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Master's Thesis

The Finite Embeddability Property for Some Modal Algebras

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March, 2006

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Chapter 1

Introduction

The subject of this thesis is the finite embeddability property(FEP) of various classes of modal algebras.

The FEP is a property of a *class*[1] of algebras and implies the well-known finite model property(FMP), which is often used to prove decidability of modal logics. In fact, the FEP provides a stronger decidability than the FMP in the following sense: the FEP of a class of algebras implies decidability of its universal theory while the FMP implies the decidability of its equational theory. The history of the FEP dates back to the paper by McKinsey[21], where the FEP for the closure algebras are proved, the paper by McKinsey & Tarski[23], where the proof of the FEP for Heyting algebras is given.

Modal logics are propositional logics enriched with *modal operators* such as $\Box$ and $\Diamond$. As suggested by the plural, there are various kinds of modal logics according to which conditions the modal operators meet. Moreover, there are variations as to which are based upon. Namely we can add, in effect, modal operators to arbitrary logics, classical or non-classical. Actually we consider modal logics based on classical, intuitionistic, and substructural propositional logic. We call them normal modal logics, intuitionistic modal logics, modal substructural logics, respectively.

But why do we consider the FEP, a property of *classes of algebras*, while working on *logics*? Generally we have the correspondence between a logic and a class of algebras. The most famous examples are that between classical logic and boolean algebras, and between intuitionistic logic and Heyting algebras. This allows us to consider the decidability of (theories of) a class of algebras when we really want to know the decidability of the corresponding logic. *Modal algebras* are generic term denoting algebras that is associated with modal logics. Thus our aim is to prove decidability of modal logics by showing that the corresponding classes of algebras has the FEP.

A traditional path to take to prove decidability for modal logics is to prove the FMP, to prove which a method called *filtration* is used in turn. But filtration needs Kripke semantics for the logic under consideration. Unfortunately, in general, Kripke semantics cannot be defined for substructural logics due to the failure of the duality between $\vee$ and $\wedge$ (see [32] for details). Thus decidability was proved exclusively by proof-theoretical method, i.e. by eliminating cut. Blok and van Alten's paper[4] blazed a trail for model-theoretical proof. There it is proved the variety of all (commutative integral) residuated lattices[2] has the FEP. In a way this paper rephrasing the results of Okada and Terui[26] in algebraic terms. The situation can be depicted

[1] Here the word "class" is used because the collection of algebras might fail to be a set.

[2] Throughout this thesis we mean commutative integral residuated lattice by residuated lattice.

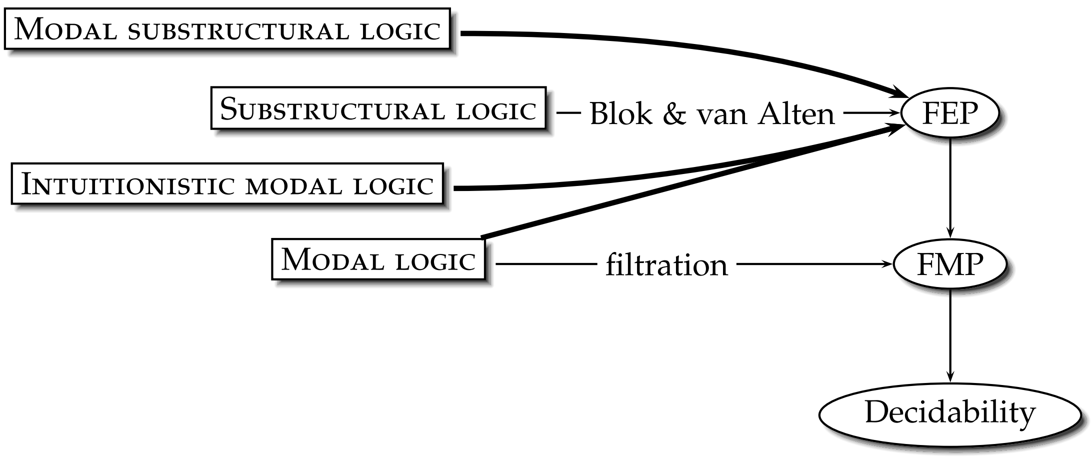

FIGURE 1. A road map to decidability

as in figure 1. The thick arrows are proved in this thesis.

Most of the FEP results so far is for the classes of algebras without modality. Our concern is to fill this gap, settling the FEP of modal algebras of various kind. We first consider normal modal algebras, then proceed to intuitionistic modal algebras and residuated lattices with modality.

The structure of the thesis is as follows: Chapter 2 introduces basic definitions. We review notions from universal algebra and fix notation. We also recall the definitions of modal and substructural logics and their semantics. Chapter 3 proves the FEP of some classes of normal modal algebras. The method used there is an amalgam of two well-known methods/results: Schütte's proof of the FMP and Jónsson-Taski's theorem. The important observation is "if the FMP of a logic can be proved via Schütte's method, then the FEP of the corresponding class of algebras can be proved". Chapter 4 shows the method of the preceding chapter extends to classes of intuitionistic modal algebras. In chapter 5, we consider a new modal logic based on FL_{ew} introduced in [31] and prove the FEP for it following Blok and van Alten[4, 1].

Chapter 2

Preliminaries

This chapter introduces important notions and notations used throughout the thesis. Main sources are [28, 6, 3].

2.1 Logic and algebra

In this section we discuss the relation between logics and (classes of) algebras. We first introduce a series of notions, starting from what an algebra is.

Definition 1 An *algebra* is a tuple $\mathbf{A} = \langle A, \langle f_i : i \in I \rangle \rangle$ where each f_i is a function from A^{ρ_i} to A and ρ_i is the arity of f_i.

We always use bold face $\mathbf{A}$ for algebra and plain A for its base set[1]. If we want to stress that the symbol is the function of an algebra $\mathbf{A}$, we write $f_i^{\mathbf{A}}$, but we usually drop it. A tuple $\langle \langle f_i : i \in I \rangle, \langle \rho_i : i \in I \rangle \rangle$, where f_i's are function symbols and ρ_i's are corresponding arities, is called *a similarity type* or simply *a type*. A function symbol of arity 0 is called a *constant*.

An algebra of type $\mathcal{F} = \langle f_i : i \in I \rangle$ is called $\mathcal{F}$-algebra, for which we also write $\langle A, f \rangle_{f \in \mathcal{F}}$. All the similarity types considered in this thesis revolves around that of boolean algebra: $\langle \wedge, \vee, ', 0, 1 \rangle$.

Definition 2 Let $\mathcal{F}$ be a similarity type. Given a set X of variables, we define the set $\mathit{Term}_{\mathcal{F}}(X)$ of *terms over* X as follows.

- Every variable x is a term.
- If $f \in \mathcal{F}$ and t_i's are terms, then $f(t_1, \ldots, t_m)$ is a term, where m is the arity of f.

An *equation* is an expression of the form $s \approx t$.

In this connection remember that we heavily abuse symbols for algebraic purpose, i.e., we use the same symbols for algebraic operation and logical connectives. For instance, $p \vee q$ can be seen as a wff or as a term over the variable set $\{p, q\}$.

We need a *valuation*[2] for algebras to be models for logics.

[1] Aliases: carrier, carrier set, domain.
[2] aka an assignment.

Definition 3 Fix a similarity type $\mathcal{F}$ and a set X of variables. An *valuation* ν on an $\mathcal{F}$-algebra $\mathbf{A}$ is a function from X to A. A valuation is extended to $Term_{\mathcal{F}}(X)$ as follows:

$$\begin{aligned}\hat{\nu}(x) &= \nu(x), \text{ where } x \in X,\\ \hat{\nu}(c) &= c^{\mathbf{A}}, \text{ where c is a constant}\\ \hat{\nu}(f(t_1,\ldots,t_m)) &= f(\hat{\nu}(t_1),\ldots,\hat{\nu}(\nu_m)).\end{aligned}$$

We can define the truth or validity of a equation in an algebra as we can do for a wff in a structure.

Definition 4 An equation $s \approx t$ is said to be *true* or to *hold* in $\mathbf{A}$ with a valuation ν, denoted $\mathbf{A}, \nu \models s \approx t$, if $\nu(s) = \nu(t)$. Furthermore $s \approx t$ is said to be *valid* in $\mathbf{A}$, denoted $\mathbf{A} \models s \approx t$, if $\mathbf{A}, \nu \models s \approx t$ for all valuation ν on $\mathbf{A}$. We also use the notation $\mathbf{A} \models E$ meaning that $\mathbf{A} \models s \approx t$ for all $s \approx t$ in the set E of equations. If $\mathbf{A} \models s \approx t$ or $\mathbf{A} \models E$, $\mathbf{A}$ is a *model* for, or satisfy $s \approx t$ or E, respectively. We write $E \models s \approx t$ if for every algebra $\mathbf{A}$ and every valuation ν, $\mathbf{A}, \nu \models E$ implies $\mathbf{A}, \nu \models s \approx t$.

We write $\mathrm{Mod}(E)$ for the class of algebras satisfying E. A class $\mathcal{K}$ of algebras is said to be *equationally definable* if there exists a set E of equations such that $\mathrm{Mod}(E) = \mathcal{K}$.

We define several operations that construct a new algebra from old one(s). First we define what it means that two algebras are similar or even the same.

