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Master's Thesis

Modal Logic of Dasein
An Intrinsic Approach to Agents' Belief

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Chapter 1

Background Knowledge

1.1 Modal and Epistemic Logics

Modal logic is the formal logic that deals with all kinds of modalities [1]. Syntactically, modal logic expands basic propositional logic by introducing various modal operators (e.g. $\Box$, $\mathcal{G}$ and $\mathcal{O}$), which are able to act upon some proposition p in order to express its modal states, such as necessity ($\Box p$, it is necessary that p), temporality ($\mathcal{G}p$, it is going to be that p) and morality ($\mathcal{O}p$, it is obligation that p). And semantically speaking, modal logic is ordinarily evaluated over relational Kripke models accredited to Kripke [2], [3].

Epistemic logic, aka modal logic for knowledge, dates back to Hintikka [4]. Equipped with an epistemic modal operator $\mathcal{K}_i$ for each agent i in the system, epistemic logic is capable of talking about agents' knowledge (e.g. $\mathcal{K}_i p$, agent i knows that p). Basically, epistemic logic is treated as a type of multi-S5 modal logic, i.e., the binary relations in the Kripke models for the epistemic modal operators are all equivalence relations, whose induced partitions represent the equivalence classes of the agents' epistemically indistinguishable possible worlds.

Another tightly relevant theme is modal logic for belief, also called as doxastic logic sometimes. Compared to modal logic for knowledge, modal logic for belief simply replaces the modal operator $\mathcal{K}_i$ with $\mathcal{B}_i$ in order to express agents' belief (e.g. $\mathcal{B}_i p$, agent i believes that p). Indubitably, modal logics for knowledge and for belief share very close philosophical motifs as well as mathematical techniques, so that they are customarily discussed together under the broad genre of epistemic logic [5]. And in terms of proof systems, briefly speaking, modal logic for belief is regarded as a type of multi-KD45 modal logic such that it replaces the axiom T in modal logic for knowledge by the axiom D, in other words intuitively, if agent i knows that p then p must actually be true in the real world, but it need not be so if agent i believes that p.

Ever since its discovery, epistemic logic has exerted itself conspicuously with vibrant adaptability, not only for elucidating philosophical puzzles [6], [7] but also for articulating a wide range of interesting studies in computer science [8], including distributed computing, network communication, artificial intelligence, etc. Furthermore, epistemic logic has also been expanded, inter alia, onto non-propositional knowledge beyond knowing-that [9], including knowing-how logic [10], knowing-why logic [11], knowing-who logic [12], etc.

We shall formally recall the basic definitions and results about doxastic logic in Chapter 3.

1.2 Existentialism and Dasein

The philosophical trend with the label 'existentialism' can be best summed up by the famous saying of Jean-Paul Sartre [13]: existence precedes essence. In order to correctly understand this declaration, here we need to take a hasty glimpse of the history of western philosophy. Ever since its origin from Ancient Greece, traditional philosophy has been the first science that accumulates into human being's true knowledge, literally as its Greek etymon 'to love wisdom'; and metaphysics is the first subject in philosophy; and ontology is the first subject in metaphysics; and thus the first philosophical question is 'What is the essence of everything that exists in this world?', for simply as Plato's allegory of the cave forcibly argues, what we directly perceive in everyday life may be nothing but unreal illusion. According to existentialism, however, the above question itself is obscure, because the meaning of 'to exist' must get clarified in advance before we could ever ask about the essence of any existing things; in fact whenever we query the essence of something, we must have implicitly presumed its existence, otherwise we would never become aware of that thing, let alone study its essence. So shall we then ask "What is the essence of 'to exist'?" in a similar way? No, because 'to exist' itself is not something that exists in this world, like a hammer or a nail. Instead we may only ask 'How do things exist in this world?', nevertheless, such a question will also be effectively meaningless if everything exists only in a single manner. Therefore, the plenary existentialism will be just a pack of nonsense, unless at least two things exist in the world by radically different means.

Let us now spotlight the founder of existentialism, Martin Heidegger, who is reckoned as one of the most significant figures in contemporary philosophy, or even to say without exaggeration, is presumably the most significant one. In his most renowned monograph *Sein und Zeit* [14], which is translated into English as *Being and Time* [15], Heidegger systematically innovates a revolutionary collection of philosophical concepts, disclosed in his unprecedented German glossary where words and phrases get creatively reassembled and reinterpreted, integrating into the so-called Heideggerian terminology [16]. Amongst his unique vocabulary — honestly speaking, it must not be entitled 'unique' any more, for it has preeminently become part of the boilerplate that is nowadays lectured to every college student who majors in philosophy — the notion of Dasein indisputably stands out as the very epicenter. Dasein, different from a hammer or a nail, exists in this world in another way of its own.

What is Dasein? As readers can imagine, an exhaustive interpretation should occupy a book's volume or so, and thus from such a censorious angle, any rudimentary explanation here must be at best primitive in some degree. Hence we have no other choice but to compromise and proceed with a primary exposition of Dasein. Translated verbatim as 'there-being', the abstract entity 'Da-sein' is typically exemplified by a human (or else, from a fashionable modern viewpoint, can also refer to an intelligent-like-a-human AI subject, which may become genuinely realized in the near future although yet not today), in the sense that he dwells amidst some cultural society, engages everything that he experiences — including his own existence — with certain meaning specific to himself, and, based on his care for such phenomenological meanings, chooses his action toward one exclusive future amongst all the possibilities that keep open to him. All in all, the principal tenet is that, according to Heidegger, all the above features of Dasein are beyond sheer features: they are essentially a priori to Dasein, constituting the ineluctable and transcendental preconditions for Dasein's existence as well as intelligibility.

Anyway, this thesis does not mean to teach Heidegger's philosophy, a mission too ambitious to squash into a research paper. Some introductory textbooks on Heidegger and Dasein — exemplary ones like Mulhall [17], Polt [18] and Vallega-Neu [19] — are supposed to play their due roles.

Chapter 2

Existing Problems and Our Solution

2.1 Balkanization in Philosophy

It is a veracious pity that, to say the truth, the immense divergence between continental and analytic philosophy has pervaded academia for over a century hitherto, but very little attempt has been carried out substantially in order to bridge the gap. While we appreciate that these two strands differ seriously in considerable facets, we also surmise that an abundance of stimulating ideas from one side can be handily borrowed by the other side quid pro quo, so long as their philosophical interest is shared in common by and large.

Despite being one of the most celebrated continental philosophers, Heidegger's thought has thus far gained meager attention from the analytic side, and in particular no examination via logic has been touched upon at all. An amazing fact ascertained by us is that, non pro rata to a vast number of voluminous treatises on Dasein, no more than a handful of works have more or less stressed some relevance between Dasein and modality [20]–[22]. Withal, an inclusive commentary on the relation between Dasein and AI is spotted in Dreyfus [23]'s notable essay, but overall, general discussion uttered purely in natural language is prone to stay impractical unless we utilize some formal logical system in order to pin down the core philosophical idea. On the flip side, ironically, the notion of 'daseinisation' has been successfully introduced into topos quantum theory [24], a type of quantum logic. Nevertheless as its name connotes, topos quantum theory is a method of mathematical physics for depicting quantum mechanics, and hence it just cosmetically adopts the designation of Dasein, but fundamentally speaking, it has nothing to do at all with what Heidegger originally meant by Dasein. As a regrettable conclusion, Heidegger's philosophy on Dasein has never been rigorously formalized by any proper logical system.

As readers can already guess from the title of this thesis, our ultimate aim is to work out an intrinsic representation of agents' knowledge or belief, and Heidegger's theory of Dasein will stand firm as the philosophical foundation for our project. Therefore, this thesis will produce a seminal cornerstone for filling up the above forsaken blank between the two hostile schools of philosophy. Our major idea will get unfolded intuitively in the following Section 2.3.

2.2 Externality of Epistemic Logic

Very popular amongst logicians, philosophers as well as computer scientists, the rich variations of epistemic logics might seem to already provide an all-encompassing account of agents' knowledge and belief. Nevertheless as far as we can see, all those classic- and neo-types of epistemic logics incur one prevalent issue: they depict agents' knowledge or belief from an omniscient, indifferent and extrinsic perspective, so that in principle, only after the entire Kripke model has been accurately procured could modal formulae subsequently get evaluated. This meticulous prerequisite for employing epistemic logic, as a matter of fact, is not often able to get fully satisfied in practice, no matter either from the standpoint of one of the agents inside the logical system or simply as an onlooker from the sidelines. Such a problem becomes even severer regarding modal logic for belief, forasmuch as someone's belief is generally apprehended as private opinions that are unobservable to others. Moreover, in doxastic logic an agent is supposed to keep vulnerable to false beliefs, but if the bona fide Kripke model is already accessible to him, then he should just become aware that he is actually believing something false in any event as such — so why does he stubbornly cling to his own false beliefs after all?

In a word, the above problem is liable to arise when there exists uncertainty about the actual Kripke model, and so a very natural response might sound like: if unsure amongst a bunch of Kripke models, why not simply connect all the models in totality by due epistemic relations so as to form a compound large Kripke model? Then for instance, a decent number of strategies in Wang and Wang [25] regarding bundled S5 modal operators are supposed to be of reference value. Such a naïve response is undeniably rational to some extent, however, we cannot help judging it to be rather far away from a satisfactory solution, much less a perfect one. Here come our decisive objections to the above proposal. Firstly, it is unclear whether this workaround truly solves the problem. In one respect, exactly because the Kripke models are for epistemic logic (rather than some other type of modal logic), we are able to introduce new epistemic relations representing uncertainty about the Kripke model, such that the compound model is still a Kripke model for epistemic logic. But meanwhile, exactly because the compound model is still a Kripke model for epistemic logic, it also succumbs to the very same kind of problem. How we should scrupulously avert such an endless cycle seems to be a headache. Secondly, even if this approach indeed works, in virtually every case except for a small number of the simplest ones, the compound Kripke model can be reasonably anticipated to blow up excessively large — out of the same rationale as the exponential blow-up regarding modal logic's complexity [26], but for now, different Kripke models are brutally glued together so that the compound Kripke model per se is already exponentially large. Thus pragmatically speaking, such convolution is hardly acceptable, either for computers or for human beings.

All things considered, the problem seems to be inherent in ordinary epistemic logic, for it barely furnishes an extrinsic account of agents' knowledge or belief. In fact we notice that, even when an agent or a bystander is uncertain about the current Kripke model, intuitively, all the different Kripke models that he thinks possible are never randomly selected like capricious lottery from the total pool of all mathematically admissible Kripke models; instead, all of his possible Kripke models are usually quite similar to each other, making up an organic family that is simply explainable by just a few lines of succinct, essential and intrinsic grounds. Inspired by such a critical observation, let us move over and seek further for a novel type of epistemic logic that embraces such kind of intrinsic approach.

2.3 Dasein as an Intrinsic Approach

How may the notion of Dasein hint at an intrinsic approach toward epistemic logic? To give an illuminating analysis, the following Example 1 further unveils the deficiency of ordinary epistemic logic exactly due to its extrinsicness — a rather counterintuitive defect which may be even disastrous to the logic's practical application:

Example 1. Consider modal logic for belief. Let p be a proposition and also let a denote Alice as an agent, then the modal formula $\mathcal{B}_a p$ intuitively says 'Alice believes that p'.

