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Master's Research Project Report

Analysis of Logical Puzzle

1710091 ZHOU Hang

Supervisor	Tojo Satoshi
Main Examiner	Tojo Satoshi
Examiners	Shirai Kiyoaki
	Iida Hiroyuki
	Ogata Kazuhiro

Graduate School of Advanced Science and Technology
Japan Advanced Institute of Science and Technology
(Information Science)

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Abstract

As a student who learning about logic, it is frequently to be asked some question as 'what can your major do?' or 'what exactly is your researching?'. Mostly being asked by my mother. That is why I consider to write an article, which is easy enough for my famliy can understand what is modal logic. And it will be even better if any audience got some fun while reading it. So the first goal of this report is giving a explain of non-classical logic, mainly the modal logic and epistemic logic to college students. It is hard for non-mathematical major student to understand what logic implies means, and why a classical implication of 'P implies Q' can transform into as relation of 'not P or Q'. Not even talk about the necessarily or the possibly in modal logic. But once we understand the fascination of non-classical logic, we can identify possible worlds which would be frequently mentioned in this report. One benefit of learning logic is that it can helps us to solve logic puzzles. There are two logic puzzles as example of explaining the connection between complex logic theory and joyful logic puzzle. After reading this paper hopefully you would have a deeper understand of what logic can do and enjoy the puzzles. This paper present basic knowledge of modal logic begin with classic logic, provide a previous studying before dynamic epistemic logic. Also introduced how to apply Kripke's semantics to some logic puzzles.

The first chapter is introduction of classic logic. Assume we are nonmathematical major college student, many of us have heard about logic without systematically learned it. We actually are using logic reasoning in our daily life without notice it. Depends on the natural language and cultural environments, the logic people using might be different. Still, there is something we can all agree about. The collection of the agreement of logic might be the origin of classic logic. This is the first step to logic, contained with messive information. In this chapter we defined the symbol to use in classic logic, the proposition p ,conjunction $\wedge$, disjunction $\vee$, negation $\neg$ and implication $\rightarrow$. Mainly, this chapter collected many symbols in classic logic in from[2][7][9]. Focusing on the notation differences. For example, as the symbol stand for negation, there are $\sim$, $\neg$ and $-$ in different references. The logician like to invent new symbol, which may cause students confused to the formulae written by different author. As classic logic can be consider as the basic of logic, a deep understanding of it is meaningful. Funny thing is, even if we have defined the only symbol to use. The classic logic may still make students confused. For example, the 'or' we use in English is not always stand for disjunction. Sometimes the 'or' we mean could be exclusive. (By the way

there is a symbol defined as exclusive or the symbol of $\oplus$.) Not only about disjunction, some people may complain that the implication is not really meaning 'if ... then ...', because it let a false proposition can implies anything. Some people may say that the rule of excluded middle is weird, there should be something is not true nor false. All the complaints are meaningful, actually, because of these complaints, people who do not like classic logic invented new logic systems. Like the complaint of implication and excluded middle, leads us to intuitionistic logic and modal logic. The people who do not like only two truth value, invented many-valued logic. After all all the logic can be connected together, at the center is the most basic logic, classic logic or say the classic propositional logic. When we get a sentence, we want to judge it is true or false. As an atom it is easy to decide which is which, But things becomes complicate when atoms connect with each other. After syntax, introduce semantics and valuation of formulae. If we defined true is 1, false is 0, change conjunction to $\times$, disjunction to +, negation to 1-. In the same time defined $1+1 = 1$, $1+0 = 1$, $1\times1 = 1$, $1\times0 = 0$, 1-1 = 0, 1-0 = 1. We can translate classic logic into calculate problem of math. It also show us that logic systems has the potential of translation. There will be a translation of one logic translate to another one, in the chapter of intuitionistic logic. We collect tautology in classic logic and call them as theorem, aware that a tautology is just like a algebraic equation, we do not have to give each element value to tell a sentence is true or false. We can prove them. So the following section introduces proof theory. In this section focus on the tableaux tree and natural deduction. While tableaux give us 9 trees that corresponding to logic connective, the natural deduction offered us the 10 rules of introduce and eliminated connectives. Briefly mentioned soundness $\vdash \varphi \Rightarrow \Vdash \varphi$ and completeness $\Vdash \varphi \Rightarrow \vdash \varphi$. Soundness means: for whatever is provable, it is true. Completeness means: for whatever is true, it is provable.

After learning the strengths and weaknesses of classic logic, we can understand the motivation of inventing new logic. As the implication $\rightarrow$ of classic logic did not satisfied Lewis, who invented Lewis systems, which then developed into modal logic systems.[1][5][6] So the second chapter is introducing modal logic. Modal logic can be simply regared as classic logic add two new with operators. $\Box$ is respond to necessary and the other $\Diamond$ stands for probability. $\Box p$ is read as 'it is necessarily p', $\Diamond p$ is read as 'it is possible p'. Then introduce the modal logic with Kripke's possible world semantics. Modal logic was invented on the motivation of capturing possibly and necessary. As we mentioned the implication of classic logic allows people write sentence like 'The sun rise up from west, therefore human can lay eggs.' With the operator of necessary, we can make sentences to claim a requirement is necessary to lead to the result. Adding figure of possible worlds also gives visualized explanation of Kripke's semantics. The intuitionistic logic is similar to modal logic, it has it's own negation and implication. There exist a translation called Gödel–McKinsey–Tarski translation, let us be able to translate intuitionistic logic to modal logic system S4. The translation so called Gödel–McKinsey–Tarski translation [5].

The third chapter is for intuitionistic logic, which introduce another two operator $\sqsupset$ and $\neg$. The $p \sqsupset q$ means a proof of p can also prove q. The $\neg\, p$ means there is no proof of p. Latter we can see that intuitionistic logic can be translate into S4. The $p \sqsupset q$ translate to $\Box(p \to q)$, $\neg\, p$ translate to $\Box(\neg p)$.

The forth chapter is introduction of epistemic logic, which is developed on modal logic, adding a new set A of agents to model $\mathfrak{M}$.[8] The model $\mathfrak{M}$ is a structure $< \mathfrak{F}, V >$, which $\mathfrak{F}$ is a frame, V is a valuation. The frame $\mathfrak{F}$ is $< W, R >$, which W is the set of possible worlds, R is the accessibility between two worlds. Introduced the basic syntax, semantics which is the same as system S5. The differences are in modal logic we use operator $\Box$ and $\Diamond$, here in epistemic logic we use K and B, which stand are the first letter of Know and Belief. But all in all, it just a syntax difference, they still share the same Kripke's semantics.

Finally, we reach to logic puzzles. Two every classic puzzle when logician talking about modal logic, the 'Muddy children' and 'Sum and Product'. Showing that without communicate with information directly. Inform each with their own statement, or playing with knowledge and ignorance can also lead us to the result.

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Chapter 1

Classic Logic

Classical propositional logic, may using in your daily life without being aware. Like 'I need to pass this class to get my credit', on a airline you may be ask 'Beef or chicken?' for your meal, and 'Both man and woman are mortal'. At the same time, classic logic is different from our intuition this section would focus on the differences and give the definition of classic logic as a formalized language.

1.1 Syntax

The *syntax* means the symbols using in a logic system. In classic logic, first we need to decide a symbol for propositions. Normally use letter the initial of proposition 'p', adding with numbers formed 'p_0, p_1, p_2...' so on. In some articles, they may use 'p, q, r...' instead. Usually, lower-case is for *atomic propositions* also called *atoms*, which means there are unit element without *connectives* in it.

The connectives are operators in classic logic following with *conjunction*, *negation* and *implication*. This part is differently described in every article. Basically, the logician wants to keep the definition of basic connectives as simple as possible. So in some reference, it only contained with three connectives or even two. The left connectives can be seem as short-writes of combination of others.

The symbol of conjunction '$\land$', read as 'and'.

The symbol of negation '$\neg$' or '$\sim$', read as 'not'.

The symbol of material implication '$\to$' or '$\supset$', read as 'if ... then' or 'implies'.

Then we can add a symbol of disjunction '$\lor$', can be seem as 'or'. But when we talking about 'or', sometimes we mean 'It has to be one of them',

Short-write	Equivalent
$p \vee q$	$\neg(\neg p \wedge \neg q)$
$p \wedge q$	$\neg(\neg p \vee \neg q)$
$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
$\neg p$	$p \rightarrow \perp$

Table 1.1: Other opterators

And	Not	Implication	iff
$\wedge$	$\neg$	$\rightarrow$	$\leftrightarrow$
&	~	$\Rightarrow$	$\Leftrightarrow$
$\cdot$	-	$\supset$	$\equiv$

Table 1.2: Different symbols

sometimes we mean 'It can be either of them and even all of them'. In short, the 'or' we used in daily life can be exclusively or not. As it may cause confusion, some logician do not take it as a basic connective. In classic logic, the disjunction is inclusive. Actually, $p \vee q$ is a short-write of $\neg(\neg p \wedge \neg q)$.