Definition 5 Suppose $\mathbf{A}, \mathbf{B}$ are $\mathcal{F}$-algebras. A mapping $h : A \to B$ is a *homomorphism* if for all $f \in \mathcal{F}$, and all $a_1, \ldots, a_n \in A$ (n is the arity of f),

$$h(f^{\mathbf{A}}(a_1,\ldots,a_n)) = f^{\mathbf{B}}(h(a_1),\ldots,h(a_n)),$$

and for any constant c,

$$h(c^{\mathbf{A}}) = c^{\mathbf{B}}.$$

$\mathbf{B}$ is said to be a *homomorphic image* of $\mathbf{A}$ if there is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{B}$.

Definition 6 An *isomorphism* is a bijective homomorphism . We say that two algebras are *isomorphic* if there exists an isomorphism between them.

Next construction is to find a “self-contained” smaller structure.

Definition 7 Let $\mathbf{A}$ be an $\mathcal{F}$-algebra. If a subset B of A is closed under every operation $f \in \mathcal{F}$, then $\langle B, f^{\mathbf{A}}\restriction_B\rangle_{f\in\mathcal{F}}$ is called a *subalgebra* of $\mathbf{A}$.

We can construct a new algebra by concatenating a series of algebras into one.

Definition 8 Let $\langle \mathbf{A}_i\rangle_{i\in I}$ be a family of algebras. We define the *product* $\prod_{i\in I} \mathbf{A}_i$ of this family as the algebra $\mathbf{A} = \langle A, f^{\mathbf{A}}\rangle_{f\in\mathcal{F}}$ where $A = \prod_{i\in I} A_i$ and the operation $f^{\mathbf{A}}$ is defined componentwise: for $\langle a_i^1\rangle_{i\in I}, \ldots, \langle a_i^n\rangle_{i\in I} \in \prod_{i\in I} A_i$, where n is the arity of f,

$$f^{\mathbf{A}}(\langle a_i^1\rangle_{i\in I}, \ldots, \langle a_i^n\rangle_{i\in I}) = \langle f^{\mathbf{A}_i}(a_i^1, \ldots, a_i^n)\rangle_{i\in I}\rangle_{i\in I}.$$

When all the $\mathbf{A}_i$’s are the same, say $\mathbf{A}$, then the product is called a power of $\mathbf{A}$ and denoted $\mathbf{A}^I$ instead of $\prod_{i\in I} A$.

Special interest is paid to those classes of algebras that are closed under the above three operations.

Definition 9 A class $\mathcal{K}$ of algebras is a variety if it is closed under taking subalgebra, homomorphic images, and products. That is, if $\mathbf{B}$ is a subalgebra of $\mathbf{A} \in \mathcal{K}$, then $\mathbf{B} \in \mathcal{K}$; if $\mathbf{B}$ is a homomorphic image of $\mathbf{A} \in \mathcal{K}$, then $\mathbf{B} \in \mathcal{K}$; if $\mathbf{A}_i \in \mathcal{K}$ for all $i \in I$, then $\prod_{i \in I} \mathbf{A}_i \in \mathcal{K}$.

Another characterization is given in classic Birkhoff's theorem:

Theorem 10 [Birkhoff] *A class of algebras is equationally definable if and only if it is a variety.*

Now we can state what is exactly meant by "a logic is associated with a class of algebras".

Definition 11 A logic L is associated with(*complete with respect to*) a class $\mathcal{K}$ of algebras if the following holds.

A wff α is provable in $L \Leftrightarrow$ For arbitrary $\mathbf{A} \in \mathcal{K}$, $\mathbf{A} \models \alpha \geqslant 1$,

where 1 is the constant denoting top element[3].

2.2 Decidability

Let us define what it means that a set is decidable. We shall fix a model of computation, *Turing Machines*[4], and stipulate that the intuitive notion of computability" coincides with the computability by Turing machines[5]. In a word, a set is decidable if its membership can be determined by some Turing machine. The presentation is largely based on [37]. We use the symbol ␣ for the special *blank* symbol. In fact we never use Turing machine in this thesis, so that this section can be skipped. We state the definitions for completeness.

Definition 12 A *Turing machine* is a septuple $\langle Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r \rangle$, where

- Q is the finite set of *states*;
- Σ is the finite set of the *input alphabet* and $␣ \notin \Gamma$;
- Γ is the finite set of the *tape alphabet*, where $␣ \in \Sigma$ and $\Sigma \subseteq \Gamma$;
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the *transition function*.
- $q_0 \in Q$ is the *start state*;
- $q_a \in Q$ is the *accept state*;
- $q_r \in Q$ is the *reject state* and $q_a \neq q_r$.

A Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r \rangle$ works as follows: At first, M is in state q_0 and the tape[6] contains the input $w_1 \cdots w_n \in \Sigma^*$[7] on the leftmost n squares and the rest is filed

[3] 1 may or may not be in the language. As far as logics under consideration are concerned, it does not matter much because $0 \to 0$ is a substitute.

[4] Turing machine was introduced in [39]. The book by Copeland [9] contains Turing's original paper besides a helpful introduction, commentary and corrections to it. It was Church who introduced the term "Turing machine" in [8]. Several other models of computation were considered around the same time and turned out to be equivalent. The survey by Gandy [14] provides an enjoyable historical account.

[5] This is what is called the *Church-Turing thesis*.

[6] We consider a tape is infinitely long to the right and divided into *squares*. If the machine is scanning the leftmost square and the transition is L, then we assume that it just stays there and does not fall out.

[7] Σ^* denotes the set of all the finite strings over Σ.

with blank symbols. M moves according to δ. Suppose, for instance, M is scanning w in state q and that $\delta(q, w) = \langle q', w', R\rangle$. Then M erases w and writes w', moves its head to the right, and changes its state to q'. Similarly for L. It goes on until it will halt (or it may not stop forever). An input is said to be *accepted* if the machine halts in state q_a, and to be *rejected* if the machine halts in state q_r.

We now shall be a little more formal. A *configuration* of a Turing machine at a given time is a triple consisting of the current state, the current tape contents, and the current head location. We write uqv to denote the configuration such that the state is q, the contents uv, and the first symbol of v being scanned. A configuration C_1 *yields* C_2 if the Turing machine can change C_1 to C_2 legally in a single step.

Let $a, b, c \in \Gamma$, $u, v \in \Gamma^*$, and $q_i, q_j \in Q$. Then a configuration uaq_ibv yields uq_jacv if $\delta(q_i, b) = \langle q_j, c, L\rangle$. For the other case of moving rightward, ua_qibv yields $uacq_jv$ if $\delta(q_i, b) = \langle q_j, c, R\rangle$. Cares must be taken when the machine is at extremes, i.e., it is at the left-hand end of the tape or scanning leftmost printed square. When it is at the left-hand end,

$$q_ibv \text{ yields } \begin{cases} q_icv & \text{if} \quad \delta(q_i, b) = \langle q_j, c, L\rangle \\ cq_jv & \text{if} \quad \delta(q_i, b) = \langle q_j, c, \rangle \end{cases}$$

The machine being the right-hand extreme is equivalent to scanning the blank symbol. That is, uaq_i equals to $uaq_{i\sqcup}$, which can be coped with in the way described above.

We say a set A is *decided* by M when both of the following hold[8]:

$a \in A$ iff M accepts a;

and

$a \notin A$ iff M rejects a.

Then decidability can be defined.

Definition 13 A set is *decidable* if it is decided by some Turing machine.

In the following we talk about decidability of a set of equations(theory) or a set of wffs(logic). To be more explicit as regards the latter, we say that a logic is decidable if we have a Turing machine that answers "yes" iff a given input is a theorem of the logic. When we try to implement some machine deciding, say, a set of formulas, we must devise a way of finitary representation for them. For instance we must finitarily represent an infinite number of propositional variables. This can be done by using $p0, p1, p00, p01, \ldots$ instead of $p, q, r, \ldots$. Another word for *decidable* is *recursive*.

Characteristic of a deciding Turing machine is that it eventually halts on every input. There exist undecidable sets, as proved by Turing and others, which do not have a machine deciding their membership. For some sets, however, we sometimes have a machine that "partially" decides them: We may have a machine M for a set A such that

$a \in A$ iff M accepts A.

This machine answers "yes" correctly if $a \in A$, but *may not halt* when $a \notin A$. In this case, we say M *recognizes* A and A is *recognizable*. Of course the crucial point is that such a machine may not halt. If we give an input to a deciding machine, then we can be sure that it will stop some day. If the machine running is a recognizer, we may have to wait for eternity with the

[8]Here we assume $A \subseteq \Sigma^+$, where Σ is the input alphabet of M.

machine buzzing about. To put it another way, recognizing process is that of listing all and only the elements of the set. If A is recognizable, we have a machine that prints all and only the members of A. If $a \in A$, then we shall find a in the list some time. But if $a \notin A$, we shall never find it and be doomed to endless searching till the day we die. Hence *recognizable* is also termed *recursively enumerable*.

To conclude this section, we show the following important theorem concerning decidability and recognizability[9].

Theorem 14 *A set is decidable iff the set and its complement is decidable.*

Proof. The direction from left to right is immediate because the complement of a decidable set is decidable (by considering a machine that "flips" its answer). For the reverse, suppose we have recognizing machines M_1 and M_2 for A and its complement $\overline{A}$, respectively. Then we construct a machine deciding A that performs as follows given an input a:

1. simulates M_1 on a and M_2 on a in parallel[10];
2. accepts a if M_1 accepts a; rejects a if M_2 accepts a.

The correctness of M is easy to see. Moreover, M halts in finitely many steps because M_1 or M_2 does by assumption. Thus M decides A. ❏

2.3 Adding modalities

In this section we consider propositional logics with additional unary logical connectives, called *modalities*, such as $\Box$ and $\Diamond$. These *modal logics* were originally meant to analyze the notions of necessity and possibility: $\Box p$ means "necessarily p holds", while $\Diamond p$ is "possibly p holds". Today various interpretations for $\Box$ and $\Diamond$ are known, and accordingly there are a huge number of modal logics. What we introduce here are only standard ones. For more comprehensive accounts, see [3] or [7].

