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An algebraic approach to the disjunction property of substructural logics

By Daisuke Souma

A thesis submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
Graduate Program in Information Science

Written under the direction of
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Professor Atsushi Ohori
Professor Tetsuo Asano

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Chapter 1

Introduction

1.1 Introduction

1.1.1 Backgrounds and purpose

The proof theoretic methods and algebraic methods are two important basic ways in study of logic. The former focuses on finite syntactically structures and the latter take methods like set theory and so on. So relation of these had not discussed. But for example in these years there are studies in which the cut elimination theorem is used in proving the finite model property and the cut elimination is proved by algebraic methods.

The study of logics by algebraic methods studied actively from 1950 to 1960. After that Kripke semantics become mainstream of semantical study. The conventional algebraic study turn off study of logics and develop as universal algebra. In 1990s , it is used for study of substructural logic and modal logic.

Roughly substructural logics are logics obtained from intuitionistic logic **LJ** and classical logic **LK** by deleting structural rules. The study starts from the study of categorial grammar by Lambek and it get active in 1990.

An algebraic study for substructural logics has been developed remarkably in these years. Also, collaborations of logicians with algebraists who interested in ordered algebraic structures are on-going. A syntactical proof of the cut elimination is not necessarily easy to understand for algebraist. Recently we get purely algebraic proof of cut elimination theorem.

In this thesis we principally take up residuated lattices which does not necessarily assume integrality [1]. This corresponds to substructural logic $\mathbf{FL_e}$.

1.1.2 Outline of this thesis

This thesis consists of 5 chapters. In chapter 5 we give an algebraic proof of the disjunction property of $\mathbf{FL_e}$, $\mathbf{FL_e}[\mathrm{E_k}]$ and $\mathbf{FL_e}[\mathrm{DN}]$, which is our main theorem. Chapters 2,3 and 4 are the chapters for the preparation.

[1] The integrality is a property that the maximum element is equal to an identity element in a monoid.

At first we introduce logics which is principally taken up in this thesis. Substructural logic $\mathbf{FL_e}$ and logics over $\mathbf{FL_e}$ which is obtained from $\mathbf{FL_e}$ by adding some axiom. Next we define a commutative residuated lattices (CRLs) which is the algebraic model for $\mathbf{FL_e}$, and we show some properties for CRLs. Third we explain relations between logics and algebras. The completeness theorem is one of well known result. Finally we show main theorems. We extend Maksimova's theorem. Then, we show disjunction property (d.p.) for some logics over $\mathbf{FL_e}$ on a characterization of logics with d.p..

1.2 Notation of algebras

Definition 1.2.1 (partial order) A structure $\mathbf{A} = \langle \mathrm{A}, \leq \rangle$ is a *partially ordered set* (p.o.set) if it satisfies the following. For all $x, y, z \in \mathrm{A}$.

(P1) $x \leq x$.

(P2) If $x \leq y$ and $y \leq x$ then $x = y$.

(P3) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Moreover if a p.o.set $\mathbf{A} = \langle \mathrm{A}, \leq \rangle$ satisfies

(P4) $x \leq y$ or $y \leq x$ for any $x, y \in \mathrm{A}$

then $\mathbf{A}$ is a *totally ordered set*.

Definition 1.2.2 (lattice) A structure $L = \langle \mathrm{L}, \cap, \cup \rangle$ is a *lattice*if it satisfies the following. For all $x, y, z \in \mathrm{L}$.

(L1) $x \cap x = x$, $x \cup x = x$.

(L2) $x \cap (y \cap z) = (x \cap y) \cap z$, $x \cup (y \cup z) = (x \cup y) \cup z$.

(L3) $x \cap y = y \cap x$, $x \cup y = y \cup x$.

(L4) $x \cap (x \cup y) = x$, $x \cup (x \cap y) = x$.

Let $\mathbf{L} = \langle \mathrm{L}, \cap, \cup \rangle$ be a lattice. Define a binary relation $\leq$ by

$$x \leq y \Leftrightarrow x \cap y = x.$$

Then, we can show that $\leq$ is a partial order. We note that $x \cap y = x$ is equivalent to the condition $x \cup y = y$. Thus each lattice induces always a partial order on it.

Definition 1.2.3 (monoid) A structure $\mathbf{A} = \langle \mathrm{A}, \cdot, 1 \rangle$ is a *monoid* if it satisfies the following. For all $x, y, z \in \mathrm{A}$.

(M1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M2) there exists some $e \in \mathrm{A}$ such that $e \cdot a = a \cdot e = a$.

We can show that such an element e exists uniquely, if any. Therefore, this element e is called *identity element*. In the following, we consider only commutative monoids, i.e. monoids satisfying $x \cdot y = y \cdot x$ for all x, y.

Chapter 2

Sequent calculus of the substructural logic $\mathbf{FL_e}$

In this section, we introduce a substructural logic $\mathbf{FL_e}$ and its extensions.

2.1 Sequent calculus LJ for intuitionistic logic

First, we introduce sequent calculus **LJ** for intuitionistic logic. We use $\wedge$(conjunction), $\vee$(disjunction), $\supset$(implication) and $\neg$(negation) as logical connectives. By using these connectives we define *formulas* inductively as follows.

Definition 2.1.1 (formula) Formulas are defined inductively as follows.

i. all propositional variables and propositional constants($\top, \bot$) are formulas,

ii. if α, β are formulas then $\alpha \wedge \beta, \alpha \vee \beta, \alpha \supset \beta$ and $\neg\alpha$ are formulas.

A *sequent* is an expression of the following form.

$\alpha_1, \alpha_2, \ldots, \alpha_m \to \beta$
(where $\alpha_1, \ldots, \alpha_m, \beta$ are formulas and $m \geq 0$. β can be empty.)

Here after we use capital Greek letters $\Gamma, \Delta, \ldots$ for finite sequences of formulas, separated by commas. Next we define the sequent calculus **LJ**.

Initial sequent The *initial sequents* of **LJ** are following.

1. $\alpha \to \alpha$

2. $\Gamma \to \top$

3. $\bot, \Gamma \to \gamma$

Inference rules

Weakening rule:

$$\frac{\Gamma \to \beta}{\alpha, \Gamma \to \beta}\ (\textit{left-weakening}) \quad \frac{\Gamma \to}{\Gamma \to \alpha}\ (\textit{right-weakening})$$

Contraction rule:

$$\frac{\alpha, \alpha, \Gamma \to \beta}{\alpha, \Gamma \to \beta}\ (\textit{contraction})$$

Exchange rule:

$$\frac{\Gamma, \alpha, \beta, \Delta \to \gamma}{\Gamma, \beta, \alpha, \Delta \to \gamma}\ (\textit{exchange})$$

Cut rule:

$$\frac{\Gamma \to \alpha \quad \alpha, \Delta \to \beta}{\Gamma, \Delta \to \beta}\ (\textit{cut})$$

Logical rule:

$$\frac{\alpha, \Gamma \to \gamma}{\alpha \wedge \beta, \Gamma \to \gamma}\ (\textit{left-}\wedge\textit{1}) \quad \frac{\Gamma, \beta, \Gamma \to \gamma}{\alpha \wedge \beta, \Gamma \to \gamma}\ (\textit{left-}\wedge\textit{2})$$

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}\ (\textit{right-}\wedge)$$

$$\frac{\alpha, \Gamma \to \gamma \quad \beta, \Gamma \to \gamma}{\alpha \vee \beta, \Gamma \to \gamma}\ (\textit{left-}\vee)$$

$$\frac{\Gamma \to \alpha}{\Gamma \to \alpha \vee \beta}\ (\textit{left-}\vee\textit{1}) \quad \frac{\Gamma \to \beta}{\Gamma \to \alpha \vee \beta}\ (\textit{left-}\vee\textit{2})$$

$$\frac{\Gamma \to \alpha \quad \beta, \Delta \to \gamma}{\alpha \supset \beta, \Gamma, \Delta \to \gamma}\ (\textit{left-}\supset) \quad \frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}\ (\textit{right-}\supset)$$

$$\frac{\Gamma \to \alpha}{\neg\alpha, \Gamma \to}\ (\textit{left-}\neg) \quad \frac{\alpha, \Gamma \to}{\Gamma \to \neg\alpha}\ (\textit{right-}\neg)$$

A sequent $\Gamma \to \phi$ is *provable* in **LJ** if it can be obtained from initial sequents by applying rules of inference repeatedly. A formula ϕ is *provable* if a sequent $\to \phi$ is provable. A figure which shows how a given sequent $\Gamma \to \phi$ is obtained is called a *proof* of $\Gamma \to \phi$ and we say that ϕ *is provable in* **LJ**.

2.2 Substructural logics over $\mathbf{FL_e}$

In this section, we define a substructural logics over $\mathbf{FL_e}$. It is obtained from **LJ**. $\mathbf{FL_{ew}}$ is a logic obtained from **LJ** by deleting the contraction rule. $\mathbf{FL_e}$ is a logic obtained from **LJ** by deleting the contraction rule and the weakening rule.

2.2.1 weakening rule and propositional constants

Here we explain the relation between propositional constants $(\top, \bot)$ and weakening rule. In **LJ** $\Gamma \to \top$ is a initial sequent. So we can show the following.

A formula ϕ is provable $\Longleftrightarrow$ ϕ is equivalent to $\top$.

$(\Rightarrow)$

$$\frac{\phi \to \top}{\to \phi \supset \top} \qquad \frac{\dfrac{\to \phi}{\top \to \phi}}{\to \top \supset \phi}$$

$(\Leftarrow)$

$$\frac{\to \top \supset \phi \qquad \dfrac{\to \top \quad \phi \to \phi}{\top \supset \phi \to \phi}}{\to \phi}$$

Moreover we can show that $\neg\phi$ is equivalent to $\phi \supset \bot$.

$$\frac{\dfrac{\dfrac{\phi \to \phi}{\neg\phi, \phi \to \bot}}{\neg\phi, \phi}}{\phi \to \phi} \qquad \frac{\dfrac{\phi \to \phi \quad \bot \to}{\phi \supset \bot, \phi \to}}{\phi \supset \bot \to \neg\phi}$$

To show these two equivalence we need weakening rule. Because $\mathbf{FL_e}$ doesn't have weakening rules we can't show these conditions. So we introduce new propositional constants t, f and initial sequents and inference rules.

1. $\to t$
2. $f \to$

$$\frac{\Gamma, \Delta \to \gamma}{\Gamma, t, \Delta \to \gamma}\ (tw) \qquad \frac{\Gamma \to}{\Gamma \to f}\ (fw)$$

The constant t is the weakest provable formula and f is the greatest formula in a set of formulas of it's contradiction be provable. We can show that $\neg\phi$ is equivalent to $\phi \supset f$ as follows.

$$\frac{\dfrac{\dfrac{\phi \to \phi}{\neg\phi, \phi \to}}{\neg\phi, \phi \to f}}{\neg\phi \to \phi \supset f} \qquad \frac{\dfrac{\phi \to \phi \quad f \to}{\phi \supset f, \phi \to}}{\phi \supset f \to \neg\phi}$$

2.2.2 structural rule and comma

In **LJ** we can show the following.

$\phi_1, \phi_2, \ldots, \phi_m \to \psi$ is provable iff $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_m \to \psi$ is provable.

To show only-if part we need contraction rule, and to show if-part we need weakening rule. In substructural logics commas on the left-hand side of sequents don't behave as conjunctions. So we introduce new logical connective $*$ called *fusion*, which represents a comma in substructural logics. We add following rules for $*$.

$$\frac{\alpha, \beta, \Gamma \to \gamma}{\alpha * \beta, \Gamma \to \gamma} \ (\textit{left-fusion}) \quad \frac{\Gamma \to \alpha \quad \Delta \to \beta}{\Gamma, \Delta \to \alpha * \beta} \ (\textit{right-fusion})$$

$\mathbf{FL_e}[\mathrm{E_k}]$ ($\mathbf{FL_e}[\mathrm{DN}]$) is the logic obtained from $\mathbf{FL_e}$ by adding $\mathrm{E_k}$: $p^k \leftrightarrow p^{k+1}$ (*weak k-potency*) (DN: $\neg\neg p \to p$ (*double negation*), respectively) as an axiom. Similarly, we can define $\mathbf{FL_{ew}}[\mathrm{E_k}]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$.
Suppose that $k = 1$. Then $\mathrm{E_k}$ is $p \leftrightarrow p^2$. Then $p \to p^2$ means a contraction. Thus $\mathbf{FL_{ew}}[\mathrm{E_1}]$ is a intuitionistic logic and $\mathbf{FL_e}[p \to p^2]$ is a $\mathbf{FL_{ec}}$.

2.3 Logics over substructural logic $\mathbf{FL_e}$

Here we define a logic as a set of formulas which closed under modus ponens and substitution. Exactly we define following.

Definition 2.3.1 (logic) A set L of formulas is a *logic* if the followings consist

1. $\mathbf{FL_e} \subseteq L$.
2. If a formula $\phi(p)$ which include a propositional variable p is a element of L then $\phi(\psi) \in L$ for any formula ψ. $\phi(\psi)$ means that all p that appearing in ϕ is replaced with ψ.
3. If $\phi, \phi \supset \psi \in L$ then $\psi \in L$.
4. If $\phi, \psi \in L$ then $\phi \wedge \psi \in L$.

(If L is logic over $\mathbf{FL_{ew}}$ then a condition 4 is derived from a condition 2 and $\phi \supset (\psi \supset (\phi \wedge \psi)) \in \mathbf{FL_{ew}}$).

