Title	作用項書換えのための多重集合経路順序
Author(s)	谷内, 亘
Citation	
Issue Date	2025-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/19818
Rights	
Description	Supervisor: 廣川 直, 先端科学技術研究科, 修士 (情報科学)

Multiset Path Orders for Applicative Term Rewriting

2310165 Wataru Yachi

In this thesis, we introduce a higher-order version of the multiset path order where λ-abstractions and types are absent. *Reduction orders* are well-founded orders on terms. It can show termination of term rewriting systems. For example, the rewrite system $\mathcal{R}_1$

$$\begin{aligned} x + \mathsf{0} &\to x & x \times \mathsf{0} &\to \mathsf{0} \\ x + \mathsf{s}(y) &\to \mathsf{s}(x+y) & x \times \mathsf{s}(y) &\to x + (x \times y) \end{aligned}$$

is terminating if there exists some reduction order $>$ such that the following inequalities hold:

$$\begin{aligned} x + \mathsf{0} &> x & x \times \mathsf{0} &> \mathsf{0} \\ x + \mathsf{s}(y) &> \mathsf{s}(x+y) & x \times \mathsf{s}(y) &> x + (x \times y). \end{aligned}$$

One of the most popular reduction orders is the multiset path order, introduced by Dershowitz (1982). This order takes an order on function symbols, called precedence. Since the multiset path order with the precedence $\times > + > \mathsf{s}$ satisfies the above inequality constraints, the termination of $\mathcal{R}_1$ is concluded.

Applicative terms represent higher-order functions without λ-abstraction. For instance, the higher-order function map used in functional programming is modeled by rewrite system $\mathcal{R}_2$

$$\begin{aligned} \mathsf{add}\ x\ \mathsf{0} &\to x & \mathsf{app}\ \mathsf{nil}\ ys &\to ys \\ \mathsf{add}\ x\ (\mathsf{s}\ y) &\to \mathsf{s}\ (\mathsf{add}\ x\ y) & \mathsf{app}\ (\mathsf{cons}\ x\ xs)\ ys &\to \mathsf{cons}\ x\ (\mathsf{app}\ xs\ ys) \\ \mathsf{mul}\ x\ \mathsf{0} &\to \mathsf{0} & \mathsf{map}\ f\ \mathsf{nil} &\to \mathsf{nil} \\ \mathsf{mul}\ x\ (\mathsf{s}\ y) &\to \mathsf{add}\ x\ (\mathsf{mul}\ x\ y) & \mathsf{map}\ f\ (\mathsf{cons}\ x\ xs) &\to \mathsf{cons}\ (f\ x)\ (\mathsf{map}\ f\ xs). \end{aligned}$$

Unfortunately, the multiset path order cannot orient the all rules in $\mathcal{R}_2$, and it even fails to orient the curried version $\mathsf{mul}\ x\ (\mathsf{s}\ y) \to \mathsf{add}\ x\ (\mathsf{mul}\ x\ y)$ of the definition of multiplication $x \times \mathsf{s}(y) > x + (x \times y)$. If we have a reduction order that can orient the all rules in $\mathcal{R}_2$ then we can show that the following equation

$$\mathsf{map}\ f\ (\mathsf{app}\ xs\ ys) = \mathsf{app}\ (\mathsf{map}\ f\ xs)\ (\mathsf{map}\ f\ ys)$$

holds for all ground constructor terms consisting of $\mathsf{0}$, s, nil, and cons.

Remarkable character of the multiset path order is that it can show termination of term rewriting systems that represent primitive recursive functions.

However, we are confronted with some problems when adapting the multiset path order to applicative terms. Firstly, variables at head positions do not preserve the direction when they are substituted. For instance, consider the precedence $\mathsf{f} > \mathsf{g}$. Thus, $x\ y > \mathsf{f}\ y$ holds but if we substitute g into x then $\mathsf{g}\ y \not> \mathsf{f}\ y$ since $\mathsf{f} > \mathsf{g}$. Secondly, the subterm property also breaks the direction when we attach arguments to both hand sides. For example, we obtain $\mathsf{g}\ \mathsf{f} > \mathsf{f}$ by the subterm property, but $\mathsf{g}\ \mathsf{f}\ \mathsf{a} \not> \mathsf{f}\ \mathsf{a}$ due to the precedence $\mathsf{f} > \mathsf{g}$. Finally, proving well-foundedness of reduction orders are difficult. The proof method of well-foundedness of orders that have the subterm property are provided. But since our order lacks this property, we need another proof technique.

Here we summarize the problems as follows:

1. stability of head variable terms,
2. compatibility with contexts, and
3. well-foundedness proof of orders without the subterm property.

In this thesis we give the following solutions for the above problems:

1. *uncurrying* and Tanaka's method,
2. introducing an *arity assignment* to restrict the comparing of terms, and
3. borrowing the proof technique of well-foundedness from complexity analysis

Then we obtain a first path ordering on applicative terms that can prove the termination of $\mathcal{R}_2$.