cdot y \in H$ holds.

   (c) $x \in H$ implies $(x \to y) \to y \in H$ for $y \in M$:
   Before proving (c), we will first show the following.

$$u \leq w \quad \textit{implies} \quad w \to z \leq u \to z \tag{3.1}$$

   The proof of (3.1).

$$\begin{array}{rcl} w \to z \leq w \to z & \Leftrightarrow & (w \to z) \cdot w \leq z \\ & \Rightarrow & (w \to z) \cdot u \leq (w \to z) \cdot w \leq z \ (\because u \leq w) \\ & \Rightarrow & w \to z \leq u \to z \end{array}$$

   Suppose that $z \in A_m^S$ and $z \leq x$. Then, $x \to y \leq z \to y$ holds by using (3.1). Using (3.1) again, we have $(z \to y) \to y \leq (x \to y) \to y$.
   By $(z \to y) \to y \in A_{m+1}^S$, $(x \to y) \to y \in H$ holds.

(d) $H$ includes $S$:
   It is obvious by the definition of $H$.

2. $H$ is the minimum filter including $S$:
   We will show $H \subseteq F$ for any filters $F$ including $S$. First, we will demonstrate $A_i^S \subseteq F$ inductively.

   $A_0^S \subseteq F$ is obvious. Suppose that $A_k^S \subseteq F$. Take $w \in A_{k+1}^S$. By the definition, we can express $w$ as $\{(y_1 \to z_1) \to z_1\}\ \{(y_2 \to z_2) \to z_2\} \cdots \{(y_n \to z_n) \to z_n\}$ for some $y_1, y_2, \cdots, y_n \in A_k^S$ and some $z_1, z_2, \cdots, z_n \in M$. Since $y_1, y_2, \cdots, y_n \in A_k^S \subseteq F$ and the conditions 2, 3 of Proposition 5, $w \in F$ holds. Therefore, $A_{k+1}^S \subseteq F$.

   Hence, $A_i^S \subseteq F$ for any $i \geq 0$.

   Now, suppose that $x \in H$. Then, there exists $z \in A_m^S$ such that $z \leq x$. Since $A_m^S \subseteq F$, $z \in F$ and hence $x \in F$. ∎

The filter generated by $S$ is expressed as $\langle S \rangle$. For $a \in M$, $\langle \{a\} \rangle$ is denoted by $\langle a \rangle$, which is called the filter generated by $a$. We are able to describe $\langle a \rangle$ as the following.

$$\langle a \rangle = \{x | m \geq 0,\ z \in A_m^a,\ z \leq x\}$$

### 3.3.1 Comparison with filters of commutative residuated lattices

Filters of commutative residuated lattices are expressed as follows.

**Definition 8 (filters)** *A nonempty subset $F$ of a commutative residuated lattice* $\mathbf{M}$ *is a filter, if for $a, b \in M$ it satisfies*

1. $1 \in F$,

2. $a, a \to b \in F$ *implies* $b \in F$.

Now, the third condition of filters of left residuated lattices in Definition 7 is obtained as follows.

Let $\mathbf{M}$ be a commutative residuated lattice and $F$ be a filter of $\mathbf{M}$. Suppose that $a \in F$ and $b \in M$.

$$\begin{aligned} a \to b \leq a \to b \quad &\Leftrightarrow \quad (a \to b) \cdot a \leq b \\ &\Leftrightarrow \quad a \cdot (a \to b) \leq b \quad (\because \textit{commutativity}) \\ &\Leftrightarrow \quad a \leq (a \to b) \to b \\ &\Leftrightarrow \quad 1 \leq a \to ((a \to b) \to b) \\ &\Leftrightarrow \quad a \to ((a \to b) \to b) = 1 \\ &\qquad\ (\because \textit{The greatest element on } M \textit{ is } 1) \end{aligned}$$

Thus, $a \to ((a \to b) \to b) = 1 \in F$. By the assumption ($a \in F$) and the second condition of filters in Definition 8, we have $(a \to b) \to b \in F$.

Let $S$ be a subset of $M$. Now, we will define the filter generated by $S$ on $\mathbf{M}$. Similarly to the above, we also have $a \leq (a \to b) \to b$ for any $a, b \in M$. For any $n \geq 0$, any $s \in A_n^S$ and any $t \in M$, we have $s \leq (s \to t) \to t$. Thus, $\{x|m \geq 0, z \in A_m^S, z \leq x\} = \{x|z \in A_0^S, z \leq x\}$ holds. Therefore, by the definition of $A_0^S$, the filter $\langle S \rangle$ generated by $S$ can be expressed as follows.

$$\langle S \rangle = \{x|a_1 \cdots a_k \leq x \ for \ some \ a_1, \cdots, a_k \in S\}$$

In particular, the filter $\langle a \rangle$ generated by singleton set $\{a\}$ is expressed as follows.

$$\langle a \rangle = \{x|a^k \leq x \ for \ some \ positive \ integer \ k\}$$

## 3.4 Relations between filters and congruences

For left residuated lattices, there exists a lattice isomorphism between the set of all filters and the set of all congruences, (see [10]). We demonstrate this as follows.

1. Showing that both the set $\mathbf{F_M}$ of all filters of a given left residuated lattice $\mathbf{M}$ and the set $Con\mathbf{M}$ of all congruences of $\mathbf{M}$ are complete lattices.

2. Defining a map from $Con\mathbf{M}$ to $\mathbf{F_M}$ and a map from $\mathbf{F_M}$ to $Con\mathbf{M}$.

3. Showing that these maps give lattice isomorphisms between $\mathbf{F_M}$ and $Con\mathbf{M}$.

First, we define complete lattices, (see [2]).

**Definition 9 (Complete lattices)** *A poset $P$ is* **complete** *if for each subset $A$ of $P$ both $supA$ and $infA$ exist (in $P$). All complete posets are lattices, and a lattice $L$ which is complete as a poset is a* **complete lattice***.*

The elements $supA$ and $infA$ will be denoted by $\vee A$ and $\wedge A$, respectively. Next, the following is shown.

**Proposition 7** *Let $P$ be a poset. Then, $P$ is a complete lattice if $P$ has a largest element and the inf of any nonempty subset exist. Also, $P$ is a complete lattice if $P$ has has a smallest element and the sup of every nonempty subset exist.*

By using Proposition 7, we will show that the set of all filters of a given left residuated lattice is a complete lattice.

**Lemma 3** *Let $\mathbf{M}$ be a left residuated lattice and $\mathbf{F_M}$ be the set of all filters of $\mathbf{M}$. Then, $\langle \mathbf{F_M}, \vee_{\mathbf{F_M}}, \wedge_{\mathbf{F_M}} \rangle$ is a complete lattice.*

**Proof.** The greatest element is $M$.
Suppose $G = \{F_i|i \in I\} \subseteq \mathbf{F_M}$. We will show $\wedge_{\mathbf{F_M}} G = \cap G = \cap_{i \in I} F_i \in \mathbf{F_M}$.

1. $\cap_{i \in I} F_i$ is a filter.

   (a) $1 \in \cap_{i \in I} F_i$:
   Since all filters includes 1, $1 \in \cap_{i \in I} F_i$ holds.

   (b) $a, a \to b \in \cap_{i \in I} F_i$ implies $b \in \cap_{i \in I} F_i$:
   For $a, a \to b \in \cap_{i \in I} F_i$,

$$\begin{aligned} a, a \to b \in \cap_{i \in I} F_i &\Leftrightarrow a, a \to b \in F_j \quad (for\ any\ j \in I) \\ &\Rightarrow b \in F_j \quad (for\ any\ j \in I) \\ &\Leftrightarrow b \in \cap_{i \in I} F_i \end{aligned}$$

   (c) $a \in \cap_{i \in I} F_i$ implies $(a \to b) \to b \in \cap_{i \in I} F_i$ for any $b \in M$:
   (For $a \in \cap_{i \in I} F_i$),

$$\begin{aligned} a \in \cap_{i \in I} F_i &\Leftrightarrow a \in F_j \quad (for\ any\ j \in I) \\ &\Rightarrow (a \to b) \to b \in F_j \quad (for\ any\ j \in I) \\ &\Leftrightarrow (a \to b) \to b \in \cap_{i \in I} F_i \end{aligned}$$

2. $\cap_{i \in I} F_i$ is the infimum of G.
   For any $i \in I$ and any $F_i \in G$, $\cap_{i \in I} F_i \subseteq F_i$ holds. Suppose $B$ which satisfies $B \subseteq F_i$ for any $i \in I$. Then, $B \subseteq \cap_{i \in I} F_i$. Therefore, $\cap_{i \in I} F_i$ is the infimum of G. ∎

It is easy to see that $\vee_{\mathbf{F_M}}$ and $\wedge_{\mathbf{F_M}}$ are defined as follows.
For any $F, G \in \mathbf{F_M}$,

$$\begin{aligned} F \vee_{\mathbf{F_M}} G &\stackrel{\text{def}}{=} the\ filter\ generated\ by\ F \cup G \\ F \wedge_{\mathbf{F_M}} G &\stackrel{\text{def}}{=} F \cap G \end{aligned}$$

Next, by using Proposition 7, we will show that the set of all congruences is a complete lattice.