2.3.1 Modal logics

First we state the syntax of modal logics.

Definition 15

1. Each propositional variable is a wff.
2. If α and β are wffs, so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, $(\neg\alpha)$, and $\Box\alpha$.

We omit parentheses as appropriate. We abbreviate $\neg\Box\neg\alpha$ to $\Diamond\alpha$.

The system K is a starting point. The axioms of K are all substitution instances of propositional tautologies plus

$$K : \Box(p \to q) \to (\Box p \to \Box q).$$

The rules of inference are modus ponens, uniform substitution, and the following *necessitation*[11]: infer $\Box\alpha$ from α.

We consider extensions of K with combinations of several axioms.

[9] This seems due to Emil Post[33]. Similar results also by Kleene[18] and by Mostowski[24].
[10] Say, we simulate each step of M_1 and M_2 alternatingly.
[11] Also called *generalization*.

D: $\Box\alpha \to \Diamond\alpha$

T: $\Box\alpha \to \alpha$;

B: $\alpha \to \Box\Diamond\alpha$;

4: $\Box\alpha \to \Box\Box\alpha$;

5: $\Diamond\alpha \to \Box\Diamond\alpha$.

The logic obtained from K plus axioms $X_1, X_2, \ldots, X_k$ is denoted $\mathsf{K}X_1X_2\ldots X_k$. For example, adding T and B to K give KTB. Aliases $\mathsf{S4}$ and $\mathsf{S5}$ are used for $\mathsf{KT4}$ and $\mathsf{KT5}$, respectively.

We shall introduce two kinds of semantics for modal logics. One is *Kripke(relational) frames* and the other is *modal algebras*. Here we only describe the former. The latter will be treated in the next subsection.

Definition 16 A *Kripke frame* is a pair $F = \langle W, R\rangle$ of nonempty set W and a binary relation R on W.

W is often called the set of possible worlds and R accessibility relation among them. To get a Kripke model, we need one more component.

Definition 17 A *Kripke model* is a triple $M = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a Kripke frame and V, called *a valuation*, is a function from the set of propositional variables to $\mathcal{P}(W)$.

We define the relation "α is true at a in a Kripke model $M = \langle W, R, V\rangle$", symbolized $a \models \alpha$, as follows.

$$
\begin{array}{rcl}
a \models p & \Leftrightarrow & a \in V(p),\ \ p \text{ is a propositional variable} \\
a \models \alpha \wedge \beta & \Leftrightarrow & a \models \alpha \text{ and } a \models \beta \\
a \models \alpha \vee \beta & \Leftrightarrow & a \models \alpha \text{ or } a \models \beta \\
a \models \alpha \to \beta & \Leftrightarrow & a \not\models \alpha \text{ or } a \models \beta \\
a \models \neg\alpha & \Leftrightarrow & a \not\models \alpha \\
a \models \Box\alpha & \Leftrightarrow & \text{for any } b \text{ with } aRb, b \models \alpha
\end{array}
$$

If $a \models \alpha$ for any $a \in W$ and any valuation V of a frame $F = \langle W, R\rangle$, then α is called *valid* in F.

The following correspondence is known between the axioms introduced above and properties of accessibility relations.

Proposition 18 *For arbitrary $\langle W, R\rangle$, the following holds.*

- D *is valid in* $\langle W, R\rangle$ *iff* $\forall x \exists y Rxy$, *i.e.*, R *is serial;*
- T *is valid in* $\langle W, R\rangle$ *iff* $\forall x Rxx$, *i.e.*, R *is reflexive;*
- B *is valid in* $\langle W, R\rangle$ *iff* $\forall xy(Rxy \to Ryx)$, *i.e.*, R *is symmetric;*
- 4 *is valid in* $\langle W, R\rangle$ *iff* $\forall xyz(Rxy \wedge Ryz \to Rxz)$, *i.e.*, R *is transitive;*
- 5 *is valid in* $\langle W, R\rangle$ *iff* $\forall xyz(Rxy \wedge Rxz \to Ryz)$, *i.e.*, R *is Euclidean.*

The following generalization of the above is available (see e.g. p79 of [7]).

Proposition 19 *The formula of the form* $\Diamond^k\Box^l p \to \Box^m\Diamond^n p$ *is valid in* $\langle W, R\rangle$ *iff*

$$\forall xyz(R^k xy \wedge R^m xz \to \exists u(R^l yu \wedge R^n zu)),$$

where $k, l, m, n \in \mathbb{N}$.

The above formula is called *Geach formula*.

2.3.2 Modal algebras

We introduce in this subsection another kind of semantics: *algebraic semantics*. In a word algebras for modal logic, which we call modal algebras, are boolean algebras with operators. We abuse symbols of logical connective for algebraic purpose.

Definition 20 A modal algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, ', 0, \Box \rangle$ such that $\langle A, \wedge, \vee, ', 0 \rangle$ is a boolean algebra and $\Box$ is a unary operator that satisfies

- $\Box 1 = 1$;
- $\Box(a \wedge b) = \Box a \wedge \Box b$,

where we abbreviate $0'$ to 1.

Obvious analogs of T etc. are as follows. Abuse of notation again. In addition, we are sadistic to the names of axioms. We mean by T both the syntactic and algebraic version of it.

D: $\Box\alpha \leqslant \Diamond\alpha$

T: $\Box\alpha \leqslant \alpha$;

B: $\alpha \leqslant \Box\Diamond\alpha$;

4: $\Box\alpha \leqslant \Box\Box\alpha$;

5: $\Diamond\alpha \leqslant \Box\Diamond\alpha$.

Corresponding algebras are called KT-algebra etc. We can prove completeness with Lindenbaum-Tarski algebras.

Theorem 21 *A formula is provable in a modal logic* L *iff it is valid in all* L*-algebras.*

2.4 Dropping structural rules

2.4.1 Intuitionistic logic

Intuitionistic logic was originally introduced by L. E. J. Brouwer, who tried to capture mathematics as human activity, and later formalized by A. Heyting. A famous characterization of intuitionistic logic is "classical logic minus the law of the excluded middle". We introduce it as a *sequent system*. A *sequent* is of the form $A_1, \ldots, A_m \Rightarrow B$, where $m \geqslant 0$ and $A_1, \ldots, A_m, B$ are wffs. Roughly this means we can infer B from $A_1, \ldots, A_m$. In what follows, capital roman alphabets $A, B, \ldots$ are metavariables for wffs and Greek capitals $\Gamma, \Sigma, \Delta, \ldots$ for sequences of wffs. When we consider intuitionistic or substructural logics, we put $\bot$ into our language and regard $\neg A$ as the abbreviation of $A \to \bot$.

A sequent system LJ for intuitionistic logic constitutes the initial sequents[12]

- $A \Rightarrow A$

[12] This system LJ was introduced in [15] together with LK for classical logic. Hence sequent systems are also called *Gentzen-style systems*. Another style of formulation, which we shall see later, is *Hilbert-style*. They consist of a number of axioms with a few inference rules, while Gentzen-style systems comprises a few axioms and a number of inference rules.

- $\Gamma, \bot, \Delta \Rightarrow C$

and the following rules of inference:
Structural rules

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \text{Cut}$$

$$\frac{\Gamma, A, A, \Delta \Rightarrow B}{\Gamma, A, \Delta \Rightarrow B}\ \text{Contraction} \qquad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}\ \text{Exchange} \qquad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \text{Weakening}$$

Rules for logical connectives

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow C}{A \to B, \Gamma, \Delta \Rightarrow C}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee 2)$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\Rightarrow \wedge)$$

$$\frac{A, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge 1 \Rightarrow) \qquad \frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge 2 \Rightarrow)$$

2.4.2 Substructural logics

In this subsection *substructural logics* are introduced. If we discard some or all of the three structural rules except Cut, a substructural logic is obtained. First we throw everything away: Let FL be the sequent system obtained from deleting all the structural rules from LJ. This requires additional logical connectives and accordingly structural rules. Its language contains a new logical connective $\cdot$ called *multiplicative conjunction* or *fusion*, $\backslash$ called *right implication*, and $/$ called *left implication*. For more information, see [29, 32, 11].

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \beta/\alpha, \Gamma, \Sigma \Rightarrow \delta}\ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \alpha\backslash\beta, \Gamma, \Sigma \Rightarrow \delta}\ (\backslash \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ (\Rightarrow \cdot) \qquad \frac{A, B, \Gamma \Rightarrow C}{A \cdot B, \Gamma \Rightarrow C}\ (\cdot \Rightarrow)$$

A slightly different system was introduced in [20], after which the above system is called **F**ull **L**ambek calculus, FL in short. We can freely retrieve the three structural rules. We indicate by subscripts which structural rules were recovered. For example, FL_c means that we took back contraction, and FL_{ec} that we have contraction and exchange. FL_{ecw} is intuitionistic logic. What we actually consider (in chapter 5) is FL_{ew}. In this system we do not need to distinguish between $\backslash$ and $/$, so we simply write $\to$ for implication.

2.4.3 Residuated lattices

The class of algebras called residuated lattice is the algebraic counterpart of FL_{ew}. Formal definition is as follows:

Definition 22 A septuple[13] $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is a residuated lattice[14] if

1. $\langle M, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with the greatest element 1 and the least element 0;
2. $\langle M, \cdot, 1\rangle$ is a commutative monoid;
3. For any $x, y \in M$, $x \cdot y \leqslant z$ iff $x \leqslant y \rightarrow z$.