Some article may include a symbol of '$\leftrightarrow$' or '$\equiv$', means 'if and only if' or 'iff'. $p \leftrightarrow q$ can be seem as a short-write of $(p \rightarrow q) \wedge (q \rightarrow p)$.

And a symbol of '$\perp$' means 'falsity'. $\neg p$ can be seem as a short-write of $p \rightarrow \perp$.

So many different types of symbols system people are using one at a time, here I draw a table to make this part clear in my article.

Finally, we have punctuation marks '(' and ')'.

The calculate priority of each connectives: that $\neg$ is stronger than $\wedge, \vee$, $\wedge, \vee$ are stronger than $\rightarrow, \leftrightarrow$. Therefore when we write $p \wedge \neg q \rightarrow \perp$ is writing the same formula as $(p \wedge (\neg q)) \rightarrow (\perp)$.

After introduce elements in classic logic language, the grammar of this language, the law of how them form as a sentence is generated as follow:

φ and ψ are formulas, then $\varphi \wedge \psi$, $\neg\varphi$, $\varphi \rightarrow \psi$ are formulas.

The formulas in this rule is called *Well-formed formulas*, or in short *wff*. Usually the Greek letters φ, ψ, χ, ... are stand for formulas, sometimes we use capital Roman letters A, B, C,... too.

Summary the contains above, we can give definitions for propositional logic:

Definition 1.1.1. The language of propositional logic has an alphabet consisting of:

(1) proposition symbols: p_0, p_1, p_2, ...

(2) connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, $\perp$.

v	φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\neg\varphi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
v_0	1	1	1	1	0	1	1
v_1	1	0	0	1	0	0	0
v_2	0	1	0	1	1	1	0
v_3	0	0	0	0	1	1	1

Table 1.3: Truth table

(3) auxiliary symbols: (,).

Definition 1.1.2. The set $PROP$ of propositions is the smallest set X with the properties

(1) $p_i \in X(i \in N)$, $\perp \in X$.

(2) φ, $\psi \in X \Rightarrow (\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\neg\varphi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi) \in X$.

1.2 Semantics and Valuations

After formulate the language, we need to set up a rule to tell a proposition is true or false. So we can define a valuation v, which gives each atom either '1' stand for true, or '0' stand for false. The value of formulas can be calculate from every element it combined with. As there are different relation between each atom, the word *semantics* here can be seem as a rule defined how connectives effect the final value.

To give a fast understanding of semantics of propositional logic, introduce the *truth table*

The row of valuations always be ignored, however I consider that adding this row make a easy understanding of this function meaning. That is, normally we choose a valuation to satisfy a formula instead of being limited by only one valuation. As in this truth table, usually our thought in the direction of 'If we want the formula $\varphi \wedge \psi$ being true, which valuation can we choose?', rather than 'Now we have the valuation v_1, is the formula $\varphi \to \psi$ true'. In most question, we are the person who try to find out a right valuation for given formula. After all, checking a given valuation if it satisfy a formula is easier than previous task.

Base on truth table, we can find several laws of equivalent in propositional logic:

DeMorgan's laws

$$\neg(\varphi \wedge \psi) \Leftrightarrow \neg\varphi \vee \neg\psi.$$
$$\neg(\varphi \vee \psi) \Leftrightarrow \neg\varphi \wedge \neg\psi.$$

Commutative laws

$$\varphi \wedge \psi \Leftrightarrow \psi \wedge \varphi.$$
$$\varphi \vee \psi \Leftrightarrow \psi \vee \varphi.$$

Associative laws

$$\varphi \wedge (\psi \wedge \chi) \Leftrightarrow (\varphi \wedge \psi) \wedge \chi.$$
$$\varphi \vee (\psi \vee \chi) \Leftrightarrow (\varphi \vee \psi) \vee \chi.$$

Idempotent laws

$$\varphi \wedge \varphi \Leftrightarrow \varphi.$$
$$\varphi \vee \varphi \Leftrightarrow \varphi.$$

Distributive laws

$$\varphi \wedge (\psi \vee \chi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi).$$
$$\varphi \vee (\psi \wedge \chi) \Leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi).$$

Absorption laws

$$\varphi \vee (\varphi \wedge \varphi) \Leftrightarrow \varphi.$$
$$\varphi \wedge (\varphi \vee \varphi) \Leftrightarrow \varphi.$$

Double Negation laws

$$\neg\neg\varphi \Leftrightarrow \varphi.$$

Conditional laws

$$\varphi \rightarrow \psi \Leftrightarrow \neg\varphi \vee \psi.$$
$$\varphi \rightarrow \psi \Leftrightarrow \neg(\varphi \wedge \neg\psi).$$

Contrapositive laws

$$\varphi \rightarrow \psi \Leftrightarrow \neg\psi \rightarrow \neg\varphi.$$

Here have the definition for semantics

Definition 1.2.1. A mapping $v : PROP \rightarrow 0, 1$ is a valuation if
$v(\varphi \wedge \psi) = min\{v(\varphi), v(\psi)\}$
$v(\varphi \vee \psi) = max\{v(\varphi), v(\psi)\}$
$v(\neg\varphi) = 1 - v(\varphi)$
$v(\varphi \rightarrow \psi) = max\{v(\neg\varphi), v(\psi)\}$
$v(\varphi \leftrightarrow \psi) = 1$ iff $v(\varphi) = v(\psi)$
$v(\perp) = 0$

φ	ψ	$\neg\varphi$	$\varphi \to \psi$	$\neg\varphi \lor \psi$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Table 1.4: Truth table of implication

Theorem 1.2.1. If v is a mapping from the atoms into $0, 1$, satisfying $v(\bot) = 0$, then there exists a unique valuation $[\![\cdot]\!]_v$, such that $[\![\varphi]\!]_v = v(\varphi)$ for atomic φ.

As a opposite of $\bot$, which $v(\bot) = 0$, we have a symbol '$\top$' stand for a *logical truth*, a *tautology*, which $v(\top) = 1$.

Definition 1.2.2. (1) φ is a tautology if $[\![\varphi]\!]_v = 1$ for all valuations v.

(2) $\vDash \varphi$ stands for 'φ is a tautology'.

(3) Let Γ be a set of propositions, then $\Gamma \vDash \varphi$ iff for all v: $([\![\varphi]\!]_v = 1$ for all $\psi \in \Gamma) \Rightarrow [\![\varphi]\!]_v = 1$.

Specially, if we draw a truth table of $\varphi \to \psi$ and $\neg\varphi \lor \psi$ we can see that:

The value of $\varphi \to \psi$ and $\neg\varphi \lor \psi$ are always the same in any valuation. That is called as *logically equivalent*. This result is true only if φ is false or ψ is true. Marked as:

$$\psi \vDash \varphi \to \psi$$
$$\neg\varphi \vDash \varphi \to \psi$$

These are called the 'paradoxes of material implication'.

1.3 Proof Theory

1.3.1 Tableaux

Now we know how to judge a formula is satisfiable by using truth table, try out all the possible valuation of it. The approach of using truth table to judge a formula is true or false can be consider as a proof from semantical point. But that is so hard and inefficient when the formula comes complex. We need another approach to test an inference for validity, in another word, from syntactical point. Here introduce the *tableaux tree*. It looks like a real tree but upside down. Also we need to defined some rules for different operators.