Now suppose you want to identify whether $\mathcal{B}_a p$ is true or false in reality, namely, whether Alice really believes p or not. So what should you do? In practice, would you actually go to consult your Kripke model (where is it?)

and mechanically evaluate the formula $\mathcal{B}_a p$ in the current possible world? No, in general, you would not. As a sharp contrast, if you want to know when the next solar eclipse will happen and you look up in the calendar, no one will blame you for that very natural action. Nevertheless in the former case on Alice's belief, a similar action sounds terribly contrived — if not insane.

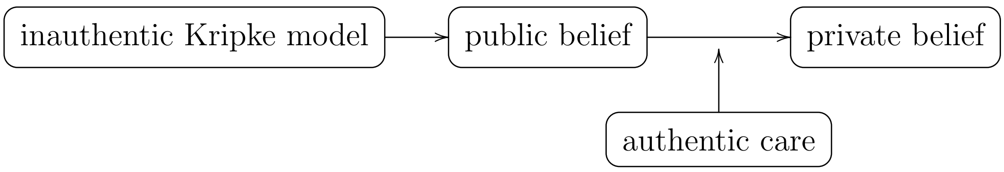
Hence instead, what should be your most intuitive action? The answer is simple: just go to meet Alice and ask her in person.

Let us cool down a while for deeper cogitation into the crux of the matter. To be honest, we concede that this most intuitive action may not always be a viable choice, nor does it always guarantee you the ultimately correct answer, either. You may not be able to find Alice or get into contact with her; when you ask Alice whether she really believes p or not, she may refuse to respond; and even if she responds, she may be lying to you. Occasionally in the last case based on other clues of information, de facto, you may even be able to infer that Alice actually is lying to you. However in theory, the method of directly asking Alice is not only the most intuitive one, but also the one with the highest priority, i.e., if Alice personally admits or denies believing p, then such testimony should always override any other indirect indication of her belief. Legitimately speaking, Alice's own confession is exactly the optimal answer to this query that you would ever hope for.

In essence, Example 1 accentuates that it is always Alice herself, rather than any Kripke model, who has the final say on whether she really believes p or not. To put it in another way, Alice instinctively bears such self-explanatory right to secretly change her mind at arbitrary moment, without having to notify anyone or any Kripke model in order to implement corresponding update. If actually so, however, then what on earth does the Kripke model reflect after all? Right now, it is a propitious time for us to turn to Dasein in search of an answer.

Heidegger ingeniously distinguishes two disparate modes in which Dasein could be: the authentic one and the inauthentic one. While in the former mode as we have unravelled in brief, a man as Dasein confronts the exterior world through his authentic care that is uniquely intelligible only to himself, in the latter mode — portrayed by Heidegger as falling into 'das Man', a German term incapable of exact translation into English, although the translation as 'the "they"' is somehow traditionally acknowledged — he acts habitually in a predictable manner in accordance with social norms and expectations, consequently dissolves himself into the 'they'. Notwithstanding a degree of pejorative sense as the word 'inauthentic' indicates, in actuality, all of us mortals unconsciously and inevitably lead such an inauthentic existence during most of time in our everyday life. (we are not always doing anything crazy!) Therefore, this mode of having-to-be-fallen also composes a sine qua non dimension of Dasein.

By far, our kernel idea has become fairly manifest: on the one hand, the Kripke model represents this inauthentic the 'they', i.e. from a stance of collective common sense, what every agent is supposed to think and act under his current circumstances, and thus the same Kripke model is publicly shared as environmental common knowledge amongst all the agents as well as outsiders; on the other hand, based on his personal care, each agent as Dasein also enjoys autocratic sovereignty over himself of liberally recasting any of his own authentic resolution. Hence, the panorama of our plan can be neatly illustrated as the following flow chart:



Nonetheless as the above flow chart reveals, in order to acquire a deterministic logic about all the agents' authentic belief, undoubtedly we still have to take an omniscient mindset and designate every agent's care in detail. It is the limit set by information theory [27] and hence is impossible to transcend by whatever miracle. So then, what are the main merits of our framework over ordinary epistemic logic? Actually, our logic's central competitiveness rests upon its lightweight, robustness, flexibility, expansibility as well as modularity.

Specially speaking, when our logic is put into practical application, different agents could possess different forms of care, and moreover, a portion of agents' private care might remain unknown. Emphatically, none of the above shortages would collapse our whole logical system, for each agent's care is an independent and detachable module that only affects his own authentic belief. In consequence, the operations of adding, deleting and altering agents' care are extremely easy to perform, simply leaving the public Kripke model intact in situ without any risk of exponential blow-up. Anyway, our logic's superiority will become fairly palpable to readers as soon as the mathematical definitions are formally established.

A final word on our logic's appellation. Although within this thesis as an elementary stage, only modal logic for agents' belief will be formally handled, it can be foreseen that the modus operandi of Dasein as authentic self should be similarly practicable for a variety of other types of modal logics as well. Hence in order to keep general, we decide to call our logic austerely as modal logic of Dasein. We shall return to this affair for future discussion in Section 8.1.

Chapter 3

Mathematical Preliminaries

This chapter concisely reviews ordinary modal logic for belief as our preliminaries. Fix a set of propositions P and a set of agents I. Our basic logical language MLB is defined as the following:

Definition 1 (Language MLB)**.** A well formed MLB-formula φ is inductively defined by the following Backus–Naur form:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{B}_i\varphi$$

where $p \in P$, $i \in I$.

By convention, $\bot$, $(\varphi \vee \psi)$ and $(\varphi \to \psi)$ are respectively defined as abbreviations of $\neg\top$, $\neg(\neg\varphi \wedge \neg\psi)$ and $\neg(\varphi \wedge \neg\psi)$, for any MLB-formulae φ and ψ.

MLB-formulae are evaluated over Kripke models for belief, which are based on KD45 Kripke frames and will be simply called as Kripke models hereinafter — that will not cause any confusion, inasmuch as no other classes of Kripke models are tackled at the same time.

Definition 2 (Kripke Model)**.** A Kripke model $\mathfrak{M}$ is a triple (S, R, V) where:

- S is a nonempty set of possible worlds.
- $R : I \to \mathcal{P}(S \times S)$ is a belief function such that for any agent $i \in I$, $R(i) \subseteq S \times S$ is a KD45, viz. serial, transitive and Euclidean binary relation over S.
  - Seriality: $\forall s \in S \exists t \in S$, $(s, t) \in R(i)$.
  - Transitivity: $\forall (s, t), (t, r) \in R(i)$, $(s, r) \in R(i)$.

- Euclideanness: $\forall (s,t), (s,r) \in R(i)$, $(t,r) \in R(i)$.

- $V : P \to \mathcal{P}(S)$ is a valuation function.

As for semantics, because we want to preserve the notation '$\vDash$' — namely the notion of satisfaction — for our authentic semantics of modal logic of Dasein defined later on in Definition 7, here we have to make use of another notation '$\Vdash$' to represent the ordinary semantics of modal logic for belief. According to our philosophical elaboration in Section 2.3, the Kripke model in Definition 2 together with the ordinary semantics stands for the 'they', where Dasein 'falls' into an inauthentic mode. Also, following practice in set theory as well as intuitionistic logic, the notation '$\Vdash$' can be read as 'force', and thus we decide to call the ordinary semantics as F-semantics, a pun for both 'falling' and 'forcing'.

Definition 3 (F-Semantics)**.** Given a Kripke model $\mathfrak{M} = (S, R, V)$ and a possible world $s \in S$. The F-semantics for any MLB-formula φ is inductively defined as the following:

$$\begin{aligned}
\mathfrak{M}, s \Vdash \top &\iff \text{always} \\
\mathfrak{M}, s \Vdash p &\iff s \in V(p) \\
\mathfrak{M}, s \Vdash \neg\varphi &\iff \text{not } \mathfrak{M}, s \Vdash \varphi \\
\mathfrak{M}, s \Vdash (\varphi \wedge \psi) &\iff \mathfrak{M}, s \Vdash \varphi \text{ and } \mathfrak{M}, s \Vdash \psi \\
\mathfrak{M}, s \Vdash \mathcal{B}_i\varphi &\iff \text{for all } t \in S \text{ such that } (s,t) \in R(i), \mathfrak{M}, t \Vdash \varphi
\end{aligned}$$

By routine, we have the following KD45 proof system `PS-F` for the F-semantics of ordinary modal logic for belief:

Definition 4 (Proof System `PS-F`)**.** The Hilbert-style proof system `PS-F` comprises the following axioms and rules:

Axioms:

TAUT	all of the propositional tautologies
K	$\mathcal{B}_i(\varphi \to \psi) \wedge \mathcal{B}_i\varphi \to \mathcal{B}_i\psi$
D	$\neg\mathcal{B}_i \perp$
4	$\mathcal{B}_i\varphi \to \mathcal{B}_i\mathcal{B}_i\varphi$
5	$\neg\mathcal{B}_i\varphi \to \mathcal{B}_i\neg\mathcal{B}_i\varphi$

Rules:

MP	$\dfrac{\varphi \to \psi \quad \varphi}{\psi}$
NEC	$\dfrac{\varphi}{\mathcal{B}_i\varphi}$, where φ is a theorem not depending on any premises

Theorem 1. *The proof system* `PS-F` *in Definition 4 is sound and strongly complete with respect to the F-semantics in Definition 3.*

Chapter 4

General Modal Logic of Dasein

Following our analysis in Section 2.3, semantics of modal logic of Dasein should surpass the inauthentic F-semantics by taking into account every agent's authentic care. As we have also mentioned, each agent's private care is actually nothing more than a dispensable module of the logic and may not be known during practical application (in which case the authentic semantics simply degenerates into the F-semantics, but only for that particular agent's own belief). Nonetheless for our tentative survey, let us appoint every agent's care straightforwardly so that we can secure a both deterministic and authentic logic.

Hence, what kind of form could an agent's care take? Generally speaking, if we do not delve into the internal mechanism of an agent's care but just want to sketch it in a functionalist way, then we may prefer to regard it expediently as a black box: we can inquire an arbitrary MLB-formula φ as the input, and then the agent conscientiously answers whether he authentically believes φ or not as the output. Thus in the most general case, an agent's authentic belief is characterized as a discretionary subset of formulae — it is not compulsorily consistent, nor is it necessarily closed under certain logical rules, although for specific applications we can ad libitum contemplate these or those ordinances which we deem reasonable.

