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Doctoral Dissertation

Many-valued logic for multi-agent system

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Abstract

In this thesis, our aim is to employ the many-valued logic to the multi-agent system. First, we extend the semantics of epistemic logic to a many-valued one. Second, we introduce many-valued semantics to express the epistemic states instead of Kripke semantics. To obtain such bases, we focus on the following issues in this thesis.

The first issue is to employ a 4-valued epistemic logic to distinguish the public/private information passing in the multi-agent system. Thus far, the agent communication has often been modeled in dynamic epistemic logic, where each agent changes his/her belief, restricting the accessibility to possible worlds in Kripke semantics. On some occasions, the recipient changes he/she belief since he/ she may not have enough background knowledge to understand it or the information may be encrypted and he/ she may not know how to decipher it. Here, we generalize those messages as private information. For this purpose, we employ 4-valued logic where each proposition is given 2 (true and false) times 2 (private or public) truth values.

The second issue is to build a n-topic semantics for the infectious logic. Beall advanced a new and interesting interpretation of Weak Kleene logic, in terms of on-topic/off-topic. In brief, Beall suggests to read the third value as *off-topic*, whereas the two classical values are read as *true and on-topic* and *false and on-topic*. Building on Beall's new interpretation, we offer an alternative semantic framework that reflects our motivations, then we provide a new interpretation of the logic of Castuskoti. Finally, we offer a general result that will allow us to make sense of a family of infectious logics in terms of Beall's on-topic/off-topic reading.

The third issue is to provide many-valued semantics instead of Kripke semantics to show the epistemic states of agents. Employing epistemic logic to express the epistemic states is often too complicated to build because we should consider all possibilities of the knowledge between agents. Here, we employ a many-valued logic to express the epistemic states of agents. We consider that there exist three kinds of epistemic states of *known, truth-value unknown*, and *content unknown*. And furthermore, we introduce two kinds of agent communication in our semantics, i.e., teaching and asking, and show how the epistemic states of agents will change.

Keywords: Epistemic logic; Many-valued logic; Multi-agent system; Infectious logic; On-topic/off-topic; Agent communication

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Chapter 1

Introduction

1.1 Background and motivation

Our motivation is to employ many-valued logic to the multi-agent system. Therefore, this thesis focuses on the following two questions.

Epistemic logic is a branch of philosophical logic that seeks to formalize the logic of discourse about knowledge. In the multi-agent system, we usually use epistemic logic to express the epistemic state of each agent. For epistemic logic is much influenced by modal logic, we usually use Kripke model as its semantics. Thus far, a bunch of researches have been studies have to extend the epistemic logic, that can be usually divided to three types.

- Offering operators to show the knowledge change, i.e., the dynamic epistemic logic, while such operators are called dynamic operators. For example, the operator $[A]$ stands for public announcement, $[L_G?A]$ stands for epistemic actions, and $[!_aA]$ stands for agent announcement.([44, 46])
- Providing modal operators to show other epistemic states than *knowledge* or *belief* that are shown by K or B. For example, Van [43] provided the operator $\mathcal{I}$ to express the *ignorance* of agents, Fan [16] used the operator $\triangle$ to show an agent knows whether a formula is true or not, and Wang [47] proposed Kh that stands for *knowing how*.
- Extending the Kripke frame (model) to obtain more information. For example, Hatano [23] added $\{C_{ab}\}_{a,b\in\mathfrak{A}}$ to normal Kripke models to express the channels between agents, Nomura [28] provided the epistemic narrow-doxastic model by adding another binary relations on possible worlds.

(Q1) Can we extend the semantics of epistemic logic to a many-valued one?

Actually, if we build the Kripke model perfectly and consider that every agent has the common sense which is shown in the model, the representation of epistemic states and the simulation of agent communication usually work very well. However, it is often too complicated to show the epistemic states of multi-agent system in the modal logic because we should consider all possibilities of knowledge between agents, e.g., agent a knows that b knows p while agent b doesn't know that agent a knows that agent b knows p. Moreover, normally an agent can hardly take care of the knowledge like $K_a K_b K_a A$. As a result, it is difficult for agents to obtain the common knowledge.

If we avoid using Kripke semantics, a natural idea is to employ many-valued logic whose values express more than classical logic. In classical logic, we assume that every proposition it may be ascribed exactly one of the two logical values, *truth* or *falsity*, called the *principle of bivalence*. Many-valued logic, as its name, provided more than two truth values. Since the birth of many-valued logic in 1920s, there have been enormous amount of many-valued logics motivated and introduced in the literature. The three-valued logics have been discussed since 1930s, by Dmitri Bochvar in [6], followed by Soren Halldén in [21] and Stephen Cole Kleene in [25] among others. Łukasiewicz introduced a n-valued logic by considering the values as $\{0, 1/n-1, 2/n-1..., 1\}$ ([26]). Around 1977, Belnap and Dunn provided a 4-valued logic called first-degree entailment logic.

(Q2) Can we express the epistemic states by many-valued logic simply instead of Kripke semantics?

In the remaining of this chapter, we introduce the solutions for these questions.

1.2 Our Proposals

In the previous section, we have seen the background of our questions. In this section, we present our proposals to solve such questions.

1.2.1 4-valued logic for agent communication (Chapter 3)

As for a solution of our question **(Q1)**, we provide a 4-valued logic which can distinguish the public/private information.

In [46], there has been distinguished the following difference in agent communication.

- public announcement: every agent receives the same information.
- whisper: other agents notice there happens an information transmission among others but the contents cannot be seen.

- channel: one to one communication, i.e., other agents cannot notice there has been an information transmission.

In addition, we would distinguish the public/private message passing, i.e., it is possible that the recipient cannot read nor understand what is written. A probable case is that the information is meaningless for the recipient because he/she does not have enough background knowledge, e.g., the message might be written in an unknown foreign language. Another probable case is that the message is encrypted and the recipient cannot decipher it; in the latter case a simple tip or a password may suffice to read it. In either way, we can generalize these cases into a category, that is, *private* information. Here, we distinguish the following two categories.

- the contents of the message is only privately understood.
- the contents of the message is publicly understood.

Here, we distinguish if the message passing is successful and the recipient surely has received the message (T/F), and if his/her belief is affected even though the message might contradict to the belief of the recipient, since the agent could not decipher the contents. Here, the communication may fail in three cases shown in Figure 1.1.

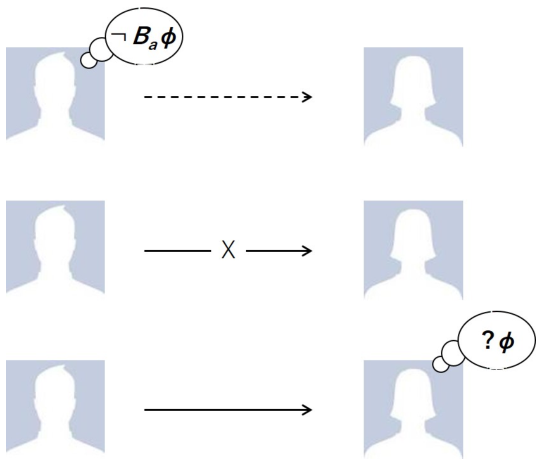

Figure 1.1: Three different miscommunication; top: the sender does not believe the contents of the information, middle: there is no channel between two agents, bottom: the recipient cannot decipher the contents.

1.2.2 Semantics for infectious logics(Chapter 4)

As for a preludes of our question **(Q2)**, first, we introduce the n-topic semantics based on Beall's off-topic interpretation, whose structure can be used to express epistemic states of n-agent later.

Beall's new interpretation of **WK** suggests to read the two classical values as *true and on-topic* and *false and on-topic*, and *off-topic* for the third value. In view of this interpretation, the semantic consequence relation of **WK** can be now understood in terms of the preservation of *true and on-topic*, not merely the preservation of *truth simpliciter* (which, in fact, is not even available under Beall's interpretation). The new interpretation also has the advantage of being able to account for the failure of Addition, or the introduction of disjunction:

$$A \vdash A \lor B.$$

Indeed, Addition is still *truth preserving*, but not on-topic preserving. Intuitively, Addition allows one to gratuitously add off-topic items to a theory, violating the requirement that theories are not about every topic expressible in their languages.

Although we are in complete agreement with Beall on his new interpretation of **WK**, we still think that there is some room for further improvement in his reading. More specifically, we have the following two motivations in mind.

(M1) First, given the distinction of truth/falsity part and on-topic/off-topic part made by Beall, it seems natural and even well-motivated to divide propositions that are off-topic into *true and off-topic* and *false and off-topic*.

(M2) Second, given that there may be a number of different topics under consideration, it seems to make a lot of sense to relativize Beall's ideas, especially the first two conditions, with respect to some topics.

Given these motivations, the aim here is threefold. First, we aim at making these two ideas more precise with the help of an alternative semantic framework. Second, by looking at our new formal framework, we will combine this with another theme from Beall on his preferred logic **FDE**, and draw a connection to the logic of Catuskoti. Finally, we will generalize our result, which will allow us to make sense of a family of infectious logics in terms of Beall's on-topic/off-topic reading.

1.2.3 Semantics for multi-agent system(Chapter 5)

As for a solution of our question **(Q2)**, we provide a 4-valued logic which can distinguish the public/private information.

Thus far, there are some studies concerning giving epistemic interpretations to many-valued logics. Dubois [15] considered that the assumption of truth-functionality is debatable because "belief is never truth-functional". His argument can be reasonably summarized as follows. If our consideration is classical, i.e., any statement A can only be either true or false even if we have no information concerning its truth-value, the statement $A \wedge \neg A$ can be unmistakably at any time predicted as being false and the statement $A \wedge \neg A$ can be unmistakably at any time predicted as being true.

On the other hand, Ciucci [9] provided three-valued logics for incomplete information and epistemic logic by giving a translation from the strong Kleene logic to the meta epistemic logic [1]. Szmuc [40] considered the third value in the paraconsistent weak Kleene logic as an epistemic interpretation. Also, in first degree entailment logic, if we consider the computer as an agent, the 4 values can be seen as the epistemic states of the agent, i.e., **true(T)** and **false(F)** mean that p is known to the agent, while **neither(N)** and **both(B)** mean that p is unknown to the agent.

However, both of the representations only show the epistemic state of a single agent, while it can be the case that a certain proposition is known to some agents while unknown to others in a multi-agent system. Here, we give a new many-valued logic semantics to express the epistemic states in the multi-agent system. We consider that each proposition is either true or false as the classical logic, while we add several additional values to show the epistemic states of agents. Therefore, all of the propositions have the same classical values for each agent, while the epistemic states is different between the agents.

1.3 Thesis Outline

The rest of this thesis is organized as follows. In Chapter 2, we introduces technical background that is needed in this thesis. In Section 2.1, we recall the basic epistemic logic as a starting point of our studies. Then, we show two extensions of epistemic logic, i.e., the ignorance logic which uses the new modal operator $\mathcal{I}$, and the semi-private announcement logic which obtains the dynamic operator $[\mathcal{A}\downarrow^a_b]$. We introduce the languages, Kripke semantics, and proof systems of the logics. In Section 2.2, we give the language and the many-valued semantics of many-valued logic, and introduce some famous many-valued logic. Then, we give the definition of infectious logic, which is an important branch of many-valued logic, and show several readings of the infectious value. Finally, in Section 2.3, we introduce two many-valued modal logic by showing the language and semantics of them.

In Chapter 3, we propose a 4-valued logic that can distinguish the private/public information passing. In Section 3.1, we introduce the idea behind this logic, i.e., consider that every proposition can be either public or private while obtaining the classical values true and false. In Sections 3.2 and 3.3, we show the syntax and semantics of this logic. In Section 3.4, we revise the belief change by private/public information passing and give an example. And then in Section 3.5, we show the recursion axiom to the ordinary dynamic epistemic logic, that is sound and complete. Finally, in Section 3.6, we give the relation between this logic to other many-valued modal logics.

In Chapter 4, we provides the semantics for infectious logic. In Section 4.1, we first make clear our intuitive idea, which is based on Beall's off-topic interpretation. Then, in Sections 4.2 and 4.3, we introduce our two-valued semantics for single topic and multi topics, and give the proofs that the semantics can be considered as same as classical logic and weak Kleene logic. In Section 4.4, we give a **FDE**-based n-topic model to show a new interpretation of the logic of Catuskoti, and give the proof that it can also be considered as **FDE** logic. Afterward, in Section 4.5, we give a many-valued semantics which can be both relating to the semantics with and without an infectious value. Finally, in Section 4.6, we show some expansions of our semantics with new connectives.

In Chapter 5, we provide many-valued semantics for multi-agent system. In Section 5.1, we propose the intuitive idea behind the semantics. In Sections 5.2 and 5.3, we introduce two pair semantics for two readings of knowledge. In Section 5.4, we propose a binary relation between agents. Finally, in Section 5.5, we combine the two semantics to express the epistemic states of multi-agent system, and then introduce two kinds of agent communication in our semantics.

Chapter 2

Preliminaries

This chapter introduces technical background that is needed in this thesis. In Section 2.1, we recall the basic epistemic logic as a starting point of our studies. Then, we show two extensions of epistemic logic, i.e., the ignorance logic which uses the new modal operator $\mathcal{I}$, and the semi-private announcement logic which obtains the dynamic operator $[A\downarrow^a_b]$. We introduce the languages, Kripke semantics, and prove systems of thelogics. In Section 2.2, we give the language and the many-valued semantics of many-valued logic, and introduce some famous many-valued logic. Then, we give the definition of infectious logic, which is an important branch of many-valued logic, and show several readings of the infectious value. Finally, in Section 2.3, we introduce two many-valued modal logic by showing the language and semantics of them.

2.1 Epistemic logic

2.1.1 Basic epistemic logic

Thus far, epistemic logic is very much influenced by the development of Kripke models of model logic, which is also considered as the possible world semantics. The idea of possible world semantics for knowledge and belief is to think of the information that an agent has in terms of the possible worlds that are consistent with the information of that agent. We say that an agent knows or believes that something is true if and only if it is true in all possible worlds that are accessible to the agent.

Normally, the language and semantics of epistemic logic are defined as following [46].

Definition 2.1 (Basic epistemic language)**.** *Let* Prop *be a countable set of propositional variables and* $\mathfrak{A}$ *be a finite set of agents, The language* $\mathcal{L}_K$ *for multi-agent epistemic logic is generated by the following Backus-Naur form:*

$$\mathcal{L}_K \ni A ::= p \mid \neg A \mid A \wedge A \mid K_a A$$

where $p \in \mathsf{Prop}$ *and* $a \in \mathfrak{A}$*.*

For every agent a, $K_a A$ is interpreted as "agent a knows that A". We define $A \vee B$ as $\neg(\neg A \wedge \neg B)$ and $A \rightarrow B$ as $\neg A \vee B$ as usual as classical logic.

Definition 2.2 (Semantics)**.** *A Kripke model* $\mathcal{M}$ *is a structure* $(W, \{R_a\}_{a \in \mathfrak{A}}, V)$*, where* W *is a set of states,* $R_a \subseteq W \times W$ *is the binary relation on agent* a *and* $V : \mathsf{Prop} \rightarrow 2^W$ *is the valuation. Then the satisfied function* $\models$ *is defined as following:*

$\mathcal{M}, w \models p$	*iff*	$w \in V(p)$
$\mathcal{M}, w \models \neg A$	*iff*	$\mathcal{M}, w \not\models A$
$\mathcal{M}, w \models A \wedge B$	*iff*	$\mathcal{M}, w \models A$ *and* $\mathcal{M}, w \models B$
$\mathcal{M}, w \models K_a A$	*iff*	*For all* u*:* $wR_a u$ *implies* $\mathcal{M}, u \models A$

We usually use the $S5$ system for knowledge. The axiomatization is as following:

All instances of tautologies	
$K_a(A \rightarrow B) \rightarrow (K_a A \rightarrow K_a B)$	distribution of K_a over $\rightarrow$
From A and $A \rightarrow B$ infers B	modus ponens
From A infers $K_a A$	necessitation of K_a
$K_a A \rightarrow A$	truth(**T**)
$K_a A \rightarrow K_a K_a A$	positive introspection(**4**)
$\neg K_a A \rightarrow K_a \neg K_a A$	negative introspection(**5**)

Table 2.1: Axiomatization of $S5$ system

2.1.2 Ignorance logic

In the epistemic logic, we usually pay attention to the "knowledge" by the operator K. However, sometimes it seems somewhat weak. For example, if we say that an agent a does not know a fact p which is written $\neg K_a p$, it may be the case that a knows that p is false which is written as $(K_a \neg p)$. Actually, it seems a bit strange because we don't need to ask someone twice like "do you know p?" and "do you know $\neg p$?" in a real case. Normally, the question "do you know p?" does not only means that "do you know p **is true**?", because the intention of the questioner is to ask "do you know **the truth value of** p?". Van [43] considers that an agent is unable to answer if he/she is **ignorant** about the value of the information it is being asked. A key property here is the state of ignorance. By the state of ignorance about A he/she refers to a mental state in which the agent is unsure about the truth value of A, so the agent does not know neither the truth value of A

nor that of $\neg A$. In the ignorance logic, a new operator $\mathcal{I}$ was added to the language and semantics of epistemic logic. $\mathcal{I}A$ is

$$M, w \models \mathcal{I}A \quad \text{iff} \quad \text{there exists } u_1, u_2 \text{ such that } wRu_1 \text{ and } wR_a u_2\text{:} \\ \mathcal{M}, u_1 \models A \text{ and } \mathcal{M}, u_2 \not\models A$$

A formula $\mathcal{I}A$ is to read as "the agent is ignorant about A", i.e., the agent is not aware of whether A or $\neg A$ is true. Actually, we can use $\Box$ to express the operator $\mathcal{I}$ as $\mathcal{I}A \leftrightarrow \neg\Box A \wedge \neg\Box\neg A$. However, we can use the operator $\mathcal{I}$ to express $\Box$ only if the frame is T-frame that is reflexive, as $\Box A \leftrightarrow A \wedge \neg\mathcal{I}A$.

The prove system is as following:

$\mathcal{I}0$	All instances of tautologies
$\mathcal{I}1$	$\mathcal{I}A \leftrightarrow \mathcal{I}\neg A$
$\mathcal{I}2$	$\mathcal{I}(A \wedge B) \rightarrow \mathcal{I}A \vee \mathcal{I}B$
$\mathcal{I}3$	$\neg\mathcal{I}A \wedge \mathcal{I}(A \wedge B) \wedge \mathcal{I}(A \rightarrow C) \wedge \mathcal{I}(D \wedge (A \rightarrow C)) \rightarrow (\neg\mathcal{I}C \wedge \mathcal{I}(B \wedge C))$
$\mathcal{I}4$	$\neg\mathcal{I}A \wedge \mathcal{I}B \rightarrow \mathcal{I}(A \wedge B) \vee \mathcal{I}(A \wedge \neg B)$
$R\mathcal{I}$	From A infer $\neg\mathcal{I}A \wedge (\mathcal{I}B \rightarrow \mathcal{I}(A \wedge B))$
MP	From A and $A \rightarrow B$ infers B
Sub	Substitution of equivalences

Table 2.2: Axiomatization of ignorance logic

Fan [16] considers in the opposite way by using the operator $\triangle$. A formula $\triangle A$ is is to read as "the agent knows whether A or $\neg A$ is true ". The semantics is defined as following:

$$\mathcal{M}, w \models \triangle A \quad \text{iff} \quad \text{for all } u_1, u_2 \text{ such that } wRu_1 \text{ and } wRu_2 : \\ M, u_1 \models A \text{ iff } \mathcal{M}, u_2 \models A$$

Here, it is easy to see that $\mathcal{I}A \leftrightarrow \neg\triangle A$ from the semantics. Therefore, we can see that the "ignorance" in the modal logic just means that "not knowing whether".

2.1.3 Semi-private announcement

Hatano [23] showed a modal epistemic language which has formalized agents' belief and channels.

Definition 2.3 (Syntax). *Let* $\mathsf{Prop} = \{p, q, \dots\}$ *be a finite set of propositional variables and* $\mathsf{G} = \{a, b, \dots\}$ *a fixed finite set of agents. The language* $\mathcal{L}_{ML_c}$ *is generated by the following Backus-Naur form:*

$$\mathcal{L}_{ML_c} \ni A ::= p \mid c_{ab} \mid \mathsf{B}_a A \mid [A{\downarrow}^a_b]A \mid \neg A \mid A \vee A$$

where $p \in \mathsf{Prop}, a \in \mathsf{G}, b \in \mathsf{G}$.

Here, c_{ab} stands for "There is a channel from agent a to agent b", $\mathsf{B}_a A$ stands for "agent a believes that A", and $[A{\downarrow}^a_b]B$ stands for "after the agent a sent a message A to the agent b via a channel, B holds"

Definition 2.4 (Semantics)**.** *A Kripke model $\mathcal{M}$ is a tuple:*

$$\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, V)$$

where W is a non-empty set of worlds, G is a non-empty set of agents, $R_{\mathsf{G}} = \{R_a \mid a \in \mathsf{G}\}$ and $R_a \subseteq W \times W$ is an accessibility of agent a on W, $C_{\mathsf{G}} = \{C_{ab} \mid a \in \mathsf{G}, b \in \mathsf{G}\}$ and $C_{ab} \subseteq W$ is a channel relation, and $V : \mathsf{Prop} \to P(W)$ is a valuation function. Here, $C_{aa} = W$ for all $a \in G$ because each agent must have a channel to itself.

Given any model $\mathcal{M}$, any world $w \in W$ and any formula A, we define the satisfaction relation inductively as follows:

$\mathcal{M}, w \models p$	iff	$w \in V(p)$
$\mathcal{M}, w \models \mathsf{c}_{ab}$	iff	$w \in C_{ab}$
$\mathcal{M}, w \models \neg A$	iff	$\mathcal{M}, w \not\models A$
$\mathcal{M}, w \models A \vee B$	iff	$\mathcal{M}, w \models A$ *or* $\mathcal{M}, w \models B$
$\mathcal{M}, w \models B_a A$	iff	*forall* $u \in W : (w, u) \in R_a$ *implies* $\mathcal{M}, u \models A$

Here, we say A is valid on $\mathcal{M}$ if $\mathcal{M}, w \models A$ for any $w \in W$, and A is valid in a class of Kripke models if A is valid on any $\mathcal{M}$ in the class. Then, it is clear that all of the axioms in the following table are valid and all of the rules preserve validity on $\mathcal{M}$.

(**Taut**)	A, A is a tautology.
($\mathbf{K_B}$)	$B_a(A \to B) \to (B_a A \to B_a B)$ $(a \in G)$
(**Selfchn**)	c_{aa} $(a \in G)$
(**MP**)	From A and $A \to B$, infer B
$\mathbf{Nec_B}$	From A, infer $B_a A$ $(a \in G)$

Table 2.3: Hilbert-style Axiomatization $\mathbf{K_c}$ of Static Logic

We often use public announcement to express the communication between agents. However, in general, most of the announcements are made between a group of agents, so that only the agent in the group can get the message, while others cannot know what they are talking. This kind of announcement is called semi-private announcement ([35]).

Here, we use the dynamic operator $[A{\downarrow}^a_b]$, which means "after the agent a sent a message A to the agent b via a channel", to express the semi-private announcement. Then, $[A{\downarrow}^a_b]B$ stands for 'after the agent a sent a message A

to the agent b via a channel, B holds". We provide the semantics of $[A{\downarrow}^a_b]B$ on a Kripke model $\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, V)$ as follows:

$$\mathcal{M}, w \models [A{\downarrow}^a_b]B \quad \text{iff} \quad \mathcal{M}^{A{\downarrow}^a_b}, w \models B$$

where $\mathcal{M}^{A{\downarrow}^a_b} = (W, R'_{\mathsf{G}}, C_{\mathsf{G}}, V)$ and $R'_i \in R'_{\mathsf{G}}$ is defined as:

- If $i = b$, for all $x \in W$,

$$R'_b(x) := \begin{cases} R_b(x) \cap [\![A]\!]_{\mathcal{M}} & \text{if } \mathcal{M}, x \models c_{ab} \wedge \mathsf{B}_a A \\ R_b(x) & \text{otherwise.} \end{cases}$$

- Otherwise, $R'_i := R_i$.

Here, $[\![A]\!]_{\mathcal{M}}$ is called the truth set of A in $\mathcal{M}$, which is defined as follows:

$$[\![A]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models A\}$$

Semantically speaking, $[A{\downarrow}^a_b]$ revises agent b's belief when agent a believes A, and there is a channel from a to b. Otherwise, agent b's belief will not be restricted([2]). It is easy to see that others than b will not revise their belief while they don't get the message A. Here, all of the agents are considered as believable and receivable, while they can only tell the truth and they receive any message made by others ([37]).