**Lemma 4** *Let* $\mathbf{M}$ *be a left residuated lattice and* $Con\mathbf{M}$ *be the set of all congruences of* $\mathbf{M}$. *Then,* $\langle Con\mathbf{M}, \vee_{Con}, \wedge_{Con} \rangle$ *is a complete lattice.*

**Proof.** The greatest element of $Con\mathbf{M}$ is $\theta$ which contains $\langle x, y \rangle$ for any $x, y \in M$.
For any $\Sigma = \{\theta_i | \theta_i \in Con\mathbf{M},\ i \in I\} \subseteq Con\mathbf{M}$, we will show $\wedge_{Con}\Sigma = \wedge_{Con\, i \in I}\theta_i = \cap_{i \in I}\theta \in Con\mathbf{M}$.

1. Reflexivity:
   For any $i \in I$, $\langle a, a \rangle \in \theta_i$ holds. Thus, $\langle a, a \rangle \in \cap_{i \in I}\theta_i$.

2. Symmetry:
   Suppose $\langle a, b \rangle \in \cap_{i \in I}\theta_i$. We have $\langle a, b \rangle \in \theta_i$ for any $i \in I$. Hence, $\langle b, a \rangle \in \theta_i$. Thus, $\langle b, a \rangle \in \cap_{i \in I}\theta_i$.

3. Transitivity:
   Suppose $\langle a, b\rangle, \langle b, c\rangle \in \cap_{i\in I}\theta_i$. Since $\langle a, b\rangle, \langle b, c\rangle \in \theta_i$ for any $i \in I$, $\langle a, c\rangle \in \theta_i$ holds. Thus, $\langle a, c\rangle \in \cap_{i\in I}\theta_i$.

4. Preservation of operators:
   Let $\oplus$ be any operators on $Con\mathbf{M}$. Take any $a_j, b_j$ as $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle \in \theta_i$ for any $i \in I$. Then, we have $\langle a_1 \oplus a_2 \oplus \cdots \oplus a_n, b_1 \oplus b_2 \oplus \cdots \oplus b_n\rangle \in \theta_i$. Thus, $\langle a_1 \oplus a_2 \oplus \cdots \oplus a_n, b_1 \oplus b_2 \oplus \cdots \oplus b_n\rangle \in \cap_{i\in I}\theta_i$ holds. ∎

In $\langle Con\mathbf{M}, \vee_{Con}, \wedge_{Con}\rangle$, $\vee_{Con}$ and $\wedge_{Con}$ are defined by the following, (see [2]).
Firstly, we define the *relational product* $r \circ s$ of two binary relations $r, s$ on $M$. It is given by: $\langle a.b\rangle \in r \circ s$ if and only if for some $c$, $\langle a, c\rangle \in r, \langle c,\ b\rangle \in s$. For any $\theta_1, \theta_2 \in Con\mathbf{M}$,

$$\begin{aligned}\theta_1 \vee_{Con} \theta_2 &\stackrel{\text{def}}{=} \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \cdots \\ \theta_1 \wedge_{Con} \theta_2 &\stackrel{\text{def}}{=} \theta_1 \cap \theta_2\end{aligned}$$

Next, we will define a map form $\mathbf{F_M}$ to $Con\mathbf{M}$ as follows. Let $\mathbf{M}$ be a left residuated lattice, and F be a filter of $\mathbf{M}$. Define $\sim_{\mathbf{F}}$ as follows,

$$x \sim_{\mathbf{F}} y \stackrel{\text{def}}{=} x \to y, y \to x \in F.$$

**Lemma 5** *The relation $\sim_{\mathbf{F}}$ is a congruence on $\mathbf{M}$.*

**Proof.** We will show that $\sim_{\mathbf{F}}$ is a congruence.

1. Reflexivity:
   Since $x \to x = 1 \in F$, $x \sim_{\mathbf{F}} x$ holds.

2. Symmetry:
   Since $x \sim_{\mathbf{F}} y$, $x \to y, y \to x \in F$ holds. Then, it is clear that $y \sim_{\mathbf{F}} x$.

3. Transitivity:
   Suppose $x \sim_{\mathbf{F}} y$ and $y \sim_{\mathbf{F}} z$. $x \to y, y \to x, y \to z, z \to y \in F$ holds. Since the condition 9 of Lemma 1, $(y \to x) \to \{(z \to y) \to (z \to x)\} = 1 \in F$ holds. Since $y \to x \in F$, $(z \to y) \to (z \to x) \in F$. Since $z \to y \in F$, we have $z \to x \in F$. Similarly, we can show $x \to z \in F$. Therefore, $x \sim_{\mathbf{F}} z$ holds.

4. Preservation of operators:

   (a) Preserving $\wedge$ (meet):
   We will show that $a_1 \sim_{\mathbf{F}} b_1, a_2 \sim_{\mathbf{F}} b_2$ implies $(a_1 \wedge a_2) \sim_{\mathbf{F}} (b_1 \wedge b_2)$, as follows.
      i. Suppose $a_2 \sim_{\mathbf{F}} b_2$. Then, $(c \wedge a_2) \sim_{\mathbf{F}} (c \wedge b_2)$ holds for any $c \in M$.
      ii. Suppose $a_1 \sim_{\mathbf{F}} b_1$. Then, $(a_1 \wedge c) \sim_{\mathbf{F}} (b_1 \wedge c)$ holds for any $c \in M$.
      iii. By $c = a_1$ on i, $(a_1 \wedge a_2) \sim_{\mathbf{F}} (a_1 \wedge b_2)$.
      By $c = b_2$ on ii, $(a_1 \wedge b_2) \sim_{\mathbf{F}} (b_1 \wedge b_2)$.
      Since $\sim_{\mathbf{F}}$ is transitive, we have $(a_1 \wedge a_2) \sim_{\mathbf{F}} (b_1 \wedge b_2)$.

The proof of i.
By the assumption, we have $a_2 \to b_2, b_2 \to a_2 \in F$.

$$\begin{aligned}
\begin{cases} c \wedge a_2 \leq a_2 \ and \\ c \wedge a_2 \leq c \end{cases} &\Rightarrow \begin{cases} (a_2 \to b_2)(c \wedge a_2) \leq (a_2 \to b_2)a_2 \ and \\ (a_2 \to b_2)(c \wedge a_2) \leq (a_2 \to b_2)c \end{cases} \\
&\Rightarrow \begin{cases} (a_2 \to b_2)(c \wedge a_2) \leq b_2 \ and \\ (a_2 \to b_2)(c \wedge a_2) \leq c \end{cases} \\
&\quad (\because (a_2 \to b_2)a_2 \leq b_2 \ and \ (a_2 \to b_2)c \leq c) \\
&\Leftrightarrow (a_2 \to b_2)(c \wedge a_2) \leq c \wedge b_2 \\
&\Leftrightarrow a_2 \to b_2 \leq (c \wedge a_2) \to (c \wedge b_2) \in F \\
&\quad (\because By\ the\ assumption\ (a_2 \to b_2 \in F)\ and \\
&\quad Proposition\ 5)
\end{aligned}$$

We also have $b_2 \to a_2 \leq (c \wedge b_2) \to (c \wedge a_2) \in F$. Hence, $(c \wedge a_2) \sim_{\mathbf{F}} (c \wedge b_2)$.
The proof of ii
By the assumption, we have $a_1 \to b_1, b_1 \to a_1 \in F$.

$$\begin{aligned}
\begin{cases} a_1 \wedge c \leq a_1 \ and \\ a_1 \wedge c \leq c \end{cases} &\Rightarrow \begin{cases} (a_1 \to b_1)(a_1 \wedge c) \leq (a_1 \to b_1)a_1 \ and \\ (a_1 \to b_1)(a_1 \wedge c) \leq (a_1 \to b_1)c \end{cases} \\
&\Rightarrow \begin{cases} (a_1 \to b_1)(a_1 \wedge c) \leq b_1 \ and \\ (a_1 \to b_1)(a_1 \wedge c) \leq c \end{cases} \\
&\quad (\because (a_1 \to b_1)a_1 \leq b_1 \ and \ (a_1 \to b_1)c \leq c) \\
&\Leftrightarrow (a_1 \to b_1)(a_1 \wedge c) \leq b_1 \wedge c \\
&\Leftrightarrow a_1 \to b_1 \leq (a_1 \wedge c) \to (b_1 \wedge c) \in F \\
&\quad (\because By\ the\ assumption\ (a_1 \to b_1 \in F)\ and \\
&\quad Proposition\ 5)
\end{aligned}$$

We also have $b_1 \to a_1 \leq (b_1 \wedge c) \to (a_1 \wedge c) \in F$. Hence, $(a_1 \wedge c) \sim_{\mathbf{F}} (b_1 \wedge c)$.