It is clear that Φ which is a set of all formulas is a logic. In addition a set of all formulas which is provable in $\mathbf{FL_e}$ is a logic. After here we express a set of all formulas which is provable in $\mathbf{FL_e}$ by $\mathbf{FL_e}$ if no confusion will occur.

A set of logics is a ordered set by relation of inclusion. The maximum logic is the Φ. Hence logics over $\mathbf{FL_e}$ means that logics exist between $\mathbf{FL_e}$ and Φ. Let $\mathcal{L} = \{L | L$ is a logic and $\mathbf{FL_e} \subseteq L\}$. Then it is clear that $L_1 \cap L_2 \in \mathcal{L}$ for any $L_1, L_2 \in \mathcal{L}$. But union $L_1 \cup L_2$ is not necessarily element of $\mathcal{L}$. So we define $L_1 \vee L_2$ as a minimum logic including $L_1 \cup L_2$. Thus $\langle \mathcal{L}, \cap, \vee, \mathbf{FL_e}, \Phi \rangle$ is a bounded lattice whose greatest element is Φ and the least element $\mathbf{FL_e}$. In this thesis we treat logics belonging $\mathcal{L}$.

Chapter 3

Commutative residuated Lattices

In this section we introduce an algebraic structures corresponding to logics over $\mathbf{FL_e}$. They are called *commutative residuated lattices* (CRLs). We show basic properties of commutative residuated lattice from the viewpoint of universal algebra.

3.1 Commutative residuated lattices

Definition 3.1.1 An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ is a CRL if $\mathbf{A}$ satisfies the following three conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee, 0, 1 \rangle$ is a lattice,

(R2) $\langle \mathrm{A}, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,

(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow x \leq y \rightarrow z$.

When $\langle \mathrm{A}, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice with the greatest element 1 and the least 0, $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ is called a *commutative integral residuated lattice* (CIRL). It is easy to see that a commutative integral residuated lattice is a *Heyting algebra* if and only if the semigroup operation $\cdot$ is equal to $\wedge$. The set A of $\mathbf{A}$ is called the *universe* of $\mathbf{A}$.

The condition (R3) of this definition is called *residuation.* This condition means that $\rightarrow$ behaves similarly to an inverse operation of $\cdot$.

3.2 Basic properties for CRLs

Subalgebras, homomorphisms and so on which play a important role in study of algebra can be introduced into CRLs. In this section we show some basic properties.

3.2.1 Homomorphism and isomorphism

Definition 3.2.1 (Homomorphism) Let $\mathbf{A} = \langle \mathrm{A}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \cdot_{\mathbf{A}}, \rightarrow_{\mathbf{A}}, 0_{\mathbf{A}}, 1_{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle \mathrm{B}, \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \cdot_{\mathbf{B}}, \rightarrow_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}} \rangle$ be CRLs. A mapping $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ is a *homomorphism* if α satisfies the following conditions.

- $\alpha(1_{\mathbf{A}}) = 1_{\mathbf{B}}$, $\alpha(0_{\mathbf{A}}) = 0_{\mathbf{B}}$,
- for any $a_1, a_2 \in \mathrm{A}$, $\alpha(a \oplus_{\mathbf{A}}) = \alpha(a_1) \oplus_{\mathbf{B}} \alpha(a_2)$, (for $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$).

Furthermore,

1. if α is an one-to-one mapping then α is called a *monomorphism* or a *embedding.*
2. if α is an onto mapping then α is called a *epimorphism* or an *onto homomorphism.*
3. if α is an one-to-one and onto mapping then α is called an *isomorphism.* If there exist an isomorphism α from $\mathbf{A}$ to $\mathbf{B}$ then $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$.

From here $\oplus$ means one of operations $\wedge$, $\vee$, $\cdot$ and $\rightarrow$.

Definition 3.2.2 (kernel and image) Let $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism. Then the *kernel of* α, written $ker(\alpha)$, and the *image of* α, written $Im(\alpha)$, are defined by

$$ker(\alpha) = \{\langle a, b \rangle \in \mathrm{A}^2 : \alpha(a) = \alpha(b)\}, \quad Im(\alpha) = \{\alpha(a) \in \mathrm{B} : a \in \mathrm{A}\}.$$

If α is a surjective then $Im(\alpha)$ is equal to $\mathbf{B}$ and we say that $\mathbf{B}$ is the homomorphic image of $\mathbf{A}$. Sometime $Im(\alpha)$ is expressed also by $\alpha(\mathbf{A})$.

3.2.2 Subalgebra and quotient algebra

Definition 3.2.3 (subalgebra) Let $\mathbf{A}$ and $\mathbf{B}$ be two CRLs. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $\mathrm{B} \subseteq \mathrm{A}$ and every operation $\oplus_{\mathbf{B}} \in \{\wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \cdot_{\mathbf{B}}, \rightarrow_{\mathbf{B}}\}$ of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$. We write simply $\mathbf{B} \leq \mathbf{A}$ when $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. A subuniverse of $\mathbf{A}$ is a subset B of A which is closed under the operations of $\mathbf{A}$, i.e. if $\oplus_{\mathbf{A}}$ is a operation of $\mathbf{A}$ and $a_1, a_2 \in \mathrm{B}$ we would require $a_1 \oplus_{\mathbf{A}} a_2 \in \mathrm{B}$.

Definition 3.2.4 Given an algebra $\mathbf{A}$ define, for every $X \subseteq \mathrm{A}$,

$$Sg(X) = \bigcap \{\mathrm{B} : X \subseteq \mathrm{B} \text{ and B is a subuniverse of } \mathbf{A}\}$$

We read $Sg(X)$ as the subuniverse generated by X.

For information on Sg, see [3].

Definition 3.2.5 (congruence) Let $\mathbf{A}$ be a CRL and let θ is equivalence relation on $\mathbf{A}$. Then θ is a *congruence* on $\mathbf{A}$ if θ satisfies the following *compatibility property*:

CP: For each operation $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$ and elements $a_1, a_2, b_1, b_2 \in \mathrm{A}$, if $a_1 \theta b_1$ and $a_2 \theta b_2$ holds then

$$(a_1 \oplus a_2)\theta(b_1 \oplus b_2)$$

holds.

We can consider that congruences on $\mathbf{A}$ are a subset of $\mathbf{A} \times \mathbf{A}$ and thus they are ordered by the set inclusion. Hence we define maximum congruence ∇ and minimum congruence Δ as follows.

$$\nabla = \{\langle a, b\rangle; a, b \in \mathrm{A}\}$$
$$\Delta = \{\langle a, a\rangle; a \in \mathrm{A}\}$$

The set of all congruences on $\mathbf{A}$ is denoted by Con $\mathbf{A}$. Then we can easily show that Con $\mathbf{A}$ is a bounded lattice which has the maximum element ∇ and the minimum element Δ. So the congruence lattice on $\mathbf{A}$ denoted by Con $\mathbf{A}$. Followings are a definition of $\wedge$ and $\vee$, where $\theta_1 \circ \theta_2$ denote the set $\{\langle a, b\rangle \mid \exists c \in \mathbf{A} \text{ s.t. } a\theta_1 c\theta_2 b\}$.

$$\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$$
$$\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \ldots.$$

Definition 3.2.6 Let $\mathbf{A}$ be a CRL and $a_1, \ldots, a_n \in \mathrm{A}$. Then $\Theta(a_1, \ldots, a_n)$ is the minimum congruence such that $a_1, \ldots, a_n$ are contained in a same equivalence class.

Proposition 3.2.1 *Let* $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ *be a homomorphism. Then* $ker(\alpha)$ *is actually a congruence on* $\mathbf{A}$*.*

Proof If $\langle a_1, a_2, b_1, b_2\rangle \in ker(\alpha)$, then

$$\begin{aligned} \alpha(a_1 \oplus_{\mathbf{A}} a_2) &= \alpha(a_1) \oplus_{\mathbf{B}} \alpha(a_2) \\ &= \alpha(b_1) \oplus_{\mathbf{B}} \alpha(b_2) \\ &= \alpha(b_1 \oplus_{\mathbf{A}} b_2) \end{aligned}$$

hence

$$\langle a_1 \oplus_{\mathbf{A}} a_2, b_1 \oplus_{\mathbf{B}} b_2\rangle \in ker(\alpha).$$

Clearly $ker(\alpha)$ is an equivalence relation, so it follows that $ker(\alpha$ is actually a congruence on $\mathbf{A}$.

Let θ is a congruence on a CRL $\mathbf{A}$. Then θ is equivalence relation. So we define a equivalence class (a/θ) include $a \in \mathrm{A}$ as follows.

$a/\theta = \{x \in \mathrm{A}; x\theta a\}$.

In addition we define quotient set A/θ as follows.

$\mathrm{A}/\theta = \{a/\theta; a \in \mathrm{A}\}$.

Definition 3.2.7 (quotient algebra) Let θ be a congruence on $\mathbf{A}$. Then the *quotient algebra of* $\mathbf{A}$ *by* θ, written $\mathbf{A}/\theta$, is the algebra whose universe is A/θ and whose operations satisfy

$a_1/\theta \oplus a_2/\theta = (a_1 \oplus a_2)/\theta$

where $a_1, a_2 \in \mathrm{A}$ and $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$.

Definition 3.2.8 (natural maps) Let $\mathbf{A}$ be an algebra and let $\theta \in \mathrm{Con}\,\mathbf{A}$. The *natural map* $\nu_\theta : \mathrm{A} \rightarrow \mathrm{A}/\theta$ is defined by $\nu_\theta(a) = a/\theta$ for any $a \in \mathrm{A}$. (When there is no ambiguity we write simply ν instead of ν_θ.)

Proposition 3.2.2 *A natural map from* $\mathbf{A}$ *to* $\mathbf{A}/\theta$ *is a onto homomorphism.*

Proof It is clear that the natural map is onto. For any $a, b \in \mathrm{A}$

$$\begin{aligned} v_\theta(a \oplus b) &= (a \oplus b)/\theta \\ &= a/\theta \oplus' b/\theta \\ &= v_\theta(a) \oplus' v_\theta(b) \end{aligned}$$

Thus v_θ is a homomorphism.

Proposition 3.2.3 (Homomorphism theorem) *Let* $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ *be an onto homomorphism. Then there is an isomorphism* β *from* $\mathbf{A}/ker(\alpha)$ *to* $\mathbf{B}$ *defined by* $\alpha = \beta \circ \nu$*, where* ν *is the natural homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}/ker(\alpha)$.

Proof First note that if $\alpha = \beta \circ \nu$ then we must have $\beta(a/\theta) = \alpha(a)$. The second of these equalities does indeed define a function β, and β satisfies $\alpha = \beta \circ \nu$. It is not difficult to verify that β is a bijection. To show that β is actually an isomorphism, suppose $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$ and $a_1, a_2 \in \mathrm{A}$. Then

$$\begin{aligned} \beta(a_1/\theta \oplus_{\mathbf{A}/\theta} a_2/\theta) &= \beta((a_1 \oplus_{\mathbf{A}} a_2)/\theta) \\ &= \alpha(a_1 \oplus_{\mathbf{A}} a_2) \\ &= \alpha(a_1) \oplus_{\mathbf{B}} \alpha(a_2) \\ &= \beta(a_1/\theta) \oplus_{\mathbf{B}} \beta(a_2/\theta). \end{aligned}$$

Let $\mathbf{A}$ be a CRL and $\phi, \theta \in \text{Con } \mathbf{A}$ and $\theta \subseteq \phi$. Then we define ϕ/θ as follows.

$$\phi/\theta = \{\langle a/\theta, b/\theta\rangle \in (\mathbf{A}/\theta)^2 : \langle a, b\rangle \in \phi\}$$

The next proposition holds.

Proposition 3.2.4 *Let $\phi, \theta \in \text{Con } \mathbf{A}$ and $\theta \subseteq \phi$. Then ϕ/θ is a congruence on $\mathbf{A}/\theta$.*

Proof Let $\langle a_1/\theta, b_1/\theta\rangle \in \phi/\theta$. Then $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in \phi$ from definition of ϕ/θ. So

$$\langle a_1 \oplus_{\mathbf{A}} a_2, b_1 \oplus_{\mathbf{B}} b_2\rangle \in \phi.$$

Hence

$$\langle (a_1 \oplus_{\mathbf{A}} a_2)/\theta, (b_1 \oplus_{\mathbf{B}} b_2)/\theta\rangle \in \phi/\theta.$$

Form this

$$\langle a_1/\theta \oplus_{\mathbf{A}/\theta} a_2/\theta, b_1/\theta \oplus_{\mathbf{B}/\theta} b_2/\theta\rangle \in \phi.$$

Thus ϕ/θ is a congruence on $\mathbf{A}/\theta$.