In the syntax including $[A{\downarrow}^a_b]B$, B is valid on the class of all finite Kripke models iff B is a theorem in $\mathbf{K_c}[\cdot{\downarrow}^{\mathbf{a}}_{\mathbf{b}}]$ of Table 2.4 as follows:

In addition to all the axioms and rules of K_c, we add:

$[A{\downarrow}^a_b]p$	$\leftrightarrow$	p
$[A{\downarrow}^a_b]c_{cd}$	$\leftrightarrow$	c_{cd}
$[A{\downarrow}^a_b]\neg B$	$\leftrightarrow$	$\neg[A{\downarrow}^a_b]B$
$[A{\downarrow}^a_b]B \vee C$	$\leftrightarrow$	$[A{\downarrow}^a_b]B \vee [A{\downarrow}^a_b]C$
$[A{\downarrow}^a_b]B_c B$	$\leftrightarrow$	$B_c B (c \neq b)$
$[A{\downarrow}^a_b]B_b B$	$\leftrightarrow$	$(c_{ab} \wedge B_a A \rightarrow B_b(A \rightarrow [A{\downarrow}^a_b]B)) \wedge$
		$(\neg(c_{ab} \wedge B_a A) \rightarrow B_b[A{\downarrow}^a_b]B)$
$(\mathbf{Nec}_{[\mathbf{A}{\downarrow}^{\mathbf{a}}_{\mathbf{b}}]})$		From B, infer $[A{\downarrow}^a_b]B$

Table 2.4: Hilbert-style Axiomatization $\mathbf{K_c}[\cdot{\downarrow}^{\mathbf{a}}_{\mathbf{b}}]$

2.2 Many-valued logic

2.2.1 Language

In the classical logic, we usually consider that the propositional language consists of a set $\{\neg, \wedge, \vee\}$ of propositional connectives and a countable set

Prop of propositional variables.

Normally, in the many-valued logic, we use the same language as the classical logic, while we usually use $\sim$ instead of $\neg$. The reason is that usually we consider that the meaning of negation in many-valued logic is different of that in the classical logic. Therefore, we give the definition of language of many-valued logic as following:

Definition 2.5 (Language). *Let* $\mathsf{Prop} = \{p, q, ...\}$ *be a finite set of propositional. The language* $\mathcal{L}$ *is generated by the following Backus-Naur form:*

$$\mathcal{L} \ni A ::= p \mid\sim A \mid A \wedge A$$

The disjunction $A \vee B$ *is defined as* $\sim (\sim A \wedge \sim B)$ *as the classical logic.*

We denote by Form the set of formulas defined as usual in $\mathcal{L}$, denote a formula of $\mathcal{L}$ by A, B, C, etc. and a subset of Form by Γ, Δ, Σ, etc.

Sometimes, other connectives are used in many-valued logics. For example, De [13] introduced several negation operators in first-degree entailment logic; in three-valued logic, the operators T and $+$ are used to show whether a formula is true or not and whether a formula is the third value or not [7] [21]. For the binary operators, $\otimes$ and $\oplus$ are usually used in first-degree entailment logic which are considered as the informational conjunction and disjunction [32]. However, in this thesis, we only take care of the connectives $\{\sim, \wedge, \vee\}$, for these connectives exist in all many-valued logic besides classical logic, and the considerations of them are almost the same.

2.2.2 Many-valued semantics

As the name of many-valued logic, normally, we pay attention to the semantics. The semantics can be shown by truth tables, however, sometimes different logics may have the same truth table while the consequence relation are different, e.g., the weak Kleene logic and the paraconsistent weak Kleene logic, or the strong Kleene logic and the logic of paradox. There are several ways to distinguish such logics, e.g., Bolc [8] introduced the definition of a propositional calculus. Here, we show a fundamental of many-valued semantic consequence relation [41].

Definition 2.6 (Many-valued semantics). *A* many-valued semantics *for the language* $\mathcal{L}$ *is a structure* $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$*, where*

- $\mathcal{V}$ *is a non-empty set of truth values,*
- $\mathcal{D}$ *is a non-empty proper subset of* $\mathcal{V}$*,*
- δ *contains, for every* n*-ary connective* $*$ *in the language, a truth-function* $\delta_* : \mathcal{V}^n \to \mathcal{V}$*.*

A interpretation *is a pair* $\langle M, \mu \rangle$*, where* M *is such a structure, and* μ *is an evaluation function from* Prop *to* $\mathcal{V}$*. Given an interpretation,* μ *is extended to a map from* Form *to* $\mathcal{V}$ *recursively, by the following clause:*

- $\mu(*(A_1, \dots, A_n)) = \delta_*(\mu(A_1), \dots, \mu(A_n))$.

Finally, $\Gamma \models^M A$ *iff for all interpretation* $\langle M, \mu \rangle$*, if* $\mu(B) \in \mathcal{D}$ *for all* $B \in \Gamma$*, then* $\mu(A) \in \mathcal{D}$*.*

Semantically speaking, $\mathcal{V}$ shows the values in the logic, $\mathcal{D}$ shows the designate values, and δ shows the truth table. Here, the designate values are considered as acceptable values which are the same as the **true** value in the classical logic.

In classical logic, the semantic consequence relation $\Gamma \models A$ holds iff A is true under all valuations that make all B in Γ true [42]. By this semantics, the semantic consequence relation $\models_{\mathbf{CL}}$ for classical logic is obtained by setting $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}\}$.

2.2.3 Some many-valued logics

Here, we introduce some famous many-valued logic using the semantics.

Strong Kleene logic is a three-valued logic that obtains the third value **i**. Normally, **i** is read as *unknown, undecided, underdetermined*, etc.

Fact 2.1 (Strong Kleene logic)**.** *The semantic consequence relation* $\models_{\mathbf{SK}}$ *for strong Kleene logic is obtained by setting* $\mathcal{V} = \{\mathbf{t}, \mathbf{i}, \mathbf{f}\}$*,* $\mathcal{D} = \{\mathbf{t}\}$*, and* δ *as the following truth table:*

	$\sim$
t	**f**
i	**i**
f	**t**

$\wedge$	**t**	**i**	**f**
t	**t**	**i**	**f**
i	**i**	**i**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**i**	**f**
t	**t**	**t**	**t**
i	**t**	**i**	**i**
f	**t**	**i**	**f**

Łukasiewicz's three-valued logic has the same semantic consequence relation as above. The difference between two logics is the implication. In strong Kleene logic, $A \to B$ is considered as $\sim A \vee B$ as the same as classical logic, while the value of $\mathbf{i} \to \mathbf{i}$ is considered as **t** in Łukasiewicz's three-valued logic.

Logic of paradox has the same truth tables as strong Kleene logic, while the designate values are different. The value **i** in this logic is normally read as *overdetermined*, etc.

Fact 2.2 (Logic of paradox)**.** *The semantic consequence relation* $\models_{\mathbf{LP}}$ *for logic of paradox is obtained as above except that we replace* $\mathcal{D} = \{\mathbf{t}\}$ *by* $\mathcal{D} = \{\mathbf{t}, \mathbf{i}\}$*.*

Weak Kleene logic, also called Bochvar logic, is a three-valued logic that obtains the third value **i**. The truth tables are all different from the above except for negation. Normally, **i** can be read as *meaningless, undefined, off-topic*, etc.

Fact 2.3 (Weak Kleene logic)**.** *The semantic consequence relation* $\models_{\mathbf{WK}}$ *for weak Kleene logic is obtained by setting* $\mathcal{V} = \{\mathbf{t}, \mathbf{i}, \mathbf{f}\}$*,* $\mathcal{D} = \{\mathbf{t}\}$*, and* δ *as the following truth table:*

	$\sim$
t	**f**
i	**i**
f	**t**

$\wedge$	**t**	**i**	**f**
t	**t**	**i**	**f**
i	**i**	**i**	**i**
f	**f**	**i**	**f**

$\vee$	**t**	**i**	**f**
t	**t**	**i**	**t**
i	**i**	**i**	**i**
f	**t**	**i**	**f**

If we consider that the third value **i** is also the designated value, we can obtain paraconsistent weak Kleene logic.

Fact 2.4 (Paraconsistent weak Kleene logic)**.** *The semantic consequence relation* $\models_{\mathbf{PWK}}$ *for paraconsistent weak Kleene logic is obtained as in* **WK** *except that we replace* $\mathcal{D} = \{\mathbf{t}\}$ *by* $\mathcal{D} = \{\mathbf{t}, \mathbf{i}\}$*.*

Belnap [4] considered a 4-valued logic called first-degree entailment logic **FDE. The four values are usually written as** $\{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$**, where b stands for *both true and false* and n stands for *neither true nor false*.**

Fact 2.5 (First-degree entailment logic)**.** *The semantic consequence relation of* $\models_{\boldsymbol{FDE}}$ *for first-degree entailment logic is obtained by setting* $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$*,* $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$*, and* δ *as the following truth table [32]:*

	$\sim$
t	**f**
b	**b**
n	**n**
f	**t**

$\wedge$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$\vee$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

Deutsch [14] considered another 4-valued logic that is usually called that is called S_{fde} **logic. The reading of values is the same as FDE, while the truth tables are different except for negation:**

Fact 2.6 (S_{fde} logic)**.** *The semantic consequence relation of* $\models_{\boldsymbol{Sfde}}$ *for* S_{fde} *logic is obtained by setting* $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$*,* $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$*, and* δ *as the following* *truth table [17]:*

	$\sim$
t	**f**
b	**b**
n	**n**
f	**t**

$\wedge$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**n**	**f**
n	**n**	**n**	**n**	**n**
f	**f**	**f**	**n**	**f**

$\vee$	**t**	**b**	**n**	**f**
t	**t**	**t**	**n**	**t**
b	**t**	**b**	**n**	**b**
n	**n**	**n**	**n**	**n**
f	**t**	**b**	**n**	**f**

Priest [33] provided a 5-valued logic called logic of Catuskoti.

Fact 2.7 (Logic of Catuskoti)**.** *The semantic consequence relation* $\models_{\boldsymbol{FDE}_A}$ *for the logic of Catuskoti [33] (*$\boldsymbol{FDE}_A$*) is obtained by setting* $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}, \mathbf{e}\}$*,* $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$*,* δ *as the following truth tables:*

	$\sim$
t	**f**
b	**b**
n	**n**
f	**t**
e	**e**

$\wedge$	**t**	**b**	**n**	**f**	**e**
t	**t**	**b**	**n**	**f**	**e**
b	**b**	**b**	**f**	**f**	**e**
n	**n**	**f**	**n**	**f**	**e**
f	**f**	**f**	**f**	**f**	**e**
e	**e**	**e**	**e**	**e**	**e**

$\vee$	**t**	**b**	**n**	**f**	**e**
t	**t**	**t**	**t**	**t**	**e**
b	**t**	**b**	**t**	**b**	**e**
n	**t**	**t**	**n**	**n**	**e**
f	**t**	**b**	**n**	**f**	**e**
e	**e**	**e**	**e**	**e**	**e**

Remark 2.1. *In this section, we use the letters* $\mathbf{b}, \mathbf{n}$ *and* $\mathbf{e}$ *to show the non-classical values in many-valued logic. Actually, the expressions are not fixed. For example, the third value can be also written as* $\mathbf{i}$ *or* $\mathbf{u}$*; in first-degree entailment logic, we may use the symbols* $\top, \bot$ *instead of* $\mathbf{b}, \mathbf{n}$*. Moreover, we can use the set of numbers* $\{1, 0.5, 0\}$ *as the value set* $\mathcal{V}$ *of a three-valued logic, or the set of pairs* $\{(1,1),(1,0),(0,1),(0,0)\}$ *as that of a 4-valued logic.*

2.2.4 Infectious logic

Normally, we say a logic is infectious if it has an infectious truth-value, that is, a truth-value such that it is assigned to a compound formula whenever it is assigned to at least one of its components. A more formal definition of infectious logic is considered as following [31]:

Definition 2.7. *A semantics* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ *for the language* $\mathcal{L}$ *is infectious iff there is an element* $x \in \mathcal{V}$ *such that for every* n*-ary connective* $*$ *in the language, with an associated truth-function* $\delta_* \in \delta$ *and for all* $v_1, ..., v_n \in \mathcal{V}$ *it holds that: if* $x \in \{v_1, ..., v_n\}$*, then* $\delta_*(v_1, ..., v_n) = x$*.*

By the definition, we can see that weak Kleene logic, S_{fde} **logic, and the logic of Catuskoti are infectious logic. Actually, the infectious logic can be seen as a non-infectious logic adds an infectious value. In fact,**

- If we add an infectious value to classical logic, we can obtain weak Kleene logic.
- If we add an infectious value to strong Kleene logic, we can obtain the S_{fde} logic.
- If we add an infectious value to FDE logic, we can obtain the logic of Catuskoti.

There are several readings of the infectious value. Kleene [25] introduced four meanings for the three-valued logic with that the predicates are partially recursive,

(i) 'true', 'false' and 'undefined' whose example is that $Q \vee R$ is 'undefined' if and only if Q and R are not defined as 'true' or 'false'. Formally, 'undefined' is to be unassigned the particular value to the predicates.

(ii) 'true', 'false' and 'unknown' which is a category means that we do not know or choose for the moment. 'unknown', however, does not exclude the other two possibilities 'true' and 'false'.

(iii) 'decidable by the algorithms to be true', 'decidable by the algorithms to be false' and 'undecidable by the algorithms whether true or false'.

(iv) 'known to be true', 'known to be false' and 'unknown whether true or false' with a fixed state of knowledge about givens.

Fitting [19] states the third value in [25] can be regarded as $\bot$ of Belnap's four-valued logic when the system is built on the bilattice structure $(\{true, false, \bot, \top\}, \leq_t, \leq_k)$. Since $true \wedge \top$ is still $true$, and also $false \wedge \top$ is still $false$ in this structure, $\top$ is not suitable for the third value on Kleene's strong three-valued logic. On the other hand, $\bot$ behaves as same as u in the above discussion.

Ferguson [18] considers the infectious value as "nonsense" or "meaningless". The idea is from the failure of addition. In classical logic, the addition rule that $A \models A \vee B$ is always valid. However, in a system of analytic implication, sometimes the formula is ill-formed, for example "$A \vee \vee$". The second $\vee$ is called "nonsense" or "meaningless". So the disjunction $A \vee B$ will be true if and only if either A or B is true, and both A and B are meaningful.

Beall [3] showed another way to express truth values of the weak Kleene logic. The terminology think of a theory in the logician's sense is considered as a set of sentences closed under a consequence relation. Beall motivates his new interpretation via the following ideas:

1. **A theory is about all and only what its elements — that is, the claims in the theory — are about.**
2. **Conjunctions, disjunctions, and negations are about exactly whatever their respective subsentences are about:**
   (a) **Conjunction $A \wedge B$ is about exactly whatever A and B are about.**
   (b) **Disjunction $A \vee B$ is about exactly whatever A and B are about.**
   (c) **Negation $\neg A$ is about exactly whatever A is about.**
3. **Theories in English are rarely about every topic expressible in English.**

Based on these ideas, Beall's new interpretation of weak Kleene logic suggests to read the two classical values as *true and on-topic* and *false and on-topic*, and *off-topic* for the third value. In view of this interpretation, the semantic consequence relation can be now understood in terms of the preservation of *true and on-topic*, not merely the preservation of *truth simpliciter* (which, in fact, is not even available under Beall's interpretation). The new interpretation also has the advantage of being able to account for the failure of Addition, or the introduction of disjunction:

$$A \vdash A \vee B.$$

2.3 Many-valued modal logic

2.3.1 4-valued modal logic

Odintsov and Wansing [30] showed that 4-valued logic can also be used in modal logic, whose operators includes $\Box$ and $\Diamond$([5]).

Definition 2.8. *To define the language of Belnap-Dunn Modal Logic $BK^{\Box}$, first the language $L^{\Box}$ was considered as following:*

$$L^{\Box} \ni A ::= p \mid \bot \mid \sim A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$$

where $\sim$ stands for "strong negation" and other operators are defined as follows:

$$\neg A := A \to \bot,\ A \leftrightarrow B := (A \to B) \wedge (B \to A)$$

$$\Diamond A := \sim \Box \sim A,\ A \Leftrightarrow B := (A \leftrightarrow B) \wedge (\sim A \leftrightarrow \sim B)$$

Then, the language $BK^{\Box}$ was defined as the least $L^{\Box}$-logic containing the following three groups of axioms:

- *Axioms of classical propositional logic in the language* $\vee, \wedge, \rightarrow, \bot$.
- *Strong negation axioms:*

$$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$$

$$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$$

$$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$$

$$\sim\sim p \leftrightarrow p, \ \textit{and} \sim \bot$$

- *Modal axioms:*

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \quad \neg \sim \Box p \leftrightarrow \Box \neg \sim p$$

The semantics of 4-valued modal logic is defined as following.

Definition 2.9 (Semantics [29,30])**.** *A* **BK***-model is a tuple* $\mathcal{M} = (W, R, V)$ *where* W *is a non-empty set of worlds,* $R \subset W \times W$ *is an accessibility relation on* W*, and* $V : \mathsf{Prop} \times W \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$ *is a valuation function. It will be convenient to have another definition close to the standard Kripke model, so we assign functions* $v^+, v^- : \mathsf{Prop} \rightarrow 2^W$ *defined as follows instead of* V*:*

$$v^+(p) = \{w | V(p, w) \in \{\mathbf{t}, \mathbf{b}\}\}$$

$$v^-(p) = \{w | V(p, w) \in \{\mathbf{f}, \mathbf{b}\}\}$$

For a **BK***-model* $\mathcal{M} = (W, R, v^+, v^-)$*, we define* $\models^+$ *and* $\models^-$ *between the worlds of* $\mathcal{M}$ *and formulas as follows:*

$$\begin{array}{lcl}
\mathcal{M}, w \models^+ p & \Leftrightarrow & w \in v^+(p) \\
\mathcal{M}, w \models^- p & \Leftrightarrow & w \in v^-(p) \\
\mathcal{M}, w \models^+ A \wedge B & \Leftrightarrow & \mathcal{M}, w \models^+ A \ \textit{and} \\
 & & \mathcal{M}, w \models^+ B \\
\mathcal{M}, w \models^- A \wedge B & \Leftrightarrow & \mathcal{M}, w \models^- A \ \textit{or} \\
 & & \mathcal{M}, w \models^- B \\
\mathcal{M}, w \models^+ A \vee B & \Leftrightarrow & \mathcal{M}, w \models^+ A \ \textit{or} \\
 & & \mathcal{M}, w \models^+ B \\
\mathcal{M}, w \models^- A \vee B & \Leftrightarrow & \mathcal{M}, w \models^- A \ \textit{and} \\
 & & \mathcal{M}, w \models^- B
\end{array}$$

$$
\begin{array}{lcl}
\mathcal{M}, w \models^{+} A \to B & \Leftrightarrow & \mathcal{M}, w \models^{+} A \Rightarrow \\
 & & \mathcal{M}, w \models^{+} B \\
\mathcal{M}, w \models^{-} A \to B & \Leftrightarrow & \mathcal{M}, w \models^{+} A \textit{ and} \\
 & & \mathcal{M}, w \models^{-} B \\
\mathcal{M}, w \not\models^{+} \bot & & \textit{always} \\
\mathcal{M}, w \models^{-} \bot & & \textit{always} \\
\mathcal{M}, w \models^{+} \sim A & \Leftrightarrow & \mathcal{M}, w \models^{-} A \\
\mathcal{M}, w \models^{-} \sim A & \Leftrightarrow & \mathcal{M}, w \models^{+} A \\
\mathcal{M}, w \models^{+} \Box A & \Leftrightarrow & \forall u \in W((w,u) \in R \Rightarrow \\
 & & \mathcal{M}, u \models^{+} A) \\
\mathcal{M}, w \models^{-} \Box A & \Leftrightarrow & \exists u \in W((w,u) \in R \textit{ and} \\
 & & \mathcal{M}, u \models^{-} A)
\end{array}
$$

2.3.2 3-valued modal logic

Correia [12] showed a Kripke semantics based on weak Kleene logic. The language is the same as normal modal logic and the semantics is defined as following.

Definition 2.10. *A Correia Kripke model $\mathcal{M}$ is a tuple $\langle W, R, V \rangle$, where W is a set of worlds, $R \subseteq W \times W$ is a binary relation on W, and $V : \mathsf{Prop} \times W \to \{0, 1, 2\}$ is a 3-valued valuation. The satisfaction $\models$ is defined as follows:*

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & V(p, w) = 1 \\
\mathcal{M}, w \models \neg A & \text{iff} & d(p, w, \mathcal{M}) \text{ and } \mathcal{M}, w \not\models A \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \Box A & \text{iff} & d(A, w, \mathcal{M}) \text{ and for all } v \in W, \\
 & & \text{if } wRv \text{ and } d(A, v, M) \text{, then } \mathcal{M}, v \models A
\end{array}
$$

where $p \in \mathsf{Prop}$, and $d(A, w, \mathcal{M})$ holds if and only if for every atom p in A, $V(p, w) \neq 2$, which means that "A is defined at world w in model $\mathcal{M}$".

Here, the value 1 was considered as *true*, 0 was considered as *false*, and 2 was considered as *undefined*.

Chapter 3

4-valued logic for agent communication

In this chapter, we propose a 4-valued logic that can distinguish the private and public information passing. In Section 3.1, we introduce the ideal behind this logic, i.e., consider that every proposition can be either public or private while obtaining the classical values true and false. In Sections 3.2 and 3.3, we show the syntax and semantics of this logic. In Section 3.4, we revise the belief change by private/public information passing and give an example. And then in Section 3.5, we show the recursion axiom to the ordinary dynamic epistemic logic, that is sound and complete. Finally, in Section 3.6, we give the relation between this logic to other many-valued modal logics.

3.1 Idea behind the logic

Consider two people Ann and Bill, who are chatting on the Internet. Ann learned a new dance and she believes that her dance is very good, so she wants to tell Bill it. Then she sends a video of her dance to Bill. However, Bill doesn't get the message that Ann's dance is good. The possible reasons are as follows:

- The Internet is not connected.
- Bill's computer is too old to watch the video.

so he cannot get the message to revise his belief. Also, it can be explained in other ways. For another example, let agents a and b be two companies. p means that "a is faced with bankruptcy". Obviously, if a and b are opponents, they won't tell it to each other if they believe p or the negation of p. Such p can be seen as a private proposition.

Hatano [23] considered the case of disconnection by channels,

while not distinguishing the different kinds of information.

Here, we use a pair (a, b) **to express the value of a proposition** A. $(a \in \{\mathbf{t}, \mathbf{f}\}, b \in \{0, 1\})$

- $A : (\mathbf{t}, 1)$ **means "**A **is true and public."**
- $A : (\mathbf{t}, 0)$ **means "**A **is true and private."**
- $A : (\mathbf{f}, 1)$ **means "**A **is false and public."**
- $A : (\mathbf{f}, 0)$ **means "**A **is false and private."**

If A **is public, other agents can get this message, and if** A **is private, others cannot revise their beliefs by this message.**

3.2 Syntax

Here, we define a new kind of 4-valued logic different from BK-model we showed in Section 2.3.1.

Definition 3.1. *Let* $\mathsf{Prop} = \{ p, q, \dots \}$ *be a finite set of propositional variables and* $\mathsf{G} = \{ a, b, \dots \}$ *a finite set of agents. a set* Form_p *of formulas of the language* $\mathcal{L}_p$ *is inductively defined as follows:*

$$\mathsf{Form}_p \ni A ::= p \mid c_{ab} \mid \mathsf{B}_a A \mid^{pub} A \mid [A{\downarrow}_b^a]A \mid \neg A \mid A \wedge A$$

where $p \in \mathsf{Prop}, a \in \mathsf{G}, b \in \mathsf{G}$.

Here, c_{ab} **means "There is a channel from agent** a **to agent** b**",** $\mathsf{B}_a\, \alpha$ **means "agent** a **believes** α**.", and** ${}^{pub}A$ **means "**A **is public".**

3.3 Semantics

First, we consider the truth tables of non-modal connectives $\neg, {}^{pub}, \wedge, \vee$ **and** $\rightarrow$**. The truth-table of** $\neg$ **is as follows. Here, we can see that the negation only changes the classical values t and f.**

A	$\neg A$
$(\mathbf{t}, 1)$	$(\mathbf{f}, 1)$
$(\mathbf{t}, 0)$	$(\mathbf{f}, 0)$
$(\mathbf{f}, 1)$	$(\mathbf{t}, 1)$
$(\mathbf{f}, 0)$	$(\mathbf{t}, 0)$

Table 3.1: Truth-table of $\neg$

A	${}^{pub}A$
$(\mathbf{t}, 1)$	$(\mathbf{t}, 1)$
$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$
$(\mathbf{f}, 1)$	$(\mathbf{t}, 1)$
$(\mathbf{f}, 0)$	$(\mathbf{f}, 1)$

Table 3.2: Truth-table of ^{pub}

The truth-table of $\neg$ is as follows. Here, we let ${}^{pub}A$ be always public, i.e., the statement that shows whether a formula is public or not is always a public message.