The operation $\rightarrow$ is called the *residual* of $\cdot$. The relation between them (item 3) is called *the law of residuation*. Often we use ac as the abbreviation for $a \cdot c$.

The definitions for these algebras as models is standard. A valuation v on a residuated lattice $\mathbf{M}$ is a mapping from the set of propositional variables to M, which is extended in the obvious way except

$$v(\bot) = 0.$$

A wff A is said to be valid in $\mathbf{M}$ if $v(A) = 1$ for any valuation v on $\mathbf{M}$.

Now we can state the connection between FL_{ew} and the variety of residuated lattices:

Proposition 23 *A wff A is provable in FL_{ew} iff A is valid in any residuated lattice.*

We note the following lemma, which will be used in chapter 5.

Lemma 24 *In any residuated lattice $\mathbf{A}$, the following hold. Let $a, b, c \in A$.*

1. $a \leqslant b$ *implies* $ac \leqslant bc$;
2. *if* $\bigvee_{i \in I} a_i$ *exists, then* $c(\bigvee_{i\in I} a_i) = \bigvee_{i \in I} ca_i$.

Proof. (1) We have $bc \leqslant bc$, so that $b \leqslant c \rightarrow bc$. With $a \leqslant b$, we have $a \leqslant c \rightarrow bc$, whence finally $ac \leqslant bc$.

(2) Since $a_i \leqslant \vee_i a_i$, $ca_i \leqslant c(\vee_i a_i)$ holds for any i by (1); i.e., $c(\vee_i a_i)$ is an upper bound for $\{ca_i : i \in I\}$.Now take any upper bound d of $\{ca_i : i \in I\}$; that is, $ca_i \leqslant d$ for all i. Then $a_i \leqslant c \rightarrow d$ for all i, so that $\vee_i a_i \leqslant c \rightarrow d$, which implies $c(\vee_i a_i) \leqslant d$. This shows that $c(\vee_i a_i)$ is $\inf\{ca_i : i \in I\} = \vee_i ca_i$. ❏

2.5 The finite embeddability property

A class $\mathcal{K}$ of algebras is said to have the finite embeddability property if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

Formally, a partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ is an algebra of the same similarity type as $\mathbf{A}$ and each function $f_i^{\mathbf{B}}$ is defined as

$$f_i^{\mathbf{B}}(b_1, \ldots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1, \ldots, b_k) & \text{if } f_i^{\mathbf{A}}(b_1, \ldots, b_k) \in B \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

[13] We use the same symbols $\wedge$, $\vee$, and $\cdot$ for algebraic purpose.

[14] Recently a more descriptive name "commutative integral residuated lattice" is preferred.

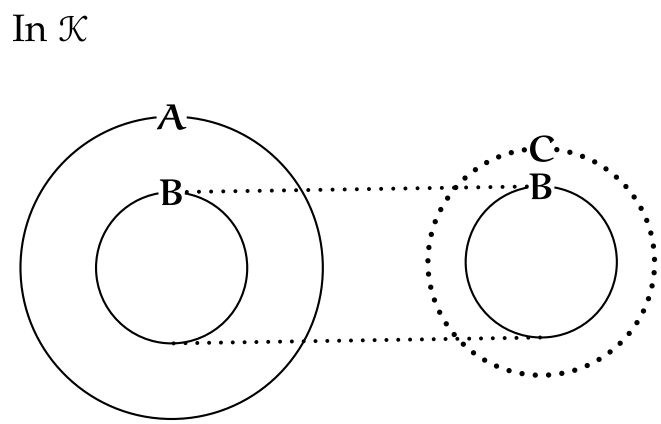

FIGURE 2. The FEP in picture

A partial subalgebra **B** is embedded[15] into A if there is an injective function h from B to A such that if $f^{\mathbf{B}}(b_1, \ldots, b_k)$ is defined, then

$$h(f^{\mathbf{B}}(b_1, \ldots, b_k)) = f^{\mathbf{A}}(h(b_1), \ldots, h(b_k))$$

holds for any $b_1, \ldots, b_k \in B$.

The task of proving the FEP can be illustrated as in figure 2: We are in the class $\mathcal{K}$ of algebras and given a finite partial subalgebra **B** of an (possibly infinite) algebra **A**. We need to find a finite algebra **C** and embed **B** into it.

The FEP is a stronger version of a well-known property of the finite model property. We briefly list the definitions of the FMP and of its variations. Here $\mathcal{K}_F$ denotes the class of the finite members of $\mathcal{K}$.

Definition 25 Let $\mathcal{K}$ be a class of algebras. We say $\mathcal{K}$ has

1. the finite model property(FMP) if $\mathcal{K}_F \models s = t$ implies $\mathcal{K} \models s = t$ for any identity $s = t$,
2. the strong FMP(SFMP) if $\mathcal{K}_F \models \sigma$ implies $\mathcal{K} \models \sigma$ for any quasi-identity σ,
3. the universal FMP(UFMP) if $\mathcal{K}_F \models \varphi$ implies $\mathcal{K} \models \varphi$ for any universal sentence φ.

Note that the FEP implies the UFMP, the UFMP implies the SFMP, and the SFMP implies the FMP. We sketch the proof of FEP $\to$ UFMP. Suppose $\mathcal{K}$ has the FEP and $\mathcal{K} \not\models \forall x_1 \cdots x_n \varphi$.Then we have some algebra **A** and its valuation v such that $\mathbf{A}, v \not\models \forall x_1 \cdots x_n \varphi$. That is, $A, v \not\models \varphi[a_1, \ldots, a_n]$[16] for some $a_1, \ldots, a_n \in A$. Let B be the elements relevant to this refutation of $\forall \varphi$: if $\psi(x_1, \ldots, x_n)$ is a subterm of φ, then $v(\psi[a_1, \ldots, a_n]) \in B$. Then **B** forms a partial subalgebra of **A**. We can embed **B** into some finite algebra $\mathbf{C} \in \mathcal{K}$. Then $\mathbf{C} \not\models \forall \varphi$ and we are done.

Each version of the FMP implies some kind of decidability with increasing strength.

Proposition 26 *Let $\mathcal{K}$ be a finitely axiomatizable class of algebras. If $\mathcal{K}$ has the FMP(SFMP, UFMP), then its equational (resp. quasi-equational, universal) theory is decidable.*

[15]We can define more generally the notion of embedding between algebras as an injective homomorphism, and the restriction "if $f^{\mathbf{B}}(b_i)$ is defined" can be dropped when we consider ordinary (non-partial) algebras.

[16]$\varphi[a_1, \ldots, a_n]$ denotes the expression resulting from substituting (the constant denoting) a_i for x_i.

Chapter 3

The FEP for normal modal algebras

In this chapter we prove the FEP of various classes of normal modal algebras. The method is an amalgam of two classic construction: Schütte's method of proving the FMP and Jónsson-Tarski's theorem. In fact we use (an algebraic analog of) Schütte's method to construct a Kripke frame from a partial algebra, then embed it into the complex algebra of the constructed frame.

We review the ways of coming and going between two kinds of semantics and then summarize two methods to prove the FMP of modal logics: One is the famous *filtration method* and the other is less-known method that was originally used by Schütte to prove the FMP of intuitionistic logic. Then we go on to prove the FEP of the variety of all modal algebras. Furthermore the FEP of the subvarieties corresponding to KT etc. are investigated.

3.1 Frames and algebras

In this section we collect some facts about the relationship between Kripke frames and modal algebras. Concretely we describe how to construct a modal algebra from a Kripke frame and back. This section is extremely indebted to [3][1], or simply a reproduction of it. We hope not to be sued for any violation of copyright.

First we show the way to go from the world of frames to that of algebras. This construction is called *complex algebra*[2].

Definition 27 Let $F = \langle W, R \rangle$ be a Kripke frame. The complex algebra denoted F^+ of F is the pair $\langle \mathcal{P}(W), \Box^+ \rangle$, where $\mathcal{P}(W)$ is the powerset algebra of W and $\Box^+$ is a modal operator on $\mathcal{P}(W)$ as

$$\Box^+ X = \{x \in W | \forall y \in W (Rxy \Rightarrow y \in X)\}.$$

It is easy to confirm the algebra really is the modal algebra. Next the other way around. We begin introducing a notion of filter.

Definition 28 A *filter* of a Boolean algebra $\mathbf{A} = \langle A, \vee, \wedge, ', 0 \rangle$ is a subset F of A such that

1. $0' = 1 \in F$;

2. if $a, b \in F$, then $a \wedge b \in F$: F is closed under meet;

[1]There general modal similarity type is investigated, but we confine ourselves to one unary modality case.

[2]What we define here is called the *full* complex algebra and complex algebra is any subalgebra thereof. But we simply drop *full* and are stick to it.

3. if $a \in F$ and $a \leqslant b$, then $b \in F$: F is upward closed.

A filter is *proper* if $F \neq A$. An *ultrafilter* is a proper filter satisfying one more condition

4. For every $a \in A$, either $a \in F$ or $a' \in F$.

The collection of all ultrafilters of $\mathbf{A}$ is denoted $\mathcal{U}(A)$.

We construct a frame upon $\mathcal{U}(A)$ from an algebra. This frame is called the *ultrafilter frame* of the given algebra.