Let $\mathbf{A}$ be a CRL and $\theta \in Con(\mathbf{A})$. Then we define sublattice $[\theta, \nabla]$ of Con $\mathbf{A}$ which holds a following.

$$[\theta, \nabla] = \{\phi \in \text{Con } \mathbf{A} : \theta \subseteq \phi \subseteq \nabla\}$$

Proposition 3.2.5 (Correspondence theorem) *Let $\mathbf{A}$ be a CRL and $\theta \in$ Con $\mathbf{A}$. Then a mapping α on $[\theta, \nabla]$ defined by*

$$\alpha(\phi) = \phi/\theta$$

is a isomorphism from $[\theta, \nabla]$ to Con $\mathbf{A}/\theta$

Proof First we show α is a one-to-one. Let $\phi, \psi \in [\theta, \nabla]$ $(\phi \neq \psi)$. Suppose that $\phi \not\subseteq \psi$. Then there are $a, b \in$ A such that $\langle a, b\rangle \in \phi - \psi$. Hence

$$\langle a/\theta, b/\theta\rangle \in (\phi/\theta) - (\psi/\theta)$$

So

$$\alpha(\phi) \neq \alpha(\psi)$$

Thus α is a one-to-one. Next we show α is a onto. Let $\psi \in \text{Con}\mathbf{A}/\theta$, and $\phi = ker(\nu_\psi \circ \nu_\theta)$. ν_ψ is a natural homomorphism from **Con A**$/\theta$ to (**Con A**$/\theta)/\psi$. Hence for any $a, b \in$ A

$$\begin{aligned} & \langle a/\theta, b/\theta\rangle \in \phi/\theta \\ \Leftrightarrow \quad & \langle a, b\rangle \in \phi \\ \Leftrightarrow \quad & \langle a/\theta, b/\theta\rangle \in \psi. \end{aligned}$$

So

$\psi = \phi/\theta = \alpha(\phi)$.

Thus α is a onto. Finally we show α is a isomorphism.

$$\begin{aligned}
(\phi \cap_{\mathbf{A}} \psi)/\theta &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \cap_{\mathbf{A}} \psi\} \\
&= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \text{ and } \langle \mathrm{a}, \mathrm{b}\rangle \in \psi\} \\
&= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi\} \text{ and } \{\langle \mathrm{a}/\theta, \mathrm{b}/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle \mathrm{a}, \mathrm{b}\rangle \in \psi\} \\
&= \phi/\theta \text{ and } \psi/\theta \\
&= \phi/\theta \cap_{\mathrm{Con}\ \mathbf{A}/\theta} \psi/\theta
\end{aligned}$$

$$\begin{aligned}
(\phi \vee_{\mathbf{A}} \psi)/\theta &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \vee_{\mathbf{A}} \psi\} \\
&= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \exists c_0 = a, c_1, \ldots, c_k = b \in \mathrm{A} \\
&\quad s.t.\ \langle c_i, c_{i+1}\rangle \in \phi \text{ or } \langle \mathrm{c_i}, \mathrm{c_{i+1}}\rangle \in \psi\ (0 \leq \mathrm{i} \leq \mathrm{k}-1)\} \\
&= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \exists c_0/\theta = a/\theta, c_1/\theta, \ldots, c_k/\theta = b/\theta \in \mathrm{A}/\theta \\
&\quad s.t.\ \langle c_i/\theta, c_{i+1}/\theta\rangle \in \phi/\theta \text{ or } \langle \mathrm{c_i}/\theta, \mathrm{c_{i+1}}/\theta\rangle \in \psi/\theta\ (0 \leq \mathrm{i} \leq \mathrm{k}-1)\} \\
&= \phi/\theta \vee_{\mathrm{Con}\ \mathbf{A}/\theta} \psi/\theta
\end{aligned}$$

Thus $\alpha(\phi \oplus_{\mathbf{A}} \psi)/\theta = \phi/\theta \oplus_{\mathrm{Con}\ \mathbf{A}/\theta}$ holds.

3.2.3 Direct product and subdirect product

Definition 3.2.9 (Direct product) Let $(\mathbf{A}_i)_{1 \leq i \leq n}$ is an indexed family of algebras. Define the *direct product* $\prod_{1 \leq i \leq n} \mathbf{A}_i$ to be the algebra whose universe is the set $\prod_{1 \leq i \leq n} \mathrm{A}_i$ and such that for $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$ and $a_i, a_i' \in \mathrm{A_i}$, $1 \leq i \leq n$,

$$\langle a_1, a_2, \ldots, a_n\rangle \oplus_{\prod_{1 \leq i \leq n} \mathbf{A}_i} \langle a_1', a_2', \ldots, a_n'\rangle = \langle a_1 \oplus_{\mathbf{A_1}} a_1', a_2 \oplus_{\mathbf{A_2}} a_2', \ldots, a_n \oplus_{\mathbf{A_n}} a_n'\rangle.$$

After here $x(j)$ means jth element of x.

Proposition 3.2.6 *Let $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ be algebras. Then the following isomorphic relations hold.*

1. $\mathbf{A_1} \times \mathbf{A_2} \cong \mathbf{A_2} \times \mathbf{A_1}$
2. $\mathbf{A_1} \times (\mathbf{A_2} \times \mathbf{A_3}) \cong \mathbf{A_1} \times \mathbf{A_2} \times \mathbf{A_3}$

Proof Let homomorphisms of 1 and 2 be $\alpha(\langle a_1, a_2\rangle) = \langle a_2, a_1\rangle$ and $\alpha(\langle a_1, \langle a_2, a_3\rangle\rangle) = \langle a_1, a_2, a_3\rangle$, respectively. Clearly that α_1, α_2 are isomorphisms.

The mapping $\pi_i : \prod_{1 \leq i \leq n} \mathbf{A}_i \longrightarrow \mathbf{A}_i$ $(1 \leq i \leq n)$ defined by

$$\pi_i(\langle a_1, a_2, \ldots, a_n\rangle) = a_i$$

is called the *projection map on the* ith *coordinate* of $\prod_{1\leq i\leq n} \mathrm{A}_i$. We can easily show that projection map is an onto homomorphism.

Definition 3.2.10 (Subdirect product) An algebra $\mathbf{A}$ is a *subdirect product* of an indexed family $(\mathbf{A}_i)_{i\in I}$ of algebras if

$$\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$$

and

$$\pi_i(\mathbf{A}) = \mathbf{A}_i \text{ for each } i \in I.$$

A subdirect product of $(\mathbf{A}_i)_{i\in I}$ is an algebra which is a subalgebra of $\prod_{i\in I} \mathbf{A}_i$ and satisfies the condition 2. Moreover because $\mathbf{A}$ satisfies the condition 2, it is not necessarily that $\mathbf{A}$ is isomorphic to $\prod_{i\in I} \mathbf{A}_i$. For example, if $\mathrm{A}_1 = \{a, b\}$, $\mathrm{A}_2 = \{c, d, e\}$, $\mathrm{A} = \{(a, c), (a, d), (b, c), (b, e)\}$, then it satisfies 2. But $\mathbf{A}$ is not isomorphic to $\prod_{i\in I} \mathbf{A}_i$ from $\prod_{i\in\{1,2\}} \mathrm{A}_i = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$. An intuitive meaning of subdirect products is that they are sufficiently large subalgebras among direct products.

Definition 3.2.11 The mapping $\alpha : \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$ is a subdirect embedding if α is a embedding and $\alpha(\mathbf{A})$ is a subdirect product of $\prod_{i\in I} \mathbf{A}_i$.

Proposition 3.2.7 *Let* $\theta \in \mathrm{Con}\ \mathbf{A}$ $(i \in I)$ *and* $\bigcap_{i\in I} \theta_i = \Delta$. *Then a homomorphism* $\nu : \mathbf{A} \to \prod_{i\in I} \mathbf{A}/\theta_i$ *defined by*

$$\nu(\alpha)(i) = a/\theta_i$$

is a subdirect embedding.

Proof If we define the nu by $\nu_i = \pi_i \circ \nu$ for any $i \in I$ then the ν_i is a natural homomorphism from $\mathbf{A}$ to $\mathbf{A}/\theta_i$. First we show that $\nu(\mathbf{A})$ is a subalgebra of $\prod_{i\in I} \mathbf{A}/\theta_i$.
For all $\nu(a), \nu(b) \in \nu(\mathbf{A})$ $(a, b \in \mathrm{A})$

$$\nu(a) \oplus_{\prod_{i\in I} \mathbf{A}/\theta_i} \nu(b) = \nu(a \oplus_{\mathbf{A}} b) \in \nu(\mathbf{A}).$$

Furthermore

$$\{\top_{\prod_{i\in I} \mathbf{A}/\theta_i}, \bot_{\prod_{i\in I} \mathbf{A}/\theta_i}, 1_{\prod_{i\in I} \mathbf{A}/\theta_i}, 0_{\prod_{i\in I} \mathbf{A}/\theta_i}\} = \{\nu(\top_{\mathbf{A}}), \nu(\bot_{\mathbf{A}}), \nu(1_{\mathbf{A}}, \nu(0_{\mathbf{A}})\} \subseteq \nu(\mathbf{A}).$$

Hence $\nu(\mathbf{A})$ is a subalgebra of $\prod_{i\in I} \mathbf{A}/\theta_i$.
Moreover for all $i \in I$ $\nu(\mathbf{A})$ is a subdirect product of $\prod_{i\in I} \mathbf{A}/\theta_i$ from $\nu_i(\mathbf{A}) = \mathbf{A}/\theta_i$.
Next we show that ν is an embedding. For all $a, b \in \mathrm{A}$ $(a \neq b)$

$$\langle a, b\rangle \notin \bigcap_{i\in I} \theta_i$$

from $\bigcap_{i\in I} \theta_i = \Delta$. Hence there exist some $j \in I$ such that

$$\langle a, b\rangle \notin \theta_j.$$

From this $\nu_j(a) \neq \nu_j(b)$. So $\nu(a) \neq \nu(b)$. Thus ν is an embedding.

3.3 properties of classes of CRLs

In the previous section we show some properties of algebras. In this section we show properties of classes of algebras.

3.3.1 Variety

Definition 3.3.1 (class operator) We define mappings from class K of algebras to class $I(K)$, $S(K)$, $H(K)$, $P(K)$ and $P_s(K)$ as follows.

- $\mathbf{A} \in I(K) \Leftrightarrow \mathbf{A}$ is isomorphic to some member of K.
- $\mathbf{A} \in S(K) \Leftrightarrow \mathbf{A}$ is a subalgebra of some member of K.
- $\mathbf{A} \in H(K) \Leftrightarrow \mathbf{A}$ is a homomorphic image of some member of K.
- $\mathbf{A} \in P(K) \Leftrightarrow \mathbf{A}$ is a direct product of a nonempty family of algebras in K.
- $\mathbf{A} \in P(K) \Leftrightarrow \mathbf{A}$ is a subdirect product of a nonempty family of algebras in K.

Let O_1 and O_2 are two operators on classes of algebras.We write O_1O_2 for the composition and $\leq$ denotes the usual partial order, i.e. $O_1 \leq O_2$ if $O_1(K) \subseteq O_2(K)$ for every class K of algebras.

Definition 3.3.2 (idempotent operator, closed class) Let K be a class of CRLs and O be a operator on class of CRLs. Then O is a *idempotent* if $O^2 = O$, and K is *closed* under O if $O(K) \subseteq K$.

Proposition 3.3.1 *Following inequalities hold.*

$$SH \leq HS$$
$$PS \leq SP$$
$$PH \leq HP$$

Also the operators H, S, and IP are idempotent.

Proof Suppose $\mathbf{A} = SH(K)$. Then for some $\mathbf{B} \in K$ and onto homomorphism $\alpha : \mathbf{B} \to \mathbf{C}$, we have $\mathbf{A} \leq \mathbf{C}$. Thus $\alpha^{-1}(\mathbf{A}) \leq \mathbf{B}$, and as $\alpha(\alpha^{-1}(\mathbf{A})) = \mathbf{A}$, we have $\mathbf{A} \in HS(K)$.
If $\mathbf{A} \in PS(K)$ then $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ for suitable $\mathbf{A}_i \leq \mathbf{B}_i \in K$, $i \in I$. As $\prod_{i \in I} \mathbf{A}_i \leq \prod_{i \in I} \mathbf{B}_i$, we have $\mathbf{A} \in SP(K)$.
Next if $\mathbf{A} \in PH(K)$, then there are algebras $\mathbf{B}_i \in K$ and epimorphisms $\alpha_i : \mathbf{B}_i \to \mathbf{A}_i$ such that $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$. It is easy to check that the mapping $\alpha : \prod_{i \in I} \mathbf{B}_i \to \prod_{i \in I} \mathbf{A}_i$ defined by $\alpha(b)(i) = \alpha_i(b(i))$ is an epimorphism; hence $\mathbf{A} \in HP(K)$.
We show $H = H^2$. $H \subseteq H^2$ is clear. If $\mathbf{A} \in H^2(K)$ then there exist onto homomorphisms $\alpha : \mathbf{B} \to \mathbf{C}$, $\beta : \mathbf{C} \to \mathbf{A}$ and $\mathbf{B} \in K$. So $\beta \circ \alpha$ is an onto homomorphism. Thus $\mathbf{A} \in H(K)$.
We can show that S, and IP are idempotent in the same way.

A class of algebra (variety) defined by following is especially major class.

Definition 3.3.3 (variety) Let K is nonempty class of algebra. K is a *variety* if it is closed under class operators S, H, P.

If K is a class of algebras let $V(K)$ denote the smallest variety containing K. We say that $V(K)$ is the *variety generated by* K. If K consists of a single member $\mathbf{A}$ then we write simply $V(\mathbf{A})$.

Proposition 3.3.2 (Tarski) $V = HSP$.

Proof Since $HV = SV = IPV = V$ and $I \leq V$, it follows that $HSP \leq HSPV = V$. From above lemma we see that $H(HSP) = HSP$, $S(HSP) \leq HSSP = HSP$, and

$$\begin{aligned} P(HSP) &\leq HPSP \\ &\leq HSPP \\ &\leq HSIPIP \\ &= HSIP \\ &\leq HSHP \\ &\leq HHSP \\ &= HSP. \end{aligned}$$

Hence for any K, $HSP(K)$ is closed under H, S, and P. As $V(K)$ is the smallest class containing K and closed under H, S, and P, we must have $V = HSP$.