There is, however, one intuitive attribute of any specific agent i's care even as a nearly omnipotent black box: when we ask agent i whether he believes φ or not, the formula φ naturally never contains any modal operator $\mathcal{B}_i$ which is not inside another $\mathcal{B}_j$'s scope. For an everyday instance, suppose that Alice is denoted as an agent $a \in I$ and that $p, q \in P$ are two propositions. If we want to know whether Alice believes $\mathcal{B}_a p$ or not, intuitively, we will not bother to ask her 'do you believe that you believe p?'; instead, we will directly ask 'do you believe p?'. Even if we want to know whether Alice believes $\mathcal{B}_a p \wedge q$ or not, it still sounds rather awkward (and arguably ambiguous) to ask 'do you believe that you believe p and that q?'; the much more natural way is to first off ask whether she believes p, and if 'yes', then ask again whether she believes q. In another aspect, when Alice peruses the formula $\mathcal{B}_a p \wedge q$ by herself, she is also very likely to at first regard the '$\mathcal{B}_a p$' part as an independent whole; and if she really believes p, then in the next step, she will naturally ignore this part and just inspect whether she believes q. Nevertheless by contrast, further suppose that Charlotte is denoted as an agent $c \in I$, then in order to know whether Alice believes $\mathcal{B}_c \mathcal{B}_a p$ or not there is simply no other way except for directly asking her 'do you believe that Charlotte believes that you believe p?'. And it is not difficult to note where the pivotal difference lies: in the formula $\mathcal{B}_c \mathcal{B}_a p$, the 'Alice' referred to by the modal operator $\mathcal{B}_a$ inside $\mathcal{B}_c$'s scope actually signifies the 'Alice' in Charlotte's view rather than the real Alice herself, and thus is no longer the authentic Alice but instead the inauthentic counterpart.

In sum, the above simple cases doubtlessly shed light on how we flesh and blood manage our belief every day: we tend to remember solely each 'atomic' belief, into which complicated belief will gradually get reduced. While which should count as the 'atomic' belief remains to be a degree of freedom, on the whole, such general orientation keeps to be observed — at least for the sake of less memory burden! And peculiarly, an agent normally pays no deliberate heed at all toward his own higher-order belief, which is supposed to stay plainly transparent to himself all the time. In other words, an agent i's care is generally outward-directed and hence deals without any appearance of $\mathcal{B}_i$ that is not within another $\mathcal{B}_j$'s scope. Based on the above intuitive analysis, it is now the opportune time to formalize our idea into a stringent definition of care in general, together with the authentic semantics of general modal logic of Dasein.

Definition 5 (Sublanguage MLB_i)**.** For any fixed agent $i \in I$, language MLB_i is a sublanguage of language MLB, such that $\mathsf{MLB}_i = \{\varphi \in \mathsf{MLB} \mid \mathcal{B}_i \text{ in } \varphi$ only appears within another $\mathcal{B}_j$'s scope, where $j \in I\}$.

Definition 6 (Kripke Model with General Care)**.** A Kripke model with general care $\mathfrak{G}$ is a pair $(\mathfrak{M}, \gamma)$ where:

- $\mathfrak{M} = (S, R, V)$ is a Kripke model.
- $\gamma : I \to \mathcal{P}(\mathsf{MLB})$ is a general care function such that for any agent $i \in I$, $\gamma(i) \subseteq \mathsf{MLB}_i$.

Intuitively, each agent i's general care $\gamma(i)$ is just a black box, dictating the subset of MLB_i-formulae that he really believes, without any appearance of the modal operator $\mathcal{B}_i$ outside of any other $\mathcal{B}_j$'s scope. Instead, when manipulating self-higher-order belief involving a plurality of superposed $\mathcal{B}_i$s, the formula in process will always get reduced down to the 'atomic' belief. Such intuition is strictly captured by the following Definition 7 through simultaneously defining both the general authentic semantics — which, in comparison with the usage of the notation '$\Vdash$' for the inauthentic F-semantics in Definition 3, uses the notation '$\vDash$' as satisfaction and so is exactly 'the' semantics — and the corresponding reduction together as a mutually inductive definition, but anyhow it is easy to see that such mutual induction is indeed well founded and thus well defined:

Definition 7 (General Semantics and Reduction)**.** Given a Kripke model with general care $\mathfrak{G} = (\mathfrak{M}, \gamma) = ((S, R, V), \gamma)$ and a possible world $s \in S$. The general semantics for any MLB-formula φ is inductively defined as the following:

$$
\begin{aligned}
\mathfrak{G}, s \vDash \top &\iff \text{always} \\
\mathfrak{G}, s \vDash p &\iff s \in V(p) \\
\mathfrak{G}, s \vDash \neg\varphi &\iff \text{not } \mathfrak{G}, s \vDash \varphi \\
\mathfrak{G}, s \vDash (\varphi \wedge \psi) &\iff \mathfrak{G}, s \vDash \varphi \text{ and } \mathfrak{G}, s \vDash \psi \\
\mathfrak{G}, s \vDash \mathcal{B}_i\varphi &\iff \mathsf{\rho}(\mathfrak{G}, s, i, \varphi) \in \gamma(i)
\end{aligned}
$$

where the reduced formula $\mathsf{\rho}(\mathfrak{G}, s, i, \varphi) \in \mathsf{MLB}_i$ is inductively defined as the following:

$$
\begin{aligned}
\mathsf{\rho}(\mathfrak{G}, s, i, \top) &= \top \\
\mathsf{\rho}(\mathfrak{G}, s, i, p) &= p \\
\mathsf{\rho}(\mathfrak{G}, s, i, \neg\varphi) &= \neg\mathsf{\rho}(\mathfrak{G}, s, i, \varphi) \\
\mathsf{\rho}(\mathfrak{G}, s, i, (\varphi \wedge \psi)) &= (\mathsf{\rho}(\mathfrak{G}, s, i, \varphi) \wedge \mathsf{\rho}(\mathfrak{G}, s, i, \psi)) \\
\mathsf{\rho}(\mathfrak{G}, s, i, \mathcal{B}_i\varphi) &= \begin{cases} \top, & \text{if } \mathfrak{G}, s \vDash \mathcal{B}_i\varphi \\ \bot, & \text{otherwise} \end{cases} \\
\mathsf{\rho}(\mathfrak{G}, s, i, \mathcal{B}_j\varphi) &= \mathcal{B}_j\varphi, \text{ where } j \neq i
\end{aligned}
$$

So, here we are. Since for the most general purpose, no extra restraint is put onto the black box of agent i's care $\gamma(i)$, at present actually, none of the axioms K, D, 4 and 5 or the rule NEC (cf. Definition 4) keep valid with respect to the general semantics in the above Definition 7. (The axioms TAUT and the rule MP, inherited from classical propositional logic, unmistakably remain to be valid.) This does not sound like any sort of good news.

Still, may we strive to prove a few universal results regarding the general semantics:

Proposition 2. *For any fixed agent $i \in I$, any Kripke model with general care $\mathfrak{G} = (\mathfrak{M}, \gamma) = ((S, R, V), \gamma)$ and any possible world $s \in S$, if $\gamma(i) \neq \varnothing$, then the axiom 4: $\mathcal{B}_i\varphi \to \mathcal{B}_i\mathcal{B}_i\varphi$ is valid at $\mathfrak{G}, s$ if and only if $\mathfrak{G}, s \vDash \mathcal{B}_i\top$.*

Proof. Since $\gamma(i) \neq \varnothing$, there exists some MLB_i-formula $\varphi \in \gamma(i)$ such that $\mathfrak{G}, s \vDash \mathcal{B}_i\varphi$ (for note that as φ is an MLB_i-formula, we have $\rho(\mathfrak{G}, s, i, \varphi) = \varphi$). For arbitrary MLB-formula ψ, suppose $\mathfrak{G}, s \vDash \mathcal{B}_i\psi$, then by definition of the general semantics, $\mathfrak{G}, s \vDash \mathcal{B}_i\psi \to \mathcal{B}_i\mathcal{B}_i\psi \iff \mathfrak{G}, s \vDash \mathcal{B}_i\mathcal{B}_i\psi \iff \rho(\mathfrak{G}, s, i, \mathcal{B}_i\psi) \in \gamma(i) \iff \top \in \gamma(i) \iff \mathfrak{G}, s \vDash \mathcal{B}_i\top$. □

Proposition 3. *For any fixed agent $i \in I$, any Kripke model with general care $\mathfrak{G} = (\mathfrak{M}, \gamma) = ((S, R, V), \gamma)$ and any possible world $s \in S$, if $\gamma(i) \neq \mathsf{MLB}_i$, then the axiom 5: $\neg\mathcal{B}_i\varphi \to \mathcal{B}_i\neg\mathcal{B}_i\varphi$ is valid at $\mathfrak{G}, s$ if and only if $\mathfrak{G}, s \vDash \mathcal{B}_i\neg\neg\top$.*

Proof. Since $\gamma(i) \neq \mathsf{MLB}_i$, there exists some MLB_i-formula $\varphi \notin \gamma(i)$ such that $\mathfrak{G}, s \vDash \neg\mathcal{B}_i\varphi$ (for note that as φ is an MLB_i-formula, we have $\rho(\mathfrak{G}, s, i, \varphi) = \varphi$). For arbitrary MLB-formula ψ, suppose $\mathfrak{G}, s \vDash \neg\mathcal{B}_i\psi$, then by definition of the general semantics, $\mathfrak{G}, s \vDash \neg\mathcal{B}_i\psi \to \mathcal{B}_i\neg\mathcal{B}_i\psi \iff \mathfrak{G}, s \vDash \mathcal{B}_i\neg\mathcal{B}_i\psi \iff \rho(\mathfrak{G}, s, i, \neg\mathcal{B}_i\psi) \in \gamma(i) \iff \neg\bot \in \gamma(i) \iff \mathfrak{G}, s \vDash \mathcal{B}_i\neg\neg\top$. □

In addition, the following Proposition 4 rigidly formalizes our intuitive comprehension about the general semantics of modal logic of Dasein: each agent's authentic care is modular, whose influence never overflows the private territory of his own belief.

Proposition 4. *For any fixed agent $i \in I$ and any MLB_i-formula φ, agent i's general care $\gamma(i)$ has no influence at all over φ's evaluation.*

Proof. From Definition 7, it is very obvious. □

After all, the semantics in Definition 7 is general to the extreme and so not too many fancy tales can be narrated just about itself. We have decided to present this general semantics ab initio for both theoretical simplicity and applicative versatility, and starting from the next Chapter 5, we shall take the intensional characterization of agents' care into consideration in order to consummate a specific working semantics of modal logic of Dasein, to which our remaining technical concentration will be chiefly devoted.

Chapter 5

A Working Semantics

5.1 Care and Semantics

In order to shape the semantics into a more tangible and maneuverable form, we cannot content ourselves with describing agents' care in a simply extensional way as a black box function $\boldsymbol{\gamma}$. Rather, we must endeavor to delineate the intrinsic causes of agents' belief via the notion of care, i.e., an agent's care should not only inform us of his authentic belief, but also expound the motives why he believes this and does not believe that.

Furthermore, it seems a very natural choice to explicate agents' intrinsic care by some logical system as well, for we can then expect an agent's belief to automatically become closed under certain logical rules. However still, there exist countless logical systems, and furthermore as we have mentioned, one agent's care need even not be in the same form as another agent's, namely, different agents could choose to evince their care through different logical systems. The general instruction persisting as such, for the time being, let us labor to devise a minimally basic working semantics of modal logic of Dasein, without any external help from other logical systems. Accordingly, only currently available resources are allowed to exploit. So what do we have now? Fortunately, we already retain one another logical system at hand — the inauthentic F-semantics.