Definition 29 Let $\mathbf{A} = \langle A, \wedge, \vee, ', 0, \Box \rangle$ be a modal algebra. The *ultrafilter frame* of $\mathbf{A}$, $\mathbf{A}_+$ in symbol, is the frame $\langle \mathcal{U}(A), R_+ \rangle$, where R is a binary relation on $\mathcal{U}(A)$ defined as, for $X, Y \in \mathcal{U}(A)$,

$$XR_+Y \text{ iff } \text{ for all } a \in A, \Box a \in X \Rightarrow a \in Y.$$

With these, we can state the following theorem of fundamental importance to our proof following in later sections.

Theorem 30 [Jónsson-Tarski] *Let $\mathbf{A}$ be a modal algebra. The function $r : A \to \mathcal{P}(\mathcal{U}(A))$ defined*

$$r(a) = \{u \in \mathcal{U}(a) | a \in U\}$$

is an embedding of $\mathbf{A}$ into $(\mathbf{A}_+)^+$. The algebra $(\mathbf{A}_+)^+$ is called the (canonical) embedding algebra *of $\mathbf{A}$ and written $\mathfrak{Em}\mathbf{A}$.*

3.2 The FMP

3.2.1 The FMP and decidability

In this subsection we give a brief summary of why we should make such a fuss about the FMP and the like: Simply put, the FMP implies decidability. Necessarily the proof is all sketchy. See (again) [3] for more general account.

Definition 31 A logic is *finitely axiomatizable* if it has a finite axiomatization.

First, the following is a standard observation. Roughly, a proof is a sequence of formulae each of which is an axiom or follows from inference rules and the previously occurring formula(e). The length of a proof is the number of formulae appearing in it.

Lemma 32 *The set of theorems of a finitely axiomatizable logic is recursively enumerable.*

Proof. We can list all the proofs one by one, say, according to its length. Namely, we first list the proofs of length 1 (i.e., axioms), then the proofs of length 2 and so on. We put the last line of each proof into the list. This way we can list all and only the theorems of a given logic. ❑

Thus a finitely axiomatizable logic is always recognizable. By theorem 14, we have only to show that its complement, i.e., the set of non-theorems are also recursively enumerable.

Theorem 33 *If a finitely axiomatizable logic has the finite model property, then the logic is decidable.*

Proof. We prove that the set of non-theorems is recursively enumerable. Then by theorem 14 and the preceding lemma, we have the theorem. To do this, we list all finite Kripke models according to its size. Since the number of axioms is finite, each model can be checked in finitely many steps if it is a model of the given logic. We also enumerate formulae and check if it is falsifiable in the listed models. By assumption there is a finite falsifying model for each unprovable formula. This means all non-theorems appear in this enumeration. ❑

3.2.2 Filtration

In this subsection we review the classic filtration method[3] of proving the FMP. More precisely we consider a particular kind of filtration called the largest filtration. In the following we assume Gentzen-style formulation of logics. First we need to construct a Kripke model to filtrate; We introduce a *canonical model* of a given modal logic.

Definition 34 Let Φ be the set of all formulae of a logic L and $U, V \subseteq \Phi$. A pair $\langle U, V\rangle$ is consistent in L if there exist no formulae $A_1, \ldots, A_m \in U$ and $B_1, \ldots, B_n \in V$ such that $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is provable in L. Furthermore $\langle U, V\rangle$ is *maximally consistent* if it is consistent and $U \cup V = \Phi$.

Every consistent pair is can be extended to maximally consistent one:

Lemma 35 *If $\langle U, V\rangle$ is consistent, then there exists a maximally consistent $\langle U', V'\rangle$ such that $U \subseteq U'$ and $V \subseteq V'$.*

Now let us define a *canonical* Kripke model $M_L = \langle W_L, R_L, \models_L\rangle$ as follows[4].

$$\begin{aligned} W_L &= \{U \subseteq \Phi | \langle U, \Phi - U\rangle \text{ is maximally consistent in } L\}. \\ UR_LV &\Leftrightarrow U_\Box \subseteq V. \\ U \models_L p &\Leftrightarrow p \in U \text{ for each propositional variable } p. \end{aligned}$$

Here $U_\Box = \{A \in \Phi : \Box A \in U\}$. Note that W_L may be infinite in general.

A maximally consistent set is the syntactic counterpart of ultrafilter. The following will be reminiscent of the definition of an ultrafilter.

Lemma 36 *For any $U \in W_L$, the following holds.*

1. *If $A_1, \ldots, A_m \in U$ and $A_1, \ldots, A_m \Rightarrow B$ is provable in L, then $B \in U$*
2. *For any formula A, exactly one of A or $\neg A$ is in U.*

This lemma propagates in a way over logical connectives. We state this for later comparison.

Corollary 37 *For any $U \in W_L$, we have the following.*

1. $A \wedge B \in U \Leftrightarrow A \in U$ *and* $B \in U$;
2. $A \vee B \in U \Leftrightarrow A \in U$ *or* $B \in U$;
3. $A \to B \in U \Leftrightarrow A \notin U$ *or* $B \in U$;
4. $\neg A \in U \Leftrightarrow A \notin U$;
5. $\Box A \in U \Leftrightarrow$ *for any* $V \in W_L$ *with* URV, $A \in V$.

[3]Historical information on filtration (and canonical models) can be found in pp.159f of [7].
[4]Here we identify the valuation and its extension.

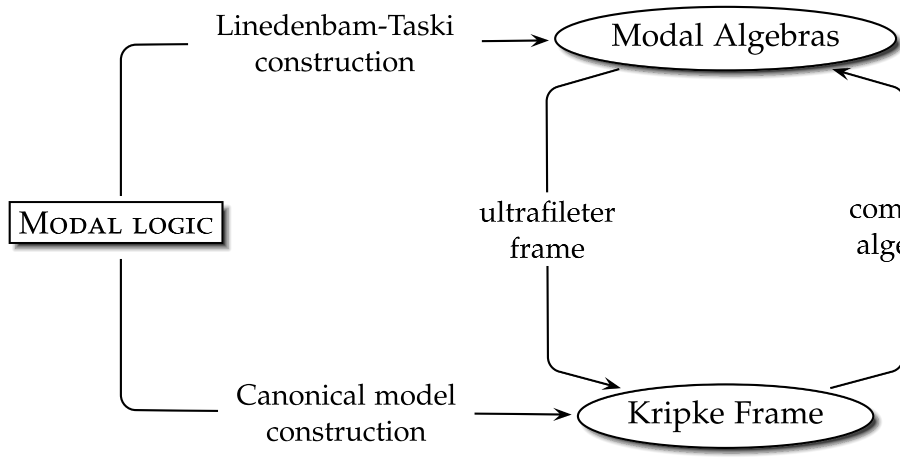

Figure 3. Two kinds of semantics

An important property of a canonical model is that the construction gives a *right* model[5]. For instance, a canonical model of KT is a reflexive Kripke model. Another feature is the following holds in a canonical model:

$$\text{For any } U \in W_L, U \models_L A \Leftrightarrow A \in U$$

Using these, we can prove the completeness of modal logics with respect to Kripke frames (called Kripke completeness). We say that a sequent $X_1, \ldots, X_m \Rightarrow Y_1, \ldots, Y_n$ is valid in a frame if $X_1 \wedge \cdots \wedge X_m \to Y_1 \vee \cdots \vee Y_n$[6] is valid in the frame.

Theorem 38 *A sequent $\Gamma \Rightarrow \Delta$ is provable in K iff $\Gamma \Rightarrow \Delta$ is valid in all Kripke frames.*

Proof. Let $\Gamma \Rightarrow \Delta$ is not provable in K. Then $\langle \Gamma, \Delta \rangle$ is consistent and hence can be extended to a maximally consistent pair $\langle U, V \rangle$. Suppose $\Gamma = X_1, \ldots, X_m$ and $\Delta = Y_1, \ldots, Y_n$. By the property of canonical model,

$$U \models_L X_i \text{ and } U \not\models_L Y_j, 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n,$$

since X_i's are in U and Y_j's are not in U. Thus we found a countermodel for $\Gamma \Rightarrow \Delta$, and hence it is not valid in all Kripke frames. Usual induction proves the other direction, i.e., soundness. ❑

A little digression. As we have just seen, canonical model construction gives a uniform way to prove Kripke completeness. On the other hand, we have *Lindenbaum-Tarski construction* to prove the completeness with respect to algebraic semantics. Moreover, we can go back and forth between Kripke frames and modal algebras as described in an earlier section. These facts can be summarized as in 3.

Now we turn to filtration. The idea is to regard two worlds as indistinguishable if the formulae in $\Psi(A)$ which hold there are the same. This is called a filtration (via $\Psi(A)$) of a Kripke model. Filtration can be defined on any Kripke model, but we concentrate on the filtration of a canonical model.

[5]This is the case at least when we consider normal modal logics. See section 4.2 of [3].
[6]If $n = 0$, then we consider $\neg \bigwedge X_i$.

Let A be a formula, $\Psi(A)$ the set of all subformulae of A. Define $\sim$ an equivalence relation on W_L as

$$U \sim V \Leftrightarrow \text{ for any formula } C \in \Psi(A), U \models_L C \text{ iff } V \models_L C.$$

We build a Kripke model on $W_L/\sim$, the set of all the equivalence classes of $\sim$. We write $[U]$ for the equivalence class of U. Define a relation R^+ as

$$[U]R^+[V] \Leftrightarrow \text{ for any } \Box C \in \Psi(A), U \models_L \Box C \text{ implies } V \models_L C.$$

This is called the largest filtration of R. Further define a valuation $\models^+$ as

$$[U] \models^+ p \Leftrightarrow U \models_L p \text{ for any } p \in \Psi(A)$$

Important to note is that $W_L/\sim$ is finite because $\Psi(A)$ is finite.