(b) Preserving $\vee$ (join):
We will show that $a_1 \sim_{\mathbf{F}} b_1, a_2 \sim_{\mathbf{F}} b_2$ implies $(a_1 \vee a_2) \sim_{\mathbf{F}} (b_1 \vee b_2)$, similarly $\wedge$.

i. Suppose $a_2 \sim_{\mathbf{F}} b_2$. Then, $(c \vee a_2) \sim_{\mathbf{F}} (c \vee b_2)$ holds for any $c \in M$.
ii. Suppose $a_1 \sim_{\mathbf{F}} b_1$. Then, $(a_1 \vee c) \sim_{\mathbf{F}} (b_1 \vee c)$ holds for any $c \in M$.
iii. By $c = a_1$ on i, $(a_1 \vee a_2) \sim_{\mathbf{F}} (a_1 \vee b_2)$.
By $c = b_2$ on ii, $(a_1 \vee b_2) \sim_{\mathbf{F}} (b_1 \vee b_2)$.
Since $\sim_{\mathbf{F}}$ is transitive, we have $(a_1 \vee a_2) \sim_{\mathbf{F}} (b_1 \vee b_2)$.

The proof of i

By the assumption, we have $a_2 \to b_2, b_2 \to a_2 \in F$.

$$
\begin{aligned}
\left\{\begin{array}{l} a_2 \to b_2 \leq a \to b_2 \ and \\ a_2 \to b_2 \leq c \to c = 1 \end{array}\right. \left\{\begin{array}{l} \\ \\ \end{array}\right. &\Leftrightarrow \begin{array}{l} (a_2 \to b_2)a_2 \leq b_2 \ and \\ (a_2 \to b_2)c \leq c \end{array} \\
&\Rightarrow \left\{\begin{array}{l} (a_2 \to b_2)a_2 \leq c \vee b_2 \ and \\ (a_2 \to b_2)c \leq c \vee b_2 \end{array}\right. \\
&\quad (\because b_2 \leq c \vee b_2 \ and \ c \leq c \vee b_2) \\
&\Leftrightarrow (a_2 \to b_2)c \vee (a_2 \to b_2)a_2 \leq c \vee b_2 \\
&\Leftrightarrow (a_2 \to b_2)(c \vee a_2) \leq c \vee b_2 \\
&\Leftrightarrow a_2 \to b_2 \leq (c \vee a_2) \to (c \vee b_2) \in F \\
&\quad (\because By \ the \ assumption \ (a_2 \to b_2 \in F) \\
&\quad and \ Proposition \ 5)
\end{aligned}
$$

We also have $b_2 \to a_2 \leq (c \vee b_2) \to (c \vee a_2) \in F$. Hence, $(c \vee a_2) \sim_{\mathbf{F}} (c \vee b_2)$.
The proof of ii
By the assumption, we have $a_1 \to b_1, b_1 \to a_1 \in F$.

$$
\begin{aligned}
\left\{\begin{array}{l} a_1 \to b_1 \leq a_1 \to b_1 \ and \\ a_1 \to b_1 \leq c \to c = 1 \end{array}\right. &\Leftrightarrow \left\{\begin{array}{l} (a_1 \to b_1)a_1 \leq b_1 \ and \\ (a_1 \to b_1)c \leq c \end{array}\right. \\
&\Rightarrow \left\{\begin{array}{l} (a_1 \to b_1)a_1 \leq b_1 \vee c \ and \\ (a_1 \to b_1)c \leq b_1 \vee c \end{array}\right. \\
&\quad (\because b_1 \leq b_1 \vee c \ and \ c \leq b_1 \vee c) \\
&\Leftrightarrow (a_1 \to b_1)a_1 \vee (a_1 \to b_1)c \leq b_1 \vee c \\
&\Leftrightarrow (a_1 \to b_1)(a_1 \vee c) \leq b_1 \vee c \\
&\Leftrightarrow a_1 \to b_1 \leq (a_1 \vee c) \to (b_1 \vee c) \in F \\
&\quad (\because By \ the \ assumption \ (a_1 \to b_1 \in F) \\
&\quad and \ Proposition \ 5)
\end{aligned}
$$

We also have $b_1 \to a_1 \leq (b_1 \vee c) \to (a_1 \vee c) \in F$. Hence, $(a_1 \vee c) \sim_{\mathbf{F}} (b_1 \vee c)$.

(c) Preserving the operation $(\to)$ :
We will show that $a_1 \sim_{\mathbf{F}} b_1, a_2 \sim_{\mathbf{F}} b_2$ implies $(a_1 \to a_2) \sim_{\mathbf{F}} (b_1 \to b_2)$. By the assumption, we have $a_1 \to b_1, b_1 \to a_1, a_2 \to b_2, b_2 \to a_2 \in F$. First we will show (3.2) and (3.3).
Here, (3.2) holds, as follows.

$$
\begin{aligned}
&(a_1 \to a_2) \to \{(b_1 \to a_1) \to (b_1 \to a_2)\} = 1 \in F \\
&(\because The \ condition \ 9 \ of \ Lemma \ 1) \\
\Rightarrow \ &(a_1 \to a_2) \to (b_1 \to a_2) \in F \qquad (3.2) \\
&(\because By \ the \ assumption \ (b_1 \to a_1 \in F) \ and \ Proposition \ 6)
\end{aligned}
$$

Here, (3.3) holds, as follows.

$$\begin{aligned}
& (a_2 \to b_2) \to \{(b_1 \to a_2) \to (b_1 \to b_2)\} = 1 \in F \\
& (\because \mathit{The\ condition}\ 9\ \mathit{of\ Lemma}\ 1) \\
\Rightarrow\ & (b_1 \to a_2) \to (b_1 \to b_2) \in F \qquad (3.3) \\
& (\because \mathit{By\ the\ assumption}\ (a_2 \to b_2 \in F)\ \mathit{and\ the\ definition\ of} \\
& \mathit{filters})
\end{aligned}$$

By using (3.2) and (3.3), we have the following.

$$\begin{aligned}
& \{(b_1 \to a_2) \to (b_1 \to b_2)\} \to [\{(a_1 \to a_2) \to (b_1 \to a_2)\} \\
& \to \{(a_1 \to a_2) \to (b_1 \to b_2)\}] = 1 \in F \\
& (\because \mathit{The\ condition}\ 9\ \mathit{of\ Lemma}\ 1) \\
\Rightarrow\ & \{(a_1 \to a_2) \to (b_1 \to a_2)\} \to \{(a_1 \to a_2) \to (b_1 \to b_2)\} \in F \\
& (\because (3.3)\ \mathit{and\ the\ definition\ of\ filters}) \\
\Rightarrow\ & (a_1 \to a_2) \to (b_1 \to b_2) \in F \\
& (\because (3.2)\ \mathit{and\ the\ definition\ of\ filters})
\end{aligned}$$

We also have $(b_1 \to b_2) \to (a_1 \to a_2) \in F$. Hence, $(a_1 \to a_2) \sim_{\mathbf{F}} (b_1 \to b_2)$.