The rules for the double negation:

$$\begin{tikzcd} \neg\neg\varphi \arrow[d] \\ \varphi \end{tikzcd}$$

The rules for the conjunction:

$$\begin{tikzcd} \varphi \wedge \psi \arrow[d] \\ \varphi \arrow[d] \\ \psi \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \neg(\varphi \wedge \psi) \arrow[dl] \arrow[dr] & \\ \neg\varphi & & \neg\psi \end{tikzcd}$$

The rules for disjunction:

$$\begin{tikzcd}[column sep=small] & \varphi \vee \psi \arrow[dl] \arrow[dr] & \\ \varphi & & \psi \end{tikzcd} \qquad \begin{tikzcd} \neg(\varphi \vee \psi) \arrow[d] \\ \neg\varphi \arrow[d] \\ \neg\psi \end{tikzcd}$$

The rules for the material implication:

$$\begin{tikzcd}[column sep=small] & \varphi \rightarrow \psi \arrow[dl] \arrow[dr] & \\ \neg\varphi & & \psi \end{tikzcd} \qquad \begin{tikzcd} \neg(\varphi \rightarrow \psi) \arrow[d] \\ \varphi \arrow[d] \\ \neg\psi \end{tikzcd}$$

The rules for the logical equation:

$$\begin{tikzcd}[column sep=small] & \varphi \leftrightarrow \psi \arrow[dl] \arrow[dr] & \\ \varphi \arrow[d] & & \neg\varphi \arrow[d] \\ \psi & & \neg\psi \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \neg(\varphi \leftrightarrow \psi) \arrow[dl] \arrow[dr] & \\ \varphi \arrow[d] & & \neg\varphi \arrow[d] \\ \neg\psi & & \psi \end{tikzcd}$$

With these rules we can prove the laws we mentioned before. For example, the DeMorgan's law of $\neg(\varphi \wedge \psi) \Leftrightarrow \neg\varphi \vee \neg\psi$. So we can construct a tree as following:

$$
\begin{tikzcd}
 & \neg(\varphi \wedge \psi) \arrow[d] & \\
 & \neg(\neg\varphi \vee \neg\psi) \arrow[d] & \\
 & \neg\neg\varphi \arrow[d] & \\
 & \neg\neg\psi \arrow[d] & \\
 & \varphi \arrow[d] & \\
 & \psi \arrow[dl] \arrow[dr] & \\
\neg\varphi \arrow[d] & & \neg\psi \arrow[d] \\
\bot & & \bot
\end{tikzcd}
$$

The result shows that for formula $\neg(\varphi \wedge \psi) \wedge \neg(\neg\varphi \vee \neg\psi)$ every branch of tableaux tree comes to contradiction. So the original formula is true, we proved one of the DeMorgan's laws.

The tableaux tree gives us a intuitive feeling of what logic proof is doing. Following the basic rules, remove the operators in formulae, search every possible result. A tableaux tree is *complete* only when all the rules can be applied has been applied. In the example, every formula has been decomposed to the unit as $\varphi, \psi, (\neg\varphi)$, and $(\neg\psi)$.

A branch is *closed* only when there is a contradiction in form as φ and $\neg\varphi$ from the note it start to the note it ends. In that case, we draw a $\bot$ or $\times$ at the bottom. Otherwise the branch is *open*.

Now back to the example of proving DeMorgan's law, we can say the tableaux tree is complete and closed. Thus we showed $\nvdash \neg(\varphi\wedge\psi)\wedge\neg(\neg\varphi\vee\neg\psi)$, therefore $\neg(\varphi \wedge \psi) \vdash (\neg\varphi \vee \neg\psi)$.

1.3.2 Natural deduction

There is another syntactical proof designed by Gentzen. It has it's own rules as follow:

$$
\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{I} \qquad \frac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{E} \quad \frac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{E}
$$

$$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{I} \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\mathrm{I} \qquad \dfrac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}\ \vee\mathrm{E}$$

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\ \to\mathrm{I} \qquad \dfrac{\varphi \quad \varphi \to \psi}{\psi}\ \to\mathrm{E}$$

$$\dfrac{\bot}{\varphi}\ \bot \qquad \dfrac{\begin{array}{c}[\neg\varphi]\\ \vdots \\ \bot\end{array}}{\varphi}\ \mathrm{RAA}$$

Take the rule of $\to$E as a example. It read as 'from φ and $\varphi \to \psi$, conclude ψ'. The proposition above the line are *premises*, the one below the line is the *conclusion*. In the rules, the upcase letter 'I' is stand for *introduce*, the letter 'E' is stand for *eliminated*. The '[]' is means the hypothesis is *cancelled*.

Use this approach we can prove previous laws in propositional logic.

The commutative laws:

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]}{\psi}\ \wedge\mathrm{E} \qquad \dfrac{[\varphi \wedge \psi]}{\varphi}\ \wedge\mathrm{E}}{\psi \wedge \varphi}\ \wedge\mathrm{I}}{\varphi \wedge \psi \to \psi \wedge \varphi}\ \to\mathrm{I}$$

$$\dfrac{\dfrac{[\varphi \vee \psi] \qquad \dfrac{[\varphi]}{\psi \vee \varphi}\ \vee\mathrm{I} \qquad \dfrac{[\psi]}{\psi \vee \varphi}\ \vee\mathrm{I}}{\psi \vee \varphi}\ \wedge\mathrm{I}}{\varphi \vee \psi \to \psi \vee \varphi}\ \to\mathrm{I}$$

Definition 1.3.1. The relation $\Gamma \vdash \varphi$ between sets of propositions and propositions is defined by: there is a derivation with conclusion φ and with all (uncancelled) hypotheses in Γ.

We read $\Gamma \vdash \varphi$ as 'φ is *derivable* from Γ'. The symbol $\vdash$ is called *turnstile*. If Γ is an empty set, we write $\vdash \varphi$, and say that φ is a theorem.

Lemma 1.3.1. (1) $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$.

(2) $\Gamma \vdash \varphi$, $\Gamma' \vdash \varphi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi$.

(3) $\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.

(4) $\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \to \psi$.
(5) $\Gamma \vdash \varphi$, $\Gamma' \vdash \varphi \to \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi$.
(6) $\Gamma \vdash \bot \Rightarrow \Gamma \vdash \varphi$
(7) $\Gamma \cup \{\neg\varphi\} \vdash \bot \Rightarrow \Gamma \vdash \varphi$

1.4 Language and Logic System

As a formula language

Definition 1.4.1. A language comprises
(1) a syntax, which includes a set of sentences
(2) a semantics, which includes
(2.1) a set of valuations of the sentences (the *admissible valuations*)
(2.2) a relation between admissible valuations and sentences (*satisfaction*)

Notes that
$\mathcal{L}$ stand for *language*,
LS stand for *logic system*,
Γ is a set of sentences(formulas),
$\Vdash$ is a *semantic consequence relation* of $\mathcal{L}$,
$\vdash$ is a *logical consequence relation* of LS.
In language:

Definition 1.4.2. φ is a *valid sentence*(tautology, logical truth) of language $\mathcal{L}$, if all admissible valuations of $\mathcal{L}$ satisfy φ. ($\Vdash \varphi$)

Definition 1.4.3. φ is a *satisfiable sentence* of language $\mathcal{L}$, exactly if some admissible valuations of $\mathcal{L}$ satisfy φ.

Definition 1.4.4. φ *implies* ψ in language $\mathcal{L}$, exactly if every admissible valuations of $\mathcal{L}$ which satisfies φ also satisfies ψ. ($\varphi \Vdash \psi$)

Definition 1.4.5. A set Γ of $\mathcal{L}$-sentence is *satisfiable*, exactly if some admissible valuation of $\mathcal{L}$ satisfies every $\varphi \in S$. (Briefly: 'satisfies Γ')

Definition 1.4.6. A set Γ of $\mathcal{L}$-sentence is *implies* φ, exactly if every admissible valuation of $\mathcal{L}$ that satisfies Γ also satisfies φ. ($\Gamma \Vdash \varphi$)

In logic system:

Definition 1.4.7. φ is a *theorem* of LS, iff φ is a logical consequence of the empty set in LS. ($\vdash \varphi$)

Between language and logic system:

Definition 1.4.8. LS is *statement sound* for $\mathcal{L}$, iff all theorems of LS are valid sentences of $\mathcal{L}$.

Definition 1.4.9. LS is *statement complete* for $\mathcal{L}$, iff all valid sentences of $\mathcal{L}$ are theorems of of LS.

Definition 1.4.10. LS is *argument sound* for $\mathcal{L}$, iff $\Gamma \Vdash \varphi$ whenever $\Gamma \vdash \varphi$, for all finite sets Γ of sentences of $\mathcal{L}$ and all sentence φ of $\mathcal{L}$.

Definition 1.4.11. LS is *argument complete* for $\mathcal{L}$, iff $\Gamma \vdash \varphi$ whenever $\Gamma \Vdash \varphi$, for all finite sets Γ of sentences of $\mathcal{L}$ and all sentences φ of $\mathcal{L}$.

Definition 1.4.12. LS is *strongly sound* for $\mathcal{L}$, iff $\Gamma \Vdash \varphi$ whenever $\Gamma \vdash \varphi$, for all sets Γ of sentences of $\mathcal{L}$ and all sentence φ.

Definition 1.4.13. LS is *strong complete* for $\mathcal{L}$, iff $\Gamma \vdash \varphi$ whenever $\Gamma \Vdash \varphi$, for all sets Γ of sentences of $\mathcal{L}$ and all sentence φ.

Chapter 2

Modal Logic

Modal logic was built for the requirement of describing necessity and possibility in logic language. An example of the disadvantage of classic logic is the 'material implication'. Despite we read the $\to$ as 'if ... then', it does not work as what we want it to be. For a false proposition implies any proposition, a true proposition can be implied by any proposition. Like saying 'If Tokyo was in America then there are rabbits live on the moon.' and 'The sun rise from east, so if I had blue eyes then the sun will rise from east.' are logically true, but we may not accept the relation these sentences built between the propositions. Tokyo is not necessarily needed to be in America to make rabbits live on the moon. That is where the idea of necessity and possibility of logic comes from.