We can show by induction that for any wff α

$$[a] \models^+ \alpha \Leftrightarrow a \models_L \alpha.$$

This provides us the FMP of K.

Theorem 39 *A sequent $\Gamma \Rightarrow \Delta$ is provable in K iff it is valid in all finite Kripke frames. Namely, K has the FMP.*

PROOF. If α is not provable in K, then it is false in the canonical model and so in the filtration of it. The other direction follows from completeness. ❏

In a similar fashion we can prove completeness and the FMP of the other modal logics we have introduced.

3.2.3 Schütte's method

In [36], Schütte introduced a method of proving the FMP of intuitionistic logic (see chapter 4). In this subsection we introduce the version of the method accommodated to modal logic by Sato in his dissertation[35]. As we shall see later, almost the same argument as above method works for the FMP of intuitionistic logic: construct a canonical model and then filtrate it. While this usual path takes two steps, Schütte's method directly constructs a finite model of a given formula in a single step (see figure 4).

The method is a modification of completeness proof using canonical model. There one defined "maximal consistency" over the set of all formulae. On the other hand, given a wff A for which we want a falsifying finite model, Schütte's method defines "maximal $\Psi(A)$-consistency" over the set of all subformulae $\Psi(A)$ of A. To put it another way, canonical model construction uses "global" maximal consistency while Schütte's method uses "local" maximal consistency.

Now for the details.

Definition 40 Let $\Psi(A)$ be the set of all subformulae of A. A pair $\langle U, V\rangle$ is *maximally* $\Psi(A)$-*consistent* in L if it is consistent and $U \cup V = \Psi(A)$.

We can get maximal $\Psi(A)$-consistency from immature pair as maximal consistency:

Lemma 41 *Any consistent $\langle U, V\rangle$, where $U, V \subseteq \Psi(A)$ can be extended to a maximally $\Psi(A)$-consistent $\Psi U', V'$ with $U \subseteq U'$ and $V \subseteq V'$.*

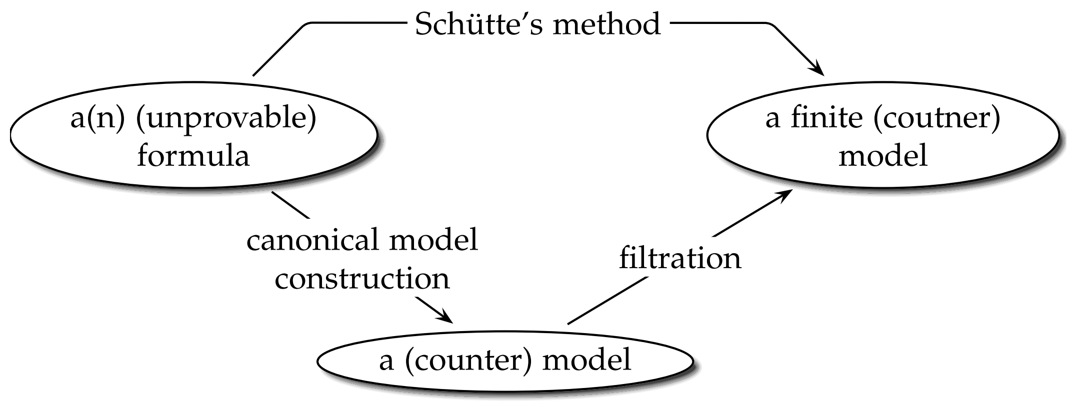

FIGURE 4. Canonical way or a shortcut

Then we define a Kripke model as follows. Note that W_0 is finite because $\Psi(A)$ is finite.

$$\begin{array}{rcl} W_0 &=& \{U \subseteq \Psi(A) | \langle U, \Psi(A) - U\rangle \text{ is maximally } \Psi(A)\text{-consistent in } L\}. \\ UR_0V &\Leftrightarrow& U_\Box \subseteq V. \\ U \models_0 p &\Leftrightarrow& p \in U \text{ for each propositional variable } p. \end{array}$$

Then the following result holds in parallel with corollary 37. Note the restriction of "if $\ldots \in \Psi(A)$".

Lemma 42 *For any $U \in W_0$, we have the following.*

1. *if $A \wedge B \in \Psi(A)$, $A \wedge B \in U \Leftrightarrow A \in U$ and $B \in U$;*
2. *if $A \vee B \in \Psi(A)$, $A \vee B \in U \Leftrightarrow A \in U$ or $B \in U$;*
3. *if $A \to B \in \Psi(A)$, $A \to B \in U \Leftrightarrow A \notin U$ or $B \in U$;*
4. *if $\neg A \in \Psi(A)$, $\neg A \in U \Leftrightarrow A \notin U$;*
5. *if $\Box A \in \Psi(A)$, $\Box A \in U \Leftrightarrow$ for any $V \in W_L$ with URV, $A \in V$.*

Although we omit the proof again, this lemma enables us to prove, for any $\alpha \in \Psi(A)$ and any $U \in W_0$,

$$U \models_0 \alpha \Leftrightarrow \alpha \in U.$$

Now we can give another proof of the FMP by mimicking the completeness proof. Suppose we have a sequent $\Gamma \Rightarrow \Delta$ which is not provable in K. Using $\bigwedge_i X_i \to \bigvee_j Y_j$ as A, we follow the above recipe and obtain the Kripke model as described. This model is a finite counter model for $\Gamma \Rightarrow \Delta$.

The advantage of Schütte's method is of course its immediateness: we get completeness and the FMP in a single step. As mentioned earlier, there is a close connection between this method and the usual division of labor of canonical model and filtration. In building canonical models, we are broad-minded and consider consistent sets over all formula of the logic. In Schütte's method, we become a little more thrifty and consider consistent sets over $\Psi(A)$, which is all and only formulae relevant to the truth value of A. This thrift allows us to obtain a finite model at once. Everything has its defects, however. In canonical model, we have R_L once and for all. But if we want to apply Schütte's method to other modal logics, we have to tweak the definition of the binary relation R_+, reflecting the property (or the properties) of R in some way or other. The following tweak is known to go well (see [28]). The right-hand side is the redefined R_+:

K, KT $U_\Box \subseteq V$;

$S4$ $U_\Box \subseteq V_\Box$;

KTB $U_\Box \subseteq V$ and $V_\Box \subseteq U$;

$S5$ $U_\Box = V_\Box$.

Compare corollary 37 and lemma 42 again and recall the restriction of "if $\ldots \in \Psi(A)$". Actually this is the cause of the above disadvantage. In later sections we show that "algebraizing" Schütte's method gives the proof of the FEP. There this "if..." restriction also causes irritating limitations to our method.

3.3 The FEP

3.3.1 Basic

Let $\mathcal{K}$ be the class of normal modal algebras, $\mathbf{A} \in \mathcal{K}$, and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$ which contains 0 and 1.

Definition 43 Let U, V subsets of A. The pair $\langle U, V\rangle$ is *overflowing* in $\mathbf{A}$ iff there exist $u_1, \ldots, u_m \in U$ and $v_1, \ldots, v_n \in V$ such that

$$u_1 \wedge \cdots \wedge u_m \leqslant v_1 \vee \cdots \vee v_n.$$

Otherwise $\langle U, V\rangle$ is *saturable* in $\mathbf{A}$. Furthermore we say that $\langle U, V\rangle$ is B-*saturated* when $\langle U, V\rangle$ is saturable and $U \cup V = B$, where $B \subseteq A$.

Lemma 44 *Any saturable pair $\langle U_0, V_0\rangle$ with $U_0, V_0 \subseteq B$ can be extended to the* B-saturated *pair $\langle U, V\rangle$ such that $U_0 \subseteq U$ and $V_0 \subseteq V$.*

Proof. Let $B = \{b_1, \ldots, b_k\}$. We construct a sequence $\langle U_m\rangle (0 \leqslant m \leqslant k)$ of subsets of B such that $U_0 \subseteq U_m$ for each m. Given a saturable pair $\langle U_m, V_m\rangle$, we define U_{m+1} as follows. First observe that either $\langle U_m, V_m \cup \{b_{m+1}\}\rangle$ or $\langle U_m \cup \{b_{m+1}\}, V_m\rangle$ is saturable; for suppose otherwise, i.e., there exist $x_1, \ldots, x_p, y_1, \ldots, y_q \in U_m$ and $z_1, \ldots, z_r, w_1, \ldots, w_s \in V_m$ for which the following hold.

$$\begin{aligned} x_1 \wedge \cdots \wedge x_p &\leqslant z_1 \vee \cdots \vee z_m \vee b_{m+1} \\ b_{m+1} \wedge y_1 \wedge \cdots \wedge y_q &\leqslant w_1 \vee \cdots \vee w_s \end{aligned}$$

Here we can assume b_{m+1} appears in the above inequalities because $z_1 \vee \cdots \vee z_m \leqslant$

$z_1 \vee \cdots \vee z_m \vee b_{m+1}$ and $b_{m+1} \wedge y_1 \wedge \cdots \wedge y_q \leqslant y_1 \wedge \cdots \wedge y_q$. Then we have:

$$\begin{aligned} x_1 \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge y_q &\leqslant (z_1 \vee \cdots \vee z_r \vee b_{m+1}) \wedge (y_1 \wedge \cdots \wedge y_q) \\ &= ((z_1 \vee \cdots \vee z_r) \wedge (y_1 \wedge \cdots \wedge y_q)) \\ &\qquad \vee (b_{m+1} \wedge y_1 \cdots \wedge y_q) \\ &\leqslant ((z_1 \vee \cdots \vee z_r) \wedge (y_1 \wedge \cdots \wedge y_q)) \\ &\qquad \vee (w_1 \cdots \wedge w_s) \\ &\leqslant (z_1 \vee \cdots \vee z_r) \vee (w_1 \vee \cdots \vee w_s), \end{aligned}$$

which contradicts the assumption that $\langle U_m, V_m \rangle$ is saturable.