(d) Preserving the operator $(\cdot)$ :
We will show that $a_1 \sim_{\mathbf{F}} b_1, a_2 \sim_{\mathbf{F}} b_2$ implies $(a_1 \cdot a_2) \sim_{\mathbf{F}} (b_1 \cdot b_2)$. First, we will show (3.4) and (3.5).
Here, (3.4) holds, as follows.

$$\begin{aligned}
a_1 \to b_1 \leq a_1 \to b_1 \ &\Leftrightarrow\ (a_1 \to b_1) \cdot a_1 \leq b_1 \\
&\Rightarrow\ (a_1 \to b_1) \cdot a_1 \cdot a_2 \leq b_1 \cdot a_2 \\
&\Leftrightarrow\ (a_1 \to b_1) \leq a_1 \cdot a_2 \to b_1 \cdot a_2 \\
&\Leftrightarrow\ 1 \leq (a_1 \to b_1) \to (a_1 \cdot a_2 \to b_1 \cdot a_2) \\
&\Leftrightarrow\ (a_1 \to b_1) \to (a_1 \cdot a_2 \to b_1 \cdot a_2) = 1 \in F \\
&\Rightarrow\ a_1 \cdot a_2 \to b_1 \cdot a_2 \in F \qquad (3.4) \\
&\quad\ (\because a_1 \to b_1 \in F\ \mathit{and\ the\ definition\ of\ filters})
\end{aligned}$$

Here, (3.5) holds, as follows.

$$\begin{aligned}
a_2 \to b_2 \leq a_2 \to b_2 &\Leftrightarrow (a_2 \to b_2) \cdot a_2 \leq b_2 \\
&\Rightarrow b_1 \cdot (a_2 \to b_2) \cdot a_2 \leq b_1 \cdot b_2 \\
&\Leftrightarrow b_1 \cdot (a_2 \to b_2) \leq a_2 \to b_1 \cdot b_2 \\
&\Leftrightarrow b_1 \leq (a_2 \to b_2) \to (a_2 \to b_1 \cdot b_2) \\
&\Leftrightarrow 1 \leq b_1 \to \{(a_2 \to b_2) \to (a_2 \to b_1 \cdot b_2)\} \\
&\Leftrightarrow b_1 \to \{(a_2 \to b_2) \to (a_2 \to b_1 \cdot b_2)\} = 1 \in F \\
&\Rightarrow b_1 \to (a_2 \to b_1 \cdot b_2) \in F \\
&\qquad (\because a_2 \to b_2 \in F \textit{ and Proposition } 6) \\
&\Rightarrow b_1 \cdot a_2 \to b_1 \cdot b_2 \in F \qquad (3.5) \\
&\qquad (\because \textit{The condition } 8 \textit{ of Lemma } 1)
\end{aligned}$$

By using (3.4) and (3.5), we have the following.

$$\begin{aligned}
&(b_1 \cdot a_2 \to b_1 \cdot b_2) \to \{(a_1 \cdot a_2 \to b_1 \cdot a_2) \to \\
&(a_1 \cdot a_2 \to b_1 \cdot b_2)\} = 1 \in F \ (\because \textit{The condition } 9 \textit{ of Lemma } 1) \\
\Rightarrow\ &(a_1 \cdot a_2 \to b_1 \cdot a_2) \to (a_1 \cdot a_2 \to b_1 \cdot b_2) \in F \\
&(\because (3.5) \textit{ and the definition of filters}) \\
\Rightarrow\ &a_1 \cdot a_2 \to b_1 \cdot b_2 \in F \\
&(\because (3.4) \textit{ and the definition of filters})
\end{aligned}$$

We also have $b_1 \cdot b_2 \to a_1 \cdot a_2 \in F$. Hence, $a_1 \cdot a_2 \sim_{\mathbf{F}} b_1 \cdot b_2$. ∎

We will define a map from $Con\mathbf{M}$ to $\mathbf{F_M}$ as follows. Let $\mathbf{M}$ be a left residuated lattice and $\theta$ be a congruence of $\mathbf{M}$. Define $F_\theta$, as follows.

$$F_\theta \stackrel{\text{def}}{=} \{a | \langle a, 1\rangle \in \theta\}$$

**Lemma 6** *$F_\theta$ is a filter.*

**Proof.**

1. $1 \in F_\theta$:
   By $\langle 1, 1\rangle \in \theta$, we have $1 \in F_\theta$.

2. $a, a \to b \in F_\theta$ implies $b \in F_\theta$:

$$\begin{aligned}
a \in F_\theta &\Leftrightarrow \langle a, 1\rangle \in \theta \\
&\Rightarrow \langle a \to b, 1 \to b\rangle \in \theta \\
&\Rightarrow \langle a \to b, b\rangle \in \theta \\
&\Rightarrow \langle b, 1\rangle \in \theta \ (\because \textit{By } \langle a \to b, 1\rangle \in \theta,\ \theta \textit{ is both transitive and symmetric}) \\
&\Leftrightarrow b \in F_\theta
\end{aligned}$$

3. $a \in F_\theta$ implies $(a \to b) \to b \in F_\theta$:

$$\begin{aligned} a \in F_\theta &\Leftrightarrow \langle a, 1\rangle \in \theta \\ &\Rightarrow \langle a \to b, 1 \to b\rangle \in \theta \\ &\Rightarrow \langle a \to b, b\rangle \in \theta \\ &\Rightarrow \langle (a \to b) \to b, b \to b\rangle \in \theta \\ &\Rightarrow \langle (a \to b) \to b, 1\rangle \in \theta \\ &\Leftrightarrow (a \to b) \to b \in F_\theta \end{aligned}$$

∎

The following Theorem 2 shows the existence of the lattice isomorphism between the set $\mathbf{F_M}$ of all filters of a left residuated lattice $\mathbf{M}$ and the set $Con\mathbf{M}$ of all congruences of $\mathbf{M}$.

**Theorem 2** *Let $\boldsymbol{M}$ be a left residuated lattice. Then, there exits a lattice isomorphism between the set of all filters of* $\mathbf{M}$ *and the set of all congruences of* $\mathbf{M}$.

**Proof.** Let the set of all filters of $\mathbf{M}$ be $\mathbf{F_M}$ and the set of all congruences of $\mathbf{M}$ be $Con\mathbf{M}$. By Lemma 3,4, $\mathbf{F_M}$ and $Con\mathbf{M}$ are complete lattices. We define a map $\alpha$ from $\mathbf{F_M}$ to $Con\mathbf{M}$ and a map $\beta$ from $Con\mathbf{M}$ to $\mathbf{F_M}$.

$$\begin{aligned} \alpha(F) &= \sim_{\mathbf{F_M}} \\ \beta(\theta) &= F_\theta \end{aligned}$$

1. $\alpha$ is one-to-one.
   For any $F, G \in \mathbf{F_M}$, we will show $F \neq G$ implies $\alpha(F) \neq \alpha(G)$.
   Without losing the generality, we can suppose $a \in F \backslash G$. Since $a \to 1 = 1, 1 \to a = a \in F$, $\langle a, 1\rangle \in \alpha(F)$ holds. On the other hand, since $1 \to a = a \notin G$, $\langle a, 1\rangle \in \alpha(G)$ does not hold. Hence, $\alpha(F) \neq \alpha(G)$.

2. $\alpha$ is onto.
   For any $\theta \in Con\mathbf{M}$, we will show $\theta = \alpha(\beta(\theta))$.

   (a) $\theta \subseteq \alpha(\beta(\theta))$:
   For any $\langle a, b\rangle \in \theta$,

$$\begin{aligned} \langle a, b\rangle \in \theta &\Rightarrow \langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta \\ &\Leftrightarrow a \to b, b \to a \in \beta(\theta) \\ &\Leftrightarrow \langle a, b\rangle \in \alpha(\beta(\theta)) \end{aligned}$$

   (b) $\alpha(\beta(\theta)) \subseteq \theta$:
   For any $\langle a, b\rangle \in \alpha(\beta(\theta))$,

$$\begin{aligned} \langle a, b\rangle \in \alpha(\beta(\theta)) &\Leftrightarrow a \to b, b \to a \in \beta(\theta) \\ &\Leftrightarrow \langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta \end{aligned}$$

Here, for $\langle a \to b, 1\rangle \in \theta$,

$$\begin{aligned}
\langle a \to b, 1\rangle \in \theta &\Leftrightarrow \langle 1, a \to b\rangle \in \theta \\
&\Rightarrow \langle 1 \cdot a, (a \to b) \cdot a\rangle \in \theta \\
&\Rightarrow \langle 1 \cdot a, \{(a \to b) \cdot a\} \cap b\rangle \in \theta \quad (\because (a \to b) \cdot a \leq b) \\
&\quad\; \textit{and} \\
&\quad\; \langle \{(a \to b) \cdot a\} \cap b, a \cap b\rangle \in \theta \\
&\Rightarrow \langle a, a \cap b\rangle \in \theta
\end{aligned}$$

We also have $\langle b, a \cap b\rangle \in \theta$ by using $\langle a, b\rangle \in \theta$. Hence, $\langle a, b\rangle \in \theta$.