Thanks to proximate adjacency between the inauthentic semantics and the authentic one, as a complimentary virtue, it is also pretty easy and straightforward to represent agents' care through the F-semantics. Hence as a further simplification, let us additionally assume that regarding any input MLB_i-formula φ, agent i will never trouble to penetrate into the structure of φ, i.e., he will always appraise φ as an entirety. Nonetheless, he is free to modify φ through dressing it up with accessory appendages that are assigned privately according to his authentic care, after which it will then get checked with respect to the F-semantics whether agent i believes such modified formula or not. In one word, the following definitions of care and semantics formalize our above intuition:

Definition 8 (Language MLB')**.** Let p_x be an arbitrarily fixed fresh proposition such that $p_x \notin P$. A well formed MLB'-formula φ is inductively defined by the following Backus–Naur form:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{B}_i\varphi$$

where $p \in P \cup \{p_x\}$, $i \in I$.

Definition 9 (Uniform Substitution)**.** For any MLB'-formula φ and any MLB-formula ψ, let $\varphi[\psi]$ denote the resulting MLB-formula of uniformly substituting p_x in φ with ψ.

Definition 10 (Kripke Model with Care)**.** A Kripke model with care $\mathfrak{C}$ is a pair $(\mathfrak{M}, \mathsf{\kappa})$ where:

- $\mathfrak{M} = (S, R, V)$ is a Kripke model.
- $\mathsf{\kappa} : I \to \mathsf{MLB}'$ is a care function.

Intuitively, for each agent $i \in I$, his care $\mathsf{\kappa}(i)$ as an MLB'-formula unequivocally enunciates how he would modify any input MLB_i-formula φ, where the freshly introduced proposition p_x serves as an ad hoc placeholder for φ (thus practically speaking, $\mathsf{\kappa}(i)$ should always include p_x in order to be meaningful by any means, although we have yet not officially proclaimed such a constraint). The corresponding semantics then goes as the following Definition 11, while the reduction just remains the same as in Definition 7:

Definition 11 (Semantics and Reduction)**.** Given a Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \mathsf{\kappa}) = ((S, R, V), \mathsf{\kappa})$ and a possible world $s \in S$. The semantics for any MLB-formula φ is inductively defined as the following:

$$\begin{aligned}
\mathfrak{C}, s \vDash \top &\iff \text{always} \\
\mathfrak{C}, s \vDash p &\iff s \in V(p) \\
\mathfrak{C}, s \vDash \neg\varphi &\iff \text{not } \mathfrak{C}, s \vDash \varphi \\
\mathfrak{C}, s \vDash (\varphi \wedge \psi) &\iff \mathfrak{C}, s \vDash \varphi \text{ and } \mathfrak{C}, s \vDash \psi \\
\mathfrak{C}, s \vDash \mathcal{B}_i\varphi &\iff \mathfrak{M}, s \Vdash \mathcal{B}_i\mathsf{\kappa}(i)[\mathsf{\rho}(\mathfrak{C}, s, i, \varphi)]
\end{aligned}$$

where the reduced formula $\mathsf{\rho}(\mathfrak{C}, s, i, \varphi) \in \mathsf{MLB}_i$ is inductively defined as the following:

$$
\begin{aligned}
\mathsf{\rho}(\mathfrak{C}, s, i, \top) &= \top \\
\mathsf{\rho}(\mathfrak{C}, s, i, p) &= p \\
\mathsf{\rho}(\mathfrak{C}, s, i, \neg\varphi) &= \neg\mathsf{\rho}(\mathfrak{C}, s, i, \varphi) \\
\mathsf{\rho}(\mathfrak{C}, s, i, (\varphi \wedge \psi)) &= (\mathsf{\rho}(\mathfrak{C}, s, i, \varphi) \wedge \mathsf{\rho}(\mathfrak{C}, s, i, \psi)) \\
\mathsf{\rho}(\mathfrak{C}, s, i, \mathcal{B}_i\varphi) &= \begin{cases} \top, & \text{if } \mathfrak{C}, s \vDash \mathcal{B}_i\varphi \\ \bot, & \text{otherwise} \end{cases} \\
\mathsf{\rho}(\mathfrak{C}, s, i, \mathcal{B}_j\varphi) &= \mathcal{B}_j\varphi, \text{ where } j \neq i
\end{aligned}
$$

Assuredly, the semantics in Definition 11 is locally a specific case of the general semantics in Definition 7, as the following Proposition 5 demonstrates:

Proposition 5. *For any Kripke model with care* $\mathfrak{C} = (\mathfrak{M}, \mathsf{\kappa}) = ((S, R, V), \mathsf{\kappa})$ *and any possible world* $s \in S$*, there exists a Kripke model with general care* $\mathfrak{G} = (\mathfrak{M}, \mathsf{\gamma}) = ((S, R, V), \mathsf{\gamma})$ *which carries the same underlying Kripke model* $\mathfrak{M}$*, such that for any* MLB*-formula* φ*,* $\mathfrak{C}, s \vDash \varphi \iff \mathfrak{G}, s \vDash \varphi$*.*

Proof. For any agent $i \in I$, let $\mathsf{\gamma}(i) = \{\varphi \in \mathsf{MLB}_i \mid \mathfrak{M}, s \Vdash \mathcal{B}_i\mathsf{\kappa}(i)[\varphi]\}$, and then the claim can be directly proved by induction. □

Also in parallel with Proposition 4 for the general semantics, now we have the following Proposition 6 as its counterpart:

Proposition 6. *For any fixed agent* $i \in I$ *and any* MLB_i*-formula* φ*, agent* i*'s care* $\mathsf{\kappa}(i)$ *has no influence at all over* φ*'s evaluation.*

Proof. From Definition 11, it is not difficult to reason that $\mathsf{\kappa}(i)$ will never be used during the evaluation of φ. □

And perhaps more prominently, the following Proposition 7 affirms that in practical application of modal logic of Dasein, when there is a lack of information about any agent i's authentic care, simply setting $\mathsf{\kappa}(i) = p_x$ will exactly downgrade his own belief back toward inauthenticity of the 'they':

Proposition 7. *For any Kripke model with care* $\mathfrak{C} = (\mathfrak{M}, \mathsf{\kappa}) = ((S, R, V), \mathsf{\kappa})$ *and any agent* $i \in I$*, if* $\mathsf{\kappa}(i) = p_x$*, then for any* MLB*-formula* φ *and any possible world* $s \in S$*,* $\mathfrak{C}, s \vDash \mathcal{B}_i\varphi \iff \mathfrak{M}, s \Vdash \mathcal{B}_i\varphi$*.*

Proof. By induction on the structure of φ. From the definition we have $\mathfrak{C}, s \vDash \mathcal{B}_i\varphi \iff \mathfrak{M}, s \Vdash \mathcal{B}_i\mathsf{\kappa}(i)[\mathsf{\rho}(\mathfrak{C}, s, i, \varphi)] \iff \mathfrak{M}, s \Vdash \mathcal{B}_i\mathsf{\rho}(\mathfrak{C}, s, i, \varphi)$. We claim that, for any possible world $t \in S$ such that $(s, t) \in R(i)$, $\mathfrak{M}, t \Vdash \varphi \iff \mathfrak{M}, t \Vdash \mathsf{\rho}(\mathfrak{C}, s, i, \varphi)$. If $\mathsf{\rho}(\mathfrak{C}, s, i, \varphi) = \varphi$, the claim is obvious. Otherwise, suppose $\mathcal{B}_i\psi$ is a subformula of φ and is not within any $\mathcal{B}_j$'s scope, where $j \in I$. Since $(s, t) \in R(i)$, $\mathfrak{M}, t \Vdash \mathcal{B}_i\psi \iff \mathfrak{M}, s \Vdash \mathcal{B}_i\psi$, then by induction hypothesis, $\mathfrak{M}, s \Vdash \mathcal{B}_i\psi \iff \mathfrak{C}, s \vDash \mathcal{B}_i\psi \iff \mathsf{\rho}(\mathfrak{C}, s, i, \mathcal{B}_i\psi) = \top \iff \mathfrak{M}, t \Vdash \mathsf{\rho}(\mathfrak{C}, s, i, \mathcal{B}_i\psi)$, also because $\mathcal{B}_i\psi$ is not within any $\mathcal{B}_j$'s scope, the claim keeps holding after arbitrary propositional composition. Finally, we conclude that $\mathfrak{M}, s \Vdash \mathcal{B}_i\varphi \iff \mathfrak{M}, s \Vdash \mathcal{B}_i\mathsf{\rho}(\mathfrak{C}, s, i, \varphi) \iff \mathfrak{C}, s \vDash \mathcal{B}_i\varphi$. $\square$

5.2 Restrictions on Care

Now that we have endorsed this specific version of care in Definition 10 together with semantics in Definition 11, nonetheless in fact, our logic remains a little bit too general to validate any of the axioms K, D, 4 and 5 or the rule NEC. The nub of the problem still resides in the care's concrete form. Whereas some of MLB'-formulae as care echo a fairly intuitive intention of the agent (we shall soon witness several intriguing examples in the next Chapter 6), others do not. As a very simple counterexample, the care $\mathsf{\kappa}(i) = \neg p_x$ sounds extraordinarily outrageous: whatever the whys and wherefores, agent i chooses to regard any input φ exactly as its negation! Maybe such care might display its usefulness in some special scenario, but supposedly not in everyday life. In conclusion, we regard it plausible for the MLB'-formulae of agents' care to safely stay positive.

Definition 12 (Positive Formula and Negative Formula)**.** Positive and negative MLB'-formulae are simultaneously defined by the following pair of mutual inductions:

- $\top$ is positive.
- For any proposition $p \in P \cup \{p_x\}$, p is positive.
- If φ is negative, then $\neg\varphi$ is positive.
- If both φ and ψ are positive, then $(\varphi \wedge \psi)$ is positive.
- If φ is positive, then for any agent $i \in I$, $\mathcal{B}_i\varphi$ is positive.

- If φ is positive, then $\neg\varphi$ is negative.
- If both φ and ψ are negative, then $(\varphi \wedge \psi)$ is negative.
- If φ is negative, then for any agent $i \in I$, $\mathcal{B}_i\varphi$ is negative.

The above Definition 12 is quite standard in the literature, and as a gentle reminder, no formulae are both positive and negative, but there exist some formulae that are neither positive nor negative, e.g. $(\top \wedge \bot)$.

Besides having to be positive, we also feel certainly convinced by another natural restriction over an agent i's care: no other propositions except for p_x can appear in the formula $\kappa(i)$. That is to say, agent i's care should simply modify the input formula φ without depending on any specific proposition, and thus only the dummy proposition p_x as the placeholder for φ is permitted. Putting these two restrictions together result in the following Definition 13:

Definition 13 (Restrictions on Care)**.** Henceforward, for any agent $i \in I$, the following supplementary restrictions on his care $\kappa(i)$ in Definition 10 will take effect:

- For any proposition $p \in P$, p does not appear in $\kappa(i)$.
- $\kappa(i)$ is positive.

Without redundant mentioning, these restrictions will constantly keep in operation in the rest of this thesis.