3.4 Free algebra

Definition 3.4.1 Let X be a set of (distinct) objects called *variables*. The set $T(X)$ of *terms over* X is the smallest set such that

1. $X \cup \{0, 1\} \subseteq T(X)$.
2. If $p_1, p_2 \in T(X)$ and $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$ then the "string" $p_1 \oplus p_2 \in T(X)$.

For $p \in T(X)$ we often write p as $p(x_1, \ldots, x_n)$ to indicate that the variables occurring in p are among $x_1, \ldots, x_n$.

Definition 3.4.2 Given a term $p(x_1, \ldots, x_n)$ over some set X and given an algebra $\mathbf{A}$ we define a mapping $p^{\mathbf{A}} : \mathrm{A}^n \to \mathbf{A}$ as follows:
(1) if p is a variable x_i, then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$$

for $a_1, \ldots, a_n \in \mathrm{A}$, i.e., $p^{\mathbf{A}}$ is the ith projection map;
(2) if p is of the form $p_1(x_1, \ldots, x_n) \oplus p_2(x_1, \ldots, x_n)$ then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = p_1^{\mathbf{A}}(x_1, \ldots, x_n) \oplus_{\mathbf{A}} p_2^{\mathbf{A}}(x_1, \ldots, x_n).$$

In particular if p is $\oplus$ then $p^{\mathbf{A}}$ is $\oplus_{\mathbf{A}}$. The expression $p^{\mathbf{A}}$ is called the *term function* on $\mathbf{A}$ corresponding to the term p. (Often we will drop the superscript $\mathbf{A}$).

The next proposition gives some useful properties of term functions.

Proposition 3.4.1 *For any algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *we have the following.*
(a) Let p be an n-ary term, let $\theta \in \mathrm{Con}\ \mathbf{A}$, and suppose $\langle a_i, b_i \rangle \in \theta$ for $1 \leq i \leq n$. Then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) \theta p^{\mathbf{A}}(b_1, \ldots, b_n).$$

(b) If p is an n-ary term and $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then

$$\alpha(p^{\mathbf{A}}(a_1, \ldots, a_n)) = p^{\mathbf{B}}(\alpha(a_1), \ldots, \alpha(a_n))$$

for $a_1, \ldots, a_n \in \mathrm{A}$.
(c) Let S be a subset of A. Then

$$\mathrm{Sg}(S) = \{p^{\mathbf{A}}(a_1, \ldots, a_n) : p \textit{ is an n-ary term, } n \leq \omega, \textit{ and } a_1, \ldots, a_n \in S\}.$$

Proof Given a term p define the length $l(p)$ of p to be a number of occurrences of n-ary operation symbols in p for $n \geq 1$. Note that $l(p) = 0$ iff $p \in X \cup \{0, 1\}$.
(a) We proceed by induction on $l(p)$. If $l(p) = 0$, then either $p = x_i$ for some i, whence

$$\langle p^{\mathbf{A}}(a_1, \ldots, a_n), p^{\mathbf{A}}(b_1, \ldots, b_n) \rangle = \langle a_i, b_i \rangle \in \theta$$

or $p = a$ for some $a \in \{0, 1\}$, whence

$$\langle p^{\mathbf{A}}(a_1, \ldots, a_n), p^{\mathbf{A}}(b_1, \ldots, b_n) \rangle = \langle a^{\mathbf{A}}, b^{\mathbf{A}} \rangle \in \theta.$$

Now suppose $l(p) > 0$ and the assertion holds for every term q with $l(p) < l(q)$. Then we know p is of the form

$$p_1(x_1, \ldots, x_n) \oplus p_2(x_1, \ldots, x_n),$$

and as $l(p_i) < l(p)$ we must have, for $i \in \{1, 2\}$,

$$\langle p_i^{\mathbf{A}}(a_1, \ldots, a_n), p_i^{\mathbf{A}}(b_1, \ldots, b_n) \in \theta;$$

hence

$$\langle (p_1^{\mathbf{A}}(a_1, \ldots, a_n) \oplus_{\mathbf{A}} p_2^{\mathbf{A}}(a_1, \ldots, a_n)), (p_1^{\mathbf{A}}(b_1, \ldots, b_n) \oplus_{\mathbf{A}} p_2^{\mathbf{A}}(b_1, \ldots, b_n)) \rangle \in \theta,$$

and consequently

$$\langle p^{\mathbf{A}}(a_1, \ldots, a_n), p^{\mathbf{A}}(b_1, \ldots, b_n) \in \theta.$$

(b) The proof of this is an induction argument on $l(p)$.
(c) Referring to chapter II §3 of [3] one can give an induction proof, for $k \geq 1$, of

$$E^k(S) = \{p^{\mathbf{A}}(a_1, \ldots, a_n) : p \text{ is an n-ary term}, l(p) \leq k, n < \omega, a_1, \ldots, a_n \in S\},$$

and thus

$$\begin{aligned} Sg(S) &= \textstyle\bigcup_{k<\infty} E^k(S) \\ &= \{p^{\mathbf{A}}(a_1, \ldots, a_n) : p \text{ is an n-ary term}, n < \omega, a_1, \ldots, a_n \in S\}. \end{aligned}$$

One can, in a natural way, transform the set $T(X)$ into an algebra.

Definition 3.4.3 (term algebra) Given X, if $T(X) \neq \emptyset$ then the *term algebra* over X, written $\mathbf{T}(X)$, has as its universe the set $T(X)$, and operations satisfy

$$p_1 \oplus_{\mathbf{T}(X)} p_2 = p_1 \oplus p_2$$

for $\oplus \in \{\wedge, \vee, \cdot, \rightarrow\}$ and $p_1, p_2 \in T(X)$.

Definition 3.4.4 (universal mapping property) Let K be a class of algebras and let $\mathbf{U}(X)$ be an algebra which is generated by X. If for every $\mathbf{A} \in K$ and for every map

$$\alpha : X \rightarrow \mathrm{A}$$

there is a homomorphism

$$\beta : \mathbf{U}(X) \rightarrow \mathbf{A}$$

which extends α (i.e., $\beta(x) = \alpha(x)$ for $x \in X$), then we say $\mathbf{U}(X)$ has the *universal mapping property for K over X*, X is called a set of *free generators* of $\mathbf{U}(X)$, and $\mathbf{U}(X)$ is said to be *freely generated* by X.

Lemma 3.4.2 *Suppose $\mathbf{U}(X)$ has the universal mapping property for K over X. Then if we are given $\mathbf{A} \in K$ and $\alpha : X \rightarrow \mathrm{A}$, there is a unique extension β of α such that β is a homomorphism from $\mathbf{U}(X)$ to $\mathbf{A}$.*

Proof This follows simply from noting that a homomorphism is completely determined by how it maps a set of generators from the domain.

The next result says that for a given cardinal m there is, up to isomorphism, at most one algebra in a class K which has the universal mapping property for K over a set of free generators of size m.

Proposition 3.4.3 *Suppose* $\mathbf{U}_1(X_1)$ *and* $\mathbf{U}_2(X_2)$ *are two algebras in a class* K *with the universal mapping property for* K *over the indicated sets. If* $\mid X_1 \mid = \mid X_2 \mid$*, then* $\mathbf{U}_1(X_1) \cong \mathbf{U}_2(X_2)$.

Proof First note that the identity map

$$\iota_j : X_j \to X_j \ (j = 1, 2),$$

has as its unique extension to a homomorphism from $\mathbf{U}_j(X_j)$ to $\mathbf{U}_j(X_j)$ the identity map. Now let

$$\alpha : X_1 \to X_2$$

be a bijection. Then we have a homomorphism

$$\beta : \mathbf{U}_1(X_1) \to \mathbf{U}_2(X_2)$$

extending α, and a homomorphism

$$\gamma : \mathbf{U}_2(X_2) \to \mathbf{U}_1(X_1)$$

extending α^{-1}. As $\beta \circ \gamma$ is an endomorphism of $\mathbf{U}_2(X_2)$ extending ι_2, it follows by lemma 3.4.2 that $\beta \circ \gamma$ is the identity map on $\mathbf{U}_2(X_2)$. Likewise $\gamma \circ \beta$ is the identity map on $\mathbf{U}_1(X_1)$. Thus β is a bijection, so $\mathbf{U}_1(X_1) \cong \mathbf{U}_2(X_2)$.

Proposition 3.4.4 *For any set* X *of variables, where* $x \neq \emptyset$ *if there are no constants, the term algebra* $\mathbf{T}(X)$ *has the universal mapping property for the class of all algebras over* X.

Proof Let $\alpha : X \to \mathrm{A}$. Define

$$\beta : T(X) \to \mathrm{A}$$

recursively by

$$\beta(x) = \alpha(x)$$

for $x \in X$, and

$$\beta(p_1 \oplus p_2) = \beta(p_1) \oplus_{\mathbf{A}} \beta(p_2)$$

for $p_1, p_2 \in T(X)$. Then $\beta(p(x_1, \ldots, x_n)) = p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n))$, and β is the desired homomorphism extending α.

Thus given any class K of algebras the term algebras provide algebras which have the universal mapping property for K. To study properties of classes of algebras we often try to find special kinds of algebras in these classes which yield the desired information. In order to find algebras with the universal mapping property for K which give more insight into K we will introduce K-free algebras. Unfortunately not every class K contains algebras with the universal mapping property for K. Nonetheless we will be able to show that any class closed under I, S, and P contains its K-free algebras. There is reasonable difficulty in providing transparent descriptions of K-free algebras for most K. However, most of the applications of K-free algebras come directly from the universal mapping property, the fact that they exist in varieties, and their relation to identities holding in K. A proper understanding of free algebras is essential in our development of universal algebra (see [3]).

Definition 3.4.5 Let K be a family of algebras. Given a set X of variables define the congruence $\theta_K(X)$ on $\mathbf{T}(X)$ by

$$\theta_K(X) = \cap\Phi_K(X)$$

where

$$\Phi_K(X) = \{\phi \in \text{Con } \mathbf{T}(X) : \mathbf{T}(X)/\phi \in IS(K)\};$$

and then define $\mathbf{F}(\overline{X})$, *the K-free algebra over* $\overline{X}$, by

$$\mathbf{F}_K(\overline{X}) = \mathbf{T}(X)/\theta_K(X),$$

where

$$\overline{X} = X/\theta_K(X).$$

Proposition 3.4.5 (Birkhoff) *Suppose* $\mathbf{T}(X)$ *exists. Then* $\mathbf{F}_K(\overline{X})$ *has the universal mapping property for* K *over* $\overline{X}$.

Proof Given $\mathbf{A} \in K$ let α be a map from $\overline{X}$ to A. Let $\nu : \mathbf{T}(X) \to \mathbf{F}_K(\overline{X})$ be the natural homomorphism. Then $\alpha \circ \nu$ maps X into A, so by the universal mapping property of $\mathbf{T}(X)$ there is a homomorphism $\mu : \mathbf{T}(X) \to \mathbf{A}$ extending $\alpha \circ \nu \upharpoonright_X$. From the definition of $\theta_K(X)$ it is clear that $\theta_K(X) \subseteq ker(\mu)$ (as $ker(\mu) \in \Phi_K(X)$). Thus there is a homomorphism $\beta : \mathbf{F}_K(\overline{X}) \to \mathbf{A}$ such that $\mu = \beta \circ \nu$ as $ker(\nu) = \theta_K(X)$. But then, for $x \in X$,

$$\begin{aligned} \beta(\overline{x}) &= \beta \circ \nu(x) \\ &= \nu(x) \\ &= \alpha \circ \nu(x) \\ &= \alpha(\overline{x}), \end{aligned}$$

so β extends α. Thus $\mathbf{F}_K(\overline{X})$ has the universal mapping property for K over $\overline{X}$.

Corollary 3.4.6 *IF K is a class of algebras and $\mathbf{A} \in K$, then for sufficiently large$X$, $\mathbf{A} \in H(\mathbf{F}_K(\overline{X}))$.*

Proof Choose $|X| \geq |A|$ and let

$$\alpha : \overline{X} \to \mathrm{A}$$

be a surjection. Then let

$$\beta : \mathbf{F}_K(\overline{X}) \to \mathbf{A}$$

be a homomorphism extending α.

Proposition 3.4.7 (Birkhoff) *Suppose $\mathbf{T}(X)$ exists. Then for $K \neq \emptyset$, $\mathbf{F}_K(\overline{X}) \in ISP(K)$. Thus if K is closed under I, S, and P, in particular if K is a variety, then $\mathbf{F}_K(\overline{X} \in K$.*

Proof As

$$\theta_K(X) = \cap \Phi_K(X)$$

it follows that

$$\mathbf{F}_K(\overline{X}) = \mathbf{T}(X)/\theta_K(X) \in IP_s(\{\mathbf{T}(X)/\theta : \theta \in \Phi_K(X)\}),$$

so

$$\mathbf{F}_K(\overline{X}) \in IP_S IS(K),$$

and thus by proposition 3.3.1 and the fact that $P_S \leq SP$,

$$\mathbf{F}_K(\overline{X}) \in ISP(K).$$

Definition 3.4.6 An *identity* over X is an expression of the form

$$p \approx q$$

where $p, q \in T(X)$. Let $Id(X)$ be the set of identities over$X$. An algebra $\mathbf{A}$ satisfies an identity

$$p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$$

(or the identity *is true in* $\mathbf{A}$, or *holds in* $\mathbf{A}$), abbreviated by

$$\mathbf{A} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n),$$

or more briefly

$$\mathbf{A} \models p \approx q,$$

if for every choice of $a_1, \ldots, a_n \in$ A we have

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n).$$

A class K of algebras satisfies $p \approx q$, written

$$K \models p \approx q,$$

if each member of K satisfies $p \approx q$. If Σ is a set of identities, we say K satisfies Σ, written

$$K \models \Sigma,$$

if $K \models p \approx q$ for each $p \approx q \in \Sigma$. Given K and X let

$$Id_K(X) = \{p \approx q \in Id(X) : K \models p \approx q\}.$$

We use the symbol $\not\models$ for "does not satisfy."