Remark 3.1. *Actually, we can also let ${}^{pub}A$ be public if and only if A is public. It means that the statement whether A is public or not is depended on whether A is public or not.*

For the 4-valued logic, we consider the truth table of conjunction $\wedge$. Let A and B be two propositions. Then the proposition $A \wedge B$ is true if and only if A is true and B is true. And $A \wedge B$ is public if and only if A is public and B is public. We define the truth-table of $\wedge$ as follows.

$\wedge$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{t}, 1)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$
$(\mathbf{f}, 1)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$

Table 3.3: Truth-table of $\wedge$

We define the function $\vee$ as follows as usual:

$$A \vee B := \neg(\neg A \wedge \neg B)$$

Then, the truth-table of $\vee$ is shown as follows:

$\vee$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{t}, 1)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$
$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$
$(\mathbf{f}, 1)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{f}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$

Table 3.4: Truth-table of $\vee$

Here, we should take notice of the truth-table of $\vee$. In this paper, we define that if $A \vee B$ is public if and only if A is public and B is public. In other words, if A is private, $A \wedge B$ and $A \vee B$ are all private even if B is public. It is because that if we cannot tell A to others, anything related to A like $A \wedge B$ or $A \vee B$ also cannot be told to others.

Finally, we define the "$\rightarrow$" as follows:

$$A \rightarrow B := \neg A \vee B$$

The truth-table of $\rightarrow$ is shown in as follows.

$\rightarrow$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{t}, 1)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 1)$	$(\mathbf{f}, 0)$
$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{f}, 0)$	$(\mathbf{f}, 0)$
$(\mathbf{f}, 1)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 1)$	$(\mathbf{t}, 0)$
$(\mathbf{f}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$	$(\mathbf{t}, 0)$

Table 3.5: Truth-table of $\rightarrow$

Notice that $A \rightarrow B$ is public if and only if A is public and B is public, which is similar to the operator $\vee$.

Here, we use Kripke semantics with our syntax.

Definition 3.2. *A Kripke model $\mathcal{M}$ is a tuple:*

$$\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, V)$$

where W is a non-empty set of worlds, G is a non-empty set of agents, $R_{\mathsf{G}} = \{R_a \mid a \in \mathsf{G}\}$ and $R_a \subseteq W \times W$ is an accessibility of agent a on W, $C_{\mathsf{G}} = \{C_{ab} \mid a \in \mathsf{G}, b \in \mathsf{G}\}$ and $C_{ab} \subseteq W$ is a channel relation such that $C_{aa} = W$ for all $a \in \mathsf{G}$, and $V : \mathsf{Prop} \times W \rightarrow \{(\mathbf{t}, 1), (\mathbf{t}, 0), (\mathbf{f}, 1), (\mathbf{f}, 0)\}$ is the valuation function. In many cases it is convenient to replace the four-valued V by two function, so we assign functions $v^t, v^p : \mathsf{Prop} \rightarrow 2^W$ defined as follows to express V:

$$v^t(p) = \{w | V(p, w) \in \{(\mathbf{t}, 1), (\mathbf{t}, 0)\}\}$$

$$v^p(p) = \{w | V(p, w) \in \{(\mathbf{t}, 1), (\mathbf{f}, 1)\}\}$$

Given any model $\mathcal{M}$, any world $w \in W$, and any formula A, we define *the satisfaction relation $\mathcal{M}, w \models^t A$ and $\mathcal{M}, w \models^p A$ as follows:*

$$
\begin{array}{lll}
\mathcal{M}, w \models^t p & \text{iff} & w \in v^t(p) \\
\mathcal{M}, w \models^t A \wedge B & \text{iff} & \mathcal{M}, w \models^t A \textit{ and } \mathcal{M}, w \models^t B \\
\mathcal{M}, w \models^t A \vee B & \text{iff} & \mathcal{M}, w \models^t A \textit{ or } \mathcal{M}, w \models^t B \\
\mathcal{M}, w \models^t A \rightarrow B & \text{iff} & \mathcal{M}, w \not\models^t A \textit{ or } \mathcal{M}, w \models^t B \\
\mathcal{M}, w \models^t \neg A & \text{iff} & \mathcal{M}, w \not\models^t A \\
\mathcal{M}, w \models^t {}^{pub}A & \text{iff} & \mathcal{M}, w \models^p A \\
\mathcal{M}, w \models^t B_a A & \text{iff} & \textit{for all } u \in W : (w, u) \in R_a \textit{ implies } \mathcal{M}, u \models^t A \\
\mathcal{M}, w \models^t \mathsf{c}_{ab} & \text{iff} & w \in C_{ab} \\
\mathcal{M}, w \models^p p & \text{iff} & w \in v^p(p) \\
\mathcal{M}, w \models^p A \wedge B & \text{iff} & \mathcal{M}, w \models^p A \textit{ and } \mathcal{M}, w \models^p B \\
\mathcal{M}, w \models^p A \vee B & \text{iff} & \mathcal{M}, w \models^p A \textit{ and } \mathcal{M}, w \models^p B \\
\mathcal{M}, w \models^p A \rightarrow B & \text{iff} & \mathcal{M}, w \models^p A \textit{ and } \mathcal{M}, w \models^p B \\
\mathcal{M}, w \models^p \neg A & \text{iff} & \mathcal{M}, w \models^p A \\
\mathcal{M}, w \models^p {}^{pub}A & & \textit{always} \\
\mathcal{M}, w \models^p B_a A & \text{iff} & \mathcal{M}, w \models^p A \\
\mathcal{M}, w \models^p \mathsf{c}_{ab} & & \textit{always}
\end{array}
$$

Here, $\mathcal{M}, w \models^p B_a A$ iff *$\mathcal{M}, w \models^p A$ means that if A is public, the message that agent a believes A is also public and vice versa.*

Remark 3.2. *Here, we define that the channel constant c_{ab} is always public. Actually, it is just for convenience. We can also consider that some channel constants are private by adding another function to our Kripke model.*

Also, let the value of A be (a, b) in world w , we can define $\models^t$ and $\models^p$ in the other way as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \models^t A & \text{iff} & a := \mathbf{t} \\
\mathcal{M}, w \models^p A & \text{iff} & b := 1
\end{array}
$$

Semantically speaking, in a model $\mathcal{M}$, $\mathcal{M}, w \models^t A$ means A is true in world w, and $\mathcal{M}, w \models^p A$ means A is public in world w.

Then we give some propositions. First, we give the proposition about channel which is also provided in [22].

Proposition 3.1. *For any $a \in \mathsf{G}$, $\mathcal{M}, w \models^t c_{aa}$ in any Kripke model $\mathcal{M}$ and any world w.*

Proof. Fix any $a \in \mathsf{G}$, any model $\mathcal{M}$ and any world w in $\mathcal{M}$. We show $\mathcal{M}, w \models^t c_{aa}$, i.e., $w \in C_{aa}$. By definition, $C_{aa} = W$ and it follows that $w \in C_{aa}$. □

For we add an operator ^{pub}, we give some propositions about this operator.

Proposition 3.2. *For any formula A, $\mathcal{M}, w \models^t {}^{pub}\neg A \leftrightarrow {}^{pub}A$ in any Kripke model $\mathcal{M}$ and any world w.*

Proposition 3.3. *For any formula A and B, $\mathcal{M}, w \models^t {}^{pub}(A \wedge B) \leftrightarrow ({}^{pub}A \wedge {}^{pub}B)$ in any Kripke model $\mathcal{M}$ and any world w.*

Proposition 3.4. *For any formula A and any agent a, $\mathcal{M}, w \models^t {}^{pub}A \leftrightarrow {}^{pub}B_a A$ in any Kripke model $\mathcal{M}$ and any world w.*

Proposition 3.5. *For any formula A, $\mathcal{M}, w \models^t {}^{pub}\,{}^{pub}A$ in any Kripke model $\mathcal{M}$ and any world w.*

Proposition 3.6. *For any formula A and any agent a, b, $\mathcal{M}, w \models^t {}^{pub}\, c_{ab}$ in any Kripke model $\mathcal{M}$ and any world w.*

Proof. The proofs above are similar and easy to see from the semantics we provided. Here, we give the proof of Proposition 3.2. Fix any $a \in \mathsf{G}$, any model $\mathcal{M}$ and any world w in $\mathcal{M}$. We show $\mathcal{M}, w \models^t {}^{pub}\neg A \leftrightarrow {}^{pub}A$.

$\mathcal{M}, w \models^t {}^{pub}\neg A \leftrightarrow {}^{pub}A$
iff $(\mathcal{M}, w \models^t {}^{pub}\neg A$ and $\mathcal{M}, w \models^t {}^{pub}A)$ or
$(\mathcal{M}, w \not\models^t {}^{pub}\neg A$ and $\mathcal{M}, w \not\models^t {}^{pub}A)$
iff $(\mathcal{M}, w \models^p \neg A$ and $\mathcal{M}, w \models^p A)$ or $(\mathcal{M}, w \not\models^p \neg A$ and $\mathcal{M}, w \not\models^p A)$
iff $(\mathcal{M}, w \models^p A$ and $\mathcal{M}, w \models^p A)$ or $(\mathcal{M}, w \not\models^p A$ and $\mathcal{M}, w \not\models^p A)$
iff $\mathcal{M}, w \models^p A$ or $\mathcal{M}, w \not\models^p A$

We can see the $\mathcal{M}, w \models^p A$ or $\mathcal{M}, w \not\models^p A$ is always true. □

3.4 Multi-agent communication

In this paper, we use the same dynamic operator $[A\downarrow^a_b]$ as [23], which means "after agent a sends a message A to agent b via a channel", and $[A\downarrow^a_b]B$ means "after agent a sends a message A to agent b via a channel, B holds". Here, the communication $[A\downarrow^a_b]$ will success only if the following hold:

- **There is a channel from agent a to agent b.**
- **Agent a believes the content of the message A.**
- **The message A is public.**

In [23], all of the message is regarded as public message, which can be told to others. Here, we use 4-valued logic which can express

whether a formula is public or not, so even if agent a believe A and there is a channel from a to b, the communication will fail if A is private, which is different from [23].

The semantics of $[A\downarrow^a_b]B$ on a Kripke model $\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, v^+, v^-)$ is given as follows:

$$\begin{array}{lll} \mathcal{M}, w \models^t [A\downarrow^a_b]B & \text{iff} & \mathcal{M}^{A\downarrow^a_b}, w \models^t B \\ \mathcal{M}, w \models^p [A\downarrow^a_b]B & \text{iff} & \mathcal{M}^{A\downarrow^a_b}, w \models^p B \end{array}$$

where $\mathcal{M}^{A\downarrow^a_b} = (W, R'_{\mathsf{G}}, C_{\mathsf{G}}, v^t, v^p)$ and $R'_i \in R'_{\mathsf{G}}$ is defined as:

- If $i = b$, for all $x \in W$,

$$R'_b(x) := \begin{cases} R_b(x) \cap [\![A]\!]_{\mathcal{M}} & \text{if } \mathcal{M}, x \models^t c_{ab} \wedge \mathsf{B}_a A \\ & \textit{and } \mathcal{M}, x \models^p A \\ R_b(x) & \textbf{otherwise.} \end{cases}$$

- Otherwise, $R'_i := R_i$.

The truth set $[\![A]\!]_{\mathcal{M}}$ is defined by:

$$[\![A]\!]_{\mathcal{M}} = \{\, w \in W \mid \mathcal{M}, w \models^t A \,\}.$$

Semantically speaking, after $[A\downarrow^a_b]$, agent b will revise his/ her belief if there is a channel from agent a to b, agent a believes the content of the message A, and A is public. Otherwise, agent b will not revise his/ her belief. Other agents than b will not change beliefs because they get no message.

Example 3.1. *Here, we give an example to show the belief change after a semi-announcement. Consider a Kripke model* $\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, v^t, v^p)$ *that is shown as the Figure 3.1.*

As the configuration above, p is only true in w_1, w_2, w_3 and is only public in w_1, w_2, w_4. According to the definition of $B_a A$, we can see that agent a believe p in world w_1, w_2 and w_3, while not believing anything in world w_4. Agent b doesn't believe anything in any world. There are channels from agent a to b in w_1, w_2, w_4, while no channel exists in w_3.

Now, consider the model $\mathcal{M}^{A\downarrow^a_b}$ which shows the new accessibility relation after the action agent a sends the message A to agent b.

- In world w_1, agent a believes p so a can send the message, there is a channel from agent a to b so the message can be sent to b, and as p is public so b can understand the message p. As the result, b will revise his/ her belief to believe p.

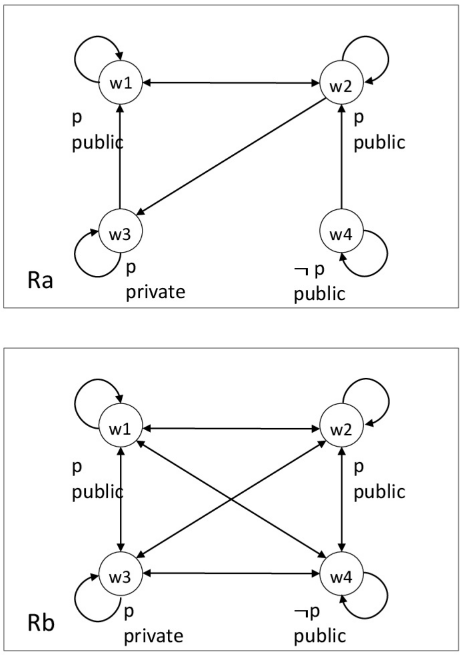

Figure 3.1: Accessibility relations of agents a and b.

- **In world w_2, agent a believes p so a can send the message, and as p is public so b can understand the message p. However, there isn't a channel from agent a to b so the message cannot be sent to b. So as the result, b won't revise his/ her belief.**

- **In world w_3, agent a believes p so a can send the message, and there is a channel from agent a to b so the message can be sent to b. However, as p is private so b cannot understand the message p. As the result, b won't revise his/ her belief.**

- **In world w_4, there is a channel from agent a to b so the message can be sent to b, and as p is public so b can understand the message p. However, agent a doesn't believe p so a cannot send the message. As the result, b won't revise his/ her belief.**

We can see that after the action $[A{\downarrow}^a_b]$ which means that agent a tells b the message p, agent b becomes to believe p only in world w_1. In other worlds, agent b doesn't change his/ her belief. So in the new model $\mathcal{M}^{A\downarrow^a_b} = (W, R'_{\mathsf{G}}, C_{\mathsf{G}}, v^t, v^p)$, $R'_b := W \times W / \{(w_1, w_3), (w_1, w_4)\}$ shown in Figure 3.2.

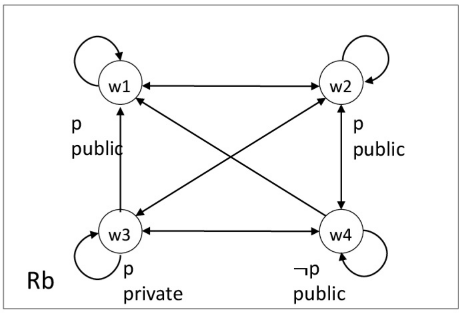

Figure 3.2: Accessibility relation of agent b after the announcement.

3.5 Hilbert-style Axiomatization

Here, we provide a similar with Hatano [22] to give the proof of completeness and soundness. Although we gave two satisfaction relations $\models^t$ and $\models^p$, our goal is to show that $\vdash_{K_p} A$ iff $\models^t A$ for all formula A. First, we present the sound and complete Hilbert-style axiomatization K_p without the dynamic operator.

(**Taut**)	A, A is a tautology.
(**$\mathbf{K_B}$**)	$B_a(A \to B) \to (B_aA \to B_aB)$
(**Selfchn**)	c_{aa}
(**MP**)	From A and $A \to B$, infer B
$\mathbf{Nec_B}$	From A, infer B_aA
$\mathbf{Pub_\neg}$	${}^{pub}\neg A \leftrightarrow {}^{pub}A$
$\mathbf{Pub_\wedge}$	${}^{pub}(A \wedge B) \leftrightarrow ({}^{pub}A \wedge {}^{pub}B)$
$\mathbf{Pub_B}$	${}^{pub}A \leftrightarrow {}^{pub}B_aA$
$\mathbf{Pub_{chn}}$	${}^{pub}c_{ab}$
Pub	${}^{pub\ pub}A$

Here, $a, b \in G$.

Table 3.6: Hilbert-style Axiomatization $\mathbf{K_p}$ of 4-valued logic

First, we give some definition of canonical model.

Definition 3.3. *For the axiomatic extension K_p, the canonical model $\mathcal{M}^{K_p} = (W^{K_p}, R_{\mathsf{G}}^{K_p}, C_{\mathsf{G}}^{K_p}, v_t^{K_p}, v_p^{K_p})$ is defined by:*

- $W^{K_p} := \{\Gamma \mid \Gamma$ *is a maximal K_p-consistent set*$\}$.
- $\Gamma R_a^{K_p} \Delta$ *iff $B_aA \in \Gamma$ implies $A \in \Delta$ for all formulas A.*

- $\Gamma \in C_{ab}^{K_p}$ *iff* $c_{ab} \in \Gamma$.
- $\Gamma \in v_t^{K_p}(p)$ *iff* $p \in \Gamma$.
- $\Gamma \in v_p^{K_p}(p)$ *iff* ${}^{pub}p \in \Gamma$.

Then we can show the Lindenbaum's Lemma and Truth Lemma.

Lemma 3.1 (Lindenbaum's Lemma)**.** *If* Γ *is any* K_p*-consistent set, then there exists a maximal* K_p*-consistent set* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$.

Lemma 3.2 (Truth Lemma)**.** *Give any formula* A *and any maximal* K_p*-consistent set* Γ, $\mathcal{M}^{K_p} \models^t A$ *iff* $A \in \Gamma$.

Proof. Proved by induction on A. Here, we only show the case for A be the form ${}^{pub}B$ because others are almost the same as [22]. Our goal is to show that $\mathcal{M}^{K_p} \models^t {}^{pub}B$ iff $B \in \Gamma$. We show it by induction on B.

- If B is a proposition p, $\mathcal{M}^{K_p} \models^t {}^{pub}p$ iff $\Gamma \in v_p^{K_p}(p)$, therefore ${}^{pub}p \in \Gamma$ by Definition 3.3.
- If B is the form $C \wedge D$, $\mathcal{M}^{K_p} \models^t {}^{pub}(C \wedge D)$ iff $\mathcal{M}^{K_p} \models^t {}^{pub}C$ and $\mathcal{M}^{K_p} \models^t {}^{pub}D$ which can be proved by Proposition 3.3, and we have ${}^{pub}C \in \Gamma$ and ${}^{pub}D \in \Gamma$ by I.H. Therefore, we can show ${}^{pub}(C \wedge D) \in \Gamma$.
- If B is the form B_aC, $\mathcal{M}^{K_p} \models^t {}^{pub}B_aC$ iff $\mathcal{M}^{K_p} \models^t {}^{pub}C$ which can be proved by Proposition 3.4, and we have ${}^{pub}C \in \Gamma$ by I.H. Therefore, we can show ${}^{pub}B_aC \in \Gamma$..
- If B is the form ${}^{pub}C$, $\mathcal{M}^{K_p} \models^t {}^{pubpub}C$ is always true by Proposition 3.5, and ${}^{pubpub}C \in \Gamma$ for Γ is a maximal K_p-consistent set. Therefore, we have $\mathcal{M}^{K_p} \models^t {}^{pubpub}C$ iff ${}^{pub\ pub}C \in \Gamma$.
- If B is the form ${}^{pub}c_{ab}$, $\mathcal{M}^{K_p} \models^t {}^{pub}c_{ab}$ is always true by Proposition 3.6, and ${}^{pub}\ c_{ab} \in \Gamma$ for Γ is a maximal K_p-consistent set. Therefore, we have $\mathcal{M}^{K_p} \models^t {}^{pub}c_{ab}$ iff ${}^{pub}c_{ab} \in \Gamma$.

Therefore, we prove our lemma. □

Then, we use filtration technique to obtain the completeness of K_O **with respect to finite models.**

Definition 3.4. *Given any formula* $A \in \mathcal{L}_p$*, we define the subformulas* $Sub(A) : \mathsf{Form}_p \to \mathcal{P}(\mathsf{Form}_p)$ *by:*

$$
\begin{array}{lll}
Sub(p) & := & \{p\}, \\
Sub(c_{ab}) & := & \{c_{ab}\}, \\
Sub(\neg A) & := & \{\neg A\} \cup Sub(A), \\
Sub({}^{pub}A) & := & \{{}^{pub}A\} \cup Sub(A), \\
Sub(A \wedge B) & := & Sub(A) \cup Sub(B) \cup \{A \wedge B\}, \\
Sub(B_aA) & := & \{B_aA\} \cup Sub(A),
\end{array}
$$

We also define $Sub(\cdot)$ for the set Γ of formulas as follows:

$$Sub(\Gamma) := \bigcup_{A \in \Gamma} Sub(A).$$

We say that the set Γ of formulas is closed under taking subformulas if $Sub(A) \subseteq \Gamma$ for all formulas $A \in \Gamma$.

Definition 3.5. *Let $\mathcal{M}$ be a Kripke model and Γ be a finite set of formulas that is closed under taking subformulas. We define the equivalence relation $\sim_\Gamma$ on W by:*

$$w \sim_\Gamma v \textit{ iff } (M, w \models^t A \textit{ iff } \mathcal{M}, v \models^t A) \textit{ for all } A \in \Gamma.$$

We define the equivalence class of $w \in W$ with respect to $\sim_\Gamma$ by:

$$[w] := \{v \in W \mid w \sim_\Gamma v\}.$$

Definition 3.6. *Let $\mathcal{M}$ be a Kripke model and Γ be a finite set of formulas that is closed under taking subformulas. The model $\mathcal{M}^\Gamma := (W^\Gamma, R^\Gamma_{\mathsf{G}}, C^\Gamma_{\mathsf{G}}, v^\Gamma_t, v^\Gamma_p)$ is a filtration of $\mathcal{M}$ through Γ if it satisfies the following conditions:*

- $W^\Gamma := W/\sim_\Gamma = \{[w] \mid w \in W\}$.
- $[w]R^\Gamma_a[w']$ *iff* wR_av *fro some* $w' \in [w]$ *and* $v' \in [v]$.
- $[w] \in C^\Gamma_{ab}$ *iff* $w \in C_{ab}$.
- $[w] \in v^\Gamma_t(p)$ *iff* $w \in v_t(p)$.
- $[w] \in v^\Gamma_p(p)$ *iff* $w \in v_p(p)$.

We can see that if Γ is finite then W^Γ is also finite.

Theorem 3.1 (Filtration Theorem)**.** *Let $\mathcal{M} = (W, R_{\mathsf{G}}, C_{\mathsf{G}}, V)$ be a model and Γ be a finite set of formulas that is closed under taking subformulas. For any $w \in W$ and any formulas $A \in \Gamma$,*

$$M, w \models^t A \textit{ iff } M^\Gamma, [w] \models^t A.$$

Theorem 3.2. *Let A be a formulas in $Form_p$ and $\mathfrak{M}$ be the class of all finite Kripke models.*

$$\mathfrak{M} \models^t A \textit{ iff } \vdash_{K_p} A.$$

Proof. It is easy to see soundness, so here we focus on the completeness with respect to the class of all finite Kripke models. We establish the completeness of K_p by the filtration. Our goal is to show that if $\nvdash_{K_p} A$, then $\mathfrak{M} \not\models^t A$. By canonical model for K_p and Lemmas 3.1 and 3.2, we can obtain that $\mathcal{M}^{K_p} \not\models^t A$. Since the domain of the canonical model $\mathcal{M}^{K_p}$ is infinite, we use the filtration to boil the model down to a finite model. By Theorem 3.1 to $\mathcal{M}^{K_p} \not\models A$, we obtain $(\mathcal{M}^{K_p})^\Gamma, [w] \not\models^t A$,i.e., $\mathfrak{M} \not\models^t A$. □

Theorem 3.3. *K_p is decidable.*

Proof. By the Theorem 3.2, we have that if $\nvdash_{K_p} A$, then there exists a model $\mathcal{M}$ such that $\mathcal{M} \not\models^t A$, i.e., the model $\mathcal{M}$ is a finite counter model. Therefore, K_p is decidable. □

Then, we present the Hilbert-style Axiomatization $\mathbf{K_p}[\cdot\downarrow^{\mathbf{a}}_{\mathbf{b}}]$ with dynamic operator.