Quite encouragingly, under such intuitive restrictions in the above Definition 13, the axioms 4 and 5 as well as the rule NEC now become valid — although not yet for the axioms D or K.

Proposition 8. *For any* MLB*-formula φ, if φ is valid with respect to the semantics in Definition 11, then φ is also valid with respect to the F-semantics in Definition 3.*

Proof. To a contradiction suppose not, namely, there exists a Kripke model $\mathfrak{M} = (S, R, V)$ and a possible world $s \in S$ such that $\mathfrak{M}, s \not\Vdash \varphi$. Consider the Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \kappa)$, where $\kappa(i) = p_x$ for any agent $i \in I$, then by Proposition 7 we have $\mathfrak{C}, s \nvDash \varphi$, contradicting that φ is valid. $\square$

Lemma 9. *For any two functions $\beth, \daleth : I \to \mathcal{P}(\mathsf{MLB})$ such that for each agent $i \in I$, the set of* MLB*-formulae $\Gamma_i = \{\mathcal{B}_i\varphi \mid \varphi \in \beth(i)\} \cup \{\neg\mathcal{B}_i\psi \mid \psi \in \daleth(i)\}$ is satisfiable under the F-semantics, and any two sets of propositions $Q, Q' \subseteq P$ such that $Q \cap Q' = \varnothing$, the set of* MLB*-formulae $\bigcup_{i\in I} \Gamma_i \cup \{q \mid q \in Q\} \cup \{\neg q' \mid q' \in Q'\}$ is also satisfiable under the F-semantics.*

Proof. This result wholly pertains to ordinary KD45 modal logic and is actually not difficult to note just by intuition. Hence we simply give out a somewhat brief and informal account for the lemma as follows.

For each agent $i \in I$ as Γ_i is satisfiable under the F-semantics, by Axiom of Choice we can arbitrarily fix a Kripke model $\mathfrak{M}_i = (S_i, R_i, V_i)$ and a possible world $s_i \in S_i$ such that $\mathfrak{M}_i, s_i \Vdash \Gamma_i$, while also make sure that $S_i \cap S_j = \varnothing$ for any $i, j \in I$ such that $i \neq j$. We then take another fresh possible world s so that $s \notin S_i$ for any $i \in I$, and construct the Kripke model $\mathfrak{M} = (S, R, V)$ as the following:

- $S = \bigcup_{i \in I} S_i \cup \{s\}$.
- For any agent $i \in I$, $R(i) = \bigcup_{j \in I} R_j(i) \cup \{(s, t_i) \mid t_i \in S_i, (s_i, t_i) \in R_i(i)\}$.
- For any proposition $q \in Q$, $V(q) = \bigcup_{i \in I} V_i(q) \cup \{s\}$; for any other proposition $p \in P \setminus Q$, $V(p) = \bigcup_{i \in I} V_i(p)$.

Since for any agent $i \in I$, the set of MLB-formulae Γ_i is entirely about i's belief, it is very easy to confirm that $\mathfrak{M}, s \Vdash \bigcup_{i \in I} \Gamma_i \cup \{q \mid q \in Q\} \cup \{\neg q' \mid q' \in Q'\}$. $\square$

Proposition 10. *For any MLB-formula φ, if φ is valid with respect to the semantics in Definition 11, then for any Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \kappa) = ((S, R, V), \kappa)$, any possible world $s \in S$ and any agent $i \in I$, the reduced MLB-formula $\rho(\mathfrak{C}, s, i, \varphi)$ is also valid with respect to the F-semantics in Definition 3.*

Proof. To a contradiction suppose not, namely, there exists a Kripke model $\mathfrak{M}' = (S', R', V')$ and a possible world $s' \in S'$ such that $\mathfrak{M}', s' \nVdash \rho(\mathfrak{C}, s, i, \varphi)$. Then for any other agent $j \in I$ such that $j \neq i$, the set of MLB-formulae $\{\mathcal{B}_j \psi \mid \mathfrak{M}', s' \Vdash \mathcal{B}_j \psi\} \cup \{\neg \mathcal{B}_j \psi \mid \mathfrak{M}', s' \Vdash \neg \mathcal{B}_j \psi\}$ is satisfiable under the F-semantics, and the set of MLB-formulae $\{p \mid p \in P, \mathfrak{M}', s' \Vdash p\} \cup \{\neg p \mid p \in P, \mathfrak{M}', s' \Vdash \neg p\}$ is satisfiable under the F-semantics as well. Also, the set of MLB-formulae $\{\mathcal{B}_i \psi \mid \mathfrak{M}, s \Vdash \mathcal{B}_i \psi\} \cup \{\neg \mathcal{B}_i \psi \mid \mathfrak{M}, s \Vdash \neg \mathcal{B}_i \psi\}$ is satisfiable under the F-semantics, and thus by Lemma 9, the union of all the above sets of MLB-formulae is still satisfiable under the F-semantics, in a Kripke model $\mathfrak{M}^\star = (S^\star, R^\star, V^\star)$ at a possible world $s^\star \in S^\star$.

We now construct the Kripke model with care $\mathfrak{C}^\star = (\mathfrak{M}^\star, \kappa^\star)$ by letting $\kappa^\star(i) = \kappa(i)$, and $\kappa^\star(j) = p_x$ for any other agent $j \in I$ such that $j \neq i$. Since for any MLB-formula ψ, $\mathfrak{M}^\star, s^\star \Vdash \mathcal{B}_i \psi \iff \mathfrak{M}, s \Vdash \mathcal{B}_i \psi$, it is then straightforward to prove by induction that for any MLB-formula ψ, $\mathfrak{C}^\star, s^\star \vDash \mathcal{B}_i \psi \iff \mathfrak{C}, s \vDash \mathcal{B}_i \psi$, namely $\mathfrak{C}^\star, s^\star \vDash \mathcal{B}_i \psi \iff \rho(\mathfrak{C}, s, i, \mathcal{B}_i \psi) = \top \iff \mathfrak{M}', s' \Vdash \rho(\mathfrak{C}, s, i, \mathcal{B}_i \psi)$. And by Proposition 7, for any MLB-formula ψ and any agent $j \in I$ such that $j \neq i$, $\mathfrak{C}^\star, s^\star \vDash \mathcal{B}_j\psi \iff \mathfrak{M}^\star, s^\star \Vdash \mathcal{B}_j\psi \iff \mathfrak{M}', s' \Vdash \mathcal{B}_j\psi$. Also for any proposition $p \in P$, $\mathfrak{C}^\star, s^\star \vDash p \iff \mathfrak{M}^\star, s^\star \Vdash p \iff \mathfrak{M}', s' \Vdash p$. Therefore, we finally conclude that for any MLB-formula ψ, $\mathfrak{C}^\star, s^\star \vDash \psi \iff \mathfrak{M}', s' \Vdash \rho(\mathfrak{C}, s, i, \psi)$, hence $\mathfrak{C}^\star, s^\star \nvDash \varphi$, contradicting that φ is valid. □

Lemma 11. *Given an MLB'-formula φ and, with respect to the language MLB', a Kripke model $\mathfrak{M} = (S, R, V)$. The following hold:*

- *If φ is positive, and for any proposition $p \in P \cup \{p_x\}$ that appears in φ, $V(p) = S$, then for any possible world $s \in S$, $\mathfrak{M}, s \Vdash \varphi$.*
- *If φ is negative, and for any proposition $p \in P \cup \{p_x\}$ that appears in φ, $V(p) = S$, then for any possible world $s \in S$, $\mathfrak{M}, s \nVdash \varphi$.*

Proof. The proof is straightforward via mutual induction. Only one slight caveat is worthwhile mentioning: the case for $\mathcal{B}_i\varphi$ when φ is negative resorts to seriality of the binary relation $R(i)$ (cf. Definition 2). □

Theorem 12. *The axioms 4 and 5 as well as the rule NEC are valid.*

Proof. The rule NEC directly follows from Proposition 10 and Lemma 11. Then the axioms 4 and 5 respectively follow from Proposition 2 and Proposition 3, together with assistance of Proposition 5 as well as the rule NEC. □

The axiom D, being one specific formula rather than an axiom schema, is relatively insignificant to a proof system: it can be easily added or removed according to our wish. So finally, what about the axiom K? It is invalid, and that is it.

Frankly speaking, commonly recognized as the problem of logical omniscience, ordinary modal logic for belief is accused of being ideally too powerful: the axiom K compels every agent to believe all the logical consequences of his own current belief, so that an agent must never maintain inconsistent beliefs on whatever occasion — a sin recurrently committed just by each of us! On the other hand, our modal logic of Dasein luckily evades such a problem, but the more important fact is that we never intend to do so: we innocently develop the semantics based on Heidegger's philosophy, and thence logical omniscience naturally gets resolved ex gratia. What a delicious windfall! Next, we are about to scrutinize a few specific examples in the following Chapter 6, whence a more coherent insight into how the axiom K fails can be learnt with ease.

Chapter 6

Pragmatic Examples

Example 2. Let us come back to the instance of Alice and Charlotte and, in this time, narrate a short yet lively story about friendship and solicitude. Alice is shy and unconfident, while Charlotte, Alice's best friend, is outgoing and considerate.

Alice recently got interviewed for a job and is now waiting for notification of the result, nevertheless, she does not sincerely believe that she could pass the interview. On the other side, based on years of acquaintance with Alice, Charlotte insists that Alice's competence suits the requirement of the job adequately, so she staunchly believes that Alice definitely deserves the job. She has also told her belief to Alice in order to relieve Alice's anxiety, thereupon in summary, it is no secret at all that Alice and Charlotte cherish markedly antithetical beliefs: both of their beliefs are simply shared as common knowledge, laying the communal background as the 'they'. Hence in order to formalize such a situation via logic, let the agents a and c respectively denote Alice and Charlotte, and the proposition p denote that 'Alice has succeeded in the interview and will be offered the job', then our current Kripke model $\mathfrak{M} = (S, R, T)$ just looks like the following:

$$\mathfrak{M}: \begin{tikzcd} \boxed{s:p} \arrow[loop above, "R(c)"] \arrow[r, shift left, "R(a)"] & \boxed{t:\neg p} \arrow[loop above, "R(a)"] \arrow[l, shift left, "R(c)"] \end{tikzcd}$$

Regardless of whether the real world is s or t — namely, whether Alice really passes the interview or not — we have $\mathfrak{M}, u \Vdash \neg\mathcal{B}_a p \wedge \mathcal{B}_c p$ ubiquitously for any possible world $u \in S$. Thus perfectly, the inauthentic beliefs fit with our intuition, but what about their authentic selves qua Dasein?