We can reformulate the above definition of satisfaction using the notion of homomorphism.

Lemma 3.4.8 *If K is a class of algebras and $p \approx q$ is an identity over X, then*

$$K \models p \approx q$$

iff for every $\mathbf{A} \in K$ and for every homomorphism $\alpha : \mathbf{T}(X) \to \mathbf{A}$ we have

$$\alpha(p) = \alpha(q)$$

Proof ($\Rightarrow$) Let $p = p(x_1, \ldots, x_n)$, $q = q(x_1, \ldots, x_n)$. Suppose $K \models p \approx q$, $\mathbf{A} \in K$, and $\alpha : \mathbf{T}(X) \to \mathbf{A}$ is a homomorphism. Then

$$\begin{aligned} & p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) = p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ \Rightarrow \quad & \alpha(p^{\mathbf{T}(X)}(x_1, \ldots, x_n)) = \alpha(q^{\mathbf{T}(X)}(x_1, \ldots, x_n)) \\ \Rightarrow \quad & \alpha(p) = \alpha(q). \end{aligned}$$

($\Leftarrow$) For the converse choose $\mathbf{A} \in K$ and $a_1, \ldots, a_n \in$ A. By the universal mapping property of $\mathbf{T}(X) \to \mathbf{A}$ such that

$$\alpha(x_i) = a_i, \quad 1 \leq i \leq n.$$

But then

$$\begin{aligned} p^{\mathbf{A}}(a_1, \ldots, a_n) &= p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= \alpha(p) \\ &= \alpha(q) \\ &= q^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= q^{\mathbf{A}}(a_1, \ldots, a_n), \end{aligned}$$

so $K \models p \approx q$.

Next we see that the basic class operators preserve identities.

Proposition 3.4.9 *For any class K, all of the classes K, $I(K)$, $S(K)$, $H(K)$, $P(K)$ and $V(K)$ satisfy the same identities over any set of variables X.*

Proof Clearly K and $I(K)$ satisfy the same identities. As

$$I \leq IS,\ I \leq H, \text{ and } I \leq IP,$$

we must have

$$Id_K(X) \supseteq Id_{S(K)}(X),\ Id_{H(K)}(X), \text{ and } Id_{P(K)}(X).$$

For the remainder of the proof suppose

$$K \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n).$$

Then if $\mathbf{B} \leq \mathbf{A} \in K$ and $b_1, \ldots, b_n \in \mathrm{B}$, then as $b_1, \ldots, b_n \in \mathbf{A}$ we have

$$p^{\mathbf{A}}(b_1, \ldots, b_n) = q^{\mathbf{A}}(b_1, \ldots, b_n);$$

hence

$$p^{\mathbf{B}}(b_1, \ldots, b_n) = q^{\mathbf{B}}(b_1, \ldots, b_n),$$

so

$$\mathbf{B} \models p \approx q.$$

Thus

$$Id_K(X) = Id_{S(K)}(X).$$

Next suppose $\alpha : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism with $\mathbf{A} \in K$. If $b_1, \ldots, b_n \in \mathrm{B}$, choose $a_1, \ldots, a_n \in \mathrm{A}$ such that

$$\alpha(a_1) = b_1, \ldots, \alpha(a_n) = b_n.$$

Then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n)$$

implies

$$\alpha(p^{\mathbf{A}}(a_1, \ldots, a_n)) = \alpha(q^{\mathbf{A}}(a_1, \ldots, a_n));$$

hence

$$p^{\mathbf{B}}(b_1, \ldots, b_n) = q^{\mathbf{B}}(b_1, \ldots, b_n)$$

Thus

$$\mathbf{B} \models p \approx q,$$

so

$$Id_K(X) = Id_{H(K)}(X).$$

Lastly, suppose $\mathbf{A}_i \in K$ for $i \in I$. Then for $a_1, \ldots, a_n \in \mathrm{A} = \prod_{i\in I} \mathrm{A}_i$ we have

$$p^{\mathbf{A}_i}(a_1(i), \ldots, a_n(i)) = q^{\mathbf{A}_i}(a_1(i), \ldots, a_n(i));$$

hence

$$p^{\mathbf{A}}(a_1, \ldots, a_n)(i) = q^{\mathbf{A}}(a_1, \ldots, a_n)(i)$$

for $i \in I$, so

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n).$$

Thus

$$Id_K(X) = Id_{P(K)}(X).$$

As $V = HSP$ by 3.3.2, the proof is complete.

Now we will formulate the crucial connection between K-free algebras and identities.

Lemma 3.4.10 *Given a class K of CRLs and terms $p, q \in T(X)$ we have*

$$\begin{array}{ll} & K \models p \approx q \\ \Leftrightarrow & \mathbf{F_K}(\bar{X}) \models p \approx q \\ \Leftrightarrow & \bar{p} = \bar{q} \ \text{ in } \ \mathbf{F_K}(\bar{X}) \\ \Leftrightarrow & \langle p, q\rangle \in \theta_K(X). \end{array}$$

Proof Let $\mathbf{F} = \mathbf{F}_K(\overline{X})$, $p = p(x_1, \ldots, x_n)$, $q = q(x_1, \ldots, x_n)$, and let

$$\nu : \mathbf{T}(X) \to \mathbf{F}$$

be the natural homomorphism. Certainly $K \models p \approx q$ implies $\mathbf{F} \models p \approx q$ as $\mathbf{F} \in ISP(K)$. Suppose next that $\mathbf{F} \models p \approx q$. Then

$$p^{\mathbf{F}}(\overline{x}_1, \ldots, \overline{x}_n) = q^{\mathbf{F}}(\overline{x}_1, \ldots, \overline{x}_n),$$

hence $\overline{p} = \overline{q}$. Now suppose $\overline{p} = \overline{q}$ in $\mathbf{F}$. Then

$$\nu(p) = \overline{p} = \overline{q} = \nu(q),$$

so

$$\langle p, q\rangle \in ker(\nu) = \theta_K(X).$$

Finally suppose $\langle p, q\rangle \in \theta_K(X)$. Given $\mathbf{A} \in K$ and $a_1, \ldots, a_n \in \mathrm{A}$ choose $\alpha : \mathbf{T}(X) \to \mathbf{A}$ such that $\alpha(x_i) = a_i$, $1 \leq i \leq n$. As $ker(\alpha) \in \Phi_K(X)$ we have

$$ker(\alpha) \supseteq ker(\nu) = \theta_K(X),$$

so it follows that is a homomorphism $\beta : \mathbf{F} \to \mathbf{A}$ such that $\alpha = \beta \circ \nu$. Then

$$\alpha(p) = \beta \circ \nu(p) = \beta \circ \nu(q) = \alpha(q).$$

Consequently

$$K \models p \approx q$$

by reformulation of definition of satisfaction.

Chapter 4

Relation between logics and algebras

In this section we show some relations between logics over $\mathbf{FL_e}$ and CRLs.

4.1 Relation between $\mathbf{FL_e}$ and CRL.

Definition 4.1.1 ϕ is *valid* in a CRL $\mathbf{A}$ if $v(\phi) \geq 1$ for every valuation v.

Definition 4.1.2 (valuation) Let $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ is a CRL. A *valuation* v is a mapping from set of all propositional variable to A. Furthermore this v is extended to a mapping from set of all formulas to A as follows.

1. $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$
2. $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$
3. $v(\phi \supset \psi) = v(\phi) \rightarrow v(\psi)$
4. $v(\phi * \psi) = v(\phi) \cdot v(\psi)$
5. $v(\neg\phi) = v(\phi) \rightarrow 0$

Proposition 4.1.1 (valid) *Let ϕ be a formula. A formula ϕ is valid in a CRL* $\mathbf{A}$ *if $v(\phi) \geq 1$ for every valuation v on* $\mathbf{A}$.

For a given CRL $\mathbf{A}$, and let $L(\mathbf{A})$ be the set of all formulas such that $v(\phi)$ for any valuation v on $\mathbf{A}$.

Proposition 4.1.2 *For each CRL* $\mathbf{A}$, $L(\mathbf{A})$ *is a logic.*

Proof Let $\phi(p)$ be a formula containing a propositional variable p and $\phi(p) \in L(\mathbf{A})$. By our assumption, $v(\phi(p)) \geq 1$ for any valuation v. Now, consider any substitution instance $\phi(\alpha)$ of $\phi(p)$ and any valuation w on $\mathbf{A}$. Let $w(\alpha) = a$. Then, define a valuation v' by $v'(p) = a$ and $v'(q) = w(q)$ if q is different from p. Then, $q \leq v'(\phi(p)) = w(\phi(\alpha))$. Therefore, $\phi(\alpha)$ is valid in $\mathbf{A}$. Thus $\phi(\alpha) \in L(\mathbf{A})$.
Next let $\phi, \phi \supset \psi \in L(\mathbf{A})$. Then for any valuation v $v(\phi) \geq 1$, $v(\phi \supset \psi) \geq 1$. By the definition of valuations we can show $v(\phi) \leq v(\psi)$. So from $v(\phi) \geq 1$ and $v(\phi) \leq v(\psi)$ $v(\psi) \geq 1$. Hence $v(\psi) \geq 1$ for any valuation v. Thus $\psi \in L(\mathbf{A})$. Thus $L(\mathbf{A})$ is closed under modus ponens.
Finally let $\phi, \psi \in L(\mathbf{A})$. Then $v(\phi) \geq 1$, $v(\psi) \geq 1$ for any valuation v. By the definition of valuations $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$. So $v(\phi) \wedge v(\psi) \geq 1$ from $v(\phi) \geq 1$ and $v(\psi) \geq 1$. Hence $v(\phi \wedge \psi) \geq 1$ for any valuation. Thus $\phi \wedge \psi \in L(\mathbf{A})$.
Therefore $L(\mathbf{A})$ is a logic.

The logic $L(\mathbf{A})$ is called the logic characterized by $\mathbf{A}$.

Proposition 4.1.3 *For any logic L over $\mathbf{FL_e}$ there exists some CRL $\mathbf{A}$ such that*

$$L = L(\mathbf{A}).$$

(Out line of proof) We show this by constructing the Lindenbaum algebra of L. First we define a binary relation $\equiv$ between formula ϕ and ψ as follows.

$$\phi \equiv \psi \Leftrightarrow \phi \supset \psi \in L \text{ and } \psi \supset \phi \in L$$

$\phi \equiv \psi$ means that ϕ and ψ are logically equivalent. It is clear that $\equiv$ is an equivalence relation. We can show moreover that $\equiv$ is a congruence relation, i.e. if $\phi \equiv \psi$, $\phi' \equiv \psi'$ then $\phi \oplus \phi' \equiv \psi \oplus \psi'$ for any logical connectives $\oplus$.
Next by using this congruence relation $\equiv$, construct the quotient set $\Phi/\equiv$ where Φ is a set of all formulas. We write $[\phi]$ equivalence class including ϕ. We can show that $\mathbf{A} = \langle \Phi/\equiv, \cap, \cup, \cdot, \rightarrow, [\top], [\bot] \rangle$ is a CRL where $\cap, \cup, \cdot, \rightarrow$ are defined as follows.

$$\begin{aligned}
[\phi] \cup [\psi] &= [\phi \wedge \psi] \\
[\phi] \cap [\psi] &= [\phi \vee \psi] \\
[\phi] \cdot [\psi] &= [\phi * \psi] \\
[\phi] \rightarrow [\psi] &= [\phi \supset \psi]
\end{aligned}$$

Finally we show that L and $L(\mathbf{A})$ correspond to each other, i.e.

$$\phi \in L \Leftrightarrow \text{for any valuation on } \mathbf{A},\ v(\mathbf{A}) \geq [1]$$

This algebra $\mathbf{A}$ is called the *Lindenbaum algebra* of a logic L.

Proposition 4.1.4 (completeness theorem) *For any formula ϕ, ϕ is provable in $\mathbf{FL_e}$ if and only if for any CRL $\mathbf{A}$ and for any valuation v on $\mathbf{A}$, $v(\phi) \geq 1$.*

(Outline of proof) We show only-if part.
We define a valuation v for a sequent $\alpha_1, \ldots, \alpha_n \to \beta$ as follows.

$$v(\alpha_1, \ldots, \alpha_n \to \beta) = v(\alpha_1 * \ldots * \alpha_n) \to \alpha(\beta)$$

However if left-hand side of a sequent is empty then $v(\to \beta) = 1 \to v(\beta)$ and if right-hand side of a sequent is empty then $v(\alpha_1, \ldots, \alpha_n \to) = v(\alpha_1 * \ldots * \alpha_n) \to 0$.
We show this by induction on the construction of a proof of a formula ϕ. That is, for a given valuation v, every sequent S in a proof $v(S) \geq 1$. First we show base case. Initial sequents are satisfies following condition. For example,

1. $v(\alpha \to \alpha) \geq 1$,
2. $v(\Gamma \to \top) \geq 1$,
3. $v(\Gamma, \bot, \Delta \to \gamma) \geq 1$.

Second we show induction case. Let for each inference rule upper sequents S_1 and S_2 are satisfies $v(S_1) \geq 1$ and $v(S_2) \geq 1$. Then lower sequent S is satisfies $v(S) \geq 1$.

