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On the Representability of Relation Algebras

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1 Introduction

The theory of binary relations was developed by A.de Morgan, C.Peirce and E.Schröder. In this theory, a relation is a binary relation (i.e., a subset of the Caretesian product) of a particular fixed set. It can be used to express mathematical relationships, consanguinity relationships, affinity relationships and so on. Various kind of logical systems for the calculus of binary relations have been introduced and studied, most of them with an algebraic flavour. Half a century ago, A.Tarski proposed an equational axiomatization for these algebraic systems. It is the beginning of relation algebras. We will follow Tarski's approach in this report.

In this thesis, we introduce the main results of relation algebras, mainly in connection with their representablility. First of all, we introduce Boolean algebras as a previus step. Then we connect relation algebras with binary relations. We also mention some algebraic properties of relation algebras. Next, we disscuss the representability of relation algebras. A.Tarski asked whether every relation algebra is isomorphic to the algebra of a binary relation. Unfortunately R.Lyndon has proved by a complicated counterexample that the answer is negative in general. In recent years, simpler counterexamples have been found, even finite ones, and it has also been showed more examples of representable relation algebras. These results are introduced in the second half of the thesis.

2 Research work

1. From binary relations to relation algebras

We firstly introduce binary relations. A relation is a binary relation (i.e., a subset of the Caretesian product) of a particular fixed set. Binary relations are closed under Boolean operators. Moreover they are also closed under the operators of composition and inverse. A distinguished binary relation is the identity. When we consider some binary relations closed under the previous operations, what we obtain is a proper relation algebra.

Definition 1 (Proper relation algebras(PRA))

Let B and U be sets. An algebra $\mathcal{S} = (S, \cup, \cap, \setminus, \emptyset, U, Id_B, |, ^{-1})$ *is a proper relation algebra with base set B if it satisfies*

(1) $S \neq \emptyset$,

(2) $S \subseteq \wp(B \times B)$,

(3) $(S, \cup, \cap, \setminus, \emptyset, U)$ *is a field of sets*,

(4) $Id_B = \{(b, b) | b \in B\} \in S$,

(5) $s \in S \Rightarrow s^{-1} = \{(c, b) | (b, c) \in s\} \in S$,

(6) $r, s \in S \Rightarrow r|s = \{(b, c) | \exists d((b, d) \in r \text{ and } (d, c) \in s)\} \in S$.

Then we will discuss the variety of relation algebras which is bigger than the class of proper relation algebras. We ask how we can axiomatise the class of algebras. The following definition has become standard.

Definition 2 (Relation Algebras(RA))

An algebra $\mathcal{A} = (A, +, \cdot, -, 0, 1, 1', \breve{}, ;)$ *is a relation algebra if it satisfies*

(1) $(A, +, \cdot, -, 0, 1)$ *is a Boolean algebra*,

(2) $(x; y); z = x; (y; z)$,

(3) $(x + y); z = x; z + y; z$,

(4) $x; 1' = x$,

(5) $\breve{\breve{x}} = x$,

(6) $(x + y)\breve{} = \breve{x} + \breve{y}$,

(7) $(x;y)\breve{} = \breve{y};\breve{x}$,

(8) $(x;-(\breve{x};y)) + y = y$.

We also survey some properties of relation algebras.

2. Representation of relation algebras

We will discuss the representability of relation algebras. A relation algebra is said to be representable if there is an isomorphism among it and a proper relation algebra. The representability problem, is suggested by A.Tarski, asks whether every relation algebra is isomorphic to a proper relation algebra. For Boolean algebras we know that there is a representation theorem which state that every Boolean algebra is isomorphic to a field of sets.

A first positive answer to the question was given by J.C.C.McKinsey who proved the representation theorem for atomic , complte and simple relation algebra with an extra condition.

But R.Lyndon has shown by a complicated counterexample that the answer is generally negative. Later, R.Mckenzie has found a minimal non-repesentable relation algebra. This algebra is now called McKenzie algebra and has only four atoms. In additon, it has been shown that the class of representable relation algebras is not axiomatizable by finitely many equations.

Next, we introduce some representable relation algebras. It turned out that it is interestiong to study the property of binary relations in order to construct representable relation algebras to focus on the special properties of binary relations. In this point of view, it is useful to analyze functional elements of relation algebras. They are algebraic counterparts of many-to-one binary relation. Specifically, it is known that any atomic relation algebra whose atoms are functional is representable. Moreover, it is also known that any relation algebra in which the unit 1 is the sum of finitely many functional elements is representable. The previous results also holds when we replace functional elements with point elements or subidentiti elements. Here Point elements are algebraic counterparts of the element of identity relation and subidentity elements are algebraic counterparts of a subset of the identity relation.

In the near future, we have some plans to study the representability problem in connection with relation algebras generated by special elements. We also want to survey what is known about the class of representable relation algebras. We also want to analyze and compare these notions from a more general perspective in a wider sense.