For a potential case, let us further assume that at this minute, Alice privately makes her decision to completely trust Charlotte because of their affectionate intimacy, and for this reason, she converts her sentiment and starts to believe p. In other words, Alice takes her secret care to believe whatever Charlotte believes, i.e., let $\kappa(a) = \mathcal{B}_c p_x$ in the Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \kappa)$. Now for any possible world $u \in S$, we actually have $\mathfrak{C}, u \vDash \mathcal{B}_a p$ because $\mathfrak{M}, u \Vdash \mathcal{B}_a \mathcal{B}_c p$, and likewise $\mathfrak{C}, u \vDash \mathcal{B}_a \mathcal{B}_a p$. However on the other hand, from Alice's point of view, she categorically believes that Charlotte yet discerns nothing at all about her instantaneous care $\kappa(a)$ (unless later on Alice voluntarily tells her care to Charlotte), and indeed for any possible world $u \in S$, we have $\mathfrak{M}, u \Vdash \mathcal{B}_c \neg \mathcal{B}_a p$ so $\mathfrak{M}, u \Vdash \mathcal{B}_a \mathcal{B}_c \mathcal{B}_c \neg \mathcal{B}_a p$ so $\mathfrak{C}, u \vDash \mathcal{B}_a \mathcal{B}_c \neg \mathcal{B}_a p$. In total, we have $\mathfrak{C}, u \vDash \mathcal{B}_a p \wedge \mathcal{B}_a \mathcal{B}_a p \wedge \mathcal{B}_a \mathcal{B}_c \neg \mathcal{B}_a p$ for any possible world $u \in S$, namely, Alice not only believes p and believes she believes p herself, but also believes Charlotte falsely believes that she does not believe p.

Now on second thought, what would ordinary modal logic for belief say about our above Example 2? Veritably for the current case, our target formula $\mathcal{B}_a p \wedge \mathcal{B}_a \mathcal{B}_a p \wedge \mathcal{B}_a \mathcal{B}_c \neg \mathcal{B}_a p$ could also be satisfied in ordinary modal logic for belief, however in order to achieve this, most probably the original Kripke model $\mathfrak{M}$ has to be multiplied by an action model, so that the eventual Kripke model $\mathfrak{M}'$ is doomed to blow up far larger. It might be argued that in this very simple example, the size of the Kripke model is just not worth such a heavy factor. Nevertheless to say the least, besides succinctness, a rather more favorable advantage of our intrinsic approach of modal logic of Dasein over ordinary modal logic for belief is our logic's maneuverability: in ordinary modal logic for belief, the original Kripke model $\mathfrak{M}$ itself transforms materially into $\mathfrak{M}'$ by belief revision, and hence basically speaking, it is always a disgustingly daunting task to withdraw a belief change, say, when a sequence of belief changes 1, 2, 3 have been applied chronologically but now only the change 2 needs to be revoked. Noticeably, our modal logic of Dasein elegantly saves such trouble, for it is straightforward to freely revise an agent's private care whenever we would like and as many times as we want, while the public Kripke model $\mathfrak{M}$ keeps untouched as the 'they'. Also note that till now, nothing is said at all about Charlotte's care $\kappa(c)$: it may well take some form other than the default $\kappa(c) = p_x$, but anyway we do not care as long as we are concerned simply with Alice's belief.

While Example 2 focuses on the semantical part regarding the Kripke model, our next Example 3 clearly exposes the syntactical difference in logical systems and conclusively exhibits how the axiom K fails in modal logic of Dasein:

Example 3. A very docile child (agent c) is always submissive to either his mother's (agent m) or his father's (agent f) wills, and thus we could specify his care as $\kappa(c) = \mathcal{B}_m p_x \vee \mathcal{B}_f p_x$. However in some possible world s belonging to some Kripke model $\mathfrak{M}$, suppose his parents are quarrelling with unmitigated disagreement about some proposition p, say $\mathfrak{M}, s \Vdash \mathcal{B}_m p \wedge \mathcal{B}_f \neg p$, then we will have $\mathfrak{C}, s \vDash \mathcal{B}_c p \wedge \mathcal{B}_c \neg p$ where the Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \kappa)$, namely, the child is now in the grip of exactly opposite beliefs. Nevertheless crucially, having inconsistent beliefs does not entail believing inconsistency itself: in fact we also have $\mathfrak{C}, s \vDash \neg\mathcal{B}_c \perp$ because $\mathfrak{M}, s \Vdash \neg\mathcal{B}_m \perp \wedge \neg\mathcal{B}_f \perp$ by the axiom D of the F-semantics (cf. Definition 4).

In short, the child's current belief is paraconsistent and the axiom K does not hold any longer, for we have $\mathfrak{C}, s \nvDash \mathcal{B}_c(p \to \perp) \wedge \mathcal{B}_c p \to \mathcal{B}_c \perp$. Moreover, the above scene incisively uncovers one frequent cause for our inconsistent beliefs: due to conflicting sources of information. As Harman [28] trenchantly points out, logic is not normative regarding belief revision: when we detect two of our beliefs to be inconsistent, logic can at most suggest us to discard either one of them, but it says nothing about which one. Ergo without further evidence, we might hang on to both beliefs for a time, whilst by rationality, our belief surely keeps under guard against absurdity as well as the cataclysmic explosion that would have been following.

As a closing remark for the above Example 3, what if in the following similar case, an agent i also relies on other two agents j and k as the authoritative criteria for his belief, but in the meantime puts more confidence on j than k? Then imaginably, his care may take the natural form as $\kappa(i) = \mathcal{B}_j p_x \vee (\neg\mathcal{B}_j \neg p_x \wedge \mathcal{B}_k p_x)$, namely, he will only assimilate k's belief that is consistent with j's. Moreover from such a case, it is strongly implied that rather than confine our expressivity, on the contrary, the intuitive restrictions in Definition 13 can methodically navigate us in some sense toward facilely finding out the most appropriate expression of an agent's care. Roughly speaking, in concordance with different kinds of applicatory environments, there exist a great deal of multifarious forms of care which the agents are able to take.

Chapter 7

Axiomatization

7.1 Closure Principle

While Theorem 12 avers that the axioms 4 and 5 as well as the rule NEC remain to be valid, on the other hand, we have also noted that the axiom D and especially the axiom K become invalid. Thus we naturally want to ask: can the axiom K be replaced by some weaker yet valid form? Actually even now, an agent's belief still has to be closed under logical entailment, that is, if agent i believes φ and φ logically entails ψ, then agent i must also believe ψ, as shown by the following Closure Principle (abbreviated as the rule CP):

Definition 14 (Rule CP)**.** The rule CP refers to the following:

$$\text{CP} \quad \frac{\varphi \to \psi}{\mathcal{B}_i\varphi \to \mathcal{B}_i\psi}, \text{ where } \varphi \to \psi \text{ is a theorem depending on no premises}$$

Candidly, the rule CP nevertheless continues suffering to a certain extent from the worry of logical omniscience; in fact, whether CP should be approved or not has engendered extensive controversy amongst epistemologists over these decades [29], [30]. In spite of assorted philosophical contentions, at a minimum, our modal logic of Dasein prepares a clean retreat from the axiom K downward to the rule CP and thus separates these two in a more distinct way. But first and foremost, it is imperative to prove that the rule CP is indeed valid.

Lemma 13. *Given an* MLB'*-formula* φ *and, with respect to the language* MLB'*, two Kripke models* $\mathfrak{M}_1 = (S, R, V_1)$ *and* $\mathfrak{M}_2 = (S, R, V_2)$*. The following monotonic properties hold:*

- *If φ is positive, and for any proposition $p \in P \cup \{p_x\}$ that appears in φ, $V_1(p) \subseteq V_2(p)$, then for any possible world $s \in S$, $\mathfrak{M}_1, s \Vdash \varphi \implies \mathfrak{M}_2, s \Vdash \varphi$.*

- *If φ is negative, and for any proposition $p \in P \cup \{p_x\}$ that appears in φ, $V_1(p) \subseteq V_2(p)$, then for any possible world $s \in S$, $\mathfrak{M}_2, s \Vdash \varphi \implies \mathfrak{M}_1, s \Vdash \varphi$.*

Proof. It is a standard result and can be shown by induction without difficulty. □

Theorem 14. *The rule CP is valid.*

Proof. It directly follows from Proposition 10 and Lemma 13. □

7.2 Reduction and Substitution

Besides the rule CP, it is also easy to notice just from the definition of the semantics that, regarding the inductive definition of the reduction, there naturally exist another group of obviously valid formulae, which we shall summarize under the rubric of the axiom RED. To begin with, let us define a few auxiliaries that are very helpful.

Definition 15 (Irreducible Formula)**.** Irreducible MLB-formulae are inductively defined as the following:

- $\top$ is irreducible.
- For any proposition $p \in P$, p is irreducible.
- If φ is irreducible, then $\neg\varphi$ is irreducible.
- If both φ and ψ are irreducible, then $(\varphi \wedge \psi)$ is irreducible.
- For any agent $i \in I$, if $\varphi \in \mathsf{MLB}_i$, then $\mathcal{B}_i\varphi$ is irreducible.

An MLB-formula is reducible if and only if it is not irreducible.

Definition 16 (Axiom RED)**.** For any fixed agent $i \in I$ and any reducible MLB-formula $\mathcal{B}_i\varphi$, suppose $\mathcal{B}_i\varphi_1, \mathcal{B}_i\varphi_2, \ldots, \mathcal{B}_i\varphi_n$ is a both complete and non-repetitive list of φ's subformulae which are in such a form (i.e. led by a modal operator $\mathcal{B}_i$) and which are not within any other $\mathcal{B}_j$'s scope inside φ, where $n \in \omega$, $j \in I$. Let there be a corresponding list of MLB-formulae $\psi_1, \psi_2, \ldots, \psi_n$, where ψ_m can be either $\top$ or $\bot$ for any $1 \leqslant m \leqslant n$ (ψ_m is a variable and is yet not determined). With respect to any determined list of $\psi_1, \psi_2, \dots, \psi_n$, let ψ be the resulting MLB_i-formula of uniformly substituting every appearance of $\mathcal{B}_i\varphi_m$ in φ that is not within any other $\mathcal{B}_j$'s scope with ψ_m for all $1 \leqslant m \leqslant n$ (hence ψ is dependent on $\psi_1, \psi_2, \dots, \psi_n$). Then the axiom RED (with respect to $\mathcal{B}_i\varphi$) refers to the following, where the big disjunction $\bigvee$ runs all over the possible combinations (precisely speaking, a total of 2^n different possibilities) of the list of formula variables $\psi_1, \psi_2, \dots, \psi_n$:

$$\text{RED} \qquad \mathcal{B}_i\varphi \leftrightarrow \bigvee_{1}^{2^n}(\bigwedge_{m=1}^{n}(\mathcal{B}_i\varphi_m \leftrightarrow \psi_m) \wedge \mathcal{B}_i\psi)$$