Therefore, $\alpha(\beta(\theta)) = \theta$ holds. This says that any $\theta \in Con\mathbf{M}$ is in the range of a map $\alpha$. Thus, $\alpha$ is onto.

3. $\alpha$ is order preserving.
We will show that for any $F, G \in \mathbf{F_M}$, $F \subseteq G$ if and only if $\alpha(F) \subseteq \alpha(G)$.

   (a) Only if part:
   For any $\langle a, b\rangle \in \alpha(F)$,

$$\begin{aligned}
\langle a, b\rangle \in \alpha(F) &\Leftrightarrow a \to b, b \to a \in F \\
&\Rightarrow a \to b, b \to a \in G \quad (\because F \subseteq G) \\
&\Leftrightarrow \langle a, b\rangle \in \alpha(G)
\end{aligned}$$

   (b) If part:
   We show that $F \nsubseteq G$ implies $\alpha(F) \nsubseteq \alpha(G)$. For $a \in F \backslash G$, $\langle 1, a\rangle \in \alpha(F)$ holds, while $\langle 1, a\rangle \in \alpha(G)$ does not hold. Hence, $\alpha(F) \nsubseteq \alpha(G)$.

4. $\alpha$ is a homomorphism.

   (a) Preserving $\vee_{\mathbf{F_M}}$:
   Suppose that any $F, G \in \mathbf{F_M}$. We will show that $\alpha(F \vee_{\mathbf{F_M}} G)$ is the least upper bound of $\alpha(F)$ and $\alpha(G)$ on $Con\mathbf{M}$. We have already known that $Con\mathbf{M}$ is a complete lattice which has $\wedge_{Con}$, $\vee_{Con}$, by Lemma 4.

      i. We show that $\alpha(F) \subseteq \alpha(F \vee_{\mathbf{F_M}} G)$ and $\alpha(F) \subseteq \alpha(F \vee_{\mathbf{F_M}} G)$.
      We have $F \subseteq F \vee_{\mathbf{F_M}} G$ and $G \subseteq F \vee_{\mathbf{F}} G$. Since $\alpha$ is order preserving, $\alpha(F) \subseteq \alpha(F \vee_{\mathbf{F_M}} G)$ and $\alpha(F) \subseteq \alpha(F \vee_{\mathbf{F_M}} G)$ hold.

      ii. Next, suppose that both $\alpha(F) \subseteq X$ and $\alpha(G) \subseteq X$ for $X \in Con\mathbf{M}$. We show that $\alpha(F \vee_{\mathbf{F_M}} G) \subseteq X$.
      Since $\alpha$ is onto and one-to-one, there exists a $H$ which satisfies $\alpha(H) = X$ for $H \in \mathbf{F_M}$. By $\alpha(F) \subseteq \alpha(H), \alpha(G) \subseteq \alpha(H)$ and order-isomorphism of $\alpha$, we have $F \subseteq H$ and $G \subseteq H$. Hence, $F \vee_{\mathbf{F_M}} G \subseteq H$. Since $\alpha$ is order-isomorphic, we have $\alpha(F \vee_{\mathbf{F_M}} G) \subseteq \alpha(H) = X$.

   By using i, ii, $\alpha(F \vee_{\mathbf{F_M}} G) = \alpha(F) \vee_{Con} \alpha(G)$ holds.

(b) Preserving $\wedge_{\mathbf{F_M}}$:
Suppose any $F, G \in \mathbf{F_M}$. Since $\wedge_{\mathbf{F}}$ and $\wedge_{Con}$ are defined by $\cap$, we have $\alpha(F \wedge_{\mathbf{F_M}} G) = \alpha(F \cap G)$ and $\alpha(F) \wedge_{Con} \alpha(G) = \alpha(F) \cap \alpha(G)$.
Now, we will show $\alpha(F \cap G) = \alpha(F) \cap \alpha(G)$.

i. $\alpha(F \cap G) \subseteq \alpha(F) \cap \alpha(G)$:

$$\begin{cases} F \cap G \subseteq F \text{ and} \\ F \cap G \subseteq G \end{cases} \Leftrightarrow \begin{cases} \alpha(F \cap G) \subseteq \alpha(F) \text{ and} \\ \alpha(F \cap G) \subseteq \alpha(G) \end{cases}$$
$$\Leftrightarrow \alpha(F \cap G) \subseteq \alpha(F) \cap \alpha(G)$$

ii. $\alpha(F) \cap \alpha(G) \subseteq \alpha(F \cap G)$:
For any $a \in \alpha(F) \cap \alpha(G)$,

$$\begin{aligned} a \in \alpha(F) \cap \alpha(G) &\Leftrightarrow a \in \alpha(F) \text{ and } a \in \alpha(G) \\ &\Rightarrow b \in F,\ b \in G \ (\text{for } b \text{ such that } a = \alpha(b)) \\ &\quad (\because \alpha \text{ is one} - \text{to} - \text{one and onto}) \\ &\Leftrightarrow b \in F \cap G \\ &\Leftrightarrow \alpha(b) \in \alpha(F \cap G) \\ &\Leftrightarrow a \in \alpha(F \cap G) \end{aligned}$$

Hence, $\alpha(F \wedge_{\mathbf{F}} G) = \alpha(F) \wedge_{Con} \alpha(G)$. ∎

By the above theorem, we can consider that each filter of **M** corresponds to a congruence of **M** and vice versa.

### 3.4.1 Comparison with commutative residuated lattices

On a commutative residuated lattice **M**, we also have a lattice isomorphism between the set of all filters of **M** and the set of all congruences of **M**, similarly to Theorem 2. We are able to use the same proof as the above, since the class of left residuated lattices includes the class of commutative residuated lattices.

## 3.5 Characterization of subdirectly irreducible left residuated lattices

In this section, we will characterize subdirectly irreducible left residuated lattices.

Let **M** and $\mathbf{N_i}$ for each $i \in I$ be left residuated lattices. By a subdirect representation of **M** with factors $\mathbf{N_i}$, we mean an embedding $f$ :$\mathbf{M} \rightarrow \Pi_{i \in I}\mathbf{N_i}$ such that each $f_i$ defined by $f_i = p_i \circ f$ is onto $\mathbf{N_i}$ for each $i \in I$. Here, $p_i$ denotes the i-th projection. A left residuated lattice **M** is *subdirectly irreducible* if it is non-degenerate and for any subdirect representation $f$ :$\mathbf{M} \rightarrow \Pi_{i \in I}\mathbf{N_i}$, there exists a $j$ such that $f_j$ is an isomorphism of **M** onto $\mathbf{N_j}$.

From Birkohoff's subdirect representation theorem it follows that every left residuated lattice has a subdirect representation with subdirectly irreducible left residuated lattice. ( see [7]). Note that when a left residuated lattice $\mathbf{M}$ is subdirectly represented by a set$\{\mathbf{N_j}\}_{j\in J}$ of left residuated lattices, the logic $L(\mathbf{M})$ determined by $\mathbf{M}$ can be expressed as $\cap_{j\in J} L(\mathbf{N_j})$.

By Theorem 2 and the following lemma (see [2]), it is easy to show the following corollary.

**Lemma 7** *Let* $\mathbf{M}$ *be a left residuated lattice. Define* $\Delta$ *by* $\Delta = \{\langle a, a\rangle | a \in M\}$. *An left residuated lattice* $\mathbf{M}$ *is subdirectly irreducible if and only if there is the minimum congruence in* $Con\mathbf{M} - \{\Delta\}$.

**Corollary 1** *A left residuated lattice is subdirectly irreducible if and only if it has the second smallest filter, i.e. the smallest filter among all filters except* $\{1\}$.

Next, we will show the following by corollary 1.