Next we show if-part.
We show the contraposition of if-part. Suppose that for a given a formula ϕ such that ϕ is not provable in $\mathbf{FL_e}$. Then by using Lindenbaum algebra of $\mathbf{FL_e}$ there exist some CRL $\mathbf{A}$ and some valuation v such that $v(\phi) \not\geq 1$.

From a proposition 4.1.4 we transcribe a proposition 4.1.2 as follows.

Proposition 4.1.5 *A logic $L(\mathbf{A})$ characterized by a CRL $\mathbf{A}$ is a logic over $\mathbf{FL_e}$.*

Proof It is clear from a proposition 4.1.4 that $\mathbf{FL_e} \subseteq L(\mathbf{A})$ for all CRL $\mathbf{A}$.

4.2 Algebraic operations and inclusion relation among logics

In previous section,we show that the $L(\mathbf{A})$ is a logic $L(\mathbf{A})$ over $\mathbf{FL_e}$ for each CRL $\mathbf{A}$, and conversely every logic L over $\mathbf{FL_e}$ can be represented as $L(\mathbf{A})$ for some CRL $\mathbf{A}$. Hereinafter we show the relations between three basic algebraic operations and logic.

Proposition 4.2.1 (subalgebras) *Let $\mathbf{A}$ and $\mathbf{B}$ are CRLs and $\mathbf{A} \leq \mathbf{B}$. Then $L(\mathbf{B}) \subseteq L(\mathbf{A})$ hold.*

Proof Suppose that$\mathbf{A} \leq \mathbf{B}$. Then any valuation on $\mathbf{A}$ can be considered to be the restriction of a valuation on $\mathbf{B}$ of A. So if ϕ is a element of $L(\mathbf{B})$, i.e. $v(\phi) \geq 1$ for any valuation v, then $u(\phi) \geq 1$ for any valuation u on $\mathbf{A}$. Thus $L(\mathbf{B}) \subseteq L(\mathbf{A})$.

Proposition 4.2.2 (quotient algebras) *Let* $\mathbf{A}$ *is CRL and* θ *is a congruence on* $\mathbf{A}$*. Then* $L(\mathbf{A}) \subseteq L(\mathbf{A}/\theta)$

Proof Let $\phi(p_1, \ldots p_n)$ be a formula, where $p_1, \ldots p_n$ are all propositional variables appearing in ϕ. And for some formula $\phi(p_1, \ldots p_n)$ we express a replacing logical connectives $\wedge$, $\vee$, $*$, $\supset$ with $\cap$, $\cup$, $\cdot$, $\rightarrow$ and a replacing propositional variables p_i with x_i by $f_\phi(x_1, \ldots, x_n)$. Then this is a element of CRL.
Suppose that $\phi(p_1, \ldots p_n) \in L(\mathbf{A})$. In other words for any valuation v on $\mathbf{A}$

$$v(\phi) = f_\phi^{\mathbf{A}}(v(p_1), \ldots, v(p_n)) \geq 1_{\mathbf{A}}.$$

Let θ is a congruence on $\mathbf{A}$. Then we can get

$$\begin{aligned} f_\phi^{\mathbf{A}}(x_1, \ldots, x_n)/\theta &= f_\phi^{\mathbf{A}/\theta}(x_1/\theta, \ldots, x_n/\theta) \\ &\geq 1_{\mathbf{A}}/\theta \\ &= 1_{\mathbf{A}/\theta}. \end{aligned}$$

This holds for any $x_1/\theta, \ldots, x_n/\theta \in \mathbf{A}/\theta$. So $\phi \in L(\mathbf{A}/\theta)$. Thus $L(\mathbf{A}) \subseteq L(\mathbf{A}/\theta)$.

Proposition 4.2.3 (homomorphisms) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *are CRLs and* $\alpha : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism. Then the following holds.*

1. *If* α *is surjective then* $L(\mathbf{A}) \subseteq L(\mathbf{B})$*.*

2. *If* α *is injective then* $L(\mathbf{B}) \subseteq L(\mathbf{A})$*.*

3. *If* α *is bijective then* $L(\mathbf{A}) = L(\mathbf{B})$*.*

Proof They follow from previous two propositions. In fact if α is surjective then $\mathbf{A}/ker(\alpha) \simeq \mathbf{B}$ and if α is injective then $\mathbf{A} \simeq Im(\alpha) \leq \mathbf{B}$. Moreover 3 is clear from 1 and 2.

Proposition 4.2.4 (direct products) $L(\prod_{i \in I} \mathbf{A}_i) = \bigcap_{i \in I} L(\mathbf{A}_i)$

Proof We show this only for the case of $I = \{1, 2\}$.
$(\subseteq)$
$\alpha_1 : \mathbf{A}_1 \times \mathbf{A}_2 \longrightarrow \mathbf{A}_1$, $\alpha_2 : \mathbf{A}_1 \times \mathbf{A}_2 \longrightarrow \mathbf{A}_2$ are onto homomorphism. So from previous proposition $L(\mathbf{A}_1 \times \mathbf{A}_2) \subseteq L(\mathbf{A}_1)$, $L(\mathbf{A}_1 \times \mathbf{A}_2) \subseteq L(\mathbf{A}_2)$. Thus $L(\mathbf{A}_1 \times \mathbf{A}_2) \subseteq L(\mathbf{A}_1) \cap L(\mathbf{A}_2)$.
$(\supseteq)$
Let $\phi \in L(\mathbf{A}_1)$ and $\phi \in L(\mathbf{A}_2)$. In other words Let $v_1(\phi) \geq 1_{\mathbf{A}_1}$ and $v_2(\phi) \geq 1_{\mathbf{A}_2}$ for any valuation v_1 and v_2 on $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively. Since any valuation v on $\mathbf{A}_1 \times \mathbf{A}_2$ can be

expressed as $v(\phi) = \langle v_1(\phi), v_2(\phi)\rangle$ for valuations v_1 and v_2 on $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. So from assumption

$$\begin{aligned} v(\phi) &= \langle v_1(\phi), v_2(\phi)\rangle \\ &\geq \langle 1_{\mathbf{A}_1}, 1_{\mathbf{A}_2}\rangle \\ &= 1_{\mathbf{A}_1\times\mathbf{A}_2}. \end{aligned}$$

Hence $\phi \in L(\mathbf{A}_1 \times \mathbf{A}_2)$.

Above propositions intuitively mean that if an algebra become bigger then a logic become smaller and if an algebra become smaller then a logic become bigger.

4.3 Logics over $\mathbf{FL_e}$ and varieties of CRLs

In previous two sections we discuss relations between logics and algebras. In this section we discuss relation between logics and classes of CRLs.

4.3.1 From logic to variety

Definition 4.3.1 Let L be a logic over $\mathbf{FL_e}$. We define a class V_L of CRLs by

$V_L = \{\mathbf{Q} : L \subseteq L(\mathbf{Q})\}$.

Proposition 4.3.1 *For every logic over* $\mathbf{FL_e}$ *a class* V_L *of CRLs is a variety.*

Proof It is enough to show that V_L is closed under homomorphic images, subalgebras, direct products.
(homomorphic images)
Let $\mathbf{A} \in V_L$.Then $L \subseteq L(\mathbf{A})$. If $\alpha(\mathbf{A})$ is a homomorphic image of $\mathbf{A}$ then by proposition 4.2.3 we can get $L(\mathbf{A}) \subseteq L(\alpha(\mathbf{A}))$. So $L \subseteq L(\alpha(\mathbf{A}))$. Thus we can get $\alpha(\mathbf{A}) \in V_L$.
(subalgebras)
Let $\mathbf{A} \in V_L$ and $\mathbf{B} \leq \mathbf{A}$. Then by proposition 4.2.1

$L \subseteq L(\mathbf{A}) \subseteq L(\mathbf{B})$

Thus $\mathbf{B} \in V_L$.
(direct products)
Let $\mathbf{A} \in V_L$ for each $i \in I$. Then by proposition 4.2.4 we can get $L(\prod_{i\in I}\mathbf{A}_i) = \bigcap_{i\in I} L(\mathbf{A}_i)$. Moreover from our assumption $2L \subseteq L(\mathbf{A}_i)$ for any $i \in I$. So we can get

$L \subseteq \bigcap_{i\in I} L(\mathbf{A}_i) = L(\prod_{i\in I}\mathbf{A}_i)$.

Thus $\prod_{i\in I}\mathbf{A}_i \in V_L$.

4.3.2 From varieties to logics

Definition 4.3.2 Let K be a class of CRLs. K is an *equational class* if there exists some sets Σ of identities, i.e.

$$K = \{\mathbf{A} : \mathbf{A} \models s \approx t \text{ for any } s \approx t \in \Sigma\}.$$

We note that all identities can be expressed of a form $1 \leq r$ for a term r. Because for any identity $s \approx t$

$$\begin{aligned} s \approx t &\Leftrightarrow s \leq t \textit{ and } t \leq s \\ &\Leftrightarrow 1 \leq s \to t \textit{ and } 1 \leq t \to s \\ &\Leftrightarrow 1 \leq (s \to t) \cap (t \to s). \end{aligned}$$

The next proposition is a well-known theorem by Birkhoff.

Proposition 4.3.2 (Birkhoff's theorem) *K is an equational class if and only if K is a variety.*

Actually a class of all CRLs is an equational class by following identities

1. $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$
2. $x \to (y \vee z) = (x \to y) \wedge (x \to z)$
3. $(x \cdot (x \to y)) \vee y = y$
4. $(x \to (x \cdot y)) \wedge y = y$

together with monoid and lattice identities.

Let V is a variety of CRLs. Then by Birkhoff's theorem, there exists a set Σ of identities. We define L_V following

$$L_V = \{\tau(\phi) : \phi \geq 1 \in \Sigma\}$$

Here a τ is a inverse mapping of valuation v defined by definition 4.1.2, i.e. $\tau((x \wedge y) \vee z) \equiv (p \wedge q) \vee r$.
A intuitive image of L_V is a set of all formula $\psi(t)$ for all equation t which satisfies $V \models t \geq 1$.
Then a following proposition holds.

Proposition 4.3.3 *For any variety V of CRLs L_V is a logic over* $\mathbf{FL_e}$.

Proof We show that L_V is closed under modus ponens and substitution and $\mathbf{FL_e} \subseteq L_V$.
(substitution)
Let $\phi(p_1, \ldots p_n)$ be a formula ($p_1, \ldots p_n$ are all propositional variable appearing in ϕ). And for some formula $\phi(p_1, \ldots p_n)$ we express a replacing logical connectives $\wedge$, $\vee$, ast, $supset$ and propositional variables p_i with $\cap$, $\cup$, $\cdot$, $\to$, x_i respectively by $f_\phi(x_1, \ldots, x_n)$. Then $f_\phi(x_1, \ldots, x_n)$ is a element of CRL. Let $\phi(p_1, \ldots p_n) \in L_V$. Then $\phi(p_1, \ldots p_n) \equiv \tau(f_\phi(x_1, \ldots, x_n))$ and $f_\phi(x_1, \ldots, x_n) \geq 1$. $f_\phi(x_1, \ldots, x_n) \geq 1$ means that for any CRL $\mathbf{A} inV$ and for any $a_1, \ldots, a_n \in$ A

$$f_\phi(x_1, \ldots, x_n) \geq 1_\mathbf{A}.$$

So for any substitution $\phi(\phi_1, \ldots, \phi_n)$ of ϕ

$$\phi(\phi_1, \ldots, \phi_n) = \tau(f_\phi(v(\phi_1), \ldots, v(\phi_n))) \text{ (v is a valuation)}$$

Suppose that $v(\phi_i) = b_i$ then

$$f_\phi(v(\phi_1), \ldots, v(\phi_n))) = f_\phi(b_1, \ldots, b_n) \geq 1_\mathbf{A}$$

Thus closed under substitution.
(modus ponens)
Let $\phi, \phi \supset \psi \in L_V$. Then there exist some s, t such that

$$\tau(s) = \phi, \tau(t) = \psi \text{ and } s \geq 1, s \to t \geq 1.$$

Hence

$$1 \leq s \to t \Leftrightarrow s \leq .$$

So $1 \leq t$. Thus $\psi \in L_V$.
($\mathbf{FL_e} \subseteq L_V$)
Let $\phi \in \mathbf{FL_e}$. Then from completeness theorem for any CRL and for any valuation v, $v(\phi) \geq 1$. For any CRL $\mathbf{A} \in V$

$$v(\phi) \geq 1_\mathbf{A}.$$

L_V is a logic which is a set of any formula ψ satisfying $V \models v(\psi)$. So

$$\phi \in L_V$$

Thus $\mathbf{FL_e} \subseteq L_V$.

Chapter 5

Disjunction property for logics over $\mathbf{FL_e}$

5.1 well-connectedness and disjunction property

First we give some definitions.

Definition 5.1.1 (well-connectedness) A CRL $\mathbf{A}$ is *well-connected* if for any $x, y \in \mathbf{A}$ from $x \vee y \geq 1$ there follows $x \geq 1$ or $y \geq 1$.

Definition 5.1.2 (disjunction property) A logic L has the *disjunction property* if for any formula ϕ and ψ the condition that $\phi \vee \psi$ is provable implies that at least one of the formulas ϕ and ψ is provable.

5.2 Main theorem and its proof

Theorem 5.2.1 (Maksimova) *Suppose that a logic L over* **Int** *is complete with respect to a class K of Heyting algebras. Then, the following are equivalent;*

i. L has the disjunction property,

ii. For all Heyting algebras $\mathbf{A}, \mathbf{B} \in K$ there exist a well-connected Heyting algebra $\mathbf{C}$ such that L is valid in $\mathbf{C}$, and there is a surjective homomorphism from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$.