Now we define $U_{m+1} = U_m$ and $V_{m+1} = V_m \cup \{b_{m+1}\}$ when $\langle U_m, V_m \cup \{b_{m+1}\} \rangle$ is saturable. Otherwise we define $U_{m+1} = U_m \cup \{b_{m+1}\}$ and $V_{m+1} = V_m$. Then put $U = U_k$ and $V = V_k$. The pair $\langle U, V \rangle$ satisfies the required condition and we are done. ❏

We sometimes say U is B-saturated if $\langle U, B - U \rangle$ is B-saturated. From here on we say simply saturated sets for the sake of brevity. Note this is reasonable because A-saturated set is just an ultrafilter in $\mathbf{A}$.

We next consider the set M_B of all B-saturated sets. That is, we define

$$M_B = \{U \subseteq B | \langle U, B - U \rangle \text{ is } B\text{-saturated}\}$$

Here we always assume that $0, 1 \in B$. Note $0 \notin U$ and $1 \in U$ for any $U \in M_B$[7]. First we state some properties of saturated sets. Resemblances to lemma 36, corollary 37, and lemma 42 is obvious.

Lemma 45 *The following hold for any $U \in M_B$.*

1. *If $a_1, \ldots, a_n \in U$, $b \in B$, and $a_1 \wedge \cdots \wedge a_n \leqslant b$, then $b \in U$.*
2. *Exactly one of a and a' is in U whenever $a, a' \in B$.*

Proof. Let $U \in M_B$. Suppose to the contrary that we have $a_i \in U$ for each i, $a_1 \wedge \cdots \wedge a_n \leqslant b$, and $b \notin U$. Then $b \in B - U$. Since we have $a_1 \wedge \cdots \wedge a_n \leqslant b$, $\langle U, B - U \rangle$ is overflowing, contradiction. Thus $b \in U$.

For the latter part, first assume we have $a, a' \in U$. Then U is overflowing since $a \wedge a' \leqslant 0$. Next assume $a, a' \notin U$, i.e., $a, a' \in B - U$. Since $1 \leqslant a \vee a'$ U is overflowing. ❏

Again we have the following propagation. The notation $U_\Box$ is the same as before, i.e., $U_\Box = \{a \in B | \Box a \in U\}$

Corollary 46 *For any $U \in M_B$ and $a, b \in B$, the following hold.*

1. *if $a \wedge b \in B$, $a \wedge b \in U$ iff $a \in U$ and $b \in U$.*
2. *if $a \vee b \in B$, $a \vee b \in U$ iff $a \in U$ or $b \in U$.*
3. *if $a' \in B$, $a' \in U$ iff $a \notin U$.*
4. *if $\Box a \in B$, $\Box a \in U$ iff for any $V \in M_B$, URV implies $a \in V$, where the relation URV on M_B is defined to be $U_\Box \subseteq V$.*

[7]We assume a pair is overflowing if either of the component is empty.

Proof.

1. (⇒)Let $a \wedge b \in U$. Since $a \wedge b \leqslant a, b$, $a, b \in U$ by the previous lemma. (⇐)Suppose $a \wedge b \notin U$. Then $a \wedge b \in B - U$. If a and b are both in U, then U is overflowing since $a \wedge b \leqslant a \wedge b$. Contradiction.

2. (⇒)Suppose $a \vee b \in U$ and $a, b \notin U$. Then $a, b \in B - U$, but this contradicts the consistency of U because $a \vee b \leqslant a \vee b$. Hence $a \in U$ or $b \in U$. (⇐) Let $a \in U$ without loss of generality. Since $a \leqslant a \vee b$, $a \vee b \in U$ by the above lemma.

3. Obvious from the previous lemma.

4. (⇒) Let $\Box a \in U$, which means $a \in U_\Box$. Then $a \in V$ for any V with URV by the definition of R. (⇐)Assume $\Box a \notin U$. Then $\langle U_\Box, \{a\}\rangle$ is saturable; otherwise there exist $b_1, \ldots, b_n \in U_\Box$ such that
$$b_1 \wedge \cdots \wedge b_k \leqslant a,$$
which in turn implies
$$\Box b_1 \wedge \cdots \wedge \Box b_k \leqslant \Box a.$$
This implies $\Box a \in U$ by the previous lemma, contradicting our assumption. So we can extend $\langle U_\Box, \{a\}\rangle$ to $\langle U', V'\rangle$ such that $U' \in M_B$, $U_\Box \subseteq U'$, and $a \in V'$. Thus we have U' with $a \notin U'$ and URU'.

❏

We embed **B** into the power set algebra $\mathcal{P}(M_B)$ with the modal operator defined as

$$\Box X = \{U \in M_B \mid \text{for any } V \in M_B,\ URV \text{ implies } V \in X\}.$$

It is easy to see that $\Box$ is normal on $\mathcal{P}(M_B)$. Note that M_B is finite and so is $\mathcal{P}(M_B)$.
Let h be the map from **B** to $\mathcal{P}(M_B)$ which sends b to $\{U \mid b \in U \in M_B\}$.

Theorem 47 *The mapping h is an embedding of **B** into $\langle \mathcal{P}(M_B), \Box\rangle$.*

Proof. First we show h is injective. Let a and b be two distinct elements of B. Then either $a \nleqslant b$ or $b \nleqslant a$. Without loss of generality, we assume $a \nleqslant b$, so that $\langle\{a\}, \{b\}\rangle$ is saturable and so can be extended to $\langle U, V\rangle$. Clearly $a \in U$ and $b \notin U$, i.e., $U \in h(a)$ and $U \notin h(b)$. Hence $h(a) \neq h(b)$ in either case.

Next we prove that h preserves all existing operations. By the previous remark $h(0) = \emptyset$ and $h(1) = M_B$. In what follows note that we only consider the operations defined in **B**.

For meet:

$$\begin{aligned} U \in h(a \wedge b) &\Leftrightarrow a \wedge b \in U \\ &\Leftrightarrow a \in U \text{ and } b \in U \text{ (by the corollary)} \\ &\Leftrightarrow U \in h(a) \text{ and } U \in h(b) \\ &\Leftrightarrow U \in h(a) \cap h(b) \end{aligned}$$

Thus we have $h(a \wedge b) = h(a) \cap h(b)$.

For join:

$$\begin{aligned} U \in h(a \vee b) &\Leftrightarrow a \vee b \in U \\ &\Leftrightarrow a \in U \text{ or } b \in U \text{ (by the corollary)} \\ &\Leftrightarrow U \in h(a) \text{ or } U \in (b) \\ &\Leftrightarrow U \in h(a) \cup h(b) \end{aligned}$$

For complement:

$$\begin{aligned} U \in h(a') &\Leftrightarrow a' \in U \\ &\Leftrightarrow a \notin U \text{ (by the corollary)} \\ &\Leftrightarrow U \notin h(a_i) \\ &\Leftrightarrow U \in h(a_i)^c \text{ (}^c \text{ is set complement)} \end{aligned}$$

For box:

$$\begin{aligned} U \in h(\Box a) &\Leftrightarrow \Box a \in U \\ &\Leftrightarrow \text{ for all } V \in M_B, URV \Rightarrow a \in V \text{ (by the corollary)} \\ &\Leftrightarrow \text{ for all } V \in M_B, URV \Rightarrow V \in h(a) \\ &\Leftrightarrow U \in \Box h(a) \end{aligned}$$

❏

Hence we have:

Theorem 48 *The class of normal modal algebras has the finite embeddability property.*

In view of proposition 26, we have the following decidability result.

Corollary 49 *The universal theory of normal modal algebras is decidable.*

3.3.2 More modal algebras

Our proof above extends to some other modal algebras. There we first construct the set M_B of all saturated sets of a given finite partial algebra B, and then embed B into the powerset algebra of M_B. Our fundamental difference from Jónsson-Tarski's theorem is that we use saturated sets in place of ultrafilters while *the construction from frames to algebras is the same* (see also next section). The latter means that the powerset algebra is of the appropriate kind *once we get the appropriate frame*. Note the property of a saturated set stated in corollary 46 is crucial in the proof of embedding. Thus the essence of our proof is

1. the set of saturated sets endowed with R comprises an *appropriate* frame;

2. Corollary 46 holds for R under discussion.

Hence our proofs below only show that R satisfies the necessary property (cf. proposition 18) if we start from a different modal algebra. As noted earlier, the restriction of "if..." in corollary 46 induces some difficulty in fully applying the method to other modal algebras.

The above covers the cases of KT and $K4 + \Box^n x \to \Box x$, besides K, with appropriate closure under $\Box$.

Lemma 50 *If* B *is a* $KT(K4 + \Box^n x \to \Box x)$*-algebra, then the constructed algebra is* $KT(K4 + \Box^n x \to \Box x)$*-algebra.*

Proof. We show that the relation R satisfies the desired property in each case.

KT We need the reflexivity of R, i.e., $U_\Box \subseteq U$. If $a \in U_\Box$, then $\Box a \in U$. Since $\Box a \leqslant a$ by T, $a \in U$ by lemma 45. Thus $U_\Box \subseteq U$.