**Lemma 8** *A left residuated lattice* $\mathbf{M}$ *is subdirectly irreducible if and only if there exists an element* $c(< 1)$ *such that for any* $x < 1$ *there exists* $z \in A^x_m$ $(m \geq 0)$ *for which* $z \leq c$ *holds.*

**Proof.** By Corollary 1, it is enough to show that a left residuated lattice has the second smallest filter if and only if there exists an element $c(< 1)$ such that for any $x < 1$ there exists $z \in A^x_m$ $(m \in I)$ for which $z \leq c$ holds.

1. Only if part:
   Let $F_0$ be the minimum filter which includes $\{1\}$ properly. Since $F_0$ is not $\{1\}$, we can suppose that there exists an element $c \in F_0$ $(c < 1)$. Let $G_x$ be the filter generated by $x$ for any $x \in M$ $(x < 1)$. We can write $G_x$ as follows.

   $$G_x = \{u | m \geq 0, z \in A^x_m, z \leq u\}$$

   By $F_0 \subseteq G_x$, $c \in F_0$ implies $c \in G_x$. Hence, $z \leq c$. Therefore, we have that there exists $c(< 1)$ such that for any $x < 1$ there exists $z \in A^x_m$ $(m \geq 0)$ for which $z \leq c$ holds.

2. If part:
   Take $c$ which satisfies the assumption. Let $F_c$ be the filter generated by $c$, which can be written as follows.

   $$F_c = \{x | l \geq 0, z' \in A^c_l, z' \leq x\}$$

   Take any filter $F$ except $\{1\}$. We will show $F_c \subseteq F$.

   Take any $w \in F \setminus \{1\}$. $A^w_n \subseteq F$ holds for any $n \geq 0$. By the assumption, $z \leq c$ holds for some $z \in A^w_m$ $(m \geq 0)$. Thus, $z \in F$. Hence, $c \in F$ by Proposition 5 and the assumption. Since $F_c$ is the minimum filter including $c$, $F_c \subseteq F$ holds. Therefore, $F_c$ is the minimum filter which includes $\{1\}$ properly. ∎

### 3.5.1 Comparison with commutative residuated lattices

We also have Corollary 1 on commutative residuated lattices. Now, the subdirectly irreducible of commutative residuated lattices are expressed as follows.

**Proposition 8** *A commutative residuated lattice* **M** *is subdirectly irreducible if and only if there exists an element $c(<1)$ such that for any $x<1$ there exists a positive integer $m$ for which $x^m \leq c$ holds.*

In section 3.3.1, the filter generated by $x$ is defined as $G'_x = \{u | k \geq 0,\ x^k \leq u\}$. By using this filter, we can show the above proposition in the similar way as Lemma 8.

## 3.6 Characterization of simple left residuated lattices

In this section, we will discuss a *simple* left residuated lattice which is a special type of subdirectly irreducible left residuated lattices. A left residuated lattice **M** is simple if it is a non-degenerate left residuated lattice which has only two filters $\{1\}$ and $M$ itself. It is easy to see that for any filter $F$ of a given left residuated lattice $\mathbf{M}^*$ the quotient algebra $\mathbf{M}^*/\mathbf{F}$ is simple if and only if $F$ is a maximal filter. Now, we can give that the following characterization of simple left residuated lattices holds.

**Lemma 9 (Simple left residuated lattices)** *A left residuated lattice* **M** *is simple if and only if for any $x<1$ in $M$ there exists a positive integer $m$ such that $0 \in A^x_m$.*

**Proof.**

1. Only if part:
   For each $x<1$, let $H_x$ be the filter generated by any $x$. We have $H_x = \{u | m \geq 0, z \in A^x_m, z \leq u\}$ . By the assumption, there exist only two filters $\{1\}$ and $M$ on **M**. Hence, $H_x = M \ni 0$. By taking 0 for $z$ we have $0 \in A^x_m$.

2. If part:
   Let $F$ be an arbitrary filter except $\{1\}$. Thus, there exists at least one element in $F \backslash \{1\}$. Take $x \in F$ $(x \neq 1)$. Let $H_x$ be the filter generated by $x$. $H_x = \{u | n \geq 0, z \in A^x_n, z \leq u\}$. We have $H_x \subseteq F$. By the assumption ($0 \in A^x_m$ for some $m \geq 0$), $0 \in H_x$ holds. Thus, $0 \in F$. It follows $F = M$. Therefore, $M$ is the single filter different from $\{1\}$. Thus, **M** is a simple left residuated lattice.

### 3.6.1 Comparison with commutative residuated lattices

**Proposition 9 (Simple commutative residuated lattices)** *A commutative residuated lattice* **M** *is* **simple** *if and only if for any $x<1$ in $M$ there exists a positive integer $m$ such that $x^m = 0$.*

On commutative residuated lattice, we have already known that the filter $H'_x$ generated by $x$ is $H'_x = \{u | k \geq 0,\ x^k \leq u\}$. By using $H'_x$ instead of $H_x$ in the proof of Lemma 9, we are able to obtain Proposition 9.

# Chapter 4

# Left residuated lattices with $C_n$

In this section, we will introduce the condition $C_n$ (see [12]), and discuss left residuated lattices with $C_n$. When a left residuated lattice $\mathbf{M}$ satisfies $C_n$, we can show that filters of $\mathbf{M}$ can be defined in the same as those in a commutative residuated lattices. Thus, we can give a simple characterization of subdirectly irreducible left residuated lattices and simple left residuated lattices with $C_n$, as shown in 4.2.

Now, let us define $\mathcal{NC}_n$ to be the variety of left residuated lattices with $C_n$. In 4.3, we show that $\{\mathcal{NC}_n\}_n$ forms an infinite ascending chains.

## 4.1 The conditions $C_n$ and n-weak exchange

We introduce first $C_n$ for $n \geq 1$. On left residuated lattice $\mathbf{M}$, $C_n$ denote the condition that, for any $x, y \in M,\ n \geq 1$,

$$y^n \leq (y \to x) \to x \quad (C_n)$$

Next, we introduce another condition $WE_n$, called *n-weak exchange*, by: for any $x, y \in M,\ n \geq 1$,

$$y^n x \leq xy \quad (WE_n)$$

We will show the following Lemma 10.

**Lemma 10** *For any $n$, $C_n$ is equivalent to $WE_n$, i.e.*
*$C_n$ holds in $\mathbf{M}$ if and only if $WE_n$ holds in $\mathbf{M}$, for any left residuated lattice $\mathbf{M}$.*

**Proof.**

1. Only if part:

$$\begin{aligned} y^n x &\leq y^n(y \to xy) \ (\because x \leq y \to xy) \\ &\leq xy \ (\because \textit{By the assumption},\ y^n \leq (y \to xy) \to xy) \end{aligned}$$

2. If part:
   By the assumption, we have $y^n(y \to x) \le (y \to x)y$.

$$\begin{array}{rcl} y^n(y \to x) & \le & (y \to x)y \\ & \le & x \end{array}$$

   Thus, we have $y^n \le (y \to x) \to x$. ∎

## 4.2 Properties of left residuated lattices with $C_n$

We will show that filters of left residuated lattices with $C_n$ can be defined in the same way as filters of commutative residuated lattices.

**Proposition 10 (filters of left residuated lattices with $C_n$)** *A nonempty subset $F$ of a left residuated lattices with $C_n$ $\mathbf{M}$ is a filter, if and only if $F$ satisfies the following,*

1. $1 \in F$,
2. $a, a \to b \in F$ *implies* $b \in F$.

**Proof.** It suffices to show that the condition 3 of Definition 7 in section 3.3 (i.e. $a \in F$ implies $(a \to b) \to b \in F$) is obtained by the above two conditions 1,2.

$$\begin{array}{rcl} a \in F & \Rightarrow & a^n \in F \\ & \Rightarrow & (a \to b) \to b \in F \\ & & (\because a^n \le (a \to b) \to b \quad \textit{and} \quad a, a \to b \in F \textit{ implies } b \in F) \end{array}$$

∎

As a corollary of Proposition 10, we can show immediately the following.

**Proposition 11 (Subdirectly irreducible left residuated lattices with $C_n$)** *Let* $\mathbf{M}$ *be a left residuated lattice with $C_n$.* $\mathbf{M}$ *is subdirectly irreducible if and only if there exists an element $c(< 1)$ such that for any $x < 1$ there exists a positive integer $m$ for which $x^m \le c$ holds.*

**Proposition 12 (Simple left residuated lattices with $C_n$)** *Let* $\mathbf{M}$ *be a left residuated lattice with $C_n$.* $\mathbf{M}$ *is simple if and only if for any $x < 1$ in $M$ there exists a positive integer $m$ such that $x^m = 0$.*

We will show the following.