In addition to all the axioms and rules of $\mathbf{K_p}$, we add:

$[A\downarrow^a_b]p$	$\leftrightarrow$	p
$[A\downarrow^a_b]c_{cd}$	$\leftrightarrow$	c_{cd}
$[A\downarrow^a_b]\neg B$	$\leftrightarrow$	$\neg[A\downarrow^a_b]B$
$[A\downarrow^a_b]^{pub}B$	$\leftrightarrow$	${}^{pub}B$
$[A\downarrow^a_b](B \wedge C)$	$\leftrightarrow$	$[A\downarrow^a_b]B \wedge [A\downarrow^a_b]C$
$[A\downarrow^a_b]B_cB$	$\leftrightarrow$	$B_c[A\downarrow^a_b]B(c \neq b)$
$[A\downarrow^a_b]B_bB$	$\leftrightarrow$	$((c_{ab} \wedge B_aA \wedge^{pub} A) \rightarrow B_b(A \rightarrow [A\downarrow^a_b]B))$
		$\wedge(\neg(c_{ab} \wedge B_aA \wedge^{pub} A) \rightarrow B_b[A\downarrow^a_b]A)$
$(\mathbf{Nec}_{[\mathbf{A}\downarrow^{\mathbf{a}}_{\mathbf{b}}]})$		From A, infer $[A\downarrow^a_b]A$

Table 3.7: Hilbert-style Axiomatization $\mathbf{K_p}[\cdot\downarrow^{\mathbf{a}}_{\mathbf{b}}]$ of 4-valued logic

Since we can regard the system $\mathbf{K_p}[\cdot\downarrow^{\mathbf{a}}_{\mathbf{b}}]$ as an axiomatic extension of K_p, we define a derivation and a theorem in $\mathbf{K_p}[\cdot\downarrow^{\mathbf{a}}_{\mathbf{b}}]$.

Definition 3.7. *The translation t : $\mathsf{Form}_{K_p[\cdot\downarrow^a_b]} \rightarrow \mathsf{Form}_{K_p}$ is defined by:*

$t(p)$	$:=$	$p,$
$t(c_{ab})$	$:=$	$c_{ab},$
$t(\neg A)$	$:=$	$\neg t(A),$
$t({}^{pub}A)$	$:=$	${}^{pub}t(A),$
$t(A \wedge B)$	$:=$	$t(A) \wedge t(B),$
$t(B_aA)$	$:=$	$B_at(A),$
$t([A\downarrow^a_b]p)$	$:=$	$p,$
$t([A\downarrow^a_b]c_{cd})$	$:=$	$c_{cd},$
$t([A\downarrow^a_b]\neg B)$	$:=$	$\neg t([A\downarrow^a_b]B),$
$t([A\downarrow^a_b]^{pub}B)$	$:=$	${}^{pub}t([A\downarrow^a_b]B),$
$t([A\downarrow^a_b](B \wedge C))$	$:=$	$t([A\downarrow^a_b]B) \wedge t([A\downarrow^a_b]C),$
$t([A\downarrow^a_b]B_cB)$	$:=$	$B_ct([A\downarrow^a_b]B), (c \neq b)$
$t([A\downarrow^a_b]B_bB)$	$:=$	$((c_{ab} \wedge B_at(A) \wedge^{pub} t(A)) \rightarrow B_b(t(A) \rightarrow t([A\downarrow^a_b]B))),$
		$\wedge(\neg(c_{ab} \wedge B_at(A) \wedge^{pub} t(A)) \rightarrow B_bt([A\downarrow^a_b]A)),$
$t([A\downarrow^a_{b[A\downarrow^c_d]}]B)$	$:=$	$t([A\downarrow^a_b]t([A\downarrow^c_d]B)).$

Lemma 3.3. *Given any formula $A \in \mathsf{Form}_{K_p[\downarrow^a_b]}$,*

$$\vdash_{K_p[\downarrow^a_b]} A \leftrightarrow t(A).$$

Theorem 3.4. *Let A be a formulas in $\mathsf{Form}_{K_p[\downarrow^a_b]}$ and $\mathfrak{M}$ be the class of all finite Kripke models.*

$$\mathfrak{M} \models^t A \textit{ iff } \vdash_{K_p[\downarrow^a_b]} A.$$

Proof. The soundness is clearly by the semantics of the dynamic operator $[A\downarrow^a_b]$. To show the completeness, the goal is to show that if $\mathfrak{M} \models^t A$ then $\vdash_{K_p[\downarrow^a_b]} A$. Assume that $\mathfrak{M} \models^t A$, by the soundness and Lemma 3.3, we obtain that $\mathfrak{M} \models^t t(A)$. For $t(A) \in \mathsf{Form}_p$, by the completeness of K_p we obtain $\vdash_{K_p} t(A)$. Since $\mathbf{K_p}[\cdot\downarrow^\mathbf{a}_\mathbf{b}]$ is an axiomatic extension of K_p, we also obtain that $\vdash_{K_p[\downarrow^a_b]} t(A)$. Finally, by Lemma 3.3, we obtain $\vdash_{K_p[\downarrow^a_b]} A$. □

3.6 Relation to many-valued modal logic

In this chapter, although we called our semantics as 4-valued logic, it is not a traditional many-valued logic as Kleene logic, FDE logic, etc.. Here, we consider that each proposition has 4 values, while the values could not be considered as truth-values. Actually, we built this semantics by using the similar structure of 4-valued modal logic we introduced in Section 2.3.

However, the interesting point here is that although we did not take care of the 3-valued modal logic, the semantic can be considered similarly if we read the *public* as *defined* and *private* as *undefined*. Moreover, Correia [12] considered two choices to the truth-clause for necessity as following and focused on the latter choice:

- **$\Box A$ is true at w iff A is defined at w and A is true at every world accessible from w;**
- **$\Box A$ is true at w iff A is defined at w and A is true at every world accessible from w at which A is defined.**

If we only take care of the connectives $\neg, \vee, \wedge, \rightarrow$ and the modal operator B, and consider that only true and public statements are valid, our semantics is the same as the 3-valued modal semantics that focus on the first choice.

Remark 3.3. *There exists other studies of many-valued modal logic. Santos [36] showed another kind of four-valued epistemic logic. The consideration of* *the 4 values was based on FDE logic, i.e., the readings of values were* ***true, false, both, neither****, whose idea was different from our research.*

Another research that is similar with our study is the awareness logic. Ditmarsch [45] provided the epistemic awareness model that could express the awareness besides knowledge. The state of aware is similar to our reading of public, for it is similar to our semantics that agent a is aware of $\neg A$ *iff a is aware of A, and a is aware of* $A \wedge B$ *iff a is aware of A and a is aware of B. However, the epistemic awareness model provided a more complex structure which was more expressive, while our study gives a simpler Kripke model which is similar to the weak Kleene logic.*

Chapter 4

Semantics for infectious logics

This chapter provides the semantics which is based on Beall's off-topic interpretation for infectious logic. In Section 4.1, we first make clear our intuitive idea. Then, in Sections 4.2 and 4.3, we introduce our two-valued semantics for single topic and multi topics, and give the proofs that the semantics can be considered as same as classical logic and weak Kleene logic. In Section 4.4, we give a FDE-based n-topic model to show a new interpretation of the logic of Catuskoti, and give the proof that it can also be considered as FDE logic. Then, in Section 4.5, we give a many-valued semantics which can be both relating to the semantics with and without an infectious value. Finally, in Section 4.6, we show some expansions of our semantics with new connectives.

4.1 Idea behind the semantics

There are several reading of the infectious value in the infectious logics. The reading *meaningless* or *undefined* seems that the truth values are two-dimensional for we can read the value true/false as *true/false and meaningful/defined*, however, we don't consider that a *meaningless/undefined* proposition is true or false normally. On the other hand, if we consider the Beall's off-topic interpretation [3], it seems natural and even well-motivated to divide propositions that are *off-topic* into *true and off-topic* and *false and off-topic.*

Our idea for the new semantics is rather simple:

- The idea motivated by (M1), is to add another valuation to take care of the on-topic/off-topic aspect, besides the usual classical valuation that will take care of the truth/falsity aspect. Basically, the idea is that the distinction 'truth/falsity' and 'on-topic/off-topic' should be seen as two distinct parallel dimensions available for each sentence of our language.

- **The idea motivated by (M2), is to consider that there may be a number of different topics under consideration. It seems to make a lot of sense to relativize Beall's ideas, especially the first two conditions, with respect to some topics.**

4.2 Alternative semantics (I): implementing (M1)

4.2.1 Basics

Definition 4.1. *A* two-valued interpretation *for the language $\mathcal{L}$ is a pair $\langle V_t, V_m \rangle$, where V_t : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ and V_m : $\mathsf{Prop} \to \{0, 1\}$. Valuations V_t, V_m are then extended to interpretations I_t, I_m by the following conditions.*

$$
\begin{array}{lll}
I_t(p)=\mathbf{t} & \textit{iff} & V_t(p)=\mathbf{t}, \\
I_m(p)=1 & \textit{iff} & V_m(p)=1, \\
I_t(\sim A)=\mathbf{t} & \textit{iff} & I_t(A)=\mathbf{f}, \\
I_m(\sim A)=1 & \textit{iff} & I_m(A)=1, \\
I_t(A \wedge B)=\mathbf{t} & \textit{iff} & I_t(A)=\mathbf{t} \textit{ and } I_t(B)=\mathbf{t}, \\
I_m(A \wedge B)=1 & \textit{iff} & I_m(A)=1 \textit{ and } I_m(B)=1, \\
I_t(A \vee B)=\mathbf{t} & \textit{iff} & I_t(A)=\mathbf{t} \textit{ or } I_t(B)=\mathbf{t}, \\
I_m(A \vee B)=1 & \textit{iff} & I_m(A)=1 \textit{ and } I_m(B)=1.
\end{array}
$$

Remark 4.1. *Note that this style of semantics with two components can be also found in a paper by Hans Herzberger in [24], as well as Peter Woodruff in [48].*

Now that there are four combinations for elements of Prop, we may easily turn the above two-valued semantics into a four-valued matrix.

Definition 4.2. *A* four-valued interpretation *of $\mathcal{L}$ is a function I_4 : $\mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$. Given a four-valued interpretation I_4, this is extended to a function that assigns every formula a truth value by the following truth functions:*

A	$\sim A$
t1	**f**1
t0	**f**0
f0	**t**0
f1	**t**1

$A \wedge B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**0	**f**0	**f**1
t0	**t**0	**t**0	**f**0	**f**0
f0	**f**0	**f**0	**f**0	**f**0
f1	**f**1	**f**0	**f**0	**f**1

$A \vee B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**0	**t**0	**t**1
t0	**t**0	**t**0	**t**0	**t**0
f0	**t**0	**t**0	**f**0	**f**0
f1	**t**1	**t**0	**f**0	**f**1

4.2.2 Some results

Given an interpretation of the language under consideration, we need to specify the set of designated values to define the semantic consequence relation. To this end, we introduce three different sets of designated values as follows:

- $\mathcal{D}_1 := \{\mathbf{t}1\}$;
- $\mathcal{D}_2 := \{\mathbf{t}1, \mathbf{t}0\}$;
- $\mathcal{D}_3 := \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0\}$.

Based on these sets of designated values, we define three consequence relations as follows.

Definition 4.3. *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_i A$ iff for all four-valued interpretations I_4, $I_4(A) \in \mathcal{D}_i$ if $I_4(B) \in \mathcal{D}_i$ for all $B \in \Gamma$.*

Remark 4.2. *It should be clear that we can also define the above consequence relations in terms of two valued semantics. We will, however, focus on the four-valued representation since that is more easy to connect to the three-valued logics.*

Then, we can show the facts as following.

- $\models_1$ **is weak Kleene logic.**
- $\models_2$ **is classical logic.**
- $\models_3$ **is paraconsistent weak Kleene logic.**

We first deal with the case in which $\mathbf{t}1$ is the only designated value. To this end, we prepare a lemma.

Lemma 4.1. *For all three-valued valuation v_3 for $\mathcal{L}$, there is a four-valued valuation v_4 such that for all $A \in \mathsf{Form}$, (i) $I_4(A) = \mathbf{t}1$ iff $I_3(A) = \mathbf{t}$, and (ii) $I_4(A) = \mathbf{f}1$ iff $I_3(A) = \mathbf{f}$.*

Proof. Given a three-valued valuation v_3, we define $v_4 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ as follows:

$$v_4(p) = \begin{cases} \mathbf{t}1 & v_3(p) = \mathbf{t} \\ \mathbf{f}0 & v_3(p) = \mathbf{u} \\ \mathbf{f}1 & v_3(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. For the base case, the desired result holds by the definition of v_4. For the induction step, we split the cases depending on the form of the formula A.

If A is of the form $\sim B$, then for (i), we have $I_4(A) = \mathbf{t}1$ iff $I_4(\sim B) = \mathbf{t}1$ iff $I_4(B) = \mathbf{f}1$ (by def. of I_4) iff $I_3(B) = \mathbf{f}$ (by IH) iff $I_3(\sim B) = \mathbf{t}$ (by def. of I_3) iff

$I_3(A)$=**t**. For (ii), $I_4(A)$ = **f**1 iff $I_4(\sim B)$=**f**1 iff $I_4(B)$=**t**1 (by def. of I_4) iff $I_3(B)$=**t** (by IH) iff $I_3(\sim B)$=**f** (by def. of I_3) iff $I_3(A)$=**f**.

If A is of the form $B \wedge C$, then for (i), $I_4(A)$=**t**1 iff $I_4(B \wedge C)$=**t**1 iff $I_4(B)$=**t**1 and $I_4(C)$=**t**1 (by def. of I_4) iff $I_3(B)$=**t** and $I_3(C)$=**t** (by IH) iff $I_3(B \wedge C)$=**t** (by def. of I_3) iff $I_3(A)$=**t**. For (ii), $I_4(A)$=**f**1 iff $I_4(B \wedge C)$=**f**1 iff ($I_4(B)$=**t**1 and $I_4(C)$=**f**1) or ($I_4(B)$=**f**1 and $I_4(C)$=**t**1) or ($I_4(B)$=**f**1 and $I_4(C)$=**f**1) (by def. of I_4) iff ($I_3(B)$=**t** and $I_3(C)$=**f**) or ($I_3(B)$=**f** and $I_3(C)$=**t**) or ($I_3(B)$=**f** and $I_3(C)$=**f**) (by IH) iff $I_3(B \wedge C)$=**f** (by def. of I_3) iff $I_3(A)$=**f**.

The case for disjunction is similar. □

We are now ready to prove one of the directions.

Proposition 4.1. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_1 A$ *then* $\Gamma \models_{\mathbf{WK}} A$.

Proof. Suppose $\Gamma \not\models_{\mathbf{WK}} A$. Then, there is a three-valued valuation v_3 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ such that $I_3(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_3(A) \neq \mathbf{t}$. Now, in view of (i) of Lemma 4.1, there is a four-valued valuation v_4 such that $I_4(B)$=**t**1 for all $B \in \Gamma$ and $I_4(A) \neq \mathbf{t}1$, namely $\Gamma \not\models_1 A$, as desired. □

For the other direction, we prepare another lemma.

Lemma 4.2. *For all four-valued valuation* v_4 *for* $\mathcal{L}$*, there is a three-valued valuation* v_3 *such that for all* $A \in \mathsf{Form}$*, (i)* $I_3(A) = \mathbf{t}$ *iff* $I_4(A) = \mathbf{t}1$*, and (ii)* $I_3(A) = \mathbf{f}$ *iff* $I_4(A) = \mathbf{f}1$.

Proof. Given a four-valued valuation v_4, we define v_3 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ as follows:

$$v_3(p) = \begin{cases} \mathbf{t} & v_4(p) = \mathbf{t}1 \\ \mathbf{u} & v_4(p) = \mathbf{t}0 \text{ or } v_4(p) = \mathbf{f}0 \\ \mathbf{f} & v_4(p) = \mathbf{f}1 \end{cases}$$

Then we prove the desired result by induction. □

Then, again, the proof is similar to the above case.

Proposition 4.2. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_{\mathbf{WK}} A$ *then* $\Gamma \models_1 A$

Proof. Suppose $\Gamma \not\models_1 A$. Then, there is a four-valued valuation v_4 : $\mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ such that $I_4(B)$=**t**1 for all $B \in \Gamma$ and $I_4(A) \neq \mathbf{t}1$. Now, in view of (i) of Lemma 4.2, there is a three-valued valuation v_3 such that $I_3(B)$=**t** for all $B \in \Gamma$ and $I_3(A) \neq \mathbf{t}$, namely $\Gamma \not\models_{\mathbf{WK}} A$, as desired. □

In view of the above propositions, we obtain the following.

Theorem 4.1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{WK}} A$ *iff* $\Gamma \models_1 A$.

In other words, we established that our new semantics is equivalent to the well-known three-valued semantics for the weak Kleene logic WK.

Let us now turn to the dual case of WK, namely the case for PWK in which $\mathbf{t}1$, $\mathbf{t}0$ and $\mathbf{f}0$ are taken as designated values. In fact, the proofs are basically the same with the cases for WK but with small changes, as we observe in the following.

Proposition 4.3. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_3 A$ *then* $\Gamma \models_{\mathbf{PWK}} A$.

Proof. We only note that we use (ii) of Lemma 4.1. □

Proposition 4.4. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_{\mathbf{PWK}} A$ *then* $\Gamma \models_3 A$

Proof. We only note that we use (ii) of Lemma 4.2 □

By combining these results, we obtain the following result.

Theorem 4.2. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{PWK}} A$ *iff* $\Gamma \models_3 A$.

Finally, we consider the case in which $\mathbf{t}1$ and $\mathbf{t}0$ are designated.

Lemma 4.3. *For all two-valued valuation* v_2 *for* $\mathcal{L}$, *there is a four-valued valuation* v_4 *such that for all* $A \in \mathsf{Form}$, *(i)* $I_4(A) = \mathbf{t}1$ *iff* $I_2(A) = \mathbf{t}$, *and (ii)* $I_4(A) = \mathbf{f}1$ *iff* $I_2(A) = \mathbf{f}$.

Proof. Given a two-valued valuation v_2, we define $v_4 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ as follows:

$$v_4(p) = \begin{cases} \mathbf{t}1 & v_2(p) = \mathbf{t} \\ \mathbf{f}1 & v_2(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction. □

Then, we obtain the following result.

Proposition 4.5. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_2 A$ *then* $\Gamma \models_{\mathbf{CL}} A$.

Proof. Suppose $\Gamma \not\models_{\mathbf{CL}} A$. Then, there is a two-valued valuation $v_2 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ such that $I_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_2(A) = \mathbf{f}$. Now, in view of Lemma 4.3, there is a four-valued valuation v_4 such that $I_4(B) = \mathbf{t}1$ (i.e. $I_4(B) \in \mathcal{D}_2$) for all $B \in \Gamma$ and $I_4(A) = \mathbf{f}1$ (i.e. $I_4(A) \notin \mathcal{D}_2$), namely $\Gamma \not\models_2 A$, as desired. □

For the other direction, we prepare one more lemma.

Lemma 4.4. *For all four-valued valuation* v_4 *for* $\mathcal{L}$, *there is a two-valued valuation* v_2 *such that for all* $A \in \mathsf{Form}$, *(i)* $I_2(A) = \mathbf{t}$ *iff* $I_4(A) \in \{\mathbf{t}1, \mathbf{t}0\}$, *and (ii)* $I_2(A) = \mathbf{f}$ *iff* $I_4(A) \in \{\mathbf{f}1, \mathbf{f}0\}$.

Proof. Given a four-valued valuation v_4, we define $v_2 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ as follows:

$$v_2(p)=\begin{cases}\mathbf{t} & v_4(p) \in \{\mathbf{t}1, \mathbf{t}0\}\\ \mathbf{f} & v_4(p) \in \{\mathbf{f}1, \mathbf{f}0\}\end{cases}$$

Then we prove the desired result by induction. □

Proposition 4.6. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_{\mathbf{CL}} A$ *then* $\Gamma \models_2 A$

Proof. Suppose $\Gamma \not\models_2 A$. Then, there is a four-valued valuation $v_4 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ such that $I_4(B) \in \mathcal{D}_2$ for all $B \in \Gamma$ and $I_4(A) \notin \mathcal{D}_2$. Now, in view of Lemma 4.4, there is a two-valued valuation v_2 such that $I_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_2(A) = \mathbf{f}$, namely $\Gamma \not\models_{\mathbf{CL}} A$, as desired. □

By combining these results, we obtain the following result.

Theorem 4.3. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{CL}} A$ *iff* $\Gamma \models_2 A$*.*

Remark 4.3. *The results so far show that our semantics allows us to compare three logics by a single four-valued truth table, by varying the set of designated values. This is not possible with the three-valued truth table unless we allow different sets of truth values that will allow us to define p- and q-consequence relations.*

4.2.3 Reflections on the results so far

The results so far show that our semantics allows us to compare three logics by using our four-valued truth tables for the connectives, by simply varying the set of designated values. This is not possible with the three-valued truth tables unless we pursue the *p*-consequence relation (cf. [20]), or more recently known as ST in the literature (cf. [10, 11]).

Furthermore, the four-valued semantics captures quite nicely a more refined reading of Beall's intuitive semantics: we can now properly distinguish not only 'true and on-topic' and 'false and on-topic' from off-topic sentences, but we can grant that the off-topic ones will also have their own truth value, as expected in a context where it is assumed that classical truth values are somehow maintained and attributed to each sentence, independently of issues concerning topic. That is, Beall's reading, by employing the three-valued semantics, although highly illuminating for WK, is not without problems, given that it is unable to account for such a feature of off-topic propositions. This is one of the advantages of our framework.

Now, we mentioned that the truth values have a completely classical behavior. The expansion to four combinations with on-topic and off-topic allows us to explicitly understand the working of propositions, when it comes to their truth values, completely in terms of CL. When it comes to topic relativity, on the other hand, the second component of our semantics plays the major role. In this sense, CL is still the underlying logic when we restrict ourselves exclusively to the behavior of the truth values. But that is not all: as we have showed, CL is also the resulting logic when, in the consequence relation, only truth and falsity are considered relevant, that is, when issues concerning topic are not playing any role. One can see the resulting framework as describing in general terms what happens to CL when an *addition* of further requirements of topicality is produced, and the resulting distinct systems, CL and WK, may be seen as resulting from emphasizing different aspects of the framework: to require only truth preservation, on the one hand, or truth *and* on-topic, on the other.

What this kind of reading of the framework we proposed allows us to do now opens some very interesting venues for discussion on some of the features typically said to characterize the nature of logic itself. As we have seen, the presented semantics captures the reading proposed by Beall very well; demanding on-topic preservation results in WK, and demanding mere truth preservation independently of on-topic preservation results in CL. In both cases, the connectives behave classically for the truth values and their truth conditions. *Under these circumstances*, and considering that the topic is an *addition* to the classical truth conditional apparatus, it could be claimed that CL embodies nicely the widely held idea that logic is 'topic-neutral'. That is, given that CL is the resulting system when we require that logical consequence should be 'blind' to the topics, it reflects topic neutrality. WK, on the other hand, represents a more 'material' approach to consequence relation, recording inferences that are truth preserving and are topic relative (and our generalization that follows will allow us to express this idea quite nicely).

Obviously, we are not claiming here that CL, in opposition to other systems, uniquely captures topic neutrality. Rather, what we claim is to be understood in conditional form: given the classical nature of our basic framework and of Beall's proposed reading, CL is the one that results when topics are left behind. The fact that the theories obtained by closing over CL and over WK are different may be read, as it were, as reflecting different interests concerning the role of topics: either they do play a role (WK), or else they do not (CL).

4.3 Alternative semantics (II): implementing (M2)

Let us now turn to our second motivation (M2). Again by building on Beall's three key ideas, this will amount to turn the first two ideas presented by Beall as follows (our additions are emphasized by italicizing the text).

1. A theory is about all and only what its elements — that is, the claims in the theory — are about. ***And there are n topics that the theory is about.***
2. Conjunctions, disjunctions and negations are about exactly whatever their respective subsentences are about:
   (a) Conjunction $A \wedge B$ is about exactly whatever ***topic t*** A and B are about.
   (b) Disjunction $A \vee B$ is about exactly whatever ***topic t*** A and B are about.
   (c) Negation $\sim A$ is about exactly whatever ***topic t*** A is about.
3. Theories in English are rarely about every topic expressible in English.

4.3.1 Basics

The relativisation we are suggesting can be represented by adding n valuations, instead of one valuation, by thinking that there can be n different topics that the theory under consideration will cover.