The above Definition 16, albeit ostensibly formidable, is actually quite intuitive to grasp. Nonetheless, to explain by natural language unavoidably gets knotty and lengthy, so let us simply emit an example. Suppose $p, q \in P$ are propositions and $i, j \in I$ are agents, then with respect to the reducible MLB-formula $\mathcal{B}_i(\mathcal{B}_ip \wedge \mathcal{B}_j\mathcal{B}_ip \to \mathcal{B}_iq)$, the axiom RED goes as the following:

$$\begin{aligned}\mathcal{B}_i(\mathcal{B}_ip \wedge \mathcal{B}_j\mathcal{B}_ip \to \mathcal{B}_iq) \leftrightarrow \quad & ((\mathcal{B}_ip \leftrightarrow \top) \wedge (\mathcal{B}_iq \leftrightarrow \top) \wedge \mathcal{B}_i(\top \wedge \mathcal{B}_j\mathcal{B}_ip \to \top)) \\ & \vee((\mathcal{B}_ip \leftrightarrow \top) \wedge (\mathcal{B}_iq \leftrightarrow \bot) \wedge \mathcal{B}_i(\top \wedge \mathcal{B}_j\mathcal{B}_ip \to \bot)) \\ & \vee((\mathcal{B}_ip \leftrightarrow \bot) \wedge (\mathcal{B}_iq \leftrightarrow \top) \wedge \mathcal{B}_i(\bot \wedge \mathcal{B}_j\mathcal{B}_ip \to \top)) \\ & \vee((\mathcal{B}_ip \leftrightarrow \bot) \wedge (\mathcal{B}_iq \leftrightarrow \bot) \wedge \mathcal{B}_i(\bot \wedge \mathcal{B}_j\mathcal{B}_ip \to \bot))\end{aligned}$$

Theorem 15. *The axiom RED is valid.*

Proof. The axiom RED is exactly a restatement of the reduction in Definition 11. □

It is then not difficult to notice that actually, with a finite series of consecutively applying the axiom RED, each reducible MLB-formula can eventually get reduced to be logically equivalent to an irreducible MLB-formula. Thus readers might begin to wonder why not only consider irreducible MLB-formulae, over which validity of the rules NEC and CP may be shown through more apparent justification. Howbeit things are not that simple. As initially descried by Wang and Cao [31] and further explored by Hatano and Sano [32], in general, such sort of reduction axioms cannot competently reduce every formula to the full on its own alone, independent of any ancillary rule for uniform substitution. In our current context for example, suppose $p \in P$ is a proposition and $i, j \in I$ are two different agents, then the valid MLB-formula $\mathcal{B}_j(\mathcal{B}_i\mathcal{B}_ip \vee \neg\mathcal{B}_i\mathcal{B}_ip)$ is actually irreducible, but we do not behold any credible means to deduce it without either applying the rule NEC onto the reducible

MLB-formula $\mathcal{B}_i\mathcal{B}_i p \vee \neg\mathcal{B}_i\mathcal{B}_i p$ or substituting the formula $\mathcal{B}_i\mathcal{B}_i p \vee \neg\mathcal{B}_i\mathcal{B}_i p$ inside $\mathcal{B}_j$'s scope. Hence, let the rule for uniform substitution (abbreviated as the rule SUB) be formally stated as the following:

Definition 17 (Rule SUB)**.** Let $\chi[\psi/\varphi]$ denote the result of uniformly substituting every occurrence of φ in χ with ψ. Then the rule SUB refers to the following:

$$\text{SUB} \quad \frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi[\psi/\varphi]}, \text{ where } \varphi \leftrightarrow \psi \text{ is a theorem depending on no premises}$$

Nevertheless presently, the rule SUB is by no means self-evident, either. Although we would like to conjecture that the rule SUB is valid, such a proof — if it exists — seems only more intricate than our proof for the rules NEC and CP in aggregate (cf. Theorem 12 and Theorem 14, of course also counting in all the preceding lemmata). Furthermore, even if the rule SUB is indeed valid, still, the axiom RED plus the rule SUB does not spontaneously ensure a sound and complete proof system. Although we would like to conjecture anew that for our current modal logic of Dasein, such an alternative approach to axiomatize the logic is also feasible, anyway, let us temporarily put aside all these intricacies as optional directions of future work.

7.3 Strong Completeness

Having verified the rule CP as well as the axiom RED, now we can set about piecing together a sound and strongly complete proof system for modal logic of Dasein. Hereon emerges kind of interaction amongst axioms and rules. Even without the (possible) rule SUB though, in fact, the axiom RED is already so effective that it drives us to reexamine some part of our previous principles, for reducible MLB-formulae can just be conveniently reduced in a lot of cases. Particularly speaking, not only it is fairly easy to notice that the axioms 4 and 5 can be simply deduced out and thus are no more required, but the rule CP also demands a refurbishment into the following variant of rule CP-F, which specializes in affixing the modal operator $\mathcal{B}_i$ in front of an MLB_i-formula so that the result is irreducible:

Definition 18 (Rule CP-F)**.** The rule CP-F refers to the following:

$$\text{CP-F} \quad \text{if } \varphi \to \psi \in \mathsf{MLB}_i \text{ and is a } \texttt{PS-F}\text{-theorem, then deduce } \mathcal{B}_i\varphi \to \mathcal{B}_i\psi$$

Theorem 16. *The rule CP-F is valid.*

Proof. It directly follows from Theorem 1 and Lemma 13. $\square$

Definition 19 (Proof System `PS-D`)**.** The Hilbert-style proof system `PS-D` consists of the axioms TAUT and RED, as well as the rules MP, NEC and CP-F.

Here remains one last preparation. Another minor supplemental restriction is imposed upon the logic henceforth: there are a minimum of two different agents, namely $\mid I \mid \geqslant 2$. As readers can readily envision, if there is only one agent $i \in I = \{i\}$, then he is even unable to conceive anyone else but himself in his care $\mathsf{\kappa}(i)$, and this grievous stricture will unsurprisingly impel more MLB-formulae to become valid. Rather overtly, such single-agent case scarcely conforms with Heidegger's philosophy where modal logic of Dasein is rooted in, and so in this thesis we have to righteously give up digressing into it. Above all, through ratifying this subsidiary restriction, we in the end come to the following:

Theorem 17. *The proof system* `PS-D` *in Definition 19 is sound and strongly complete with respect to the semantics of modal logic of Dasein in Definition 11.*

Proof. Soundness follows from Theorem 12, Theorem 15 and Theorem 16.

For strong completeness, we only need to show that every consistent set of MLB-formulae is satisfiable. By the axiom RED, just as we intuitively comment right after Theorem 15, it does not matter to suppose a consistent set Γ of irreducible MLB-formulae, and so in the below we shall construct a Kripke model with care $\mathfrak{C} = (\mathfrak{M}, \mathsf{\kappa}) = ((S, R, V), \mathsf{\kappa})$ and a possible world $s \in S$ such that $\mathfrak{C}, s \vDash \Gamma$.

First of all, like Lindenbaum Lemma, we decompose formulae in Γ up to propositional connectives and determine the truth value of each atomic part. This is to say, for every MLB-formula $\varphi \in \Gamma$, and every subformula of φ which is in the form of either p or $\mathcal{B}_i\psi$ and is not within any $\mathcal{B}_k$'s scope inside φ, where $p \in P$, $i, k \in I$, we choose to add either p or $\neg p$ (respectively, either $\mathcal{B}_i\psi$ or $\neg\mathcal{B}_i\psi$) into Γ so that the extended Γ is still consistent, and we inductively go through the above procedure for each subformula by certain predetermined well order. Therefore at last, we can obtain a set of MLB-formulae Γ_i for each agent $i \in I$ and two sets of propositions Q, Q' in exactly the same form as stated in Lemma 9, so that $\bigcup\limits_{i \in I} \Gamma_i \cup \{q \mid q \in Q\} \cup \{\neg q' \mid q' \in Q'\}$ is still consistent and entails Γ. Also note that any MLB-formula $\mathcal{B}_i\psi \in \Gamma_i$ or $\neg\mathcal{B}_i\psi \in \Gamma_i$ is still irreducible.

Next, by the rule NEC, $\mathcal{B}_i\top$ is a `PS-D`-theorem for any $i \in I$, and we add it into Γ_i as well. Then for each agent $i \in I$, we let $\mathsf{\kappa}(i) = \top$ if all the formulae in Γ_i are in the form of $\mathcal{B}_i\psi$; otherwise, there exist both some formula $\mathcal{B}_i\psi$ and some other formula $\neg\mathcal{B}_i\varphi$ in Γ_i, and we let $\mathsf{\kappa}(i) = \mathcal{B}_j\neg\mathcal{B}_i\neg p_x$, where $j \in I$ is an arbitrarily fixed agent such that $j \neq i$ as $\mid I \mid\geqslant 2$. In the former case we obviously have $\mathfrak{C}, s \vDash \Gamma_i$; in the latter case because every formula in Γ_i is irreducible, namely $\psi, \varphi \in \mathsf{MLB}_i$ for any $\mathcal{B}_i\psi$ or $\neg\mathcal{B}_i\varphi$ in Γ_i, by the definition of the semantics we have $\mathfrak{C}, s \vDash \Gamma_i \iff \mathfrak{M}, s \Vdash \{\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\psi \mid \mathcal{B}_i\psi \in \Gamma_i\} \cup \{\neg\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$. Also apparently, $Q \cap Q' = \varnothing$ and so $\{q \mid q \in Q\} \cup \{\neg q' \mid q' \in Q'\}$ is satisfiable, thus by Lemma 9, we only need to show that for each fixed individual agent $i \in I$ in the latter case, $\{\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\psi \mid \mathcal{B}_i\psi \in \Gamma_i\} \cup \{\neg\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$ is satisfiable under the F-semantics in some Kripke model $\mathfrak{M}_i = (S_i, R_i, V_i)$ and a possible world $s_i \in S_i$.

Now we introduce the following two nonempty sets of totally fresh possible worlds into S_i, where each different name denotes a different possible world which is also different from s_i: $\{t_\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$ and $\{r_{\langle\varphi,\psi\rangle} \mid \neg\mathcal{B}_i\varphi, \mathcal{B}_i\psi \in \Gamma_i\}$. We let $\{t_i \in S_i \mid (s_i, t_i) \in R_i(i)\} = \{s_i\}$; let $\{t_i \in S_i \mid (s_i, t_i) \in R_i(j)\} = \{t_\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$ which is nonempty, hence for any $\neg\mathcal{B}_i\chi \in \Gamma_i$ we must have $\{t_i \in S_i \mid (t_\chi, t_i) \in R_i(j)\} = \{t_\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$ as well; and for any $\neg\mathcal{B}_i\varphi \in \Gamma_i$ let $\{r_i \in S_i \mid (t_\varphi, r_i) \in R_i(i)\} = \{r_{\langle\varphi,\psi\rangle} \mid \mathcal{B}_i\psi \in \Gamma_i\}$ which is nonempty, hence for any $\mathcal{B}_i\chi \in \Gamma_i$ we must have $\{r_i \in S_i \mid (r_{\langle\varphi,\chi\rangle}, r_i) \in R_i(i)\} = \{r_{\langle\varphi,\psi\rangle} \mid \mathcal{B}_i\psi \in \Gamma_i\}$ as well. Under the above partial assignment for the relation function R_i we can already asseverate that obviously, if for any $\neg\mathcal{B}_i\varphi, \mathcal{B}_i\psi \in \Gamma_i$ we have $\mathfrak{M}_i, r_{\langle\varphi,\psi\rangle} \Vdash \neg\varphi \wedge \psi$, then we will directly have $\mathfrak{M}_i, s_i \Vdash \{\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\psi \mid \mathcal{B}_i\psi \in \Gamma_i\} \cup \{\neg\mathcal{B}_i\mathcal{B}_j\neg\mathcal{B}_i\neg\varphi \mid \neg\mathcal{B}_i\varphi \in \Gamma_i\}$. Also note that on the one hand because $\varphi, \psi \in \mathsf{MLB}_i$, no modal operator $\mathcal{B}_i$ appears in φ or ψ outside of any other $\mathcal{B}_k$'s scope, where $k \in I$; on the other hand, we have yet stipulated nothing more about the possible world $r_{\langle\varphi,\psi\rangle}$ except for its $R_i(i)$ relation; therefore, intuitively via construction very similar to the proof tactics in Lemma 9, we can smoothly attain $\mathfrak{M}_i, r_{\langle\varphi,\psi\rangle} \Vdash \neg\varphi \wedge \psi$ without any side effects, provided that the formula $\neg\varphi \wedge \psi$ itself is satisfiable under the F-semantics, namely by Theorem 1, $\neg\varphi \wedge \psi$ is consistent within the proof system `PS-F`.