**Lemma 11** *In any subdirectly irreducible left residuated lattices with $C_n$, if $x \cup y = 1$ then either $x = 1$ or $y = 1$ holds.*

**Proof.** By taking the contraposition, it suffices to show that $x, y < 1$ implies $x \cup y < 1$ in subdirectly irreducible left residuated lattices with $C_n$. Let **M** be a subdirectly irreducible left residuated lattice with $C_n$. Since **M** is subdirectly irreducible, there exists $a < 1$ such that for any $z < 1$ there exists a number $k$ satisfying $z^k \leq a$. In particular, both $x^m \leq a$ and $x^n \leq a$ hold for some positive integers $m$ and $n$. Define $s = max\,\{m, n\}$ and $t = 2s-1$. Then, clearly $x^s \leq a$ and $y^s \leq a$ hold. We can write $(x \cup y)^t$ as follows.

$$(x \cup y)^t = \underbrace{x \cdots x}_{t \text{ elements}} \cup \underbrace{x \cdots xyx \cdots x}_{t \text{ elements}} \cup \cdots \cup \underbrace{y \cdots y}_{t \text{ elements}}$$

On the right-hand side, every term has $t$ elements which consist of multiplications of $x$ and $y$.

Take any term $T$ among them. Suppose that $x$ and $y$ appear $k$ times and $j$ times, respectively, in $T$. It is clear that $k + j = t$. If $k \geq j$, then $k \geq s$. Thus,

$$T \leq x^k \leq x^s \leq a.$$

Otherwise, $j \geq s$. In this case,

$$T \leq y^j \leq y^s \leq a.$$

Thus, $(x \cup y)^t \leq a$. Therefore, $x \cup y$ cannot be equal to 1. ∎

Next, we introduce the formula Lin by,

$$Lin: \; (p \supset q) \vee (q \supset p).$$

The formula Lin is sometimes called *the (algebraic) strong de Morgan law.* Using Lemma 11, we can show the following.

**Lemma 12** *Let* **M** *be a subdirectly irreducible left residuated lattices with* $C_n$. *The formula* Lin *is valid in* **M** *if and only if* **M** *is linearly ordered.*

**Proof.** Suppose first that **M** is linearly ordered. For an arbitrary valuation $v$ on **M**, let $v(p) = a$ and v(q) =b. Then, either $a \leq b$ or $b \leq a$ holds by the assumption. It follows that either $a \to b = 1$ or $b \to a = 1$. Therefore, $v(\text{Lin}) = (a \to b) \cup (b \to a) = 1$. Hence, Lin is valid in **M**. Conversely, suppose that Lin is valid in **M**. This implies that $(a \to b) \cup (b \to a) = 1$ for all $a, b \in M$. By Lemma 11, either $a \to b = 1$ or $b \to a = 1$ holds. Thus, either $a \leq b$ or $b \leq a$. Hence, **M** is linearly ordered. ∎

## 4.3 Varieties $\mathcal{NC}_n$ of left residuated lattices with $C_n$

By Proposition 3, the class of all left residuated lattices forms variety. In the following, the variety of left residuated lattices with $C_n$, the variety of left residuated lattices and the variety of commutative residuated lattice are denoted by $\mathcal{NC}_n$, $\mathcal{N}$ and $R$, respectively. We will show the $\mathcal{NC}_n \subsetneq \mathcal{NC}_{n+1}$.

**Lemma 13** $\mathcal{NC}_1 = \mathcal{R}$.

**Proof.** By Lemma 10, the condition $C_1$ is equivalent to the condition $yx \leq xy$. It means the commutativity. ∎

**Theorem 3** $\mathcal{NC}_{n-1} \subsetneq \mathcal{NC}_n$ *holds.*

**Proof.** It is easy to show $\mathcal{NC}_{n-1} \subseteq \mathcal{NC}_n$. So, we will show $\mathcal{NC}_{n-1} \neq \mathcal{NC}_n$. By using the following *linear* left residuated lattice $\mathbf{M_n}$, we will show this.

In $\mathbf{M_n} = \langle M_n, \cap, \cup, \cdot, \rightarrow, 0, 1\rangle$ for $M_n = \{1, a, a^2, \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b, 0\}$, we define the operation $(\cdot)$ of $\mathbf{M_n}$ by the table 4.1. We also define $a^n b = ba = 0$.

Table 4.1: The definition of the operator $(\cdot)$ in $\mathbf{M_n}$

| $\cdot$ | $1$ | $a$ | $\cdots$ | $a^{n-2}$ | $a^{n-1}$ | $a^n$ | $b$ | $ab$ | $\cdots$ | $a^{n-2}b$ | $a^{n-1}b$ | $0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $\cdots$ | $a^{n-2}$ | $a^{n-1}$ | $a^n$ | $b$ | $ab$ | $\cdots$ | $a^{n-2}b$ | $a^{n-1}b$ | $0$ |
| $a$ | $a$ | $a^2$ | $\cdots$ | $a^{n-1}$ | $a^n$ | $a^n$ | $ab$ | $a^2b$ | $\cdots$ | $a^{n-1}b$ | $0$ | $0$ |
| $a^2$ | $a^2$ | $a^3$ | $\cdots$ | $a^n$ | $a^n$ | $a^n$ | $a^2b$ | $a^3b$ | $\cdots$ | $0$ | $0$ | $0$ |
| $\vdots$ | | $\vdots$ | | | $\vdots$ | | | $\vdots$ | | | $\vdots$ | |
| $a^{n-1}$ | $a^{n-1}$ | $a^n$ | $\cdots$ | $a^n$ | $a^n$ | $a^n$ | $a^{n-1}b$ | $0$ | $\cdots$ | $0$ | $0$ | $0$ |
| $a^n$ | $a^n$ | $a^n$ | $\cdots$ | $a^n$ | $a^n$ | $a^n$ | $0$ | $0$ | $\cdots$ | $0$ | $0$ | $0$ |
| $b$ | $b$ | $0$ | $\cdots$ | | | $\cdots$ | | | $\cdots$ | | | $0$ |
| $ab$ | $ab$ | $0$ | $\cdots$ | | | $\cdots$ | | | $\cdots$ | | | $0$ |
| $\vdots$ | | $\vdots$ | | | $\vdots$ | | | $\vdots$ | | | $\vdots$ | |
| $a^{n-1}b$ | $a^{n-1}b$ | $0$ | $\cdots$ | | | $\cdots$ | | | $\cdots$ | | | $0$ |
| $0$ | $0$ | $0$ | $\cdots$ | | | $\cdots$ | | | $\cdots$ | | | $0$ |

We will show that $\mathbf{M_n}$ satisfies the following conditions.

1. Firstly we will demonstrate that $M_n$ is a left residuated lattice.
   (a) It is clear that $\mathbf{M_n}$ is a bounded lattice with the greatest element 1 and the least 0.
   (b) Next, we show that $\langle M_n, \cdot, 1\rangle$ is a monoid:
   By the definition of $\mathbf{M_n}$, 1 is the identity. Next, we will check the associativity. To do so, it suffices to consider every combination described in table 4.2. We will show the associativity on each combinations.

Table 4.2: The combinations for the associativity

| $x$ | $y$ | $z$ |
|---|---|---|
| $a^{i_1}$ | $a^{i_2}$ | $a^{i_3}$ |
| $a^{i_1}$ | $a^{i_2}$ | $a^j b$ |
| $a^{i_1}$ | $a^j b$ | $a^{i_2}$ |
| $a^j b$ | $a^{i_1}$ | $a^{i_2}$ |
| $a^{j_1} b$ | $a^{j_2} b$ | $a^i$ |
| $a^{j_1} b$ | $a^i$ | $a^{j_2} b$ |
| $a^i$ | $a^{j_1} b$ | $a^{j_2} b$ |
| $a^{j_1} b$ | $a^{j_2} b$ | $a^{j_3} b$ |

$(i_1, i_2, i_3, j \in I)$

i. $x = a^{i_1},\ y = a^{i_2},\ z = a^{i_3}$:

A. $i_1 = 0$ or $i_2 = 0$ or $i_3 = 0$:
Since $a^{i_1} = 1$ or $a^{i_2} = 1$ or $a^{i_3} = 1$, the associativity holds.

B. $i_1 \geq 1$ and $i_2 \geq 1$ and $i_3 \geq 1$:

$$
\begin{aligned}
(a^{i_1} a^{i_2}) a^{i_3} &= a^{min\{i_1+i_2,n\}} a^{i_3} \\
&= \begin{cases} a^{i_1+i_2} a^{i_3}, & \text{for } i_1 + i_2 \leq n \\ a^n a^{i_3}, & \text{for } n \leq i_1 + i_2 \end{cases} \\
&= \begin{cases} a^{min\{i_1+i_2+i_3,n\}}, & \text{for } i_1 + i_2 \leq n \\ a^n, & \text{for } n \leq i_1 + i_2 \end{cases} \\
&= \begin{cases} \begin{cases} a^{i_1+i_2+i_3}, & \text{for } i_1 + i_2 + i_3 \leq n \\ a^n, & \text{for } n \leq i_1 + i_2 + i_3 \end{cases} \\ a^n, \quad \text{for } n \leq i_1 + i_2 \leq i_1 + i_2 + i_3 \end{cases} \\
&= \begin{cases} a^{i_1+i_2+i_3}, & \text{for } i_1 + i_2 + i_3 \leq n \\ a^n, & \text{for } n \leq i_1 + i_2 + i_3 \end{cases} \\
a^{i_1} (a^{i_2} a^{i_3}) &= a^{i_1} a^{min\{i_2+i_3,n\}} \\
&= \begin{cases} a^{i_1} a^{i_2+i_3}, & \text{for } i_2 + i_3 \leq n \\ a^{i_1} a^n, & \text{for } n \leq i_2 + i_3 \end{cases} \\
&= \begin{cases} a^{min\{i_1+i_2+i_3,n\}}, & \text{for } i_2 + i_3 \leq n \\ a^n, & \text{for } n \leq i_2 + i_3 \end{cases} \\
&= \begin{cases} \begin{cases} a^{i_1+i_2+i_3}, & \text{for } i_1 + i_2 + i_3 \leq n \\ a^n, & \text{for } n \leq i_1 + i_2 + i_3 \end{cases} \\ a^n, \quad \text{for } n \leq i_2 + i_3 \leq i_1 + i_2 + i_3 \end{cases} \\
&= \begin{cases} a^{i_1+i_2+i_3}, & \text{for } i_1 + i_2 + i_3 \leq n \\ a^n, & \text{for } n \leq i_1 + i_2 + i_3 \end{cases}
\end{aligned}
$$

Thus, $(a^{i_1} a^{i_2}) a^{i_3} = a^{i_1} (a^{i_2} a^{i_3})$.

ii. $x = a^{i_1},\ y = a^{i_2},\ z = a^j b$:

A. $i_1 = 0$ or $i_2 = 0$:
Since $a^{i_1} = 1$ or $a^{i_2} = 1$, the associativity holds.

B. $i_1 \geq 1$ and $i_2 \geq 1$:

$$\begin{aligned}
(a^{i_1}a^{i_2})a^j b &= a^{min\{i_1+i_2,n\}}a^j b \\
&= \begin{cases} a^{i_1+i_2}a^j b, & \text{for } i_1+i_2 \leq n \\ a^n a^j b, & \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} a^{min\{i_1+i_2+j,n\}}b, & \text{for } i_1+i_2 \leq n \\ 0, & \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} \begin{cases} a^{i_1+i_2+j}b, & \text{for } i_1+i_2+j \leq n \\ 0, & \text{for } n \leq i_1+i_2+j \end{cases} \\ 0, \quad \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} a^{i_1+i_2+j}b, & \text{for } i_1+i_2+j \leq n \\ 0, & \text{for } n \leq i_1+i_2+j \end{cases} \\
a^{i_1}(a^{i_2}a^j b) &= a^{i_1}a^{min\{i_2+j,n\}}b \\
&= \begin{cases} a^{\{i_1+i_2+j\}}b, & \text{for } i_2+j \leq n \\ a^n a^j b, & \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} a^{min\{i_1+i_2+j,n\}}b, & \text{for } i_1+i_2 \leq n \\ 0, & \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} \begin{cases} a^{i_1+i_2+j}b, & \text{for } i_1+i_2+j \leq n \\ 0, & \text{for } n \leq i_1+i_2+j \end{cases} \\ 0, \quad \text{for } n \leq i_1+i_2 \end{cases} \\
&= \begin{cases} a^{i_1+i_2+j}b, & \text{for } i_1+i_2+j \leq n \\ 0, & \text{for } n \leq i_1+i_2+j \end{cases}
\end{aligned}$$

iii. $x = a^j b$ or $y = a^j b$:
We find that the terms have $(a^j b)a^i = 0$ or $(a^{j_1}b)(a^{j_2}b) = 0$. Thus, the associativity holds.

(c) Define the operator $(\rightarrow)$:
By $x \rightarrow y = max\,\{z \in M_n : zx \leq y\}$ and the table 4.1, we can show that a left residuation is always defined.

(d) Showing $w(x \cup y)z = wxz \cup wyz$:
Before proving (d), we will demonstrate the following.

In a linear left residuated lattice $\mathbf{M}$, for any $w, x, y, z \in M$, $w(x \cup y)z = wxz \cup wyz$ holds if and only if $x \leq y$ implies $wxz \leq wyz$.

Suppose that $x \leq y$. By the assumption, $wxz \leq wyz$ holds. Thus, we have $wxz \cup wyz = wyz$. Since $x \leq y$, we obtain $wyz = w(x \cup y)z = wxz \cup wyz$.

Conversely, we suppose that $w(x \cup y)z = wxz \cup wyz$ and $x \leq y$. We have $wyz = w(x \cup y)z = wxz \cup wyz$. It means $wxz \leq wyz$.

By using the above and the table 4.1, it is easy to show that $w(x \cup y)z = wxz \cup wyz$ holds on $M_n$.

2. Next, we will demonstrate that $\mathbf{M_n}$ satisfies $M_n \in \mathcal{NC}_n$ and $M_n \notin \mathcal{NC}_{n-1}$.

   (a) $M_n \in \mathcal{NC}_n$.

      i. $x = a^{i_1}, y = a^{i_2}$ for $i_1, i_2 \geq 0$:
      It is obvious that $x^n y \leq yx$.

      ii. $x = a^{i_1}b^{j_1},\ y = a^{i_2}b^{j_2}$ for $i_1, i_2, j_2 \geq 0$ and $j_1 \geq 1$:
      We have $x^n y = (a^{i_1}b^{j_1})^n(a^{i_2}b^{j_2}) = 0$. Thus, $x^n y \leq yx$ holds.

      iii. $x = a^{i_1},\ y = a^{i_2}b^j$ for $i_1, i_2 \geq 0$ and $j \geq 1$:
      We have $x^n y = a^n b^j = 0$. Thus, $x^n y \leq yx$ holds.

   (b) $M_n \notin \mathcal{NC}_{n-1}$.
   Take $a$ and $b$ for $x$ and $y$, respectively. We have $a^{n-1}b > ba = 0$. Thus, $C_{n-1}$ does not hold. ∎

# Chapter 5

# Conclusions and remarks

In this thesis, we study logics without contraction and exchange rules, from the point of view of algebraic semantics.

In Chapter 3, we study left residuated lattices which correspond to $\mathbf{FL}'_{\mathbf{w}}$. By defining filters of left residuated lattices as in Definition 7, we show the existence of a lattice isomorphism between the set of all filters and the set of all congruences of a given left residuated lattice. We give a characterization of subdirectly irreducible left residuated lattices by Lemma 8 and simple left residuated lattices by Lemma 9. Thus, these properties which we obtained include those of commutative residuated lattices.

In Chapter 4, we study left residuated lattices with $C_n$. By using $C_n$, we define filters of left residuated lattices with $C_n$ in the same way as filters of commutative residuated lattices. Then, we show that subdirectly irreducible and simple left residuated lattices with $C_n$ can be characterized, similarly to those of commutative residuated lattices. Next, we discuss varieties $\mathcal{NC}_n$ of left residuated lattices by using $C_n$. Our result say that $\{\mathcal{NC}_n\}_n$ forms an infinite ascending chain.

Many problems on left residuated lattices remain. Here, we give two problems.

1. In this paper, we deal with left residuated lattices which have only one residuation. How can we characterize filters on bi-residuated lattices which have two residuations?

2. We have already know some relations between $\mathcal{R}$ and $\mathcal{NC}_2$ (see [4]). What relations are there between $\mathcal{NC}_n$ and $\mathcal{N}$?

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