In the same way as this, we can show the following theorem 5.2.2. So we omit a proof.

Theorem 5.2.2 *Suppose that a logic L over* $\mathbf{FL_e}$ *is complete with respect to a class K of CRLs. Then, the following are equivalent;*

i. L has the disjunction property,

ii. For all CRLs $\mathbf{A}, \mathbf{B} \in K$ there exist a well-connected CRL $\mathbf{C}$ such that L is valid in $\mathbf{C}$, and there is a surjective homomorphism from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$.

Proof ii $\to$ i. Let ii be true and ϕ ψ is not valid in L for some formulas ϕ and ψ. Show that $v(\phi \vee \psi) \neq 1$.
Because of the completeness of L we have $v_{\mathbf{A}}(\phi) \ngeq_{\mathbf{A}} 1_{\mathbf{A}}$ and $v_{\mathbf{B}}(\psi) \ngeq_{\mathbf{B}} 1_{\mathbf{B}}$ for some valuations $v_{\mathbf{A}}$ in $\mathbf{A}$ and $v_{\mathbf{B}}$ in $\mathbf{B}$. From ii there exist a well-connected CRL $\mathbf{C}$, such that L is valid in $\mathbf{C}$, and a surjective homomorphism α from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$. We define a valuation v in $\mathbf{C}$ following. For any propositional variable p, define $v(p) = a$, where a is an arbitrary element in $\alpha^{-1}(\langle v_{\mathbf{A}}(p), v_{\mathbf{B}}(p)\rangle)$. So for any variable p,

$$\alpha(v(p)) = \langle v_{\mathbf{A}}(p), v_{\mathbf{B}}(p)\rangle.$$

From properties of homomorphisms we can show inductively that for any formula δ. $\alpha(v(\delta)) = \langle v_{\mathbf{A}}(\delta), v_{\mathbf{B}}(\delta)\rangle$. In particular,

$$\alpha(v(\phi)) = \langle v_{\mathbf{A}}(\phi), v_{\mathbf{B}}(\phi)\rangle < \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}\rangle$$
$$\text{and } \alpha(v(\psi)) = \langle v_{\mathbf{A}}(\psi), v_{\mathbf{B}}(\psi)\rangle < \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}\rangle.$$

Hence, we have $v(\phi) < 1_{\mathbf{C}}$, $v(\psi) < 1_{\mathbf{C}}$ and $v(\phi \vee \psi) < 1_{\mathbf{C}}$ by the well-connectedness of $\mathbf{C}$. Thus $\phi \vee \psi$ is not valid in $\mathbf{C}$.
i $\to$ ii. If L has the disjunction property, then all free algebras of the corresponding variety $V(L) = \{\mathbf{C} \mid L \text{ is valid in } \mathbf{C}\}$ are well-connected. At the same time, if $\mathbf{A}, \mathbf{B} \in V(L)$ then $\mathbf{A} \times \mathbf{B} \in V(L)$. Any algebra in $V(L)$ is a homomorphic image of a suitable free algebra of $V(L)$. This completes the proof.

We show the disjunction property of logics over $\mathbf{FL_e}$ by using this theorem. First we show the disjunction property of $\mathbf{FL_e}$ and $\mathbf{FL_e}[\mathrm{E_k}]$.

Theorem 5.2.3 (Disjunction property for $\mathbf{FL_e}$ and $\mathbf{FL_e}[\mathrm{E_k}]$) $\mathbf{FL_e}$ *and* $\mathbf{FL_e}[\mathrm{E_k}]$ *has a disjunction property.*

To prove this theorem we construct a suitable CRL for given $\mathbf{A}$ and $\mathbf{B}$. Define a CRL $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, \to, 0, 1\rangle$ by;

1. $\mathrm{C} = \mathrm{A} \times \mathrm{B} \times \{0\} \cup \{\langle a, b, 1\rangle \mid a \geq_{\mathbf{A}} 1_{\mathbf{A}},\ b \geq_{\mathbf{B}} 1_{\mathbf{B}}\}$,
   - $\mathbf{1} = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$,
   - $\mathbf{0} = \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$.
2. Define a binary relation $\leq$ on C as follows.

$$\langle a, b, i\rangle \leq \langle a', b', j\rangle \Leftrightarrow a \leq_{\mathbf{A}} a',\ b \leq_{\mathbf{B}} b',\ i \leq j$$

3. Define $\cdot$ as follows.

$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle$$

4. Define $\rightarrow$ as follows.

   - If ($1_{\mathbf{A}} \not\leq_{\mathbf{A}} a \rightarrow_{\mathbf{A}} a'$ or $1_{\mathbf{B}} \not\leq_{\mathbf{B}} b \rightarrow_{\mathbf{A}} b'$) and $i \rightarrow j = 1$ then $\langle a, b, i\rangle \rightarrow \langle a', b', j\rangle = \langle a \rightarrow_{\mathbf{A}} a', b \rightarrow_{\mathbf{B}} b', 0\rangle$.
   - Otherwise, $\langle a, b, i\rangle \rightarrow \langle a', b', j\rangle = \langle a \rightarrow_{\mathbf{A}} a', b \rightarrow_{\mathbf{B}} b', i \rightarrow j\rangle$.

We show that $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is a CRL.

Lemma 5.2.4 *The tuple $\langle \mathrm{C}, \wedge, \vee, 0, 1\rangle$ is a lattice.*

Proof Clearly binary relation $\leq$ is a partial order on the set C.
For every $\langle a, b, i\rangle$ and $\langle a', b', j\rangle$ in C,

$$\langle a, b, i\rangle \leq \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle,$$
$$\langle a', b', j\rangle \leq \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle.$$

Let $\langle x, y, k\rangle$ in C as follows.

$$\langle x, y, k\rangle \geq \langle a, b, i\rangle,$$
$$\langle x, y, k\rangle \geq \langle a', b', j\rangle.$$

Then

$$\langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle \leq \langle x, y, k\rangle.$$

So we can show

$$\sup\{\langle a, b, i\rangle, \langle a', b', j\rangle\} = \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle.$$

Similarly $\inf\{\langle a, b, i\rangle, \langle a', b', j\rangle\} = \langle a \wedge_{\mathbf{A}} a', b \wedge_{\mathbf{B}} b', i \wedge j\rangle$. Thus C is a lattice.

Lemma 5.2.5 *The tuple $\langle \mathrm{C}, \cdot, 1\rangle$ is a commutative monoid.*

Proof For every $\langle a, b, i\rangle$ in C,

$$\langle a, b, i\rangle \cdot \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle \cdot \langle a, b, i\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle.$$

So $\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$ is a identity element.
For every $\langle a, b, i\rangle$, $\langle a', b', j\rangle$ and $\langle a'', b'', k\rangle$ in C, following equality hold.

$$\begin{aligned}
(\langle a, b, i\rangle \cdot \langle a', b', j\rangle) \cdot \langle a'', b'', k\rangle &= \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle \cdot \langle a'', b'', k\rangle \\
&= \langle (a \cdot_{\mathbf{A}} a') \cdot_{\mathbf{A}} a'', (b \cdot_{\mathbf{B}} b') \cdot_{\mathbf{B}} b'', (i \cdot j) \cdot k\rangle \\
&= \langle a \cdot_{\mathbf{A}} (a' \cdot_{\mathbf{A}} a''), b \cdot_{\mathbf{B}} (b' \cdot_{\mathbf{B}} b''), i \cdot (j \cdot k)\rangle \\
&= \langle a, b, i\rangle \cdot \langle a' \cdot_{\mathbf{A}} a'', b' \cdot_{\mathbf{B}} b'', j \cdot k\rangle \\
&= \langle a, b, i\rangle \cdot (\langle a', b', j\rangle \cdot \langle a'', b'', k\rangle)
\end{aligned}$$

Thus $\langle \mathrm{C}, \cdot, 1\rangle$ is a monoid.

Lemma 5.2.6 *The algebra* $\mathbf{C}$ *satisfies residuated law.*

Proof First we prove only-if part.
From assumption, we can easily show $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$ and $i \cdot j \leq k$.
Hence we can get $a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3$, $b_1 \leq_{\mathbf{B}} b_2 \to_{\mathbf{B}} b_3$, $i \leq j \to k$.

- If $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, j \to k\rangle$ then

  $$\langle a_1, b_1, i\rangle \leq \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, j \to k\rangle = \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

- If $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 0\rangle$
  then we can prove $a_1 \not\geq_{\mathbf{A}} 1_{\mathbf{A}}$ from $a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3$ and $a_2 \to_{\mathbf{A}} a_3 \not\geq_{\mathbf{A}} 1_{\mathbf{A}}$. Similarly we can prove $b_1 \not\geq_{\mathbf{B}} 1_{\mathbf{b}}$. So $i \neq 1$. Thus

  $$\langle a_1, b_1, i\rangle \leq \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 0\rangle = \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

Next we prove if-part.

- Let $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, j \to k\rangle$.
  Then we can prove $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$, $i \cdot j \leq k$ easily.
  Thus

  $$\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

- Let $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 0\rangle$.
  Then similarly we can prove $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$. $i \cdot j = 0 \leq k$ from $i = 0$.
  Thus

  $$\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

Lemma 5.2.7 *A mapping* α *from* $\mathbf{C}$ *to* $\mathbf{A} \times \mathbf{B}$ *defined by*

$$\alpha(\langle a, b, i\rangle) = \langle a, b\rangle$$

is a surjective homomorphism.

Proof A mapping α is clearly surjective.

$$\begin{aligned}
\alpha(\langle a, b, i\rangle \vee \langle a', b', j\rangle) &= \alpha(\langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle) \\
&= \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b'\rangle \\
&= \langle a, b\rangle \vee \langle a', b'\rangle \\
&= \alpha(\langle a, b, i\rangle) \vee \alpha(\langle a', b', j\rangle)
\end{aligned}$$

We can show $\alpha(\langle a,b,i\rangle \wedge \langle a',b',j\rangle) = \alpha(\langle a,b,i\rangle) \wedge \alpha(\langle a',b',j\rangle)$ and $\alpha(\langle a,b,i\rangle \cdot \langle a',b',j\rangle) = \alpha(\langle a,b,i\rangle) \cdot \alpha(\langle a',b',j\rangle)$ in the same way as above.
If $\langle a,b,i\rangle \to \langle a',b',j\rangle = \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', i \to j\rangle$, then

$$\begin{aligned}\alpha(\langle a,b,i\rangle \to \langle a',b',j\rangle) &= \alpha(\langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', i \to j\rangle) \\ &= \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b'\rangle \\ &= \langle a,b\rangle \to \langle a',b'\rangle \\ &= \alpha(\langle a,b,i\rangle) \to \alpha(\langle a',b',j\rangle).\end{aligned}$$

If $\langle a,b,i\rangle \to \langle a',b',j\rangle = \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', 0\rangle$, then

$$\begin{aligned}\alpha(\langle a,b,i\rangle \to \langle a',b',j\rangle) &= \alpha(\langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', 0\rangle) \\ &= \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b'\rangle \\ &= \langle a,b\rangle \to \langle a',b'\rangle \\ &= \alpha(\langle a,b,i\rangle) \to \alpha(\langle a',b',j\rangle).\end{aligned}$$

Thus α is surjective homomorphism.

Lemma 5.2.8 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy a condition* $\mathrm{E_k}$ *then the algebra* $\mathbf{C}$ *satisfies* $\mathrm{E_k}$.

Proof For every $\langle a,b,i\rangle$ in C

$$\begin{aligned}\langle a,b,i\rangle^k \to \langle a,b,i\rangle^{k+1} &= \langle a^k \to_{\mathbf{A}} a^{k+1}, b^k \to_{\mathbf{B}} b^{k+1}, i^k \to i^{k+1}\rangle \\ &= \langle a^k \to_{\mathbf{A}} a^{k+1}, b^k \to_{\mathbf{B}} b^{k+1}, i \to i\rangle \\ &= \langle 1_A, 1_B, 1\rangle.\end{aligned}$$

Similarly we can show

$$\langle a,b,i\rangle^{k+1} \to \langle a,b,i\rangle^k = \langle 1_A, 1_B, 1\rangle.$$

Thus $\mathbf{C}$ satisfies E_k.

Proof of Theorem 5.2.3 We construct a suitable algebra is enough to prove this theorem. From above lemmas we can show that $\mathbf{C}$ is a CRL which satisfies $\mathrm{E_k}$ and there is a surjective homomorphism from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$.

Generally a constructed CRL $\mathbf{C}$ is not satisfies DN. We need some modification of $\mathbf{C}$ in proving the disjunction property of $\mathbf{FL_e}$[DN].