So finally, to a contradiction suppose $\neg\varphi \wedge \psi$ is inconsistent within the proof system `PS-F`, namely $\psi \to \varphi$ is a `PS-F`-theorem. Since $\psi \to \varphi \in \mathsf{MLB}_i$, by the rule CP-F, $\mathcal{B}_i\psi \to \mathcal{B}_i\varphi$ is a `PS-D`-theorem, contradicting that $\{\neg\mathcal{B}_i\varphi, \mathcal{B}_i\psi\} \subseteq \Gamma_i$ is consistent within the proof system `PS-D`. $\square$

Chapter 8

Concluding Remarks

8.1 On Philosophy

Discontented with extrinsic, factitious and even nonsensical postulation that reigns over ordinary epistemic logic, we have brought forth modal logic of Dasein from an intrinsic approach in order to faithfully transcribe agents' interior belief. Our logic highlights the authentic self versus inauthentic the 'they' distinction, upon which Heidegger had been expatiating as one of the key routes toward an understanding of Dasein. A moderate number of animated examples are provided, which epitomize the logic's advanced usability. We also probe into a sound and strongly complete proof system, and most remarkably, our formalization lucidly discriminates between the axiom K and the rule CP, a profound result not only technically consequential but also conceptually inspirational for settling the protracted dissension over a few questions relating to epistemic closure.

First and last, this thesis purports to contribute toward healing the divisions between analytic and continental philosophy: it is a unified philosophical community that we ardently appeal for.

For future work, as we have mentioned at the end of Section 2.3, Dasein is embodied amply into multiple folds of modalities besides propositional belief, so that modal logic of Dasein can and will be generalized onto lots of other philosophical realms as well. To wit, Heidegger also bends much effort into classifying an antagonistic pair of modes when Dasein encounters any other being — the well-known 'present-at-hand' versus 'ready-to-hand'. Thus very naturally, we only need to upgrade basic propositional modal logic onward to first-order modal logic [33], and then these two modes of agents toward elements in the first-order domain could just be fluently formalized. Moreover in his later philosophy after 'die Kehre' ('the turn' in English), Heidegger [34] shifts to acute criticism that a modernized, thoroughly technological lifestyle will inexorably annihilate Dasein into mere non-Dasein. We would also like to envisage looking into this impressive and momentous assertion.

8.2 On Mathematics

When trying to axiomatize modal logic of Dasein, we have left a few conjectures regarding the rule SUB at the end of Section 7.2. Also, some regular sorts of mathematical characteristics of modal logic of Dasein remain to be studied, including bisimulation, decidability, complexity and so on. What is more, since modal logic of Dasein rejects the axiom K, it is categorized as a type of non-normal modal logic, and thus from an algebraic aspect we could also look for a commensurate neighborhood semantics to interpret our logic [35].

Furthermore as Definition 10 shows, the notion of care that accompanies a Kripke model tangibly blends syntax into semantics, such that validity of formulae, while superficially quantifying over all the Kripke models with care, effectually quantifies over all the possible formulae as care. Therefore theoretically speaking, our modal logic of Dasein is in fact a type of second-order modal logic in disguise, even though nominally there is no explicit second-order quantifier over formulae. Anyway until now, higher-order modal logic is still an almost virgin land with only a modest amount of pioneering exploration [36], and thereby we are also looking forward to further investigation into this fertile field.

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Bibliography

[1] P. Blackburn, M. de Rijke, and Y. Venema, *Modal Logic*. New York: Cambridge University Press, 2001.

[2] S. A. Kripke, "A completeness theorem in modal logic," *The Journal of Symbolic Logic*, vol. 24, no. 1, pp. 1–14, 1959. DOI: 10.2307/2964568.

[3] ——, "Semantical analysis of modal logic I: Normal modal propositional calculi," *Mathematical Logic Quarterly*, vol. 9, no. 5, pp. 67–96, 1963. DOI: 10.1002/malq.19630090502.

[4] J. Hintikka, *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Ithaca, New York: Cornell University Press, 1962.

[5] H. van Ditmarsch, W. van der Hoek, and B. Kooi, *Dynamic Epistemic Logic*. Dordrecht, The Netherlands: Springer-Verlag, 2007.

[6] J. Plaza, "Logics of public communications," in *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: Poster Session Program*, M. Emrich, M. Pfeifer, M. Hadzikadic, and Z. Ras, Eds., Charlotte, North Carolina: Oak Ridge National Laboratory, 1989, pp. 201–216.

[7] ——, "Logics of public communications," *Synthese*, vol. 158, no. 2, pp. 165–179, 2007. DOI: 10.1007/s11229-007-9168-7.

[8] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi, *Reasoning about Knowledge*. Cambridge, Massachusetts: The MIT Press, 1995.

[9] Y. Wang, "Beyond knowing that: A new generation of epistemic logics," in *Jaakko Hintikka on Knowledge and Game Theoretical Semantics*, H. van Ditmarsch and G. Sandu, Eds., Cham, Switzerland: Springer-Verlag, 2018, pp. 499–533.

[10] ——, "A logic of goal-directed knowing how," *Synthese*, vol. 195, no. 10, pp. 4419–4439, 2018. DOI: 10.1007/s11229-016-1272-0.

[11] C. Xu, Y. Wang, and T. Studer, “A logic of knowing why,” *Synthese*, vol. 198, no. 2, pp. 1259–1285, 2021. DOI: `10.1007/s11229-019-02104-0`.

[12] Y. Wang, Y. Wei, and J. Seligman, “Quantifier-free epistemic term-modal logic with assignment operator,” *Annals of Pure and Applied Logic*, vol. 173, no. 3, p. 103 071, 2022. DOI: `10.1016/j.apal.2021.103071`.

[13] J.-P. Sartre, *Existentialism is a Humanism*. New Haven, Connecticut: Yale University Press, 2007, First published 1946 in French. Translated into English by Carol Macomber.

[14] M. Heidegger, *Sein und Zeit*. Tübingen, Germany: Max Niemeyer Verlag, 1967, First published 1927.

[15] ——, *Being and Time*. Oxford: Blackwell Publishers, 1962, First published 1927 in German. Translated into English by John Macquarrie and Edward Robinson.

[16] D. O. Dahlstrom, *The Heidegger Dictionary*. London: Bloomsbury Publishing, 2013.

[17] S. Mulhall, *Routledge Philosophy GuideBook to Heidegger and Being and Time*. London: Routledge, 1996.

[18] R. Polt, *Heidegger: An Introduction*. London: Routledge, 2003, First published 1999.

[19] D. Vallega-Neu, *Heidegger’s Contributions to Philosophy: An Introduction*. Bloomington, Indiana: Indiana University Press, 2003.

[20] I. Kolev, “Modal thinking in the philosophical anthropology,” in *Proceedings of the XXII World Congress of Philosophy*, Myung-Hyun-Lee, Ed., vol. 20, Charlottesville, Virginia: Philosophy Documentation Center, 2008, pp. 129–136.

[21] J. Haugeland, *Dasein Disclosed: John Haugeland’s Heidegger*. Cambridge, Massachusetts: Harvard University Press, 2013.

[22] W. Martin, “The semantics of “Dasein” and the modality of Being and Time,” in *The Cambridge Companion to Heidegger’s Being and Time*, M. A. Wrathall, Ed., Cambridge: Cambridge University Press, 2013, pp. 100–128.

[23] H. L. Dreyfus, “Why Heideggerian AI failed and how fixing it would require making it more Heideggerian,” *Artificial Intelligence*, vol. 171, no. 18, pp. 1137–1160, 2007. DOI: `10.1016/j.artint.2007.10.012`.

[24] A. Döring, "The physical interpretation of daseinisation," in *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, H. Halvorson, Ed., Cambridge: Cambridge University Press, 2011, pp. 207–238.

[25] X. Wang and Y. Wang, "Knowledge-now and knowledge-all," in *Proceedings of the 7th International Workshop on Logic, Rationality, and Interaction*, P. Blackburn, E. Lorini, and M. Guo, Eds., Berlin, Germany: Springer-Verlag, 2019, pp. 399–412.

[26] Y. Li, Q. Yu, and Y. Wang, "More for free: A dynamic epistemic framework for conformant planning over transition systems," *Journal of Logic and Computation*, vol. 27, no. 8, pp. 2383–2410, 2017. DOI: `10.1093/logcom/exx020`.

[27] T. M. Cover and J. A. Thomas, *Elements of Information Theory: Second Edition*. Hoboken, New Jersey: John Wiley & Sons, 2006.

[28] G. Harman, *Change in View: Principles of Reasoning*. Cambridge, Massachusetts: The MIT Press, 1986.

[29] F. Dretske, "Is knowledge closed under known entailment? the case against closure," in *Contemporary Debates in Epistemology*, M. Steup and E. Sosa, Eds., Oxford: Blackwell Publishers, 2005, pp. 13–26.

[30] J. Hawthorne, "Is knowledge closed under known entailment? the case for closure," in *Contemporary Debates in Epistemology*, M. Steup and E. Sosa, Eds., Oxford: Blackwell Publishers, 2005, pp. 26–43.

[31] Y. Wang and Q. Cao, "On axiomatizations of public announcement logic," *Synthese*, vol. 190, no. 1, pp. 103–134, 2013. DOI: `10.1007/s11229-012-0233-5`.

[32] R. Hatano and K. Sano, "Recapturing dynamic logic of relation changers via bounded morphisms," *Studia Logica*, vol. 109, no. 1, pp. 95–124, 2021. DOI: `10.1007/s11225-020-09902-5`.

[33] M. Fitting and R. L. Mendelsohn, *First-Order Modal Logic*. Dordrecht, The Netherlands: Springer Science+Business Media, 1998.

[34] M. Heidegger, *Contributions to Philosophy (Of the Event): Of the Event*. Bloomington, Indiana: Indiana University Press, 2012, First published 1989 in German. Translated into English by Richard Rojcewicz and Daniela Vallega-Neu.

[35] E. Pacuit, *Neighborhood Semantics for Modal Logic*. Cham, Switzerland: Springer International Publishing, 2017.

[36] R. Muskens, "Higher order modal logic," in *Handbook of Modal Logic*, P. Blackburn, J. van Benthem, and F. Wolter, Eds., Amsterdam, The Netherlands: Elsevier, 2006, pp. 621–653.