2.1 The Lewis Systems

In classic logic, we have the material implication that makes:

(1) a false proposition implies any proposition: $\neg\varphi \to (\varphi \to \psi)$
(1) a true proposition can be implied any proposition: $\psi \to (\varphi \to \psi)$

And that makes some logician unhappy, Clarence Irving Lewis was one of them. Basic on this traditional implication, $\neg\varphi \to \psi$ and $\varphi \vee \psi$ are logically equivalent. Lewis infers that the disjunction has been given too much intentional sense. He think that we need to built a new 'implies' with meaning in 'proper'. First in 1918, Lewis gave a notion of *impossibility*: $\neg\Diamond$. And he defined the operator of strict implication: $\Rightarrow$, so that $(\varphi \Rightarrow \psi) \Leftrightarrow (\neg\Diamond\psi \Rightarrow \neg\Diamond\varphi)$. Latter in 1932, a new axiomatic base was given to five of new systems. That is the historical reason why the name of modal logic system we are going to talk about contain S4 and S5. The system from S1 to S3 are exist, but in later development, they had been replaced. Back to Lewis's new systems, there is operator of *possibility*: $\diamond$, the strict implication $\varphi \Rightarrow \psi$ is defined as $\neg\diamond(\varphi \wedge \neg\psi)$.

The axioms of system S1:
(B1) $(\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)$
(B2) $(\varphi \wedge \psi) \Rightarrow \varphi$
(B3) $\varphi \Rightarrow (\varphi \wedge \varphi)$
(B4) $((\varphi \wedge \psi) \wedge \chi) \Rightarrow (\varphi \wedge (\psi \wedge \chi))$
(B5) $\varphi \Rightarrow \neg\neg\varphi$
(B6) $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \chi)) \Rightarrow (\varphi \Rightarrow \chi)$
(B7) $(\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi$

The rules of S1:
Uniform Substitution: A valid formula remains valid if a formula is uniformly substituted in it for a propositional variable.
Substitution of Strict Equivalents: Either of two strictly equivalent formulas can be substituted for one another.
Adjunction: If φ and ψ have been inferred, then $\varphi \wedge \psi$ may be inferred.
Strict Inference: If φ and $\varphi \Rightarrow \psi$ have been inferred, then ψ may be inferred.

The remain systems are basic on S1
S2 adds axiom (B8) $\diamond(\varphi \wedge \psi) \Rightarrow \diamond\varphi$.
S3 adds axiom (A8) $(\varphi \Rightarrow \psi) \Rightarrow (\neg\diamond\varphi \Rightarrow \neg\diamond\psi)$.
S4 adds axiom (C10) $\neg\diamond\neg\varphi \Rightarrow \neg\diamond\neg\neg\diamond\neg\varphi$.
S5 adds axiom (C11) $\diamond\varphi \Rightarrow \neg\diamond\neg\diamond\varphi$. Also S5 can be seem as adding (C12) $\varphi \Rightarrow \neg\diamond\neg\diamond\varphi$ to S4.

2.2 Syntax

First we keep all the symbols in propositional logic, and add two monadic operators $\Box$ and $\diamond$:

Definition 2.2.1. The language of propositional modal logic:
(1) proposition symbols: $p_0, p_1, p_2, ...$
(2) connectives: $\wedge, \vee, \rightarrow, \neg, \leftrightarrow, \perp, \Box, \diamond$.
(3) auxiliary symbols: (,).

Definition 2.2.2. The set of propositional modal logic is the smallest set X with the properties:

(1) $p_i \in X(i \in N)$, $\perp \in X$.

(2) $\varphi, \psi \in X \Rightarrow (\varphi \wedge \psi), (\varphi \vee \psi), (\neg\varphi), (\varphi \to \psi), (\varphi \leftrightarrow \psi), (\Box\varphi), (\Diamond\varphi) \in X$.

Definition 2.2.3. The transform between two monadic operators:

$\Box\varphi \Leftrightarrow \neg\Diamond\neg\varphi$

$\Diamond\varphi \Leftrightarrow \neg\Box\neg\varphi$

In some references they may use L or K(Knowledge) as the monadic operator stand for 'necessarily',' must be', 'is bond to be', 'know'. And use M or B(Belief) stand for 'possibly', 'can be', 'may be', 'believe'.

2.3 Proof of Theorems

In system K, we have: $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$.

Proof ($\to$):

$$
\begin{prooftree}
\AxiomC{$\varphi \wedge \psi \to \varphi$}
\RightLabel{N}
\UnaryInfC{$\Box(\varphi \wedge \psi \to \varphi)$}
\RightLabel{K}
\UnaryInfC{$\Box(\varphi \wedge \psi) \to \Box\varphi$}
\AxiomC{$[\Box(\varphi \wedge \psi)]$}
\RightLabel{MP}
\BinaryInfC{$\Box\varphi$}
\AxiomC{$\varphi \wedge \psi \to \psi$}
\RightLabel{N}
\UnaryInfC{$\Box(\varphi \wedge \psi \to \psi)$}
\RightLabel{K}
\UnaryInfC{$\Box(\varphi \wedge \psi) \to \Box\psi$}
\AxiomC{$[\Box(\varphi \wedge \psi)]$}
\RightLabel{MP}
\BinaryInfC{$\Box\psi$}
\RightLabel{$\wedge$I}
\BinaryInfC{$\Box\varphi \wedge \Box\psi$}
\RightLabel{$\to$I}
\UnaryInfC{$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$}
\end{prooftree}
$$

Proof ($\leftarrow$):

$$
\begin{prooftree}
\AxiomC{$\varphi \wedge \psi \to \varphi \wedge \psi$}
\RightLabel{N}
\UnaryInfC{$\Box(\varphi \wedge \psi \to \varphi \wedge \psi)$}
\RightLabel{PC}
\UnaryInfC{$\Box(\varphi \to (\psi \to \varphi \wedge \psi))$}
\RightLabel{K}
\UnaryInfC{$\Box\varphi \to \Box(\psi \to \varphi \wedge \psi)$}
\RightLabel{K}
\UnaryInfC{$\Box\varphi \to \Box\psi \to \Box(\varphi \wedge \psi)$}
\UnaryInfC{$(\Box\varphi \to \Box\psi) \to \Box(\varphi \wedge \psi)$}
\end{prooftree}
$$

Proof ($\leftrightarrow$):

$$\dfrac{\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi \qquad (\Box\varphi \to \Box\psi) \to \Box(\varphi \wedge \psi)}{\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi}\;\text{DEF}$$

Basic on $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ we can easily prove that $\Diamond(\varphi \vee \psi) \Leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.

Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)}{\neg\Diamond\neg(\varphi \wedge \psi) \leftrightarrow (\neg\Diamond\neg\varphi \wedge \neg\Diamond\neg\psi)}\;\text{DEF}}{\neg\Diamond(\neg\varphi \vee \neg\psi) \leftrightarrow (\neg\Diamond(\neg\varphi) \wedge \neg\Diamond(\neg\psi))}\;\text{DeMorgan}}{\neg\Diamond(\varphi \vee \psi) \leftrightarrow (\neg\Diamond\varphi \wedge \neg\Diamond\psi)}\;\text{US}}{(\neg(\neg\Diamond\varphi \wedge \neg\Diamond\psi)) \leftrightarrow \Diamond(\varphi \vee \psi)}\;\text{Contrapositive}}{(\neg\neg\Diamond\varphi \vee \neg\neg\Diamond\psi)) \leftrightarrow \Diamond(\varphi \vee \psi)}\;\text{DeMorgan}}{(\Diamond\varphi \vee \Diamond\psi) \leftrightarrow \Diamond(\varphi \vee \psi)}\;\text{DNE}$$

We also can prove $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$

Proof:

$$\dfrac{\dfrac{[\Box\varphi \vee \Box\psi] \qquad \dfrac{[\Box(\varphi)]}{\Box(\varphi \vee \psi)}\;\vee\text{I} \qquad \dfrac{[\Box(\psi)]}{\Box(\varphi \vee \psi)}\;\vee\text{I}}{\Box(\varphi \vee \psi)}\;\vee\text{E}}{(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)}\;\to\text{I}$$

But as result, $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$ is not a theorem of K. The formula $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ is not a valid formula.

2.4 Kripke's Possible Worlds Semantics

The language of modal logic can be explain as a *structure* $< W, R, v >$,

Definition 2.4.1. (Modal Kripke structure)

A structure or a *model* of modal logic $< W, R, v >$, where W is a non-empty set, R is a binary relation on W, v is a valuation.