Definition 4.4. *A n-topic* model *for the language $\mathcal{L}$ is a pair $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, where $V_t : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$, $V_a : \mathsf{Prop} \to \{0, 1\}$, and $\mathfrak{A}$ is a finite and non-empty set with n elements. Valuations V_t, V_a are then extended to interpretations I_t, I_a by the following conditions:*

$I_t(p)=\mathbf{t}$ *iff* $V_t(p)=\mathbf{t}$	$I_a(p)=1$ *iff* $V_a(p)=1$
$I_t(\sim A)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{f}$	$I_a(\sim A)=1$ *iff* $I_a(A)=1$
$I_t(A \wedge B)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{t}$ *and* $I_t(B)=\mathbf{t}$	$I_a(A \wedge B)=1$ *iff* $I_a(A)=1$ *and* $I_a(B)=1$
$I_t(A \vee B)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{t}$ *or* $I_t(B)=\mathbf{t}$	$I_a(A \vee B)=1$ *iff* $I_a(A)=1$ *and* $I_a(B)=1$

Remark 4.4. *In Beall's account [3], a topic is held fixed and sentences are either on-topic or off-topic, with no possibility of distinguishing distinct topics. However, the states of on-topic and off-topic usually depend on situations. For example, in a seminar, the statements about studies are on-topic and the private statements are off-topic, while it becomes reversed in an entertainment party. In fact, when people are chatting, the main topic that they take care of changes with time normally.*

4.3.2 Some results

Definition 4.5. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$,

- $\Gamma \models_1^n A$ *iff for all n-topic models,* $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$, *if for all* $B \in \Gamma$, $I_t(B) = \mathbf{t}$ *and* $I_a(B) = 1$ *for all* $a \in \mathfrak{A}$.
- $\Gamma \models_2^n A$ *iff for all n-topic models,* $I_t(A) = \mathbf{t}$, *if for all* $B \in \Gamma$, $I_t(B) = \mathbf{t}$.
- $\Gamma \models_3^n A$ *iff for all n-topic models,* $I_t(A) \neq \mathbf{f}$ *or* $I_a(A) = 0$ *for some* $a \in \mathfrak{A}$, *if for all* $B \in \Gamma$, $I_t(B) \neq \mathbf{f}$ *or* $I_a(B) = 0$ *for some* $a \in \mathfrak{A}$.

We now turn to show the following facts.

- $\models_1^n$ **is weak Kleene logic.**
- $\models_2^n$ **is classical logic.**
- $\models_3^n$ **is paraconsistent weak Kleene logic.**

Lemma 4.5. *For all three-valued valuation* v_3 *for* $\mathcal{L}$, *there is an n-topic model* $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ *such that for all* $A \in \mathsf{Form}$,

(i) $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_3(A) = \mathbf{t}$, *and*
(ii) $I_t(A) = \mathbf{f}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_3(A) = \mathbf{f}$.

Proof. Given a three-valued valuation v_3, we define $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ as follows[1]:

$$V_t(p) := \begin{cases} \mathbf{t} & v_3(p) = \mathbf{t} \\ \mathbf{f} & v_3(p) = \mathbf{f} \\ \mathbf{t} & v_3(p) = \mathbf{u} \end{cases} \qquad V_a(p) := \begin{cases} 1 & v_3(p) \neq \mathbf{u} \\ 0 & v_3(p) = \mathbf{u} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. Since the proofs are similar, we only deal with the case for (i).

For the base case, the desired result holds by the definition of $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$. For the induction step, we split the cases depending on the form of the formula A.

[1]The value of $V_t(p)$ when $v_3(p) = \mathbf{u}$ can also be $\mathbf{f}$.

- If A is of the form $\sim B$, we have

$$\begin{aligned}
&I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{iff } I_t(\sim B) = \mathbf{t} \text{ and } I_a(\sim B) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{iff } I_t(B) = \mathbf{f} \text{ and } I_a(B) = 1 \text{ for all } a \in \mathcal{C} && \text{by def. of } I_t \text{ and } I_a \\
&\text{iff } I_3(B) = \mathbf{f} && \text{by IH} \\
&\text{iff } I_3(\sim B) = \mathbf{t} && \text{by def. of } I_3 \\
&\text{iff } I_3(A) = \mathbf{t}
\end{aligned}$$

- If A is of the form $B \wedge C$, we have

$$\begin{aligned}
&I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{iff } I_t(B \wedge C) = \mathbf{t} \text{ and } I_a(B \wedge C) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{iff } I_t(B) = I_t(C) = \mathbf{t} \text{ and } I_a(B) = I_a(C) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{iff } I_3(B) = \mathbf{t} \text{ and } I_3(C) = \mathbf{t} \\
&\text{iff } I_3(B \wedge C) = \mathbf{t} \\
&\text{iff } I_3(A) = \mathbf{t}
\end{aligned}$$

The case for disjunction is similar. □

We are now ready to prove one of the directions.

Proposition 4.7. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_1^n A$ *then* $\Gamma \models_{\mathbf{WK}} A$.

Proof. Suppose $\Gamma \not\models_{\mathbf{WK}} A$. Then, there is a three-valued valuation v_3 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ such that $I_3(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_3(A) \neq \mathbf{t}$. Now, in view of (i) of Lemma 4.5, there is an n-topic model $\langle V_t, \{V_a\}_{a \in \mathfrak{A}} \rangle$ such that $I_t(B) = \mathbf{t}$ and $I_a(B) = 1$ for all $a \in \mathcal{C}$ for all $B \in \Gamma$ and $I_t(A) \neq \mathbf{t}$ or $I_a(A) \neq 1$ for some $a \in \mathcal{C}$, namely $\Gamma \not\models_1^n A$, as desired. □

Lemma 4.6. *Given an n-topic model $\langle V_t, \{V_a\}_{a \in \mathfrak{A}} \rangle$ for $\mathcal{L}$, there is a three-valued valuation v_3 such that for all $A \in \mathsf{Form}$,*

(i) *$I_3(A)=\mathbf{t}$ iff $I_t(A) = \mathbf{t}$ and $I_a(A) = 1$ for all $a \in \mathcal{C}$, and*
(ii) *$I_3(A)=\mathbf{f}$ iff $I_t(A) = \mathbf{f}$ and $I_a(A) = 1$ for all $a \in \mathcal{C}$.*

Proof. Given an n-topic model $\langle V_t, \{V_a\}_{a \in \mathfrak{A}} \rangle$, we define $v_3 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ as follows:

$$v_3(p) := \begin{cases} \mathbf{t} & I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{f} & I_t(A) = \mathbf{f} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{u} & I_a(A) = 0 \text{ for some } a \in \mathcal{C} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. Since the proofs are similar, we only deal with the case for (i).

For the base case, the desired result holds by the definition of v_3. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then

$$
\begin{aligned}
I_3(A) = \mathbf{t} &\text{ iff } I_3(\sim B) = \mathbf{t} \\
&\text{ iff } I_3(B) = \mathbf{f} \\
&\text{ iff } I_t(B) = \mathbf{f} \text{ and } I_a(B) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{ iff } I_t(\sim B) = \mathbf{t} \text{ and } I_a(\sim B) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{ iff } I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C}
\end{aligned}
$$

- If A is of the form $B \wedge C$,

$$
\begin{aligned}
I_3(A) = \mathbf{t} &\text{ iff } I_3(B \wedge C) = \mathbf{t} \\
&\text{ iff } I_3(B) = \mathbf{t} \text{ and } I_3(C) = \mathbf{t} \\
&\text{ iff } I_t(B){=}I_t(C){=}\mathbf{t} \text{ and } I_a(B){=}I_a(C){=}1 \text{ for all } a \in \mathcal{C} \\
&\text{ iff } I_t(B \wedge C) = \mathbf{t} \text{ and } I_a(B \wedge C) = 1 \text{ for all } a \in \mathcal{C} \\
&\text{ iff } I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C}
\end{aligned}
$$

The case for disjunction is similar. □

Then, again, the proof is similar to the above case.

Proposition 4.8. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_{\mathbf{WK}} A$ *then* $\Gamma \models^n_1 A$

Proof. Suppose $\Gamma \not\models^n_1 A$. Then, there is an n-topic model $\langle V_t, \{V_a\}_{a \in \mathfrak{A}} \rangle$ such that $I_t(B) = \mathbf{t}$ and $I_a(B) = 1$ for all $a \in \mathcal{C}$ for all $B \in \Gamma$ and $I_t(A) \neq \mathbf{t}$ or $I_a(A) \neq 1$ for some $a \in \mathcal{C}$. Now, in view of (i) of Lemma 4.6, there is a three-valued valuation v_3 such that $I_3(B){=}\mathbf{t}$ for all $B \in \Gamma$ and $I_3(A){\neq}\mathbf{t}$, namely $\Gamma \not\models_{\mathbf{WK}} A$, as desired. □

In view of the above propositions, we obtain the following.

Theorem 4.4. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{WK}} A$ *iff* $\Gamma \models^n_1 A$.

Remark 4.5. *Note that by making use of (ii) of Lemmas 4.5 and 4.6, we also obtain the following result: for all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{PWK}} A$ *iff* $\Gamma \models^n_3 A$.

Lemma 4.7. *For all two-valued valuation* v_2 *for* $\mathcal{L}$*, there is an* n*-topic model* $\langle V_t, \{V_a\}_{a \in \mathfrak{A}} \rangle$ *such that for all* $A \in \mathsf{Form}$*,* $I_t(A) = I_2(A)$.

Proof. Given a two-valued valuation v_2, we define an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ as follows:

$$V_t(p) := v_2(p) \qquad V_a(p) := 1$$

Then we prove the desired result by induction on the complexity of the formula. □

Lemma 4.8. *Given an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ for $\mathcal{L}$, there is a two-valued valuation v_2 such that for all $A \in \mathsf{Form}$, $I_2(A)$=$I_t(A)$.*

Proof. Given an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, we define $v_2 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ as follows:

$$v_2(p) := V_t(p)$$

Then we prove the desired result by induction on the complexity of the formula. □

By the lemmas above, we can get the theorem following.

Theorem 4.5. $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_2^n A$ *iff* $\Gamma \models_{\mathbf{CL}} A$*.*

Proof. For the left to the right direction, suppose $\Gamma \not\models_{\mathbf{CL}} A$. Then, there is a two-valued valuation $v_2 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ such that $I_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_2(A) = \mathbf{f}$. Now, in view of Lemma 4.7, there is an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that $I_t(A) = I_2(A)$ for all $A \in \mathsf{Form}$. Therefore, we obtain that $I_t(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_t(A) = \mathbf{f}$, namely $\Gamma \not\models_2^n A$, as desired.

For the other direction, suppose $\Gamma \not\models_2^n A$. Then, there is an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that $I_t(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I(A) \neq \mathbf{t}$. Now, in view of Lemma 4.8, there is a two-valued valuation v_2 such that $I_2(A) = I_t(A)$ for all $A \in \mathsf{Form}$. Therefore, we obtain that $I_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_2(A) = \mathbf{f}$, namely $\Gamma \not\models_{\mathbf{CL}} A$, as desired. □

4.3.3 Another way to view WK

One interesting aspect of this discussion is that independently of how many distinct topics one adds, the underlying logic of truth and topic preservation is always WK. One could be led to think that topic neutrality could be reached in a straightforward way: by adding more and more topics, with the demand that logic consequence is defined as truth and on-topic preservation for all topics. That, it could be thought, would progressively make the system more and more insensible to the content specific to each such context, bringing us closer to CL and its topic neutrality.

However, that is not what happens. CL only obtains when we ***disregard*** all that is relative to a topic, and not when we bring in all that is relevant to every topic. In this sense, we could say, perhaps, in Wittgensteinian terms, that a logic that preserves truth and on-topic property is, somehow, more informative, given that it takes the topic into account, while CL, by disregarding topics, is not informative, because it holds in every topic, independently of what the topic is about.

One could perhaps also speculate whether it makes any difference for the resulting logic whether one demands that one topic is preserved, two, or all of them. The answer is that it doesn't. Our next result establishes that. In intuitive terms, it says that topic neutrality is a matter of all or nothing: either one chooses that no topic will have an influence, and has CL for that, or else one chooses at least one topic, and that is already enough to give us WK.

Given a set $\mathcal{G} \subseteq \mathcal{C}$ such that $\mathcal{G} \neq \emptyset$, we may consider the following definition of a consequence relation.

Definition 4.6. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_1^{n,\mathcal{G}} A$ *iff for all n-topic models,* $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{G}$ *if for all* $B \in \Gamma$, $I_t(B) = \mathbf{t}$ *and* $I_a(B) = 1$ *for all* $a \in \mathcal{G}$.

Then, a careful inspection of the proofs reveals that nothing specific about $\mathcal{C}$ played a role in the proofs. Therefore, by repeating the proof with suitable modifications will establish the following theorem.

Theorem 4.6. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$ *and* $\mathcal{G} \subseteq \mathfrak{A}$ *where* $\mathcal{G} \neq \emptyset$, $\Gamma \models_{\mathbf{WK}} A$ *iff* $\Gamma \models_1^{n,\mathcal{G}} A$.

In other words, WK is also obtained not only by preserving ***all*** the topics, but also by preserving ***some*** of the topics as well.

4.4 A new interpretation of the logic of Catuskoti

As we have seen, there is a sense to be made that logic is topic neutral in the context of our interpretation, if we choose our designated values so that truth is preserved independently of topics. Given that it is CL that governs the behavior of the connectives in the context of the framework we have been discussing so far, it is only natural that the resulting logic when topics are ignored is CL.

4.4.1 Basics

We now turn to reflect the above considerations in a slightly formal terms. The models will have the four-valued FDE-valuations, in place of the two-valued classical valuations.

Definition 4.7. *An* **FDE**-based n-topic model *for the language* $\mathcal{L}$ *is a pair* $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, *where* V_t : $\mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, V_a : $\mathsf{Prop} \longrightarrow \{0, 1\}$, *and* $\mathfrak{A}$ *is a finite and non-empty set with* n *elements. Valuations* V_t *are extended to interpretations* I_t *by the usual manner for* **FDE**, *and valuations* V_a *are then extended to interpretations* I_a *by the following conditions:*

$$\begin{array}{l} I_a(p)=1 \textit{ iff } V_a(p)=1 \\ I_a(\sim A)=1 \textit{ iff } I_a(A)=1 \\ I_a(A \wedge B)=1 \textit{ iff } I_a(A)=1 \textit{ and } I_a(B)=1 \\ I_a(A \vee B)=1 \textit{ iff } I_a(A)=1 \textit{ and } I_a(B)=1 \end{array}$$

Based on these models, we define two kinds of semantical consequence relation, one focusing on the preservation of truth, and the other also preserving the topics.

Definition 4.8. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$,

- $\Gamma \models^{4,n}_1 A$ *iff for all* **FDE**-*based* n-*topic models,* $I_t(A) \in \{\mathbf{t}, \mathbf{b}\}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *if for all* $B \in \Gamma$, $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$ *and* $I_a(B) = 1$ *for all* $a \in \mathcal{C}$.
- $\Gamma \models^{4,n}_2 A$ *iff for all* **FDE**-*based* n-*topic models,* $I_t(A) \in \{\mathbf{t}, \mathbf{b}\}$ *if for all* $B \in \Gamma$, $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$.

Remark 4.6. *We may also consider the more tolerant consequence relation, defined as follows:* $\Gamma \models^n_3 A$ *iff for all* n*-topic models,* $I_t(A) \neq \mathbf{f}$ *or* $I_a(A) = 0$ *for some* $a \in \mathcal{C}$ *if for all* $B \in \Gamma$, $I_t(B) \neq \mathbf{f}$ *or* $I_a(B) = 0$ *for some* $a \in \mathcal{C}$. *However, for the purpose of saving some space, we will focus on the above two consequence relations.*

4.4.2 Some results

We will now turn to connect the above consequence relations to those known in the literature. More specifically, we will observe that our consequence relations are equivalent to those for the logic of Catuskoti FDE$_\varphi$ and FDE, respectively.

- $\models^{4,n}_1$ **is FDE$_\varphi$.**
- $\models^{4,n}_2$ **is FDE.**

We will first deal with the case for $\mathbf{FDE}_\varphi$.

Lemma 4.9. *For all five-valued valuation v_5 for $\mathcal{L}$, there is an n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that for all $A \in \mathsf{Form}$,*

(i) $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_5(A) = \mathbf{t}$, *and*
(ii) $I_t(A) = \mathbf{b}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_5(A) = \mathbf{b}$, *and*
(iii) $I_t(A) = \mathbf{n}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_5(A) = \mathbf{n}$, *and*
(iv) $I_t(A) = \mathbf{f}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$ *iff* $I_5(A) = \mathbf{f}$.

Proof. Given a five-valued valuation v_5, we define an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ as follows[2]:

$$V_t(p) := \begin{cases} \mathbf{t} & v_5(p) = \mathbf{t} \\ \mathbf{b} & v_5(p) = \mathbf{b} \\ \mathbf{n} & v_5(p) = \mathbf{n} \\ \mathbf{f} & v_5(p) = \mathbf{f} \\ \mathbf{t} & v_5(p) = \mathbf{e} \end{cases} \qquad V_a(p) := \begin{cases} 1 & v_5(p) \neq \mathbf{e} \\ 0 & v_5(p) = \mathbf{e} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. Since the proofs are similar, we only deal with the case for (i).

For the base case, the desired result holds by the definition of $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, we have

$$\begin{aligned} & I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_t(\sim B) = \mathbf{t} \text{ and } I_a(\sim B) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_t(B) = \mathbf{f} \text{ and } I_a(B) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_5(B) = \mathbf{f} \\ & \text{iff } I_5(\sim B) = \mathbf{t} \\ & \text{iff } I_5(A) = \mathbf{t} \end{aligned}$$

- If A is of the form $B \wedge C$, we have

$$\begin{aligned} & I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_t(B \wedge C) = \mathbf{t} \text{ and } I_a(B \wedge C) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_t(B) = I_t(C) = \mathbf{t} \text{ and } I_a(B) = I_a(C) = 1 \text{ for all } a \in \mathcal{C} \\ & \text{iff } I_5(B) = \mathbf{t} \text{ and } I_5(C) = \mathbf{t} \\ & \text{iff } I_5(B \wedge C) = \mathbf{t} \\ & \text{iff } I_5(A) = \mathbf{t} \end{aligned}$$

[2]The value of $V_t(p)$ when $v_5(p) = \mathbf{e}$ can also be other values.

The case for disjunction is similar. □

Proposition 4.9. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_1^{4,n} A$ *then* $\Gamma \models_{\mathbf{FDE}_\varphi} A$.

Proof. Suppose $\Gamma \not\models_{\mathbf{FDE}_\varphi} A$. Then, there is a five-valued valuation v_5 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}, \mathbf{e}\}$ such that $I_5(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I_5(A) \notin \{\mathbf{t}, \mathbf{b}\}$. Now, in view of (i) and (ii) of Lemma 4.9, there is an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$ and $I_a(B) = 1$ for all $a \in \mathcal{C}$ for all $B \in \Gamma$ and $I_t(A) \notin \{\mathbf{t}, \mathbf{b}\}$ or $I_a(A) \neq 1$ for some $a \in \mathcal{C}$, namely $\Gamma \not\models_1^{4,n} A$, as desired. □

Lemma 4.10. *Given an* **FDE**-*based* n-*topic model* $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ *for* $\mathcal{L}$, *there is a five-valued valuation* v_5 *such that for all* $A \in \mathsf{Form}$,

(i) $I_5(A){=}\mathbf{t}$ *iff* $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$, *and*
(ii) $I_5(A){=}\mathbf{b}$ *iff* $I_t(A) = \mathbf{b}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$, *and*
(iii) $I_5(A){=}\mathbf{n}$ *iff* $I_t(A) = \mathbf{n}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$, *and*
(iv) $I_5(A){=}\mathbf{f}$ *iff* $I_t(A) = \mathbf{f}$ *and* $I_a(A) = 1$ *for all* $a \in \mathcal{C}$.

Proof. Given an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, we define v_5 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}, \mathbf{e}\}$ as follows:

$$v_5(p) := \begin{cases} \mathbf{t} & V_t(p) = \mathbf{t} \text{ and } V_a(p) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{b} & V_t(p) = \mathbf{t} \text{ and } V_a(p) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{n} & V_t(p) = \mathbf{t} \text{ and } V_a(p) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{f} & V_t(p) = \mathbf{f} \text{ and } V_a(p) = 1 \text{ for all } a \in \mathcal{C} \\ \mathbf{e} & V_a(p) = 0 \text{ for some } a \in \mathcal{C} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. Since the proofs are similar, we only deal with the case for (i).

For the base case, the desired result holds by the definition of v_3. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then

$$\begin{aligned} I_5(A) = \mathbf{t} &\text{ iff } I_5(\sim B) = \mathbf{t} \\ &\text{ iff } I_5(B) = \mathbf{f} \\ &\text{ iff } I_t(B) = \mathbf{f} \text{ and } I_a(B) = 1 \text{ for all } a \in \mathcal{C} \\ &\text{ iff } I_t(\sim B) = \mathbf{t} \text{ and } I_a(\sim B) = 1 \text{ for all } a \in \mathcal{C} \\ &\text{ iff } I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \end{aligned}$$

- If A is of the form $B \wedge C$,

$$\begin{aligned} I_5(A) = \mathbf{t} &\text{ iff } I_5(B \wedge C) = \mathbf{t} \\ &\text{ iff } I_5(B) = \mathbf{t} \text{ and } I_5(C) = \mathbf{t} \\ &\text{ iff } I_t(B)=I_t(C)=\mathbf{t} \text{ and } I_a(B)=I_a(C)=1 \text{ for all } a \in \mathcal{C} \\ &\text{ iff } I_t(B \wedge C) = \mathbf{t} \text{ and } I_a(B \wedge C) = 1 \text{ for all } a \in \mathcal{C} \\ &\text{ iff } I_t(A) = \mathbf{t} \text{ and } I_a(A) = 1 \text{ for all } a \in \mathcal{C} \end{aligned}$$

The case for disjunction is similar. □

Proposition 4.10. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_{\mathbf{FDE}_\varphi} A$ *then* $\Gamma \models_1^{4,n} A$

Proof. Suppose $\Gamma \not\models_1^{4,n} A$. Then, there is an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, such that $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$ and $I_a(B) = 1$ for all $a \in \mathcal{C}$ for all $B \in \Gamma$ and $I_t(A) \notin \{\mathbf{t}, \mathbf{b}\}$ or $I_a(A) \neq 1$ for some $a \in \mathcal{C}$. Now, in view of (i) and (ii) of Lemma 4.10, there is a five-valued valuation v_5 such that $I_5(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I_5(A) \notin \{\mathbf{t}, \mathbf{b}\}$, namely $\Gamma \not\models_{\mathbf{FDE}_\varphi} A$, as desired. □

In view of the above propositions, we obtain the following.

Theorem 4.7. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{FDE}_\varphi} A$ *iff* $\Gamma \models_1^{4,n} A$*.*

Remark 4.7. *In view of this result, we may also interpret* $\mathbf{FDE}_\varphi$ *in a rather different manner from Priest's interpretation presented in [33]. On the one hand, for the purpose of reflecting the reading in terms of* emptiness *for the value* **e***, the construction through the framework of plurivalent semantics due to Priest himself will be more suitable. On the other hand, based on our semantics and the result above,* $\mathbf{FDE}_\varphi$ *can be understood in a completely different way, that is, as a logic introduced via* **FDE***-based* n*-topic models, defining the consequence relation in terms of the preservation of on-topic truth.*

We now turn to the case for the second consequence relation. Again, the proof will be basically the same with the case we treated CL, but with some changes as expected.

Lemma 4.11. *For all four-valued valuation* v_4 *for* $\mathcal{L}$*, there is there is an* **FDE***-based* n*-topic model* $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ *such that for all* $A \in \mathsf{Form}$*,* $I_t(A) = I_4(A)$*.*

Proof. Given a four-valued valuation v_4, we define an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ as follows:

$$V_t(p) := v_4(p) \qquad V_a(p) := 1$$

Then we prove the desired result by induction on the complexity of the formulas. □

Lemma 4.12. *Given an* **FDE***-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ for $\mathcal{L}$, there is a four-valued valuation v_4 such that for all $A \in \mathsf{Form}$, $I_4(A)=I_t(A)$.*

Proof. Given an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$, we define $v_4 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ as follows:

$$v_4(p) := V_t(p).$$

Then we prove the desired result by induction on the complexity of the formula. □

Theorem 4.8. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{FDE}} A$ iff $\Gamma \models_2^{4,n} A$.*

Proof. For the left to the right direction, suppose $\Gamma \not\models_{\mathbf{FDE}} A$. Then, there is a four-valued valuation $v_4 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ such that $I_4(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I_4(A) \notin \{\mathbf{t}, \mathbf{b}\}$. Now, in view of Lemma 4.11, there is an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I(A) \notin \{\mathbf{t}, \mathbf{b}\}$, namely $\Gamma \not\models_2^{4,n} A$, as desired.