Theorem 5.2.9 (Disjunction property for $\mathbf{FL_e}$[DN]) $\mathbf{FL_e}$[DN] *has a disjunction property.*

Define a CRL $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, \to, 0, 1\rangle$ by;

1. $\mathrm{C} = \mathrm{A} \times \mathrm{B} \times \{1/2\} \cup \{\langle a, b, 1\rangle \mid a \geq_{\mathbf{A}} 1_{\mathbf{A}},\ b \geq_{\mathbf{B}} 1_{\mathbf{B}}\} \cup \{\langle a, b, 0\rangle \mid a \leq_{\mathbf{A}} 0_{\mathbf{A}},\ b \leq_{\mathbf{B}} 0_{\mathbf{B}}\}$,
   - $\mathbf{1} = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$,
   - $\mathbf{0} = \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$.
2. Define a binary relation $\leq$ as follows.

$$\langle a, b, i\rangle \leq \langle a', b', j\rangle \Leftrightarrow a \leq_{\mathbf{A}} a',\ b \leq_{\mathbf{B}} b',\ i \leq j$$

3. Define $\cdot$ as follows, where $i \cdot j = \min\{i, j\}$.
   - If $a \cdot_{\mathbf{A}} a' \leq 0_{\mathbf{A}}$ and $b \cdot_{\mathbf{B}} b' \leq_{\mathbf{B}} 0_{\mathbf{B}}$ and $i, j \neq 1$ then $\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', 0\rangle$.
   - If $(a \cdot_{\mathbf{A}} a' \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ or $b \cdot_{\mathbf{B}} b' \not\leq_{\mathbf{B}} 0_{\mathbf{B}})$ and $i \cdot j = 0$ then $\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', 1/2\rangle$.
   - Otherwise,
     $\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle$.
4. Define $\to$ as follows, where $i \to j = 1$ when $i \leq j$, $i \to j = 1/2$ when $i = 1$ and $j = 1/2$, and $i \to j = 0$ otherwise.
   - If $(1_{\mathbf{A}} \not\leq_{\mathbf{A}} a \to_{\mathbf{A}} a'$ or $1_{\mathbf{B}} \not\leq_{\mathbf{B}} b \to_{\mathbf{B}} b')$ and $i \to j = 1$ then $\langle a, b, i\rangle \to \langle a', b', j\rangle = \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', 1/2\rangle$.
   - If $i = 1/2$ and $\langle a', b', j\rangle \leq \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$ then $\langle a, b, i\rangle \to \langle a', b', j\rangle = \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', 1/2\rangle$.
   - Otherwise,
     $\langle a, b, i\rangle \to \langle a', b', j\rangle = \langle a \to_{\mathbf{A}} a', b \to_{\mathbf{B}} b', i \to j\rangle$.

Lemma 5.2.10 *The tuple* $\langle \mathrm{C}, \wedge, \vee, 0, 1\rangle$ *is a lattice.*

Proof Clearly $\leq$ is a partial order on the set C.
By the same way as $\mathbf{FL}_e$ and $\mathbf{FL}_e[\mathrm{E_k}]$ We can show that for every $\langle a, b, i\rangle, \langle a', b', j\rangle \in \mathrm{C}$ both $\sup\{\langle a, b, i\rangle, \langle a', b', j\rangle\}$ and $\inf\{\langle a, b, i\rangle, \langle a', b', j\rangle\}$ exist.

Lemma 5.2.11 *The tuple* $\langle \mathrm{C}, \cdot, 1\rangle$ *is a commutative monoid.*

Proof For every $\langle a, b, i\rangle$ in C, $\langle a, b, i\rangle \cdot \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle \cdot \langle a, b, i\rangle = \langle a, b, i\rangle$. So $\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$ is a identity element. Next we prove associative law.

Let $\langle a_1, b_1, i\rangle, \langle a_2, b_2, j\rangle, \langle a_3, b_3, k\rangle \in \mathrm{C}$.
Suppose that

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle,$$
$$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, (i \cdot j) \cdot k\rangle$$

then

$$\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, j \cdot k\rangle,$$
$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, j \cdot k\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), i \cdot (j \cdot k)\rangle.$$

So

$$\begin{aligned}(\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle) \cdot \langle a_3, b_3, k\rangle &= \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, (i \cdot j) \cdot k\rangle \\ &= \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), i \cdot (j \cdot k)\rangle \\ &= \langle a_1, b_1, i\rangle \cdot (\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle).\end{aligned}$$

Suppose that $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle$.

- Let $k = 1$. Then we can show

  $$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$$

  from $a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$.
  So

  $$\begin{aligned}(\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle) \cdot \langle a_3, b_3, k\rangle &= \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle \cdot \langle a_3, b_3, k\rangle \\ &= \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 1/2\rangle.\end{aligned}$$

  We can easily show $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, l\rangle$ such that $l = j$ or $l = 1/2$.
  If $l = j$ then from $i \cdot j = 0$, $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$ we can show

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, j\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 1/2\rangle.$$

  If $l = 1/2$ then from $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$ we can get

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, 1/2\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 1/2\rangle.$$

- Let $k = 1/2$. Then

  $$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle \cdot \langle a_3, b_3, 1/2\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, l\rangle$$

  such that $l = 1/2$ or $l = 0$.

  If $l = 1/2$ then

  $$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

  Let $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, 1/2\}$.
  If $i = 0$ then from $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$,

  $$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 1/2\rangle.$$

If $i \neq 0$, i.e. $j = 0$. Then by same way of former case

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 1/2\rangle.$$

If $l = 0$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Suppose that $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, 1/2\}$. So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_A a_3, b_2 \cdot_B b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$.

- Let $k = 0$. Then

$$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, l\rangle$$

such that $l = 1/2$ or $l = 0$.
If $l = 0$ then

$$\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle \text{ such that } m \in \{0, 1/2\}.$$

So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$

If $l = 1/2$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Suppose that $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, 1/2\}$. So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$
Suppose that $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle$.
If $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Then

$$\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle.$$

($m \in \{0, 1/2, 1\}$.)
So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

If $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 1/2\rangle$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Then

$$\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$$

from $i \neq 1$ and $j \neq 1$. ($m \in \{0, 1/2, \}$.)
So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 1/2\rangle.$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$.

Lemma 5.2.12 *The algebra* $\mathbf{C}$ *satisfies residuated law.*

Proof First we show only-if part.

- Let

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle,$$
  $$\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, j \to k\rangle,$$

  then clearly

  $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle$ if and only if $\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle$.

- Let

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle,$$
  $$\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle.$$

  Then

  $$a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3,\ b_1 \leq_{\mathbf{B}} b_2 \to_{\mathbf{B}} b_3,\ i \leq j \to k.$$

  If $j \to k = 0$ then

  $$i \leq j \to k \leq 1/2.$$

  If $j \to k = 1$ then

  $$a_2 \to_{\mathbf{A}} a_3 \not\geq_{\mathbf{A}} 1_{\mathbf{A}},$$
  $$b_2 \to_{\mathbf{B}} b_3 \not\geq_{\mathbf{B}} 1_{\mathbf{B}}.$$

  So

$$a_1 \not\geq_{\mathbf{A}} 1_{\mathbf{A}} \text{ and } b_1 \not\geq_{\mathbf{B}} 1_{\mathbf{B}}.$$

Thus $i \neq 1$. Hence

$$\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

- Let $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle$, i.e. $i \cdot j = 0$. So $i = 0$ or $j = 0$. If $i = 0$ then clearly

$$\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle$$

If $i \neq 0$ then $j = 0$.
When $i = 1$, we can get

$$\begin{aligned} &j \to k = 1,\\ &1_{\mathbf{A}} \leq_{\mathbf{A}} a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3,\\ &1_{\mathbf{B}} \leq_{\mathbf{B}} b_1 \leq_{\mathbf{B}} b_2 \to_{\mathbf{B}} b_3.\end{aligned}$$

So

$$\langle a_1, b_1, i\rangle \leq \langle a_2, b_2, 0\rangle \to \langle a_3, b_3, k\rangle.$$

If $i = 1/2$ then by $j \to k = 1$,

$$\langle a_1, b_1, i\rangle \leq \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

- Let $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle$, i.e. $a \cdot_{\mathbf{A}} a' \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b \cdot_{\mathbf{B}} b' \leq_{\mathbf{B}} 0_{\mathbf{B}}$ and $i, j \neq 1$.
Suppose that $k \geq 1/2$ then we can get

$$\langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle$$

from $j \to k = 1$. So

$$\langle a_1, b_1, i\rangle \leq \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

Suppose that $k = 0$.
If $j = 0$ then

$$\langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle$$

by $j \to k = 1$.
If $j = 1/2$ then $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle$ from definition. So

$$\langle a_1, b_1, i\rangle \leq \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle \leq \langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle.$$

Next we prove if-part.

- Let $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, j \to k\rangle$.
  If

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle$$
  $$\text{or } \langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle,$$

  then we can show $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$ and $0 \leq i \cdot j \leq k$.
  Thus

  $$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \leq \langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle.$$

  If

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle$$

  then

  $$a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3,\ b_1 \leq_{\mathbf{B}} b_2 \to_{\mathbf{B}} b_3,\ a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}\ b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

  So clearly $a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$. Hence $k \neq 0$. Thus

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle \leq \langle a_3, b_3, k\rangle.$$

- Let $\langle a_2, b_2, j\rangle \to \langle a_3, b_3, k\rangle = \langle a_2 \to_{\mathbf{A}} a_3, b_2 \to_{\mathbf{B}} b_3, 1/2\rangle$, i.e. $j \to k = 1$ or $j \to k = 0$.
  If $j \to k = 1$ and $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1/2\rangle$ then

  $$a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

  So

  $$a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

  Hence $k \neq 0$.
  If $j \to k = 1$ and $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle$ is otherwise then $0 \cdot i \cdot j \leq j \leq k$. Thus

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle.$$

  If $j \to k = 0$ then $j = 1/2$ and $k = 0$. So

  $$a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}},$$
  $$b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$$

  from $a_1 \leq_{\mathbf{A}} a_2 \to_{\mathbf{A}} a_3$ and $b_1 \leq_{\mathbf{B}} b_2 \to_{\mathbf{B}} b_3$ respectively. Hence

  $$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \leq \langle a_3, b_3, k\rangle.$$

Lemma 5.2.13 *A mapping α from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$ defined by*

$$\alpha(\langle a, b, i\rangle) = \langle a, b\rangle.$$

is a surjective homomorphism.

Proof We can easily show that

$$\begin{aligned}
\alpha(\langle a, b, i\rangle \oplus \langle a', b', j\rangle) &= \alpha(\langle a \oplus_{\mathbf{A}} a', b \oplus_{\mathbf{B}} b', k\rangle) \\
&= \langle a \oplus_{\mathbf{A}} a', b \oplus_{\mathbf{B}} b'\rangle \\
&= \langle a, b, i\rangle \oplus_{\mathbf{A}\times\mathbf{B}} \langle a', b', j\rangle \\
&= \alpha(\langle a, b, i\rangle) \oplus_{\mathbf{A}\times\mathbf{B}} \alpha(\langle a', b', j\rangle).
\end{aligned}$$

So α is homomorphism. A mapping α clearly holds a condition $\alpha(\mathbf{C}) = \mathbf{A} \times \mathbf{B}$. Thus α is surjective homomorphism.

Lemma 5.2.14 *If* **A** *and* **B** *satisfy a condition* DN *then* **C** *satisfies a condition* DN.

Proof Let $\langle a, b, i\rangle$ in C ($\neg\langle a, b, i\rangle$ means $\langle a, b, i\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$).
When $i = 1$,

$$\begin{aligned}
&\langle a, b, 1\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg a, \neg b, 0\rangle \\
&\langle \neg a, \neg b, 0\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg\neg a, \neg\neg b, 1\rangle \\
&\langle \neg\neg a, \neg\neg b, 1\rangle \to \langle a, b, 1\rangle = \langle \neg\neg a \to_{\mathbf{A}} a, \neg\neg b \to_{\mathbf{B}} b, 1 \to 1\rangle \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle.
\end{aligned}$$

When $i = 0$,

$$\begin{aligned}
&\langle a, b, 0\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg a, \neg b, 1\rangle \\
&\langle \neg a, \neg b, 1\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg\neg a, \neg\neg b, 0\rangle \\
&\langle \neg\neg a, \neg\neg b, 0\rangle \to \langle a, b, 0\rangle = \langle \neg\neg a \to_{\mathbf{A}} a, \neg\neg b \to_{\mathbf{B}} b, 0 \to 0\rangle \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle.
\end{aligned}$$

When $i = 1/2$,

$$\begin{aligned}
&\langle a, b, 1/2\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg a, \neg b, 1/2\rangle \\
&\langle \neg a, \neg b, 1/2\rangle \to \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle = \langle \neg\neg a, \neg\neg b, 1/2\rangle \\
&\langle \neg\neg a, \neg\neg b, 1/2\rangle \to \langle a, b, 1/2\rangle = \langle \neg\neg a \to_{\mathbf{A}} a, \neg\neg b \to_{\mathbf{B}} b, 1/2 \to 1/2\rangle \geq \\
&\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle.
\end{aligned}$$

Thus **C** satisfies DN.

Proof of Theorem 5.2.9 From these lemma we can easily show that **C** is a CRL which satisfies DN and there are a surjective homomorphism from **C** onto $\mathbf{A} \times \mathbf{B}$. Thus we construct suitable algebra.

Our proof works well also for $\mathbf{FL_{ew}}$, $\mathbf{FL_{ew}}[\mathrm{E_k}]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$. In these cases, $1_{\mathbf{A}}$ and $1_{\mathbf{B}}$ are the greatest, $0_{\mathbf{A}}$ and $0_{\mathbf{B}}$ are the leastest elements of **A** and **B**, respectively. Thus, we can obtain **C** of the following forms.

Chapter 6

Further works

We have following problems for further works.

- What kind of axioms can be preserved by construction of the algebra in Chapter 5? More precisely, for which formula ϕdoes the following hold?

  If a formula ϕ is valid in both **A** and **B** then ϕ is valid also in **C**.

  For example, how about the distributive law, and how about the axiom $\neg p^{k+1} \rightarrow \neg p^{k}$?

- The following result is shown independently by P. Minari and M. Zakharyaschev.

  If a logic L over **Int** has the disjunction property then L correspond to **Int** with formulas which include no $\vee$.

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