Some references may use letter 'M' in *Fraktur* font $\mathfrak{M}$ for model, $\mathfrak{F}$ for frame $< W, R >$, $\mathfrak{V}$, for valuation. Therefore they mark the structure $< W, R, v >$ as $\mathfrak{M} =< \mathfrak{F}, \mathfrak{V} >$, sure they are talking about the same thing.

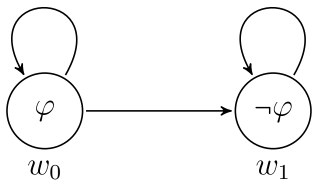

Figure 2.1: Possible worlds

In Kripke semantics, the W is a set of possible worlds w, R is binary relation of possible worlds $w \in W(R \subseteq W \times W)$, sometimes we call it as '*accessibility*'. Thus $w_0 R w_1$ means for w_0, w_1 in W, 'w_1 is *accessible* from w_0'.

The valuation v works just the same as for connectives in classic logic. For any world $w \in W$:

$v_w(\neg\varphi) = 1 \Leftrightarrow v_w(\varphi) = 0$, otherwise 0.

$v_w(\varphi \wedge \psi) = 1 \Leftrightarrow v_w(\varphi) = v_w(\psi) = 1$, otherwise 0.

$v_w(\varphi \vee \psi) = 1 \Leftrightarrow v_w(\varphi)$ or $v_w(\psi) = 1$, otherwise 0.

The important part is how it works on modal operators. For any world $w \in W$:

$v_w(\Diamond\varphi) = 1$ if, exist some $w' \in W$ such that wRw', $v_{w'}(\varphi) = 1$; otherwise 0.

$v_w(\Box\varphi) = 1$ if, for all $w' \in W$ such that wRw', $v_{w'}(\varphi) = 1$; otherwise 0.

To have a quick understanding of two modal operators, here I give a example, Let $W = w_0, w_1; w_0Rw_0, w_0Rw_1, w_1Rw_1; v_{w_0}(\varphi) = 1, v_{w_1}(\varphi) = 0$. The relation can be represented in picture as follow:

An interesting particularity of modal logic is that if w accesses to no worlds, every formula in $\Diamond\varphi$ is false, while every formula in $\Box\varphi$ is true. The reason is just like the implication in classic logic 'a false proposition can implies anything'.

2.5 Systems of Modal Logic

There are many kinds of systems in modal logic, the system K is the most basic one where the letter K stands for Kripke. The axiom of system K consist of all well-formed formula of propositional calculus.

Axiom **K** in formula:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

System K also contain following three primitive transformation rules:

(1) The Rule of Uniform Substitution(US): The result of uniformly replacing any variable or variables $p_0, p_1, ...p_n$ in a theorem by any wff $q_0, q_1, ...q_n$ respectively is itself a theorem.

An example is that for $(p \vee \neg p) \rightarrow \top$ is just the same as $(q \vee \neg q) \rightarrow \top$.

(2) The Rule of Modus Ponens, or the Rule of Detachment(MP): If φ and $\varphi \rightarrow \psi$ are theorems, so is ψ.

(3) The Rule of Necessitation(N): If φ is a theorem, so is $\Box\varphi$.

We can write these three rules as:

US: $\vdash \varphi \Rightarrow \vdash \varphi[q_0/p_0, q_1/p_1, ...q_n/p_n]$.

MP: $\vdash \varphi, \varphi \rightarrow \psi \Rightarrow \vdash \psi$.

N: $\vdash \varphi \Rightarrow \vdash \Box\varphi$.

Beside the rules US and MP are not specially made for modal logic, the rule N is a specially modal rule, also preserves K-validity.

Axiom **T** (*Axiom of Necessity*) in formula:

$$\Box\varphi \rightarrow \varphi$$

Read as 'Whatever is necessarily true, it so be true.' Therefore the axiom provide the logic frame the condition of *reflexive*, which means for all possible worlds in set W, it access itself. wRw.

System T is adding axiom **T** to system K, marked as K_ρ.

Axiom **D** in formula:

$$\Box\varphi \rightarrow \Diamond\varphi \ or \ \Diamond\top$$

Read as 'Whatever should be the case is in fact the case.' Therefore the axiom provide the logic frame the condition of *serial*, which means for any possible worlds in set W, it access to at least one world. $\forall w \exists w'(wRw')$.

System D is adding axiom **D** to system K, marked as K_η.

Axiom **4** in formula:

$$\Box\varphi \rightarrow \Box\Box\varphi$$

Read as 'Whatever is necessary true, then it is necessarily necessary true.' Therefore the axiom provide the logic frame the condition of *transitive*, which means for possible world w access to w', and possible world w' access to w'', thus world w can access to w'' directly. $wRw' \wedge w'Rw'' \Rightarrow wRw''$.

System S4 is adding axiom **4** to system T, marked as $K_{\rho\tau}$.

Axiom	Formula	Frame Condition	Accessibility constraint
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$		
T	$\Box\varphi \to \varphi$	wRw	ρ, Reflexivity
4	$\Box\varphi \to \Box\Box\varphi$	$wRw' \wedge w'Rw'' \Rightarrow wRw''$	τ, Transitivity
D	$\Box\varphi \to \Diamond\varphi$ or $\Diamond\top$	$\forall w \exists w'(wRw')$	η, Extendability
B	$\varphi \to \Box\Diamond\varphi$	$wRw' \wedge \Rightarrow w'Rw$	σ, Symmetry
5	$\Diamond\varphi \to \Box\Diamond\varphi$	$wRw' \wedge wRw'', \Rightarrow w'Rw''$	Euclid

Table 2.1: Table of Axioms

Axiom **B** (*Brouwerian axiom*) in formula:

$$\varphi \to \Box\Diamond\varphi$$

Read as 'Whatever is true, then it is necessarily possibly true.' Therefore the axiom provide the logic frame the condition of *symmetric*, which means for possible world w access to w', world w' can access to w too. $wRw' \wedge \Rightarrow w'Rw$.

System B is adding axiom **B** to system T, marked as $K_{\rho\sigma}$.

Axiom **5** (or be called as axiom **E**) in formula:

$$\Diamond\varphi \to \Box\Diamond\varphi$$

Read as 'Whatever is possibly true, then it is necessarily possibly true.' Therefore the axiom provide the logic frame the condition of *euclidean*, which means for possible world w access to w', and possible world w access to w'', thus world w' can access to w'' directly. $wRw' \wedge wRw'', \Rightarrow w'Rw''$.

System S5 is adding axiom **5** to system T, marked as $K_{\rho\sigma\tau}$. It is also can be treat as adding **B** to system S4.

Here we can draw a table for axioms:

Recalling the Lewis's systems, we can find that the systems from S1 to S3 has been replaced. And the axioms of S4:

(B1) $(\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)$
(B2) $(\varphi \wedge \psi) \Rightarrow \varphi$
(B3) $\varphi \Rightarrow (\varphi \wedge \varphi)$
(B4) $((\varphi \wedge \psi) \wedge \chi) \Rightarrow (\varphi \wedge (\psi \wedge \chi))$
(B5) $\varphi \Rightarrow \neg\neg\varphi$
(B6) $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \chi)) \Rightarrow (\varphi \Rightarrow \chi)$
(B7) $(\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi$
(C10) $\neg\Diamond\neg\varphi \Rightarrow \neg\Diamond\neg\neg\Diamond\neg\varphi$

Now has been given an alternative axiomatization of S4:

(Necessitation) If $\vdash p$ then $\vdash \Box p$
(Axiom **K**) $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(Axiom **T**) $\vdash \Box\varphi \to \varphi$
(Axiom **4**) $\vdash \varphi \to \Box\Box\varphi$

Chapter 3

Intuitionistic Logic

3.1 Introduction

As we can read a sentence because we know each word, combine all the pieces together, we then understand it. Classic logic is just like natural language, to justify a logic sentence is true or false, we have to give every element in that line an exact value. This condition is so strict that can guarantee all the judgement we make remain true. However this characteristic expose a shortcoming of classic logic, that is it can not make decision when any atomic proposition in the sentence was uncertain.

That was the motivation of non-classical logic, people wants to describe something uncertain. The intuitionistic logic is one of the non-classical logic, which focus on the relation between elements in a sentence rather than each elements true or false. We justify a logic sentence not basic on the conditions under which is true, but by which is proved.

3.2 Syntax and Possible World Semantics

Beside two symbols $\wedge$ and $\vee$ from classic logic, there are two new symbols in intuitionistic logic $\neg$ and $\sqsupset$ defined as negation and conditional. Directly,

A proof of $\varphi \wedge \psi$ is a pair comprising a proof of φ and a proof of ψ.