For the other direction, suppose $\Gamma \not\models_2^{4,n} A$. Then, there is an **FDE**-based n-topic model $\langle V_t, \{V_a\}_{a\in\mathfrak{A}}\rangle$ such that $I_t(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I(A) \notin \{\mathbf{t}, \mathbf{b}\}$. Now, in view of Lemma 4.12, there is a four-valued valuation v_4 such that $I_4(B) \in \{\mathbf{t}, \mathbf{b}\}$ for all $B \in \Gamma$ and $I_4(A) \notin \{\mathbf{t}, \mathbf{b}\}$, namely $\Gamma \not\models_{\mathbf{FDE}} A$, as desired. □

4.5 Beyond FDE family

A careful inspection of the proofs of the previous section shows that we may take K3 or LP, instead of FDE, for all the results. This may then make us wonder to which extent we can generalize the results we have established so far following the themes from Beall. The aim of this section is precisely to address this question.

4.5.1 Basics

First, we recall a rather general way, described by Priest, to obtain infectious logic building on a given many-valued semantics.

Definition 4.9 (Priest). *Let $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ be a many-valued semantics for $\mathcal{L}$. Then we define a* many-valued infectious semantics based on M, *$M^{\mathbf{e}} = \langle \mathcal{V}^{\mathbf{e}}, \mathcal{D}^{\mathbf{e}}, \delta^{\mathbf{e}} \rangle$ for $\mathcal{L}$ as follows:*

- $\mathcal{V}^{\mathbf{e}} = \mathcal{V} \cup \{\mathbf{e}\}$,
- $\mathcal{D}^{\mathbf{e}} = \mathcal{D}$,
- *$\delta^{\mathbf{e}}$ contains a function $\delta^{\mathbf{e}}_*$ that extends, if needed, each δ_* by letting $\delta^{\mathbf{e}}_*(v^{\mathbf{e}}_1, \ldots, v^{\mathbf{e}}_n) = \mathbf{e}$ iff $v^{\mathbf{e}}_i = \mathbf{e}$ for some $v^{\mathbf{e}}_i \in \mathcal{V}^{\mathbf{e}}$. Otherwise, $\delta^{\mathbf{e}}_* = \delta_*$.*

Remark 4.8. *As one can see, this is a generalization of both* **WK** *and* $\mathbf{FDE}_\varphi$. *Indeed, by taking the many-valued semantics to be the two-valued semantics for* **CL** *and the four-valued semantics for* **FDE**, *respectively, then the resulting many-valued infectious semantics will be those for* **WK** *and* $\mathbf{FDE}_\varphi$, *respectively.*

Then, note that the results established by Priest in [34] show that plurivalent semantics will allow us to make sense of the many-valued infectious semantics based on M when M is one of the many-valued semantics for the FDE-family. Our aim in this section is to provide another way to interpret the many-valued infectious semantics based on M in terms of on-topic/off-topic reading along Beall. Interestingly, our method works for *any* many-valued semantics, not restricted to the FDE-family. In what follows, we will focus on one topic case, although the same construction will also work well with more than one topic.

Definition 4.10. *Given a many-valued semantics $M = \langle \mathcal{V}, \mathcal{D}, \delta \rangle$ for the language $\mathcal{L}$, an* M-based topic model *for the language $\mathcal{L}$ is a triple $\langle M, V_t, V_a \rangle$, where V_t : $\mathsf{Prop} \longrightarrow \mathcal{V}$, V_a : $\mathsf{Prop} \longrightarrow \{0, 1\}$. Valuations V_t, V_a are then extended to interpretations I_t, I_a by the following conditions:*

$I_t(p)=V_t(p)$	$I_a(p)=1$ *iff* $V_a(p)=1$
$I_t(\sim A)=\delta_\sim(I_t(A))$	$I_a(\sim A)=1$ *iff* $I_a(A)=1$
$I_t(A \wedge B)=\delta_\wedge(I_t(A), I_t(B))$	$I_a(A \wedge B)=1$ *iff* $I_a(A)=1$ *and* $I_a(B)=1$
$I_t(A \vee B)=\delta_\vee(I_t(A), I_t(B))$	$I_a(A \vee B)=1$ *iff* $I_a(A)=1$ *and* $I_a(B)=1$

We will again define two kinds of consequence relations.

Definition 4.11. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$,*

- *$\Gamma \models^M_1 A$ iff for all M-based n-topic models, $I_t(A) \in \mathcal{D}$ and $I_a(A) = 1$ if for all $B \in \Gamma$, $I_t(B) \in \mathcal{D}$ and $I_a(B) = 1$.*
- *$\Gamma \models^M_2 A$ iff for all M-based n-topic models, $I_t(A) \in \mathcal{D}$ if for all $B \in \Gamma$, $I_t(B) \in \mathcal{D}$.*

4.5.2 Relating to the semantics with an infectious value

We will now turn to relate the first semantic consequence relation $\models_1^M$ to the semantic with an infectious value.

Lemma 4.13. *For all interpretation $\langle M^{\mathbf{e}}, \mu\rangle$, there is an M-based topic model $\langle M, V_t, V_a\rangle$ such that*

(i) *if $\mu(A) \neq \mathbf{e}$, then $I_t(A) = \mu(A)$ and $I_a(A) = 1$, and*
(ii) *if $\mu(A) = \mathbf{e}$, then $I_a(A) = 0$.*

Proof. Note first that for all interpretation, there is at least one designated value. In what follows, we will refer to this value as $\mathbf{d}$.

Given an interpretation $\langle M^{\mathbf{e}}, \mu\rangle$, we define a M-based topic model $\langle M, V_t, V_a\rangle$ as follows:

$$V_t(p)=\begin{cases}\mu(p) & \text{if } \mu(p) \neq \mathbf{e}\\ \mathbf{d} & \text{if } \mu(p) = \mathbf{e}\end{cases} \qquad V_a(p)=\begin{cases}1 & \text{if } \mu(p) \neq \mathbf{e}\\ 0 & \text{if } \mu(p) = \mathbf{e}\end{cases}$$

Then we prove the desired results by induction on the complexity of the formula.

Ad (i) For the base case, the desired result holds by the definition of V_t and V_a. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then assume $\mu(A) \neq \mathbf{e}$. Given this assumption, note that we obtain $\mu(B) \neq \mathbf{e}$. Indeed, if $\mu(B) = \mathbf{e}$, then $\mu(A) = \mu(\sim B) = \delta^*_\sim(\mu(B)) = \mathbf{e}$ by the definition of δ^*. It then remains to show that $I_t(A) = \mu(A)$ and $I_a(A) = 1$ which can be checked as follows.

$$\begin{aligned} I_t(A) = I_t(\sim B) &= \delta_\sim(I_t(B)) && \text{(by def. of } I_t)\\ &= \delta_\sim(\mu(B)) && \text{(by IH)}\\ &= \mu(\sim B) && \text{(by def. of } \mu)\\ &= \mu(A). \end{aligned}$$

  Moreover, since $\mu(B)\neq\mathbf{e}$, we have $I_a(B)=1$ by IH, and thus $I_a(\sim B)=1$, i.e. $I_a(A)=1$.
- If A is of the form $B\wedge C$, then assume $\mu(A) \neq \mathbf{e}$. Given this assumption, note that we obtain $\mu(B) \neq \mathbf{e}$ and $\mu(C) \neq \mathbf{e}$. Indeed, if $\mu(B) = \mathbf{e}$ or $\mu(C) = \mathbf{e}$, then $\mu(A) = \mu(B \wedge C) = \delta^*_\wedge(\mu(B), \mu(C)) = \mathbf{e}$ by the definition of δ^*. It then remains to show that $I_t(A) = \mu(A)$ and

$I_a(A) = 1$ which can be checked as follows.

$$\begin{aligned} I_t(A) = I_t(B \wedge C) &= \delta_\wedge(I_t(B), I_t(C)) && \text{(by def. of } I_t) \\ &= \delta_\wedge(\mu(B), \mu(C)) && \text{(by IH)} \\ &= \mu(B \wedge C) && \text{(by def. of } \mu) \\ &= \mu(A). \end{aligned}$$

Moreover, since $\mu(B){\neq}\mathbf{e}$ and $\mu(C){\neq}\mathbf{e}$, we have $I_a(B)$=1 and $I_a(C)$=1 by IH, and thus $I_a(B \wedge C)$=1, i.e. $I_a(A)$=1.

The case for disjunction is similar.

Ad (ii) For the base case, the desired result holds by the definition of V_t and V_a. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then assume $\mu(A) = \mathbf{e}$. Given this assumption, note that we obtain $\mu(B) = \mathbf{e}$. Indeed, if $\mu(B) \neq \mathbf{e}$, then by the definition of δ^*, we obtain $\mu(A) \neq \mathbf{e}$. Therefore, by IH, we obtain $I_a(B) = 0$. Therefore, by the definition of I_a, we obtain $I_a(A) = I_a({\sim} B) = 0$, as desired.
- If A is of the form $B \wedge C$, then assume $\mu(A) = \mathbf{e}$. Given this assumption, note that we obtain $\mu(B) = \mathbf{e}$ or $\mu(C) = \mathbf{e}$. Indeed, if $\mu(B) \neq \mathbf{e}$ and $\mu(C) \neq \mathbf{e}$, then by the definition of δ^*, we obtain $\mu(A) \neq \mathbf{e}$. Therefore, by IH, we obtain $I_a(B) = 0$ or $I_a(C) = 0$. Therefore, by the definition of I_a, we obtain $I_a(A) = I_a(B \wedge C) = 0$, as desired.

Again, the case for disjunction is similar. □

Proposition 4.11. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_1^M A$ then $\Gamma \models^{M^\mathbf{e}} A$.*

Proof. Suppose $\Gamma \not\models^{M^\mathbf{e}} A$. Then, there is a interpretation $\langle M^\mathbf{e}, \mu_0 \rangle$ such that $\mu_0(B) \in \mathcal{D}$ for all $B \in \Gamma$ and $\mu_0(A) \notin \mathcal{D}$. Then, by making use of (i) of Lemma 4.13, for an M-based topic model $\langle M, V_t, V_a \rangle$, $\mu_0(B) \in \mathcal{D}$ for all $B \in \Gamma$ implies that $I_t(B) \in \mathcal{D}$ and $I_a(B) = 1$ for all $B \in \Gamma$. Moreover, by making use of both (i) and (ii) of Lemma 4.13, we obtain that $\mu_0(A) \notin \mathcal{D}$ implies $I_t(A) \notin \mathcal{D}$ or $I_a(A) \neq 1$. Indeed, if $\mu(A) \neq \mathbf{e}$, then we use (i) to obtain $I_t(A) \notin \mathcal{D}$, and if $\mu(A) = \mathbf{e}$, then we use (ii) to obtain $I_a(A) \neq 1$. Therefore, we obtain that $\Gamma \not\models_1^M A$, as desired. □

Lemma 4.14. *For all M-based topic model $\langle M, V_t, V_a \rangle$, there is an interpretation $\langle M^\mathbf{e}, \mu \rangle$ such that for all $A \in \mathsf{Form}$,*

(i) *if $I_a(A) = 1$, then $\mu(A) = I_t(A)$, and*
(ii) *if $I_a(A) = 0$, then $\mu(A) = \mathbf{e}$.*

Proof. Given a M-based topic model $\langle M, V_t, V_a\rangle$, we define an interpretation $\langle M^{\mathbf{e}}, \mu\rangle$ as follows:

$$\mu(p)=\begin{cases} V_t(p) & \text{if } V_a(p)=1 \\ \mathbf{e} & \text{if } V_a(p)=0 \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula.

Ad (i) For the base case, the desired result holds by the definition of μ. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then assume $I_a(A) = 1$. Then, by the definition of I_a, we obtain that $I_a(B) = 1$. Therefore, by IH, we have that $\mu(B) = I_t(B)$, and thus $\mu(A) = \mu(\sim B) = \delta_{\sim}(\mu(B)) = \delta_{\sim}(I_t(B)) = I_t(\sim B) = I_t(A)$, as desired.
- If A is of the form $B \wedge C$, then assume $I_a(A) = 1$. Then, by the definition of I_a, we obtain that $I_a(B) = 1$ and $I_a(C) = 1$. Therefore, by IH, we have that $\mu(B) = I_t(B)$ and $\mu(C) = I_t(C)$, and thus $\mu(A) = \mu(B \wedge C) = \delta_{\wedge}(\mu(B), \mu(C)) = \delta_{\wedge}(I_t(B), I_t(C)) = I_t(B \wedge C) = I_t(A)$, as desired.

The case for disjunction is similar.

Ad (ii) For the base case, the desired result holds by the definition of μ. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then assume $I_a(A) = 0$. Then, by the definition of I_a, we obtain that $I_a(B) = 0$. Therefore, by IH, we have that $\mu(B) = \mathbf{e}$, and thus $\mu(A) = \mu(\sim B) = \delta_{\sim}(\mu(B)) = \delta_{\sim}(\mathbf{e}) = \mathbf{e}$, as desired.
- If A is of the form $B \wedge C$, then assume $I_a(A) = 0$. Then, by the definition of I_a, we obtain that $I_a(B) = 0$ or $I_a(C) = 0$. Therefore, by IH, we have that $\mu(B) = \mathbf{e}$ or $\mu(C) = \mathbf{e}$, and thus $\mu(A) = \mu(B \wedge C) = \delta_{\wedge}(\mu(B), \mu(C)) = \mathbf{e}$, as desired.

The case for disjunction is similar. □

Proposition 4.12. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models^{M^{\mathbf{e}}} A$ *then* $\Gamma \models^{M}_{1} A$.

Proof. Suppose $\Gamma \not\models^{M}_{1} A$. Then, there is an M-based topic model $\langle M, V_{t_0}, V_{a_0}\rangle$ such that $I_{t_0}(B) \in \mathcal{D}$ and $I_{a_0}(B) = 1$ for all $B \in \Gamma$, and $I_{t_0}(A) \notin \mathcal{D}$ or $I_{a_0}(A) \neq 1$. Then, by making use of (i) of Lemma 4.14, $I_{t_0}(B) \in \mathcal{D}$ and $I_{a_0}(B) = 1$ for all $B \in \Gamma$ imply that $\mu_0(B) \in \mathcal{D}$ for all

$B \in \Gamma$, and by making use of both (i) and (ii) of Lemma 4.14, $I_{t_0}(A) \notin \mathcal{D}$ or $I_{a_0}(A) \neq 1$ implies that $\mu_0(A) \notin \mathcal{D}$. That is, $\Gamma \not\models^{M^{\mathsf{e}}} A$, as desired. □

By combining Propositions 4.11 and 4.12, we obtain the following.

Theorem 4.9. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_1^M A$ *iff* $\Gamma \models^{M^{\mathsf{e}}} A$.

Remark 4.9. *This result can be interpreted as the wide applicability of Beall's reading involving on-topic/off-topic. More specifically, as far as the theory, in Beall's sense, is closed under a many-valued consequence relation, then we may tell the same story as Beall did in [3].*

Besides the two examples we already examined in some details, namely **WK** *and* $\mathbf{FDE}_\varphi$, *we can add two more examples by looking at the four-valued semantics. That is, Oller's logic can be seen as based on* **LP***-based topic models, and Ciuni, Ferguson, and Szmuc's semantics can be seen as* **WK** *or* **PWK***-based topic models, depending on the value* $n1$ *being designated or not. It is of course a separate issue if these ways of making sense of the semantics will serve for similar philosophical applications. It may well be the case that the framework of the plurivalent semantics will be more suitable for certain applications.*

4.5.3 Relating to the semantics without an infectious value

We now turn to relate the second semantic consequence relation to the original semantics M**.**

Lemma 4.15. *For all interpretation* $\langle M, \mu \rangle$, *there is an* M*-based topic model* $\langle M, V_t, V_a \rangle$ *such that for all* $A \in \mathsf{Form}$, *(i)* $I_t(A) = \mu(A)$ *and (ii)* $I_a(A) = 1$.

Proof. Given an interpretation $\langle M, \mu \rangle$, we define an M-based topic model $\langle M, V_t, V_a \rangle$ as follows:

$$V_t(p) := \mu(p) \text{ and } V_a(p) := 1.$$

Then we prove the desired result by induction on the complexity of the formula. For the base case, the desired result holds by the definition of V_t and V_a. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then

$$\begin{aligned} I_t(A) = I_t(\sim B) &= \delta_\sim(I_t(B)) && \text{(by def. of } I_t) \\ &= \delta_\sim(\mu(B)) && \text{(by IH)} \\ &= \mu(\sim B) && \text{(by def. of } \mu) \\ &= \mu(A). \end{aligned}$$

  Moreover, we have $I_a(B)$=1 by IH, and thus $I_a(\sim B)$=1, i.e. $I_a(A)$=1.
- If A is of the form $B \wedge C$, then

$$\begin{aligned} I_t(A) = I_t(B \wedge C) &= \delta_\wedge(I_t(B), I_t(C)) && \text{(by def. of } I_t) \\ &= \delta_\wedge(\mu(B), \mu(C)) && \text{(by IH)} \\ &= \mu(B \wedge C) && \text{(by def. of } \mu) \\ &= \mu(A). \end{aligned}$$

  Moreover, we have $I_a(B)$=1 and $I_a(C)$=1 by IH, and thus $I_a(B \wedge C)$=1, i.e. $I_a(A)$=1.

The case for disjunction is similar. □

Proposition 4.13. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_2^M A$ *then* $\Gamma \models^M A$.

Proof. Suppose $\Gamma \not\models^M A$. Then, there is an interpretation $\langle M, \mu_0 \rangle$ such that $\mu_0(B) \in \mathcal{D}$ for all $B \in \Gamma$ and $\mu_0(A) \notin \mathcal{D}$. Then, by making use of Lemma 4.15, for some M-based topic model $\langle M, V_t, V_a \rangle$, $\mu_0(B) \in \mathcal{D}$ for all $B \in \Gamma$ implies that $I_t(B) \in \mathcal{D}$ (and $I_a(B) = 1$) for all $B \in \Gamma$. Moreover, we obtain that $\mu_0(A) \notin \mathcal{D}$ implies $I_t(A) \notin \mathcal{D}$ (and $I_a(A) = 1$). Therefore, we obtain that $\Gamma \not\models_2^M A$, as desired. □

Lemma 4.16. *For all M-based topic model $\langle M, V_t, V_a \rangle$, there is an interpretation $\langle M, \mu \rangle$ such that for all $A \in \mathsf{Form}$, $\mu(A) = I_t(A)$.*

Proof. Given a M-based topic model $\langle M, V_t, V_a \rangle$, we define an interpretation $\langle M^{\mathbf{e}}, \mu \rangle$ as follows:

$$\mu(p) := V_t(p)$$

Then we prove the desired result by induction on the complexity of the formula. For the base case, the desired result holds by the definition of μ. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, then

$$
\begin{aligned}
\mu(A) = \mu(\sim B) &= \delta_{\sim}(\mu(B)) && \text{(by def. of } \mu) \\
&= \delta_{\sim}(I_t(B)) && \text{(by IH)} \\
&= I_t(\sim B) && \text{(by def. of } I_t) \\
&= \mu(A).
\end{aligned}
$$

- If A is of the form $B \wedge C$, then

$$
\begin{aligned}
\mu(A) = \mu(B \wedge C) &= \delta_{\wedge}(\mu(B), \mu(C)) && \text{(by def. of } \mu) \\
&= \delta_{\wedge}(I_t(B), I_t(C)) && \text{(by IH)} \\
&= I_t(B \wedge C) && \text{(by def. of } I_t) \\
&= I_t(A).
\end{aligned}
$$

The case for disjunction is similar. □

Proposition 4.14. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models^M A$ *then* $\Gamma \models_2^M A$.

Proof. Suppose $\Gamma \not\models_2^M A$. Then, there is an M-based topic model $\langle M, V_{t_0}, V_{a_0} \rangle$ such that $I_{t_0}(B) \in \mathcal{D}$ for all $B \in \Gamma$, and $I_{t_0}(A) \notin \mathcal{D}$. Then, by making use of Lemma 4.16, $I_{t_0}(B) \in \mathcal{D}$ for all $B \in \Gamma$ imply that $\mu_0(B) \in \mathcal{D}$ for all $B \in \Gamma$, and $I_{t_0}(A) \notin \mathcal{D}$ implies that $\mu_0(A) \notin \mathcal{D}$. That is, $\Gamma \not\models^M A$, as desired. □

By combining Propositions 4.13 and 4.14, we obtain the following.

Theorem 4.10. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_2^M A$ *iff* $\Gamma \models^M A$.

Remark 4.10. *This result may be interpreted as showing the advantage of M-based topic models being able to capture the original many-valued semantics M, unlike the many-valued infectious semantics $M^{\mathbf{e}}$ which requires to go non-Tarskian to recover some part of the semantic consequence relation based in M.*

4.6 Some expansions

One of the advantages of the new semantics can be observed through the applicability of our framework to expansions of $\mathcal{L}$. We will here focus on two connectives, characterized by the following truth tables, discussed by Bochvar in [7, p.91] and Halldén in [21, p.47], respectively:

A	$\mathsf{T}A$	$+A$
t	**t**	**t**
u	**f**	**f**
f	**f**	**t**

Now, these connectives seem to resist treatments in the negative plurivalent semantics. In contrast, we have a very straightforward way to capture both connectives in our new semantic framework. First, consider the following conditions in the most simple semantics M1:

- $I_t(\mathsf{T}A) = \mathbf{t}$ iff $I_t(A) = \mathbf{t}$ and $I_m(A) = 1$,
- $I_m(\mathsf{T}A) = 1$ for all $A \in \mathsf{Form}$.
- $I_t(+A) = \mathbf{t}$ iff $I_m(A) = 1$,
- $I_m(+A) = 1$ for all $A \in \mathsf{Form}$.

Equivalently, we obtain the following four-valued truth table:

A	$\mathsf{T}A$	$+A$
t1	**t**1	**t**1
t0	**f**1	**f**1
f0	**f**1	**f**1
f1	**f**1	**t**1

Then, the proof of establishing the equivalence of the original three-valued semantics and our semantics will carry over to the expanded cases. Moreover, we can obtain n connectives in the n-topic model which can be written as $+_a, +_b, ...$ or $\mathsf{T}_a, \mathsf{T}_b$ where $a, b \in \mathfrak{A}$.

Chapter 5

Semantics for multi-agent system

In this chapter, we provide many-valued semantics for multi-agent system. In Section 5.1, we propose the intuitive idea behind the semantics. In Sections 5.2 and 5.3, we introduce two pair semantics for two readings of knowledge. In Section 5.4, we propose a binary relation between agents. Finally, in Section 5.5, we combine the two semantics to express the epistemic states of multi-agent system, and then introduce two kinds of agent communication in our semantics and give an example.

5.1 Idea behind the semantics

In Chapter 4, we proposed several semantics based on Beall's off-topic interpretation. For our aim in to build the semantics for multi-agent, it is a very natural idea to consider ***n-agent*** instead of ***n-topic***. Moreover, as the reading ***off-topic*** (or ***private*** we introduced in Chapter 3) of the additional value is ***infectious***, we need to find an ***infectious*** epistemic state of agent.

Normally, we consider epistemic states as ***known*** and ***unknown***, while the readings are sometimes ambiguous. For example, consider the following situation,

Example 5.1. *Let p, q are two propositions. p is known to agent a while q is unknown to a. Assume that p is true, is $p \vee q$ known to agent a?*

Actually, agent a can acquire the knowledge that $p \vee q$ is true, which is the same in epistemic logic that $\Box p \to \Box(p \vee q)$ no matter what q is.

However, sometimes we don't say that $p \vee q$ is known to a. For instance, if we consider that q is unknown to agent a as a doesn't know the content of q, then we cannot say that a knows the content of $p \vee q$. In this case, we consider that a formula A is known to a agent if and only if all of the atoms of A are known to the agent. Therefore, the state unknown is infectious. Here, we call such *known* as *well informed* to distinguish from other *known*.

Then we can read the interpretation as follows:

- $I_a(A)=1$ as agent a is well informed of A;
- $I_a(A)=0$ as agent a is not informed of A.

