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An algebraic approach to the disjunction property of substructural logics

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Substructural logics are logic obtained from classical logic **LK** or intuitionistic logic **LJ** by deleting some of structural rules. This study is started from study of **FL** by Lambek. They include relevant logics, linear logic and BCK-logics. By introducing sequent calculi which have the cut elimination theorem we can show various syntactical results. But syntactical methods work well only for particular logics, so we cannot use them for general discussions. So we need to find useful semantical methods. Since Kripke-type semantics is quite powerful in the study of modal logics, it does not work well for substructural logics, it does not work well for substructural logics. In recent years, algebraic methods have been developed as a powerful tool for investigating substructural logics. In this thesis, we give an algebraic proof of the disjunction property of logics over $\mathbf{FL_e}$ and $\mathbf{FL_e w}$.

$\mathbf{FL_{ew}}$ is the logic obtained from the sequent calculus **LJ** for intuitionistic logic **Int** by deleting the contraction rule. Furthermore $\mathbf{FL_e}$ is the logic obtained from $\mathbf{FL_{ew}}$ by deleting the weakening rule. $\mathbf{FL_e}[\mathrm{E_k}]$ ($\mathbf{FL_e}[\mathrm{DN}]$) is an extension of $\mathbf{FL_e}$ by adding $\mathrm{E_k}$: $p^n \to p^{n+1}$(*weak k-potency*) (DN: $\neg\neg p \to p$(*double negation*), respectively) as an axiom. Similarly, we can introduce $\mathbf{FL_{ew}}[\mathrm{E_k}]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$.

Algebraic structures corresponding to logics over $\mathbf{FL_e}$ are *commutative residuated lattices* (CRL). Here a CRL is an algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ which satisfies the following three conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee, 0, 1\rangle$ is a lattice,

(R2) $\langle \mathrm{A}, \cdot, 1\rangle$ is a commutative monoid,

(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow x \leq y \rightarrow z$.

When $\langle \mathrm{A}, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with the greatest element 1 and the least 0, $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is called a commutative integral residuated lattice. It is easy to see that a commutative integral residuated lattice is a *Heyting algebra* if and only if the semigroup operation $\cdot$ is equal to $\wedge$.

A logic L has the *disjunction property* if and only if for all formulas ϕ and ψ, if $\phi \vee \psi$ is provable then either ϕ or ψ is provable. In order to give an algebraic characterization of disjunction property, a notion of well-connectedness of algebra plays an important role: a CRL $\mathbf{A}$ is said to be *well-connected*, for all $x, y \in \mathrm{A}$ if $x \vee y \geq 1$ then $x \geq 1$ or $y \geq 1$.

The next proposition is given by L. Maksimova in 1984..

Proposition 1 (Maksimova) *Suppose that a logic L over* **Int** *is complete with respect to a class K of Heyting algebras. Then, the following are equivalent;*

1. *L has the disjunction property,*
2. *For all Heyting algebras* $\mathbf{A}, \mathbf{B} \in K$ *there exists a well-connected Heyting algebra* $\mathbf{C}$ *such that L is valid in* $\mathbf{C}$*, and there is a surjective homomorphism from* $\mathbf{C}$ *onto* $\mathbf{A} \times \mathbf{B}$.

We can extend Maksimova's result to logics over $\mathbf{FL_e}$.

Theorem 2 *Suppose that a logic L over* $\mathbf{FL_e}$ *is complete with respect to a class K of CRLs. Then, the following are equivalent;*

1. *L has the disjunction property,*

2. *For all CRLs* $\mathbf{A}, \mathbf{B} \in K$ *there exists a well-connected CRL* $\mathbf{C}$ *such that* L *is valid in* $\mathbf{C}$*, and there is a surjective homomorphism from* $\mathbf{C}$ *onto* $\mathbf{A} \times \mathbf{B}$*.*

The following theorem is our main result. To prove this theorem we construct a CRL $\mathbf{C}$ which satisfies the condition of Theorem 2 for give CRL $\mathbf{A}$ and $\mathbf{B}$.

Theorem 3 *Each of* $\mathbf{FL_e}$, $\mathbf{FL_e}[\mathrm{E_k}]$ *and* $\mathbf{FL_e}[\mathrm{DN}]$ *has the disjunction property.*

Similarly, we have the following. The proofs are obtained immediately from proofs of Theorem 3, by adding the assumption that 1 and 0 are the greatest element and the least, respectively.

Theorem 4 *Each of* $\mathbf{FL_{ew}}$, $\mathbf{FL_{ew}}[\mathrm{E_k}]$ *and* $\mathbf{FL_{ew}}[\mathrm{DN}]$ *has the disjunction property.*

Note that while the disjunction property of $\mathbf{FL_e}$, $\mathbf{FL_e}[\mathrm{DN}]$, $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$ can be shown also by using syntactical methods, we have no cut-free sequent calculi for other logics in the above. Thus, our results on these logics show an importance of algebraic methods.