A proof of $\varphi \vee \psi$ is a proof of φ or a proof of ψ.

A proof of $\neg \varphi$ is a proof that there is no proof of φ.

(Which is different from a proof of that is not φ).

A proof of $\varphi \sqsupset \psi$ is a construction that, given any proof of φ , can be applied to give a proof of ψ.

The language structure of propositional intuitionistic logic can be seem as same as the structure of normal modal logic $K_{\rho\tau}$. The structure $< W, R, v >$, which,

W is a set of possible-world.

R is accessibility between each world.

v is valuation.

The accessibility relation is reflexive and transitive. Also, we have *heredity condition*. For every propositional parameter p:

for all $w \in W$, if $v_w(p) = 1$ and $wRw', v_{w'}(p) = 1$

The value of formulae is given by following conditions:

$v_w(\varphi \wedge \psi) = 1$ if $v_w(\varphi) = 1$ and $v_w(\psi) = 1$; otherwise it is 0.

$v_w(\varphi \vee \psi) = 1$ if $v_w(\varphi) = 1$ or $v_w(\psi) = 1$; otherwise it is 0.

$v_w(\neg \varphi) = 1$ if for all w' such that wRw', $v_{w'}(\varphi) = 0$; otherwise it is 0.

$v_w(\varphi \sqsupset \psi) = 1$ if for all w' such that wRw', either $v_{w'}(\varphi) = 0$ or $v_{w'} = 1$; otherwise it is 0.

3.3 Tableaux for Intuitionist logic

In order to obtain tableau for intuitionistic logic, we need to modify the node on tableau in the form of $\varphi, +w_0$ and $\varphi, -w_0$. As $\varphi, +w_0$ means in world w_0, proposition φ is true(+). Naturally, $\varphi, -w_0$ means in world w_0, proposition φ is false(-). It is because there is no traditional negation in the syntax of intuitionistic logic. And left the truth value of proposition with possible word make it clear when we try to prove formula. But if you do not like that, you can just add $\neg$ in to it, make $\neg\varphi, w_0$ be the same meaning of $\varphi, -w_0$ to eliminate the + and - mark before possible worlds.

Here gives the rules of the tableau for connectives in intuitionistic logic:

$$\begin{array}{c} \varphi \wedge \psi, +w \\ \downarrow \\ \varphi, +w \\ \psi, +w \end{array} \qquad \begin{array}{c} \varphi \wedge \psi, -w \\ \swarrow \quad \searrow \\ \varphi, -w \quad \psi, -w \end{array}$$

$$\begin{array}{c} \varphi \vee \psi, +w \\ \swarrow \quad \searrow \\ \varphi, +w \quad \psi, +w \end{array} \qquad \begin{array}{c} \varphi \vee \psi, -w \\ \downarrow \\ \varphi, -w \\ \psi, -w \end{array}$$

$$\begin{array}{c} \varphi \sqsupset \psi, +w \\ wRw' \\ \swarrow \quad \searrow \\ \varphi, -w' \quad \psi, +w' \end{array} \qquad \begin{array}{c} \varphi \sqsupset \psi, -w \\ \downarrow \\ wRw' \\ \varphi, +w \\ \psi, -w \end{array}$$

$$\begin{array}{c} \neg\varphi, +w \\ wRw' \\ \Big\downarrow \\ \varphi, -w' \end{array} \qquad\qquad \begin{array}{c} \neg\varphi, -w \\ \Big\downarrow \\ wRw' \\ \varphi, +w \end{array}$$

$$\begin{array}{c} p, +w \\ wRw' \\ \Big\downarrow \\ p, +w' \end{array}$$

In the rules of $\sqsupset$ and $\neg$, the place wRw' up may cause confusion. It is actually intends to give a symbol for the limitation of wRw'. When the wRw' is stay with top node, as in $\neg\varphi, +w$, it reads as 'for **all** w' that w access to'. In the case $\varphi \vee \psi, -w$, wRw' stay with branch, reads as 'there **exist** w' that w access to'.

The last rule is meaning that in intuitionistic logic, whatever has been proved, it remains true forever. It is called as the *heredity rule*. Note that it is only worked when the value is true(+). There is no corresponding rule for $p, -w$

As an example, here is a proof shows that $\vdash_I p \sqsupset \neg\neg p$:

$$\begin{array}{c} p \sqsupset \neg\neg p, -w_0 \\ \Bigg\downarrow \\ w_0 R w_0 \\ p, +w_0 \\ \neg\neg p, -w_0 \\ \downarrow \\ w_0 R w_1 \\ \neg p, +w_1 \\ w_1 R w_1 \\ \downarrow \\ p, -w_1 \\ w_0 R w_1 \\ \Big\downarrow \\ p, +w_1 \\ \bot \end{array}$$

We proved that $p \sqsupset \neg\neg p, -w_0$ conclude contradiction, therefore $p \sqsupset \neg\neg p$ is true for every $w \in \mathcal{D}(M)$, $\vdash_I p \sqsupset \neg\neg p$.

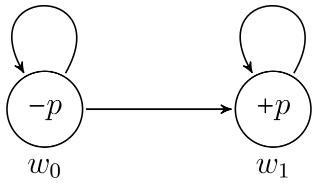

Figure 3.1: Possible worlds

3.4 Classic logic and intuitionistic logic

Intuitionistic logic can be succinctly described as classical logic without the Aristotelian law of excluded middle: (LEM)

$$\varphi \vee \rightharpoondown \varphi$$

or the classical law of double negation elimination: (DNE)

$$\rightharpoondown\rightharpoondown \varphi \sqsupset \varphi$$

but with the law of contradiction:

$$(\varphi \sqsupset \psi) \sqsupset ((\varphi \sqsupset \rightharpoondown \psi) \sqsupset \rightharpoondown \varphi)$$

and ex falso *sequitur quodlibet*:

$$\rightharpoondown \varphi \sqsupset (\varphi \sqsupset \psi)$$

An interesting property of intuitionistic logic is that it do not accept excluded middle. Some body may complain that it is because the negation defined in intuitionistic logic is not the meaning of ‘not’ of the negation we using when we are reasoning in our mind, basic on propositional logic. Still it is interesting to prove such property that against human intuition.

The proof of $\nvdash_I p \vee \rightharpoondown p$:

$$\dfrac{p \vee \rightharpoondown p, -w_0}{\dfrac{p, -w_0}{\dfrac{\rightharpoondown p, -w_0}{p, +w_1}}}$$

Basic on the proof, which gives us a settle of possible worlds we can draw as follow: We defined the $-p$ stands for $p, -w$ and $+p$ stands for $p, +w$. From the figure we can see that when in possible world w_0, it can access to world w_0 and world w_1, and in world w_1 proposition p is true. Thus $p \vee \rightharpoondown p$ is false.

$p \vee \neg p$ is meaning for any proposition p, there exist a proof of p or there exist a proof of not p. However, in real life there are many propositions we either have a proof of it nor we can disproof it. The claim is actually intend to say for every proposition p, there exist a proof of it, or there is no proof of it. That is the classic logic we have told in previous. That is where intuitionistic logic start to be different from classic logic.

3.5 Translation Intuitionistic Logic to Modal Logic S4

Let φ be a formula in intuitionistic logic, T be the function for $T(\varphi)$ is a formula in Modal logic S4. We can define the function T as follows:

$$\begin{aligned} T(p) &=_{\text{def}} \Box p \\ T(\bot) &=_{\text{def}} \bot \\ T(\varphi \wedge \psi) &=_{\text{def}} T(\varphi) \wedge T(\psi) \\ T(\varphi \vee \psi) &=_{\text{def}} T(\varphi) \vee T(\psi) \\ T(\varphi \sqsupset \psi) &=_{\text{def}} \Box(T(\varphi) \to T(\psi)) \\ T(\neg \varphi) &=_{\text{def}} \Box \neg T(\varphi) \end{aligned}$$

Such function was so called as *Gödel-Tarski translation* or *Gödel–McKinsey–Tarski translation*.

Theorem 3.5.1. Gödel–McKinsey–Tarski translation. Any propositional logic formula in the form of $\Gamma \vdash \varphi$ is valid in intuitionistic logic, if and only if when $T(\Gamma) \vdash \varphi$ is valid in modal logic S4.

In formula, it is saying **IPC**$\vdash \varphi$ if and only if **S4** $\vdash T(\varphi)$. (IPC stands for 'Intuitionistic Propositional logiC')

Theorem 3.5.2. If **S4**$\vdash \Box\varphi \vee \Box\psi$, then **S4**$\vdash \Box\varphi$ or **S4**$\vdash \Box\psi$

Theorem 3.5.3. If **IPC**$\vdash \Box\varphi \vee \Box\psi$, then **IPC**$\vdash \Box\varphi$ or **IPC**$\vdash \Box\psi$

First, we need to show that for any formula φ, $\Box T(\varphi) = T(\varphi)$ is provable in S4.