Consider it more, even though a proposition may possess an objective truth value, an agent may not acquire its truth unless he/she could interpret it. In agent communication, for example, although an agent receives a message from other agents, he/she may not be able to read it. This case happens when the message is enciphered; when recipient a possesses the key to decipher the cryptography of A as $I_a(A) = 1$, the agent could be said to be well *informed* of A. By the way, there can be unintended encryption. For non Roman-alphabet languages, there could be multiple different ways to encode their characters into *ascii* codes. If encoding and decoding are not consistent, called mis-conversion, the message becomes illegible. Such a proper decoder or a translator is regarded as a kind of decryption key.

In this case, the different consequence relations defined, for weak Kleene and classical logic do represent inferences according to different standards. According to the weak Kleene consequence relation, consequence means that truth of the messages is preserved, and that every agent is well informed about its meaning, that is, every agent is able to interpret the message. That could represent well cases where drawing inferences depend on a message being collectively known. In the case of classical logic, inferences are allowed that preserve truth, and no relation to the ability or possibility of agents to be informed about them is required. Notice that the apparatus seems to capture the fallibility of agents in being well informed about some truths, given the difference between classical logic and weak Kleene. That is, agents are only allowed to infer, collectively, from propositions they are informed about and that are true; they cannot infer merely assuming truth of propositions without knowing their meaning (which in a context involving encryption seems quite reasonable).

The key to read the received message has another view, that concerns whether the recipient is authorized to read the message. Namely, the recipient of the message may not be able to read it unless he/she is officially permitted to do so. We can implement this permission in terms of communication channel [39] so that an agent is admitted to send/read messages only via approved channels. When agent a is sent A and is authorized to access and read it as $I_a(A) = 1$, we could say again that the agent is well informed of A.

The same interpretation concerning the information of agents relative to a proposition could be used to codify the role of agents in organizations. For instance, one could have two agents that play the leading role in an organization, and they can only proceed in the business of the organization whenever both are informed of the propositions involved. That would justify restricting inferences to propositions that are true and of which both are aware of. More general scenarios may be considered with more agents involved, and, as we shall see when we introduce more structure in the set of agents, we can codify the idea that some kind of hierarchy exists between them.

Note that although this reading seems similar to the "ignorance" we introduce in the previous study, the consideration is totally different. For example, if agent a is informed of p while not informed of q, then we say that agent a is not informed of $p \wedge q$. However, if p is false, actually agent a has the ability to judge the formula $p \wedge q$ as false, so a knows whether $p \wedge q$ is true or not. In other word, "informed" is stricter than "not ignorance".

5.2 Semantics (I)

By this consideration of unknown , we give a semantics for infectious logic the same as Chapter 4.

Definition 5.1. *A* two-valued interpretation *for the language $\mathcal{L}$ is a pair $\langle V_t^w, V_a^w \rangle$, where $V_t^w : \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ and $V_a^w : \mathsf{Prop} \longrightarrow \{0, 1\}$. Valuations V_t^w, V_a^w are then extended to interpretations I_t, I_m by the following conditions.*

- $I_t^w(p)=\mathbf{t}$ *iff* $V_t^w(p)=\mathbf{t}$
- $I_a^w(p)=1$ *iff* $V_a^w(p)=1$
- $I_t^w(\sim A)=\mathbf{t}$ *iff* $I_t^w(A)=\mathbf{f}$
- $I_a^w(\sim A)=1$ *iff* $I_a^w(A)=1$
- $I_t^w(A \wedge B)=\mathbf{t}$ *iff* $I_t^w(A)=\mathbf{t}$ *and* $I_t^w(B)=\mathbf{t}$
- $I_a^w(A \wedge B)=1$ *iff* $I_a^w(A)=1$ *and* $I_a^w(B)=1$
- $I_t^w(A \vee B)=\mathbf{t}$ *iff* $I_t^w(A)=\mathbf{t}$ *or* $I_t^w(B)=\mathbf{t}$

- $I_a^w(A \vee B)$=1 *iff* $I_a^w(A)$=1 *and* $I_a^w(B)$=1

Definition 5.2. *A* four-valued interpretation *of $\mathcal{L}$ is a function I_4^w : $\mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$. Given a four-valued interpretation I_4^s, this is extended to a function that assigns every formula a truth value by the following truth functions:*

A	$\sim A$
t1	f1
t0	f0
f0	t0
f1	t1

$A{\wedge}B$	t1	t0	f0	f1
t1	t1	t0	f0	f1
t0	t0	t0	f0	f0
f0	f0	f0	f0	f0
f1	f1	f0	f0	f1

$A{\vee}B$	t1	t0	f0	f1
t1	t1	t0	t0	t1
t0	t0	t0	t0	t0
f0	t0	t0	f0	f0
f1	t1	t0	f0	f1

We introduce three different sets of designated values as follows:

- $\mathcal{D}_1^w := \{\mathbf{t}1\}$;
- $\mathcal{D}_2^w := \{\mathbf{t}1, \mathbf{t}0\}$;
- $\mathcal{D}_3^w := \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0\}$.

Based on these sets of designated values, we define three consequence relations as follows.

Definition 5.3. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_i^w A$ iff for all four-valued interpretations I_4^w, $I_4^w(A) \in \mathcal{D}_i^w$ if $I_4^w(B) \in \mathcal{D}_i^w$ for all $B \in \Gamma$, where $i \in \{1, 2, 3\}$.*

We have already shown the facts in Chapter 4 that:

- $\models_1^w$ **is the weak Kleene logic;**
- $\models_2^w$ **is the classical logic;**
- $\models_3^w$ **is the paraconsistent weak Kleene logic.**

5.3 Semantics (II)

In the previous section, we gave a semantics for the case we considered unknown be *infectious*. Actually, if we think in the different way, i.e., we consider that a formula A is known to a agent if and only if the agent knows whether A is true or not, then we can obtain a new semantics as following. Here, attention that this semantics should be similar with that of epistemic logic.

Definition 5.4. *A* two-valued interpretation *for the language $\mathcal{L}$ is a pair $\langle V_t^s, V_a^s \rangle$, where V_t^s : $\mathsf{Prop} \longrightarrow \{\mathsf{t}, \mathsf{f}\}$ and V_a^s : $\mathsf{Prop} \longrightarrow \{0, 1\}$. Valuations V_t^s, V_a^s are then extended to interpretations I_t^s, I_a^s by the following conditions.*

- $I_t^s(p)$=**t** *iff* $V_t^s(p)$=**t**
- $I_a^s(p)$=1 *iff* $V_a^s(p)$=1
- $I_t^s(\sim A)$=**t** *iff* $I_t^s(A)$=**f**
- $I_a^s(\sim A)$=1 *iff* $I_a^s(A)$=1
- $I_t^s(A \wedge B)$=**t** *iff* $I_t^s(A)$=**t** *and* $I_t^s(B)$=**t**
- $I_a^s(A \wedge B)$=1 *iff* ($I_t^s(A \wedge B)$=**t** *and* $I_a^s(A)$=1 *and* $I_a^s(B)$=1) *or* ($I_t^s(A)$=**f** *and* $I_a^s(A)$=1) *or* ($I_t^s(B)$=**f** *and* $I_a^s(B)$=1))
- $I_t^s(A \vee B)$=**t** *iff* $I_t^s(A)$=**t** *or* $I_t^s(B)$=**t**
- $I_a^s(A \wedge B)$=1 *iff* ($I_t^s(A \wedge B)$=**f** *and* $I_a^s(A)$=1 *and* $I_a^s(B)$=1) *or* ($I_t^s(A)$=**t** *and* $I_a^s(A)$=1) *or* ($I_t^s(B)$=**t** *and* $I_a^s(B)$=1)

The definition of $I_a^s(A \wedge B)$ and $I_a^s(A \vee B)$ may seem strange. Actually, we just consider it as the S5 system of the epistemic logic. For example, we consider the $I_a^s(A \wedge B) = 1$ as $\Box_a(A \wedge B) \vee \Box_a \neg(A \wedge B)$. Therefore there exists three possible cases in all S5 models that

- $A \wedge B$ **and** $\Box_a A \vee \Box_a \neg A$ **and** $\Box_a B \vee \Box_a \neg B$**;**
- $\neg A$ **and** $\Box_a A \vee \Box_a \neg A$**;**
- $\neg B$ **and** $\Box_a B \vee \Box_a \neg B$**.**

which is the same as the definition we showed above. Also, we can see the semantics more clearly by the truth table.

Definition 5.5. *A* four-valued interpretation *of $\mathcal{L}$ is a function I_4^s : Prop $\to$ $\{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$. Given a four-valued interpretation I_4^s, this is extended to a function that assigns every formula a truth value by the following truth functions:*

A	$\sim A$
t1	**f**1
t0	**f**0
f0	**t**0
f1	**t**1

$A \wedge B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**0	**f**0	**f**1
t0	**t**0	**t**0	**f**0	**f**1
f0	**f**0	**f**0	**f**0	**f**1
f1	**f**1	**f**1	**f**1	**f**1

$A \vee B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**1	**t**1	**t**1
t0	**t**1	**t**0	**t**0	**t**0
f0	**t**1	**t**0	**f**0	**f**0
f1	**t**1	**t**0	**f**0	**f**1

We introduce three different sets of designated values as follows:

- $\mathcal{D}_1^s := \{\mathbf{t}1\}$**;**
- $\mathcal{D}_2^s := \{\mathbf{t}1, \mathbf{t}0\}$**;**
- $\mathcal{D}_3^s := \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0\}$**.**

Based on these sets of designated values, we define three consequence relations as follows.

Definition 5.6. *For all $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \models_i^s A$ iff for all four-valued interpretations I_4^s, $I_4^s(A) \in \mathcal{D}_i$ if $I_4^s(B) \in \mathcal{D}_i^s$ for all $B \in \Gamma$, where $i \in \{1, 2, 3\}$.*

Then, we can show the facts that:

- $\models^s_1$ **is the strong Kleene logic;**
- $\models^s_2$ **is the classical logic;**
- $\models^s_3$ **is the logic of paradox.**

We first deal with the case in which t1 is the only designated value. To show the first fact, we prepare a lemma.

Lemma 5.1. *For all strong Kleene three-valued valuation v^s_3 for $\mathcal{L}$, there is a four-valued valuation v^s_4 such that for all $A \in \mathsf{Form}$, (i) $I^s_4(A) = \mathbf{t}1$ iff $I^s_3(A) = \mathbf{t}$, and (ii) $I^s_4(A) = \mathbf{f}1$ iff $I_3(A) = \mathbf{f}$.*

Proof. Given a three-valued valuation v^s_3, we define $v^s_4 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ as follows:

$$v^s_4(p) = \begin{cases} \mathbf{t}1 & v^s_3(p) = \mathbf{t} \\ \mathbf{f}0 & v^s_3(p) = \mathbf{b} \\ \mathbf{f}1 & v^s_3(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. For the base case, the desired result holds by the definition of v^s_4. For the induction step, we split the cases depending on the form of the formula A.

If A is of the form $\sim B$, then for (i), we have $I^s_4(A) = \mathbf{t}1$ iff $I^s_4(\sim B)=\mathbf{t}1$ iff $I^s_4(B)=\mathbf{f}1$ (by def. of I^s_4) iff $I^s_3(B)=\mathbf{f}$ (by IH) iff $I^s_3(\sim B)=\mathbf{t}$ (by def. of I^s_3) iff $I^s_3(A)=\mathbf{t}$. For (ii), $I^s_4(A) = \mathbf{f}1$ iff $I^s_4(\sim B)=\mathbf{f}1$ iff $I^s_4(B)=\mathbf{t}1$ (by def. of I^s_4) iff $I^s_3(B)=\mathbf{t}$ (by IH) iff $I^s_3(\sim B)=\mathbf{f}$ (by def. of I^s_3) iff $I^s_3(A)=\mathbf{f}$.

If A is of the form $B \wedge C$, then for (i), $I^s_4(A)=\mathbf{t}1$ iff $I^s_4(B \wedge C)=\mathbf{t}1$ iff $I^s_4(B)=\mathbf{t}1$ and $I^s_4(C)=\mathbf{t}1$ (by def. of I^s_4) iff $I^s_3(B)=\mathbf{t}$ and $I^s_3(C)=\mathbf{t}$ (by IH) iff $I^s_3(B \wedge C)=\mathbf{t}$ (by def. of I^s_3) iff $I^s_3(A)=\mathbf{t}$. For (ii), $I^s_4(A)=\mathbf{f}1$ iff $I^s_4(B \wedge C)=\mathbf{f}1$ iff $I^s_4(B)=\mathbf{f}1$ or $I^s_4(C)=\mathbf{f}1$(by def. of I^s_4) iff $I^s_3(B)=\mathbf{f}$ or $I^s_3(C)=\mathbf{f}$(by IH) iff $I^s_3(B \wedge C)=\mathbf{f}$ (by def. of I^s_3) iff $I^s_3(A)=\mathbf{f}$.

The case for disjunction is similar. □

We are now ready to prove one of the directions.

Proposition 5.1. *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models^s_1 A$ then $\Gamma \models_{SK} A$.*

Proof. Suppose $\Gamma \not\models_{SK} A$. Then, there is a three-valued valuation v^s_3 : $\mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ such that $I^s_3(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I^s_3(A) \neq \mathbf{t}$. Now, in view of (i) of Lemma 5.1, there is a four-valued valuation v^s_4 such that $I^s_4(B)=\mathbf{t}1$ for all $B \in \Gamma$ and $I^s_4(A) \neq \mathbf{t}1$, namely $\Gamma \not\models^s_1 A$, as desired. □

For the other direction, we prepare another lemma.

Lemma 5.2. *For all four-valued valuation v_4^s for $\mathcal{L}$, there is a strong Kleene three-valued valuation v_3^s such that for all $A \in \mathsf{Form}$, (i) $I_3^s(A) = \mathbf{t}$ iff $I_4^s(A) = \mathbf{t}1$, and (ii) $I_3^s(A) = \mathbf{f}$ iff $I_4^s(A) = \mathbf{f}1$.*

Proof. Given a four-valued valuation v_4^s, we define $v_3^s : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ as follows:

$$v_3^s(p) = \begin{cases} \mathbf{t} & v_4^s(p) = \mathbf{t}1 \\ \mathbf{b} & v_4^s(p) = \mathbf{t}0 \text{ or } v_4^s(p) = \mathbf{f}0 \\ \mathbf{f} & v_4^s(p) = \mathbf{f}1 \end{cases}$$

Then we prove the desired result by induction. □

Then, again, the proof is similar to the above case.

Proposition 5.2. *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_{SK} A$ then $\Gamma \models_1^s A$*

Proof. Suppose $\Gamma \not\models_1^s A$. Then, there is a four-valued valuation $v_4^s : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$ such that $I_4^s(B) = \mathbf{t}1$ for all $B \in \Gamma$ and $I_4^s(A) \neq \mathbf{t}1$. Now, in view of (i) of Lemma 5.2, there is a three-valued valuation v_3^s such that $I_3^s(B) = \mathbf{t}$ for all $B \in \Gamma$ and $I_3^s(A) \neq \mathbf{t}$, namely $\Gamma \not\models_{SK} A$, as desired. □

In view of the above propositions, we obtain the following.

Theorem 5.1. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{SK} A$ iff $\Gamma \models_1^s A$.*

In other words, this semantics is equivalent to the strong Kleene logic.

Then, consider the case for the logic of paradox, in which t1, t0 and f0 are taken as designated values. In fact, the proofs are basically the same with the cases for the strong Kleene logic.

Theorem 5.2. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{LP} A$ iff $\Gamma \models_3^s A$.*

Proof. We use (ii) of Lemma 5.1 and Lemma 5.2. □

Finally, we consider the case in which t1 and t0 are designated.

Theorem 5.3. *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{CL} A$ iff $\Gamma \models_2^s A$.*

Actually, the proof is just the same as $\models_2^w$ of Semantics (I) which we showed in [38]. The reason is that if we ignore the additional value that shows the epistemic state, it is easy to see that the two semantics are the same.

5.4 Relation in $\mathfrak{A}$

Given an model with $\mathfrak{A}$, we can naturally consider introducing relations among the elements of $\mathfrak{A}$, concerning the legibility/ interpretability of propositions.

A pair or a subset of $\mathfrak{A}$ may possess an arbitrary meaning in regard to their valuations. Remember $V_a(p) = 1$ implies p holds for $a \in \mathfrak{A}$. For example, suppose such a relation that for any p and (a, b) in this relation $V_a(p) \neq V_b(p)$. Then, agent a and b have completely disjoint sets of decryption keys, that is, a cannot decipher what b can, and vice versa. Suppose, next, such a subset $G \subset \mathfrak{A}$ that $\sum_{a \in G} V_a(p) = 1$. It means that if agents in G cooperate, every encrypted sentence becomes readable, covering any method of encryption. This means the wisdom of crowds.

However, we can naturally expect that an ordered pair of (a, b) $(a, b \in \mathfrak{A})$, or a Cartesian product of $\mathfrak{A}$ would bear useful fruits.

5.4.1 Binary relation in $\mathfrak{A}$

Definition 5.7 (Binary relation in $\mathfrak{A}$). *Let $\mathcal{R} \subseteq \mathfrak{A} \times \mathfrak{A}$. We first consider*

$$V_a(p) \leq V_b(p) \textit{ iff } (a, b) \in \mathcal{R}. \tag{5.1}$$

Then, since $V_a(p) = V_a(p)$, $\mathcal{R}$ is reflexive, and also, since $V_a(p) \leq V_b(p)$ and $V_b(p) \leq V_c(p)$ implies $V_a(p) \leq V_c(p)$, $\mathcal{R}$ becomes transitive, so that $\mathcal{R}$ becomes preorder. For convenience, we may write $a \preceq b$ for $(a, b) \in \mathcal{R}$, and thus $V_a(p) \leq V_b(p)$ iff $a \preceq b$ for (5.1).

We are motivated to define $\mathcal{R}$ as (5.1), considering the following probable readings upon $V_a(p) \leq V_b(p)$, e.g., agent b possesses more decryption keys than agent a, or b has more knowledge on decoding than a. If we regard this as a power relation, b is more authoritative than a and can access all the information which a can.

Example 5.2 (Authorized agents). *A judge in a court is more authoritative than the prosecutor and the lawyer for the defendant. The judge can collect all the information which reside in either side. Furthermore, the prosecutor is more authoritative than the police.*

On the other hand, we can consider the relation as the agent communication. For example, there are two agents a and b and at first only agent a is well informed of A, i.e., $I_a(A) = 1$ and $I_b(a) = 0$. If we have $(a, b) \in \mathfrak{A}$ which means agent a should tell agent b all

he/she is informed of, then at last both a and b are informed of A, i.e., finally $I_a(A) = I_b(a) = 1$.

We can consider various additional laws on $\mathcal{R}$. First, if we introduce the symmetric relation, $(\preceq)$ becomes the equivalence relation. If we, instead, add the anti-symmetric relation, $\mathfrak{A}$ becomes a partially-ordered set (poset). According to the above discussion on new interpretations, we assume hereafter that $\mathcal{R}$ be a poset, unless otherwise mentioned. A subset of $\mathcal{R}$ can happen to be totally (linearly) ordered, or as a special case, $\mathcal{R}$ itself may be in the linear order.

We may define a maximal element in $\mathfrak{A}$ in terms of $(\preceq)$. When there exist the maximum and minimum elements in any subset of $\mathfrak{A}$, the relation constructs a *lattice*.

Example 5.3 (Judge in court). *Let j, p and l be agents of judge, prosecutor, and lawyer, respectively. Then, for all proposition p that $V_j(p) \geq V_p(p)$ and $V_j(p) \geq V_l(p)$, however, there is no relation between $V_j(p)$ and $V_l(p)$. Here, the judge is the maximal element in terms of $(\preceq)$ relation.*

The investigator of police (agent i) collects information from the accused (a), and the official record of police investigation must be submitted to the prosecutor (p). Therefore, their power relation is locally linear, as

$$V_j(p) \geq V_p(p) \geq V_i(p) \geq V_a(p).$$

The witness (a) must tell what he/she knows to the police (i). Also a tells the story to the lawyer (l). The judge (j) hears all the stories both from the prosecutor and the lawyer, and thus, the valuation should be under the condition as lattice.

$$V_j(p) \geq \{V_p(p), V_l(p)\} \geq V_a(p).$$

5.4.2 Distinguished element in $\mathfrak{A}$

In this semantics, we presuppose a special member o in $\mathfrak{A}$ that the valuation $V_o(p) = 1$ for all $p \in \mathsf{Prop}$. The meaning of such o can be considered as an omniscient agent that is well informed of anything. Here, we do not assume that for all $\mathfrak{A}$ there exists a distinguished element. From the reading, we have that if o is such a distinguished element of $\mathfrak{A}$, then for all $i \in \mathfrak{A}$ that $(i, o) \in \mathcal{R}$. It is easy to see that a distinguish element should be the maximum element of $\mathfrak{A}$. However, a maximum element is not always a distinguish element, e.g., the judge is the maximum element in the example above, while

there may exist something that the judge is not informed (so none is informed of that naturally). Actually, if we consider the case as a novel, a distinguished element is more likely the reader that can get all the information in the story.

Remark 5.1. *McCarthy [27] also considered that there may exist a special agent that called "any fool". The knowledge of such agent obtained the common knowledge of all agents in a model, i.e., every agent should know at least what "any fool" knows. Therefore, we can see that the omniscient agent here can be seen as the opposite of the agent "any fool", for every agent should know at most what the omniscient agent knows.*

5.4.3 Some results

Here, we check that whether the valuation is closed under the binary relation. That is, we prove the proposition as following:

Proposition 5.3. *Given a set* $\mathfrak{A}$ *and* $\mathcal{R} \subseteq \mathfrak{A} \times \mathfrak{A}$*, the valuation* $V_a(a \in \mathfrak{A})$ *is under the condition that: if* $(a, b) \in \mathcal{R}$ *then for all* $p \in \mathsf{Prop}$ *that* $V_a(p) \leq V_b(p)$*, then the extension valuation* $I_a(a \in \mathfrak{A})$ *is also under the condition that: if* $(a, b) \in \mathcal{R}$ *then for all formula* A *that* $I_a(A) \leq I_b(A)$*.*

Semantically speaking, we show that the relation $\mathcal{R}$ is not only limit the value of proposition. If agent b has more authoritative than agent a ($(a, b) \in \mathfrak{A}$), then not only for all proposition but also for all statements that a is informed is informed by agent b.

Also, we investigate whether the valuation is closed under the distinguish element. That is, we prove the proposition as following:

Proposition 5.4. *Given a set* $\mathfrak{A}$ *and* $o \in \mathfrak{A}$*, the valuation* $V_o(p) = 1$ *for all* $p \in \mathsf{Prop}$*, then the extension valuation* $I_o(A) = 1$ *for all formula* A*.*

Semantically speaking, we show that the a distinguish element is not only informed of all propositions but also all statements in the model.

5.5 A many-valued semantics for multi-agent system

In the previous sections, we give two different semantics for the different readings of unknown. If we consider the two reading of

unknown as two different epistemic states, we can give a more general semantics.

Actually, we can consider the epistemic states as following:

- **The agent knows that A is true or agent knows that A is false. ($I_a^s(A) = 1$ in Semantics (I) or $I_a^w(A) = 1$ in Semantic (II))**
- **The agent doesn't know the content of A. ($I_a^w(A) = 0$ in Semantics (I))**
- **The agent considers that A is possibly true and A is possibly false, i.e., the agent knows the contents of A but doesn't know whether A is true or not. ($I_a^s(A) = 0$ in Semantic (II))**

Therefore, we combine the two semantics above and extend the valuation V_a to three-valued $\{1, 0.5, 0\}$. We read the values as following:

- **$V_a(p) = 1$: Agent a knows the content of p and whether p is true or not.**
- **$V_a(p) = 0$: Agent a doesn't know the content of p.**
- **$V_a(p) = 0.5$: Agent a knows the content of p but doesn't know whether p is true or not.**

Also, if we consider the case as programming or in the database, we can read the values as following:

- **$V_a(p) = 1$: The value of p is decided in the database (either true or false).**
- **$V_a(p) = 0.5$: p is assigned but the value of p is undecided in the database.**
- **$V_a(p) = 0$: p is not assigned in the database.**

According to the consideration above, first, we give the semantics for a single agent.