Proof ($\to$):
(Axiom T) $\vdash \Box\varphi \to \varphi$
($[T(\varphi)/\varphi]$) $\vdash \Box T(\varphi) \to T(\varphi)$

Proof ($\leftarrow$):

(Axiom N) $\vdash \varphi \Rightarrow \vdash \Box\varphi$
(Induction) $\vdash T(\varphi) \to \Box T(\varphi)$

So $\Box T(\varphi) = T(\varphi)$ is provable in S4.
The left proof of $\varphi \wedge \psi$ and $\varphi \vee \psi$ is just the same proof in section 2.3.

Chapter 4

Epistemic Logic

4.1 Introduction

Before we finally talking about logic puzzles, we need to know *epistemic logic*, which will be add a dynamic component latter in this section. The epistemic logic was developed from the modal logic and accepted logical system S5. To introduce basic epistemic logic, we need to introduce a new element for *agents*. That is why I mentioned that sometimes the modal operator $\Box$ and $\Diamond$ not only can be read as necessarily and possibly, but also know and belief. So the epistemic logic is a logic of agents' knowledge and belief, the dynamic epistemic logic is considering revision of agents' knowledge and belief.

4.2 Syntax

The language of epistemic logic is basically the same as modal logic, but adding with finite set of agents A.

Definition 4.2.1. (Basic epistemic language) P is a set of atomic propositions $p, q, r...$, A is a set of agents $a, b, c, ...$, the language $\mathcal{L}_K$ is the language for *multi-agent epistemic logic*. It is generated by following BNF(Backus Normal Form):

$$\varphi \quad ::= \quad p|\neg\varphi|(\varphi \wedge \varphi)|K_a\varphi$$

From the definition, we can see that for all the atoms in P, are formulae. From it we can develop complex formula as $\neg K_a \neg K_a(K_a p \wedge \neg p)$.

For every agent a, $K_a\varphi$ read as 'agent a knows that φ'. Simply notes that epistemic logic is just adding a unary operator of agents to let then make logic reduction. At first sight, some body may consider that $K_a(p \vee \neg p)$ is the same as $K_a p \vee K_a \neg p$. One is read as 'agent a knows that p or not p',

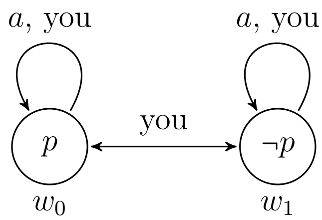

Figure 4.1: Possible worlds

while another is 'agent a knows p or agent a knows not p'. Here I would like to give an example to explain how powerful this language is.

Let us say that agent a flipped a coin, and let you guess heads(p) or tails($\neg p$). Now both of you knows that it must be heads or tails, so in formula it is $K_a(p \vee \neg p)$. Sure it is a tautology. But at next moment, you find that agent a slightly removed his hand and peeped the coin. Immediately you realize that agent a now knows what result exactly is. You know that agent a knows it is heads or he knows it is tails. In formula you know that $K_a p \vee K_a \neg p$. Now summary the situation, we can conclude that $K_{you}(p \vee \neg p)$ and $K_{you}(K_a(p) \vee K_a(\neg p))$.

We can also draw this example out: The arrow between two world w_0 and w_1 stands for you can not tell the difference between w_0 and w_1. The reflexive arrow stands for awareness, it means when you are in the world, you will notice that you are in the world. As we can see, the agent a do not have the accessibility between w_0 and w_1, which means in agent a's sight w_0 and w_1 are different. To understand what happened, we need to introduce semantics of epistemic logic.

4.3 Semantics

Definition 4.3.1. (*Kripke model*) P is a countable set of atoms, A is a finite set of agents, a Kripke model is a structure $M = < S, R^A, V^P >$,where

(1) S is a set of states(or possible worlds). The set S is also called the domain $\mathcal{D}(M)$ of M.

(2) R^A is a function, yielding for every $a \in A$ an accessibility relation $R^A(a) \subseteq S \times S$.

(3) $V^P : P \to 2^S$ is a valuation function that for every $p \in P$ yields that the set $V^P(p) \subseteq S$ of states in which p is true.

Normally, we can ignore the sets of P and A, represent the model as $< S, R, V >$ or $< W, R, v >$ in modal logic. And we just write R_a instead of

$R^A(a)$, and use sR_at in the same meaning of R_ast. In some cases we may write V_p rather than $V(p)$ for the valuation of atom p.

Definition 4.3.2. (*Epistemic formulae*) Epistemic formulae are interpreted on pairs $< M, s >$ considering of a Kripke model $M = < S, R, V >$ and a states $s \in S$. And we always assume that the state s is in the domain of $M(s \in \mathcal{D}(M))$. Given a model $M = < S, R, V >$, we define that formula φ is true in (M,s) as follows:

$$\begin{aligned} M, s \vDash p & \quad \text{iff } s \in V(p) \\ M, s \vDash (\varphi \wedge \psi) & \quad \text{iff } M, s \vDash \varphi \text{ and } M, s \vDash \psi \\ M, s \vDash \neg\varphi & \quad \text{iff not } M, s \vDash \varphi \\ M, s \vDash K_a\varphi & \quad \text{iff for all} t \text{ such that } R_ast, M, t \vDash \varphi \end{aligned}$$

The formula φ is true in (M,s), can be written as $M, s \vDash \varphi$. Instead of not $M, s \vDash \varphi$, we write $M, s \nvDash \varphi$. When $M, s \vDash \varphi$ is true for every $s \in \mathcal{D}(M)$, we can mark it as $M \vDash \varphi$, saying that φ is true in M. If $M \vDash \varphi$ for all Kripke models M, then we can say that φ is valid, write as $\vDash \varphi$ or $\mathcal{K} \vDash \varphi$.

Now we need to define some classes of models that are important in epistemic logic.

Definition 4.3.3. (Semantic Classes)

(1) The class of all Kripke models is sometimes denoted $\mathcal{K}$. Hence, $\mathcal{K} \vDash \varphi$ coincides with $\vDash \varphi$.

(2) R_a is *serial* if for all s, there exist a t, such that $sR_at.(\forall s \exists t,\ sRt)$

The class of serial Kripke models $M = < S, R, V > |$ every R_a is serial is denoted by $\mathcal{KD}$.

(3) R_a is *reflexive* if for all s, sRs. $(\forall s,\ sRs)$

The class of reflexive Kripke models $M = < S, R, V > |$ every R_a is reflexive is denoted by $\mathcal{T}$

(4) R_a is *transitive* if for all s, t, u, if sRt, tRu, then sRu.

The class of transitive Kripke models is denoted by $\mathcal{K}4$

The class of reflexive transitive Kripke models is denoted by $\mathcal{S}4$. (or $\mathcal{KT}4$)

(5) R_a is *Euclidean* if for all s, t, u, if sRt, sRu, then tRu.

The class of transitive Euclidean models is denoted by $\mathcal{K}45$

The class of serial transitive Euclidean models is denoted by $\mathcal{KD}45$

(6) R_a is an *equivalence relation* if R_a is reflexive, transitive, and *symmetric*(for all s, t, if sRt then tRs). Equivalently, R_a is an equivalence relation if R_a is reflexive, transitive and Euclidean.

The class of Kripke models with equivalence relation is denoted by $\mathcal{S}5$.

It is obvious that the classes of epistemic logic is similar to modal logic systems, which we mentioned at the beginning of this section. Sense we introduced systems from S1 to S5, in the section of intuitionistic logic we already know that it can translate into S4, the system left is S5. Therefore in epistemic logic, we will focus on S5.

Definition 4.3.4. (Bisimulation) Given two models $M = < S, R, V >$ and $M' = < S', R', V' >$. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a bisimilation iff for all $s \in S$ and $s' \in S'$, $(s, s') \in \mathfrak{R}$:

atoms $s \in V(p)$ iff $s' \in V'(p)$ for all $p \in P$

forth for all $a \in A$ and all $t \in S$, if $(s, t) \in R_a$, then there is a $t' \in S'$ such that $(s', t') \in R'_a$ and $(t, t') \in \mathfrak{R}$

back for all $a \in A$ and all $t' \in S$', if $(s', t') \in R'_a$, then there is a $t \in S$ such that $(s, t) \in R_a$ and $(t, t') \in \mathfrak{R}$ We write $(M, s) \underline{\leftrightarrow} (M', s')$ for there is a bisimulation between M and M' linking s and s'. We call (M, s) and (M', s') bisimilar.

4.4 Axiomatisation

An *axiomatisation* is a syntactic to specify a logic, by giving a core set of formulae of axioms and inference rules. Basic on these given axioms and rules, the whole logic are derivable.

Definition 4.4.1. The basic epistemic logic **K**, with an operator K_a for every $a \in A$, is comprised of all instance of propositional tautologies; the **K** axiom; and the derivation rules Modus Ponens (MP) and Necessitation (Nec).

Definition 4.4.2. We define the axioms systems as:

System **T** = System **K** + Axiom T
System **S4** = System **T** + Axiom 4
System **S5** = System **S4** + Axiom 5

Theorem 4.4.1. 1.(Soundness and completeness)

Axiom system **K** is sound and complete with respect to the semantic class $\mathcal{K}$ i.e., for every formula φ, we have $\vdash_{\mathbf{K}} \varphi$ iff $\mathcal{K} \vDash \varphi$.

The same holds for **T** w.r.t. $\mathcal{T}$, for **S4** and finally, for **S5** w.r.t. $\mathcal{S}5$

2.(Finite models and decidability)

Each of the systems mentioned above has the finite model property: any φ is satisfiable in a class χ if and only if it is satisfiable in a finite model of that class.

Moreover, all the systems mentioned are *decidable*: for any class χ mentioned, there exist a decision procedure that determines, in a finite amount of time, for any φ, whether it is satisfiable in χ or not.

Theorem 4.4.2. Axiom system **B** is strongly sound and strongly complete with respect to the semantic class $\mathcal{K}$. The same hold for **T** w.r.t. $\mathcal{T}$, for **B4** w.r.t $\mathcal{S}4$ and **S5** w.r.t. $\mathcal{S}5$.

Chapter 5

Logic Puzzle Analysis in Kripke Semantic

This chapter is introducing a type of logic puzzles that can be consider as knowledge changing by adding new information to the player gradually. Or by claiming knowledge and ignorance, how people can understand their real statement. These types of puzzles might be able to be formalized in future work. But at now, let us enjoy them and try to find out a same structure behind.

5.1 Muddy Children

When talking about modal logic or other logic for belief revision, the muddy children might be one of the most famous example. It is a story as follow:

There are three child playing on the ground. Their father comes to them and say: "At least one of you have mud on your face, go bathroom and wash it." As there is neither mirror nor pool in the ground for children to check their own face. And they can see each other's face, but can not communicate to each other. So they can not talk to the muddy child he is muddy. Still can find out whether himself is muddy or not. How can it be?

Actually we need at least adding an idea of *round* to solve this puzzle. Now we can solve the problem:

At very beginning, father said that at least one of them is muddy. So we come to round 1. In this round, if a child saw two clean face, he could immediately understand that he is the muddy child. Because at least one of children is muddy, and other children are clean, there is only one answer is that he is the muddy child. So the child who saw two clean face would go to the bathroom. The rest children keep playing on the ground. Game end in

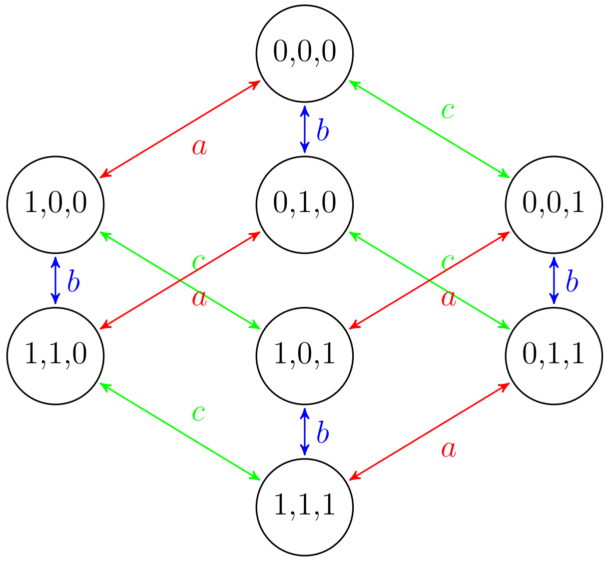

Figure 5.1: Possible worlds

round 1.

What happened if the game did not end in round 1. That means nobody saw two clean face. Which means at least there are two children with mud on their face. As the children see one dirty face and one clean face, he can not finger out if himself is clean or muddy, thus he remain silence. As we already explain in the round 1, if the muddy child is the only one, he could know it immediately, so the game would end in round 1. But the game comes to round 2, that means in the view of the muddy child there are not two clean face, which lead to 'I' am a muddy child too. Two children who saw one clean face and one muddy face could find they are the muddy children. They go the bathroom, game end in round 2.

So the same method, we survived round 1 and round 2, finally come to round 3, which means all three children saw two muddy children at very beginning. They wait for round 2, hopes that himself is the only clean child basic on the reasoning process here I gave above. But the game did not end in round 2, it leads to the last answer that all children are muddy children.

At last we know that even when people can not communicate with each other directly, they can still be informed by observing the statement of other members. It seems this puzzle share some properties with modal logic. Actually, we can draw this puzzle into a possible world figure 6.1.

Name three children as a, b and c. Inside state, the (a, b, c) is valued as 0 or 1, stand for clean or muddy. The arrow between two state means, these two states is indistinguishable as that agent. We can see this figure as it has 4 level corresponding 3 round of the game. At the top is state (0,0,0) is actually round 0, as the game start the father's announcement denied it. If the true state is (1,0,0), from this figure we can see that as the state (0,0,0) has been denied, agent a conclude the only answer is (1,0,0). While agent b is confused with (1,0,0) and (1,1,0) which he can not tell the difference in round 1. So do agent c.

5.2 Sum and Product

There is a type of puzzle about knowledge and ignorance. The puzzle 'Sum and product' is one of the famous example:

There are two different natural number a and b in the range from 0 to 100. Sam knows the sum of them, Peter knows the product. Then they have a communication as follow:

Peter: I don't know what number a or b is.

Sam: I knew that you do not find the answer.

Peter: Now I know the answer.

Sam: Now I know the answer too.

Try to find out the number a and b.

The talking between them sounds meaningless at very beginning. They are just talking about what they know and not, and by exchanging information of their states, how can they know anything new. At the moment your brain noticed prime numbers, the way to solve this puzzle is open. The remain work is handle out all the reasonable combination and eliminate them by the information given in this conversation. The first step is that it can not be two prime number, in such case Peter should know the answer at every beginning. The remain work is easy to follow.

This puzzle is like an ultimate hard version of muddy children. That is why I am considering that there exist a type of puzzles can be collected in a set, for every puzzle in that set can be translate to epistemic logic.

Chapter 6

Conclusion

It has been a long way since logic development comes from classic logic to nowadays modal logics and other logics. This report is just a brief survey of the history of logic and only focused on the logic about belief revision. Actually it would be better if audiences were familiar with electric circuit, it would be funny to explain logic as eletric circuit and could be calculated with boolean algebra which lead us to computer we have today.

The motivation of inventing new logic systems is that there is no any logic system could satisfy everyone (at least as the logic systems we have so far). Therefore the logician invent logic systems different from the ones we had. Someone did not like the idea of implication in classic logic, thought it is uncertain and misleading. So we have intuitionistic logic, which leads us to modal logic. Someone did not like the classic logic, because it can not reason with probability. So we have first-order logic, comes with symbol $\forall$ and $\exists$. Beside the epistemic logic, it is easy to point that what if the belif of agent start changing, what if the agent is not alone but with a group of other agents. Considering this, we can imagine the next topic dynamic epistemic logic is about.

At last, when we considering muddy children as agents changing their belief by given information and awareness of their state. It is actually the applying dynamic epistemic logic. By solving logic puzzle, we may consciously applying logic we have not learnt or even possible invent new logic. As inventing new tools to solve problem is the motivation of most logic we have today. Including the classic logic, which we used to prove theorem; the modal logic, which wants to express necessity and probability; the intuitionistic logic, which purpose a better implication. All these logic has been invented with it's motivation. So a logic system that can solve the type of puzzles like muddy children might be possible.

According to the similarity of modal logics and logic puzzles specially when we using kripke semantics, also be called as possbile world semantics, it is easy to see that we might be able to find a formalized method of solving logic puzzles just as how we solve logic problems. In fact, in the books writen by Hans van Ditmarsch, it was frequently discussing logics with puzzles to help audiences get a better understanding of the topics. However as far I we know, there is no such process can solve all types of logic puzzle at last in very specific purpose. Although in complexity theory, we already can transform one problem to another basic on their complexity, the relationship link from puzzles to logic remain unclear. As I mentioned before, this report only showed the similarity of logics and puzzles all basic on kripke semantics, there are no evidence for the relation goes directly between them. If There is a relation in some types of puzzles, then how can such method emply to other puzzles, and how we can draw the range of puzzles we can sovle. This remained problem needs future works to solve.

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