5.5.1 Many-valued semantics for single agent

Definition 5.8. *A many-valued semantics for the language $\mathcal{L}$ is a pair $\langle V_t, V_a \rangle$, where V_t : Prop $\longrightarrow \{\mathbf{t}, \mathbf{f}\}$ and V_a : Prop $\longrightarrow \{0, 0.5, 1\}$. Valuations V_t, V_a are then extended to interpretations I_t, I_a by the following conditions.*

- *I_t is the same as the classical logic.*
- *$I_a(p)=V_a(p)$*
- *$I_a(\sim A)=I_a(A)$*
- *$I_a(A \wedge B)=1$ iff ($I_t(A \wedge B)=\mathbf{t}$ and $I_a(A)=1$ and $I_a(B)=1$) or ($I_t(A)=\mathbf{f}$ and $I_a(A)=1$ and $I_a(B)\neq 0$) or ($I_t(B)=\mathbf{f}$ and $I_a(B)=1$ and $I_a(A)\neq 0$))*

- $I_a(A \wedge B)$=0.5 *iff* ($I_t(A \wedge B)$=**f** *and* $I_a(A)$=0.5 *and* $I_a(B)$=0.5) *or* ($I_t(A)$=**t** *and* $I_a(A)$=0.5 *and* $I_a(B) \neq 0$) *or* ($I_t(B)$=**t** *and* $I_a(B)$=0.5 *and* $I_a(A) \neq 0$))
- $I_a(A \wedge B)$=0 *iff* $I_a(A)$=0 *or* $I_a(B)$=0
- $I_a(A \vee B)$=1 *iff* ($I_t(A \vee B)$=**f** *and* $I_a(A)$=1 *and* $I_a(B)$=1) *or* ($I_t(A)$=**t** *and* $I_a(A)$=1 *and* $I_a(B) \neq 0$) *or* ($I_t(B)$=**t** *and* $I_a(B)$=1 *and* $I_a(A) \neq 0$))
- $I_a(A \vee B)$=0.5 *iff* ($I_t(A \vee B)$=**t** *and* $I_a(A)$=0.5 *and* $I_a(B)$=0.5) *or* ($I_t(A)$=**f** *and* $I_a(A)$=0.5 *and* $I_a(B) \neq 0$) *or* ($I_t(B)$=**f** *and* $I_a(B)$=0.5 *and* $I_a(A) \neq 0$))
- $I_a(A \vee B)$=0 *iff* $I_a(A)$=0 *or* $I_a(B)$=0

Definition 5.9. *A six-valued interpretation $\mathcal{L}$ is a function I_6 : $\mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0.5, \mathbf{t}0, \mathbf{f}0, \mathbf{f}0.5, \mathbf{f}1\}$. Given a six-valued interpretation I_6, this is extended to a function that assigns every formula a truth value by the following truth functions:*

A	$\sim A$	$A \wedge B$	**t**1	**t**0.5	**t**0	**f**0	**f**0.5	**f**1
t1	**f**1	**t**1	**t**1	**t**0.5	**t**0	**f**0	**f**0.5	**f**1
t0.5	**f**0.5	**t**0.5	**t**0.5	**t**0.5	**t**0	**f**0	**f**0.5	**f**1
t0	**f**0	**t**0	**t**0	**t**0	**t**0	**f**0	**f**0	**f**0
f0	**t**0	**f**0	**f**0	**f**0	**f**0	**f**0	**f**0	**f**0
f0.5	**t**0.5	**f**0.5	**f**0.5	**f**0.5	**f**0	**f**0	**f**0.5	**f**1
f1	**t**1	**f**1	**f**1	**f**1	**f**0	**f**0	**f**1	**f**1

We introduce three different sets of designated values as follows:

- $\mathcal{D}_1 := \{\mathbf{t}1\}$**, if we ask the agent that what true statement it knows;**
- $\mathcal{D}_2 := \{\mathbf{t}1, \mathbf{t}0.5, \mathbf{t}0\}$**, if we ask that what true statement is(may be answered by an omniscient agent);**
- $\mathcal{D}_3 := \{\mathbf{t}1, \mathbf{t}0.5, \mathbf{f}0.5\}$**, if we ask the agent that what statement the agent considers that may be true.**

Based on these sets of designated values, we define the consequence relations as follows.

Definition 5.10. *For all $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \models_i A$ iff for all four-valued interpretations I_6, $I_6(A) \in \mathcal{D}_i$ if $I_6(B) \in \mathcal{D}_i$ for all $B \in \Gamma$, where $i \in \{1, 2, 3\}$.*

Then, we can show the facts that:

- $\models_2$ **is the classical logic;**
- $\models_3$ **is the** S_{fde} **logic.**

The proof of $\models_2$ is just the same as above because it is just the classical logic if we only take care of V_t and I_t. To deal with the case of $\models_3$, we prepare two lemmas.

Lemma 5.3. *For all S_{fde} valuation v_{sfde} for $\mathcal{L}$, there is a six-valued valuation v_6 such that for all $A \in \mathsf{Form}$,*

- *(i) $I_6(A) = \mathbf{t}1$ iff $I_{sfde}(A) = \mathbf{t}$;*
- *(ii) $I_6(A) = \mathbf{t}0.5$ or $I_6(A) = \mathbf{f}0.5$ iff $I_{sfde}(A) = \mathbf{b}$;*
- *(iii) $I_6(A) = \mathbf{t}0$ or $I_6(A) = \mathbf{f}0$ iff $I_{sfde}(A) = \mathbf{n}$;*
- *(iv) $I_6(A) = \mathbf{f}1$ iff $I_{sfde}(A) = \mathbf{f}$;*

Proof. Given a S_{fde} valuation v_{sfde}, we define $v_6 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0.5, \mathbf{t}0, \mathbf{f}0, \mathbf{f}0.5, \mathbf{f}1\}$ as follows:

$$v_6(p) = \begin{cases} \mathbf{t}1 & v_{sfde}(p) = \mathbf{t} \\ \mathbf{f}0.5 & v_{sfde}(p) = \mathbf{b} \\ \mathbf{f}0 & v_{sfde}(p) = \mathbf{n} \\ \mathbf{f}1 & v_{sfde}(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. Since the proofs are similar, we only deal with the case for (i).

For the base case, the desired result holds by the definition of V_6. For the induction step, we split the cases depending on the form of the formula A.

- If A is of the form $\sim B$, we have

$$\begin{aligned} &I_6(A) = \mathbf{t}1 \\ &\text{ iff } I_6(\sim B) = \mathbf{t}1 \\ &\text{ iff } I_6(B) = \mathbf{f}1 \\ &\text{ iff } I_{sfde}(B) = \mathbf{f} \\ &\text{ iff } I_{sfde}(\sim B) = \mathbf{t} \\ &\text{ iff } I_{sfde}(A) = \mathbf{t} \end{aligned}$$

- If A is of the form $B \wedge C$, we have

$$\begin{aligned} &I_6(A) = \mathbf{t}1 \\ &\text{ iff } I_6(B \wedge C) = \mathbf{t}1 \\ &\text{ iff } I_6(B) = I_6(C) = \mathbf{t}1 \\ &\text{ iff } I_{sfde}(B) = \mathbf{t} \text{ and } I_{sfde}(C) = \mathbf{t} \\ &\text{ iff } I_{sfde}(B \wedge C) = \mathbf{t} \\ &\text{ iff } I_{sfde}(A) = \mathbf{t} \end{aligned}$$

The case for disjunction is similar. □

Lemma 5.4. *For all six-valued valuation v_6 for $\mathcal{L}$, there is a strong Kleene three-valued valuation v_{sfde} such that for all $A \in \mathsf{Form}$,*

- *(i)* $I_{sfde}(A) = \mathbf{t}$ *iff* $I_6(A) = \mathbf{t}1$;
- *(ii)* $I_{sfde}(A) = \mathbf{b}$ *iff* $I_6(A) = \mathbf{t}0.5$ *or* $I_6(A) = \mathbf{f}0.5$;
- *(iii)* $I_{sfde}(A) = \mathbf{n}$ *iff* $I_6(A) = \mathbf{t}0$ *or* $I_6(A) = \mathbf{f}0$;
- *(iv)* $I_{sfde}(A) = \mathbf{f}$ *iff* $I_6(A) = \mathbf{f}1$;

Proof. Given a four-valued valuation v_4^s, we define $v_3^s : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ as follows:

$$v_{fde}(p) = \begin{cases} \mathbf{t} & v_6(p) = \mathbf{t} \\ \mathbf{b} & v_6(p) = \mathbf{t}0.5 \text{ or } v_6(p) = \mathbf{f}0.5 \\ \mathbf{n} & v_6(p) = \mathbf{t}0 \text{ or } v_6(p) = \mathbf{f}0 \\ \mathbf{f} & v_6(p) = \mathbf{f}1 \end{cases}$$

Then we prove the desired result by induction. □

Actually, it is easy to see that $\models_1$ is the logic if we replace the designate values $\mathcal{D}$ of S_{fde} by $\mathcal{D} = \{\mathbf{t}\}$.

5.5.2 Many-valued semantics for multi-agent system

Consider that there are several agents, therefore there should be several valuations of V_a. Then we give a many-valued semantics for multi-agent system.

Definition 5.11. *A many-valued semantics for the language $\mathcal{L}$ is a pair $\langle V_t, \{V_a\}_{a \in Ag}\rangle$, where Ag is a non-empty set of agents, $V_t : \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ and $V_a : \mathsf{Prop} \longrightarrow \{0, 0.5, 1\}$. Valuations V_t, V_a are then extended to interpretations I_t, I_a by the following conditions.*

- I_t *is the same as the classical logic.*
- $I_a(p)=V_a(p)$
- $I_a(\sim A)=1 - I_a(A)$
- $I_a(A \wedge B)=1$ *iff* ($I_t(A \wedge B)=\mathbf{t}$ *and* $I_a(A)=1$ *and* $I_a(B)=1$) *or* ($I_t(A)=\mathbf{f}$ *and* $I_a(A)=1$ *and* $I_a(B)\neq 0$) *or* ($I_t(B)=\mathbf{f}$ *and* $I_a(B)=1$ *and* $I_a(A)\neq 0$))
- $I_a(A \wedge B)=0.5$ *iff* ($I_t(A \wedge B)=\mathbf{f}$ *and* $I_a(A)=0.5$ *and* $I_a(B)=0.5$) *or* ($I_t(A)=\mathbf{t}$ *and* $I_a(A)=0.5$ *and* $I_a(B)\neq 0$) *or* ($I_t(B)=\mathbf{t}$ *and* $I_a(B)=0.5$ *and* $I_a(A)\neq 0$))
- $I_a(A \wedge B)=0$ *iff* $I_a(A)=0$ *or* $I_a(B)=0$
- $I_a(A \vee B)=1$ *iff* ($I_t(A \vee B)=\mathbf{f}$ *and* $I_a(A)=1$ *and* $I_a(B)=1$) *or* ($I_t(A)=\mathbf{t}$ *and* $I_a(A)=1$ *and* $I_a(B)\neq 0$) *or* ($I_t(B)=\mathbf{t}$ *and* $I_a(B)=1$ *and* $I_a(A)\neq 0$))
- $I_a(A \vee B)=0.5$ *iff* ($I_t(A \vee B)=\mathbf{t}$ *and* $I_a(A)=0.5$ *and* $I_a(B)=0.5$) *or* ($I_t(A)=\mathbf{f}$ *and* $I_a(A)=0.5$ *and* $I_a(B)\neq 0$) *or* ($I_t(B)=\mathbf{f}$ *and* $I_a(B)=0.5$ *and* $I_a(A)\neq 0$))

- $I_a(A \wedge B)$=0 *iff* $I_a(A)$=0 *or* $I_a(B)$=0

We introduce some different sets of designated values as follows. To see it clearly, we write the value as $I_t, I_{a_1}, ...I_{a_n}$.

- $\mathcal{D}_1^k := \{\mathbf{t}, I_{a_1}, ..., I_{a_n} : I_{a_k} = 1\}$, **if we ask the agent k that what k knows that is true;**
- $\mathcal{D}_2^k := \{\mathbf{t}, I_{a_1}, ..., I_{a_n} : I_{a_j} \in \{1, 0.5, 0\}(j \in (1, n))\}$, **if we ask that what is true(may be answered by an omniscient agent);**
- $\mathcal{D}_3^k := \{\mathbf{t}, I_{a_1}, ..., I_{a_n} : I_{a_k} \in \{1, 0.5\}\} \cup \{\mathbf{f}, I_{a_1}, ..., I_{a_n} : I_{a_k} = 0.5\}$ **, if we ask the agent k that what k knows that is not false.**

Based on these sets of designated values, we define three consequence relations as follows.

Definition 5.12. *For all $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \models_i^k A$ iff for all interpretations I, $I(A) \in \mathcal{D}_i^k$ if $I(B) \in \mathcal{D}_i^k$ for all $B \in \Gamma$, where $i \in \{1, 2, 3\}$.*

We can show the facts that:

- $\models_2^k$ **is the classical logic.**
- $\models_3^k$ **is the S_{fde} logic.**

The proof is the same as case of a single agent.

5.5.3 Agent communication

We can see that, in the many-valued semantics for multi-agent system, the valuation shows both the classical values and the epistemic states of each agent. In other words, the valuation can be considered as a model like the Kripke model we use in the dynamic epistemic logic. Therefore, we can consider several kinds of valuation change like the dynamic operators to express the agent communication. First, we give a definition of consequence relation which is like the epistemic logic.

Definition 5.13. *Let Ag be a non-empty set of agents,* Prop *be the set of propositions and $V = \{V_t, \{V_a\}_{a \in Ag}\}$ be the valuation where V_t :* Prop $\rightarrow \{t, f\}$ *and V_a :* Prop $\rightarrow \{1, 0.5, 0\}$. *Then we can define the satisfied functions $\models_a^t$ and $\models_a^{mt}$ as following:*

$V \models_a^t A$ *iff* $I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1$

$V \models_a^{mt} A$ *iff* $(I_t(A) = \mathbf{t}$ *and* $I_a(A) = 1)$ *or* $I_a(A) = 0.5$

Semantically speaking, $V \models_a^t A$ means that agent a knows that A is true, and $V \models_a^{mt} A$ means that agent a considers that A may

be true. Actually, $\models^t_a$ stands for the $\models^k_1$ and $\models^{mt}_a$ stands for the $\models^k_3$. It is easy to see from the definition that: for all V and A, $V \models^{mt}_a A$ if $V \models^t_a A$. In other words, if agent a knows that p is true, then a considers that p may be true.

In the dynamic epistemic logic, there are several operators to show the communication between agents. [44] showed the semantics of agent announcement that one agent tells others some statement instead of the public announcement. [23] considered the channel communication as the semi-private announcement. Unlike the studies of dynamic epistemic logic above, we do not use a Kripke model so that we cannot show all of the belief changes. However, it is very simple to show a certain kind of changes of knowledge by our semantics. Moreover, we can give some new ideas by considering two kinds of agent communication: *teaching* and *asking*. Here, we consider that textitteaching is the communication from agent a to a set of agent G that a will tell every proposition he/ she knows to the member of G. We write such agent communication by the operator $\downarrow^a_G$. Semantically speaking, *teaching* is considered as the act that the teacher teaches the knowledge to the students. If G has only one member, then the act can be considered as the semi-private announcement. If G is the set of all agents, then the act can be considered as the agent announcement.

Definition 5.14. *Let the original model be V. After the act that agent a teaches the group G, the new valuation $V^{\downarrow^a_G}$ is defined as following:*

For all $p \in \mathsf{Prop}$,

- *If $i \notin G$, then $V_i^{\downarrow^a_G}(p) = V_i(p)$, and*
- *If $i \in G$, then*

$$V_i^{\downarrow^a_G}(p) := \begin{cases} 1 & \text{if } V_a(p) = 1 \\ V_i(p) & \text{otherwise.} \end{cases}$$

Semantically speaking, the students of a can acquire the knowledge that is known to a, and the epistemic states of other agents will not change.

Then, consider the situation that the student asks the teacher questions. The student possesses several propositions that he/ she doesn't know whether true or false, and if the teacher knows that, the student can obtain the answer to know that proposition. It seems that *asking* is just the opposite of *teaching* so that the results that a teaches b and b asks a are the same. However, actually sometimes the two cases are different for the epistemic states of

teacher may also be changed by asked questions. For example, a is the teacher and b is the student, let $V_a(p) = 0$ and $V_b(p) = 0.5$, b doesn't know whether p is true or not, so he/ she will ask a about p, while the teacher doesn't know even the content of p at the moment. Therefore, after *asking*, student b cannot obtain an answer, while teacher a becomes to know the content of p. Then, we define the model $V^{\uparrow^a_b}$ that show the model after b asks a questions as following:

Definition 5.15. *For all $p \in \mathsf{Prop}$,*

- *If $i \notin \{a, b\}$, then $V_i^{\uparrow^a_b}(p) = V_i(p)$;*
- *If $i = a$, then*

$$V_i^{\uparrow^a_b}(p) := \begin{cases} 1 & \text{if } V_a(p) = 0.5 \text{ and } V_b(p) = 1 \\ V_i(p) & \text{otherwise.} \end{cases}$$

  and
- *If $i = b$, then*

$$V_b^{\uparrow^a_b}(p) := \begin{cases} 0.5 & \text{if } V_a(p) = 0.5 \text{ and } V_b(p) = 0 \\ V_b(p) & \text{otherwise.} \end{cases}$$

Remark 5.2. *Here, we only consider the acts as valuation change. We don't consider it as dynamic operators because the for we don't want to change the language of our logic which is the same as classical logic. To add the dynamic operators and give proofs completeness and soundness are the remaining future works.*

5.5.4 Example

Here, we give an example to show some agent communication in our semantics. Let $Ag = \{a_1, a_2, a_3\}$ be the set of agents and $\mathsf{Prop} = \{p, q, r\}$ be the set of propositional variables. Assume that $V_t(p) = \mathbf{t}$, $V_t(q) = \mathbf{f}$, $V_t(p) = \mathbf{f}$, and the epistemic states of each agent are shown in the following table.

Then in the original model, we have that $V \models^t_{a_1} p$, $V \not\models^t_{a_1} q$ and $V \not\models^t_{a_1} r$, which mean that agent a_1 only knows that p is true. Also we have $V \models^{mt}_{a_1} p$, $V \models^{mt}_{a_1} q$, $V \not\models^{mt}_{a_1} r$, which mean that agent a_1 considers that p and q are possible true. If we take care of the formula $p \vee r$, we have $I_{a_1}(p \vee r) = 0$ so $V \not\models^t_{a_1} p \vee r$ which means that

	$V_{a_i}(p)$	$V_{a_i}(q)$	$V_{a_i}(r)$
a_1	1	0.5	0
a_2	0.5	1	0
a_3	0	0	0.5

Table 5.1: The epistemic states in V

a_1 cannot respond that $p \vee r$ is true while he/ she knows that p is true.

Then, we consider some cases of agent communication.

- **If a_1 teaches others the knowledge a_1 knows, the new epistemic states in $V^{\downarrow^{a_1}_{\{a_2,a_3\}}}$ are shown in Table 5.2:**

	$V_{a_i}(p)$	$V_{a_i}(q)$	$V_{a_i}(r)$
a_1	1	0.5	0
a_2	1	1	0
a_3	1	0	0.5

Table 5.2: The epistemic states in $V^{\downarrow^{a_1}_{\{a_2,a_3\}}}$

We can see that at first $V \not\models^{t}_{a_2} p$ and after teaching we have $V^{\downarrow^{a_1}_{\{a_2,a_3\}}} \models^{mt}_{a_2} p$, which means that a_2 becomes to know p is true after the communication. For agent a_3, he/ she becomes to know that p is true while he/ she still doesn't know the content of q because the teacher doesn't say any statement of q.

- **If a_2 asks a_1 what a_2 doesn't know, the new epistemic states in $V^{\uparrow^{a_2}_{a_1}}$ are shown in Table 5.3.**

	$V_{a_i}(p)$	$V_{a_i}(q)$	$V_{a_i}(r)$
a_1	1	0.5	0
a_2	1	1	0
a_3	0	0	0.5

Table 5.3: The epistemic states in $V^{\uparrow^{a_2}_{a_1}}$

Comparing with the states in $V^{\downarrow^{a_1}_{\{a_2,a_3\}}}$, we can see that the results of the case that a_1 teaches a_2 and the case a_2 asks a_1 are the same.

- **If a_3 asks a_1 what a_3 doesn't know, the new epistemic states shown in $V^{\uparrow^{a_3}_{a_1}}$ are shown in Table 5.4.**

	$V_{a_i}(p)$	$V_{a_i}(q)$	$V_{a_i}(r)$
a_1	1	0.5	0.5
a_2	0.5	1	0
a_3	0	0	0.5

Table 5.4: The epistemic states in $V^{\uparrow^{a_2}_{a_1}}$

Comparing with the states in $V^{\downarrow^{a_1}_{\{a_2,a_3\}}}$, we can see that the results of the case that a_1 teaches a_3 and the case a_3 asks a_1 are different, for a_1 becomes to know the content of r.

We can see that the epistemic state of the teacher will change if he/ she is asked a question that he/ she doesn't know the content. In other words, the teacher becomes to know that he/ she did not know the answer of the question.

Chapter 6

Conclusions and Further Directions

6.1 Conclusion

As we explained in our introduction, the aim of this thesis is to employ the many-valued logic to the multi-agent system. Let us turn back to the two questions:

(Q1) Can we extend the semantics of epistemic logic to a many-valued one?

(Q2) Can we express the epistemic states by many-valued logic simply instead of Kripke semantics?

We can summarize our answers for these questions and technical contributions in the following two items:

(A1) We have shown a 4-valued logic which distinguishes the ordinary truth value of each proposition as well as the information is private or public. By private information transmission, since the recipient cannot read the contents he/ she does not change his/ her belief. This unsuccessful message passing corresponds to such practical situations that the information needs other background knowledge, password, deciphering protocol, and so on.

We have reconstructed the dynamic epistemic logic including the 4-valued logic, and have introduced the two kinds of negations, the truth tables for the logical connectives, their semantics, and its Hilbert-style axiomatization. Since the recursion axioms can reduce the formulae with dynamic operators to those without them, we can ensure the completeness and soundness if we disregard the second value of the pairwise truth.

(A2) We provided several many-valued semantics instead Kripke modal to express the epistemic states.

- **First, we introduced many-valued semantics based on Beall's off-topic interpretation. We have presented a semantic framework which not only overcomes some of the limitations of the three-valued semantics that Beall made use of, but also that is able to generalize the treatment to include different topics. We have shown, furthermore, that the framework provides a stable basis from which to have CL and WK. It is just a matter of adjusting the set of designated truth values. We have seen that this mixes two different issues: the attribution of truth values to a sentence, and the issue of whether it is on-topic or off-topic, given a settled topic. Therefore, the next step is to show our framework can be employed to an FDE basis, in order to introduce the on-topic/off-topic distinction with such a logical basis. As a result, we may also have gluts and gaps being on-topic or off-topic. The resulting system may be seen as the logic of Catuskoti, which acquires a new interpretation now, as an FDE-based system with on-topic/off-topic distinction on the top of it. The finally step is to generalize the addition of the on-topic/off-topic distinction over a logic not only to FDE, but also to any many-valued based case. The simplicity in covering both the influence of the topic distinction as well as of recovering the underlying consequence relation one started with is a great advantage of the method, and shows how fruitful Beall's interpretation is, when in connection with the semantic framework advanced here. It also highlights that topic neutrality, at least in this context, is a matter relative to the logic one advances as a basic case over which the topics are inserted.**

- **Second, we showed a many-valued semantics regarding** n**-agent as** n**-topic to express the epistemic states in multi-agent system, avoiding Kripke model. We introduced two pair semantics to express the different considerations of knowledge, and showed that they could be considered as the two three-valued logic: weak Kleene logic and strong Kleene logic. We showed that we could use the relation between to limit the valuation. Then, we gave a new semantics by combining the two semantics and the two states of unknown, and extended it to express the epistemic states of multi agents. Moreover, we gave two kinds of agent communication,*teaching* and *asking*, and showed the results of them could be different.**

6.2 Further Directions

In the current stage, there are a number of directions for future research. First, we can continue the study of a many-valued epistemic logic based on Kripke model. In Chapter 3, our formalization may still have redundancy; in the case we need a password for the private information, the password itself would be formalized in the very similar way to the channel variables. However, our objective is to formalize the unsuccessful communication in general. Thus, we will further develop the distinction between miscommunication by lack of necessary information and that by unsuccessful message transmission in future.

Second, we need to look for other useful and practical reinterpretations of the multiple values. Instead of topics, or agents, the difference of language could be a target. Also, we should search for other purposeful binary relations between topics/ agents/ languages so that we can represent natural partial orders between them.

Third, in Chapter 5, we ignored the situation of misunderstanding and introduced only two kinds of agent communication. Therefore, we may add a new epistemic value to express the state of misunderstanding, and give more kinds of agent communication besides what we introduced, e.g., the action *discussing* that makes a group of agents exchange their knowledge, or *forgetting* that lets some agents lose some information they knew before, etc.

Fourth, in Chapter 4, there is still a lot of ground to be covered by applying this semantic framework. For example, to add modal expansion of the language, say temporal operators, etc., in particular by revisiting the modal expansions of WK, explored by Fabrice Correia in [12], with our new framework.

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