$K4 + \Box^n x \to \Box x$ First of all, we must close B up to n Boxes. Namely, we put $\Box^n a$ whenever $a \in B$ to get B', and then close B_1 again, and so on. Note this procedure stops in finitely many steps because with the axioms under consideration, $\Box^m a = \Box^{m+1} a$ for any $m \in \mathbb{N}$. So we start "closed" version of B, and construct the frame as in the previous section. What we want is to prove that[8] when we have URV, that is, $U_\Box \subseteq V$, we also have V_i's such that $U_\Box \subseteq V_1, V_{1\Box} \subseteq V_2, \cdots, V_{n-1\Box} \subseteq V_n = V$. Since $U_\Box \subseteq U_{\Box\Box}$ by 4, we have a sequence of inclusions, $U_\Box \subseteq U_{\Box^2}, (U_{\Box^2})_\Box \subseteq U_{\Box^3}, \cdots, (U_{\Box^{n-2}})_\Box \subseteq U_{\Box^{n-1}}, (U_{\Box^{n-1}})_\Box \subseteq U_\Box$. The last inclusion holds due to $\Box^n x \to \Box x$. Appending $U_\Box \subseteq V$ onto the above sequence shows UR^n

❏

From lemma 50 we have:

Corollary 51 *The varieties corresponding to* KT *and* $K4 + \Box^n x \to \Box x$ *have the FEP. Hence the universal theories of them are decidable.*

Tweaks on the definition of R give the FEP for some other modal algebras. When the definition of R is changed, all that matters is whether or not corollary 46 holds by the argument above. We state the definition of R with the proof that corollary gets along with the change of R. Again we need the closure under (appropriate number of) $\Box$ and/or $\neg$.
KT4(S4) Define $URV \Leftrightarrow U_\Box \subseteq V_\Box$. Then $\langle \mathcal{P}(M_B), \Box \rangle$ satisfies $\Box X \subseteq \Box\Box X$. And we need

$$\Box a \in U \text{ iff for any } V \in M_B, U_\Box \subseteq V_\Box \Rightarrow a \in V.$$

($\Rightarrow$) Assume $\Box a \in U$ and $U_\Box \subseteq V_\Box$. Then $a \in U_\Box$, so that $a \in V_\Box$. By T, $a \in V$.

($\Leftarrow$) We get U' such that $\Box a \notin U$ and $U_\Box$ as in the proof of the corollary. We want URU', i.e., $U_\Box \subseteq U'_\Box$. Suppose $a \in U_\Box$. Then $\Box a \in U$, which implies $\Box\Box a \in U$. This is equivalent to $\Box a \in U_\Box$. Since $U_\Box \subseteq U'$, we have $\Box a \in U'$, and finally $a \in U'_\Box$.
KTB Let $URV \Leftrightarrow U_\Box \subseteq V \& V_\Box \subseteq U$. For the proof of the corollary, the forward implication is obvious. We assume $\Box a \notin U$ and take U' as above. Since we have $U_\Box \subseteq U'$, we prove $U'_\Box \subseteq U$.

$$\begin{aligned} a \in U'_\Box &\Rightarrow \Box a \in U' \\ &\Rightarrow \neg\Box a \notin U' \\ &\Rightarrow \neg\Box a \notin U_\Box (U_\Box \subseteq U') \\ &\Rightarrow \Box\neg\Box a \notin U \\ &\Rightarrow \neg\Box\neg\Box a \in U \\ &\Rightarrow a \in U(B). \end{aligned}$$

KT5(S5) Let $URV \Leftrightarrow U_\Box = V_\Box$. The forward implication of the corollary uses T and is easy. For the converse, we take U' again and prove $U_\Box = U'_\Box$. First, we have $U_\Box \subseteq U'_\Box$ because

[8] cf. proposition 19.

$a \in U_\Box \Leftrightarrow \Box a \in U \Rightarrow \Box\Box a \in U \Leftrightarrow \Box a \in U_\Box \Rightarrow \Box a \in U' \Rightarrow a \in U'_\Box$. Second, we have a chain of implications as follows:

$$\begin{aligned} a \in U'_\Box &\Rightarrow \Box a \in U' \\ &\Rightarrow \neg\Box a \notin U' \\ &\Rightarrow \Box\neg\Box a \notin U'(T) \\ &\Rightarrow \neg\Box a \notin U'_\Box \\ &\Rightarrow \neg\Box a \notin U_\Box (U_\Box \subseteq U'_\Box) \\ &\Rightarrow \Box\neg\Box a \notin U_\Box(T) \\ &\Rightarrow \neg\Box\neg\Box a \in U_\Box \\ &\Rightarrow \Box a \in U_\Box(5) \\ &\Rightarrow a \in U_\Box(T). \end{aligned}$$

Thus $U'_\Box \subseteq U_\Box$.

These considerations leads to the following result.

Theorem 52 *The varieties corresponding to* S4, KTB, *and* S5 *have the FEP. Hence the universal theories of them are decidable.*

3.4 Informal discussion

3.4.1 Proof idea

We remark on some ideas behind our proof of the FEP above though they are scattered around before. As mentioned in the introduction, our proof idea is hybridizing Schütte's method and Jónsson-Taski's theorem. We shall be explicit about this. As discussed earlier, a similar notion emerges in various disguise, algebraic and syntactic, or global and local. It is something that we have added members as far as possible in a sense. They can be summarized as in table 1. The only novel notion is our *saturated set*. A maximally consistent set in canonical model construction is a syntactic analog of ultrafilter, or ultrafilter is an algebraic analog thereof. Then we have remarked that maximally $\Psi(A)$-consistent set used in Schütte's proof is a "localized" version of maximally consistent set. Now it should be clear is that we devised a notion to fill the gap of the table with *saturated set*, an algebraic analog of maximally $\Psi(A)$-consistent set.

	algebraic	**syntactic**
global	ultrafilter	maximally consistent set
local	saturated set	maximally $\Psi(A)$-consistent set

Table 1. Notions of "fat" sets

With these in mind, our proof of the FEP can be depicted as in the figure 5. The upper path is trodden by Jónsson-Tarski's theorem. Our path is the lower one. By considering saturated sets, "local" ultrafilter, we have bound the constructed frame by finite in size. This is the crucial point and the only difference from Jónsson-Tarski's theorem. *We use saturated set instead of ultrafilter*. Then from frames to algebras we travel in the usual way, i.e., complex algebra. Also compare the figure 4.

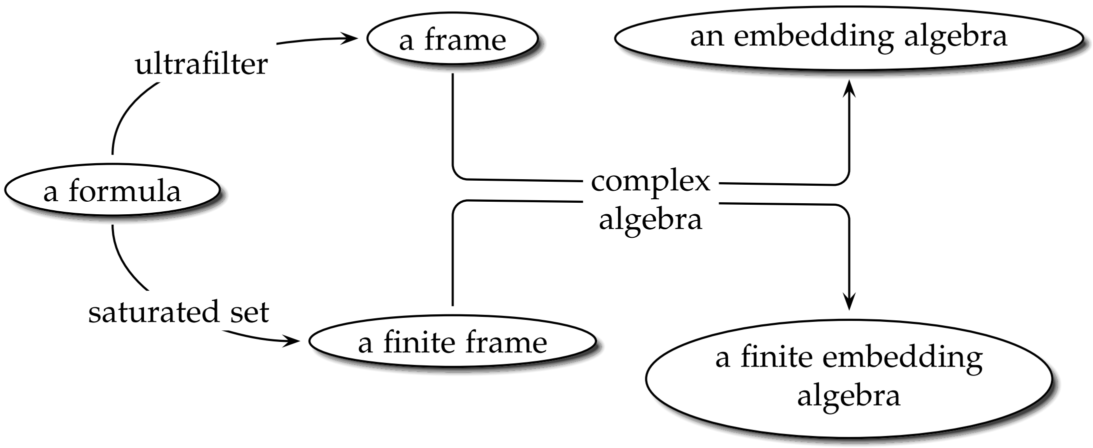

FIGURE 5. Proof idea

To repeat, saturated set is algebraic translation of the notion in Schütte's method. We use it instead of ultrafilters in Jónsson-Tarski's theorem. Then we get the FEP. This makes it clear what is meant by our proof being the hybrid of Schütte's method and Jónsson-Tarski's theorem. Thus the critical observation below:

> If the FMP of a logic can be proved via Schütte's method, then the FEP of the corresponding class of algebras can be proved.

However, it is fairly obscure what it means or when it is possible to use Schütte's method in proof of the FMP. Below we add some remarks on the relation between the largest filtration[9] and Schütte's proof.

3.4.2 A connection

We discuss the relation between the two constructed finite models: the largest filtration of the canonical model and the model constructed by Schütte's method. We present in two ways, frame-theoretic and algebraic, again and are quite repetitive. First we treat the frame-theoretic case.

In the definition of the largest filtration observe that

$$\begin{aligned} U \sim V &\Leftrightarrow \text{ for any formula } C \in \Psi(A), U \models_L C \text{ iff } V \models_L C \\ &\Leftrightarrow \text{ for any formula } C \in \Psi(A), C \in U \text{ iff } C \in V \\ &\Leftrightarrow \Psi(A) \cap U = \Psi(A) \cap V. \end{aligned}$$

and that

$$\begin{aligned} [U]R^+[V] &\Leftrightarrow \text{ for any } \Box C \in \Psi(A), U \models_L \Box C \text{ implies } V \models_L C \\ &\Leftrightarrow \text{ for any } \Box C \in \Psi(A), \Box C \in U \text{ implies } C \in V \\ &\Leftrightarrow (\Psi(A) \cap U)_\Box \subseteq (\Psi(A) \cap V). \end{aligned}$$

These mean that the largest filtration can be taken as we are in fact restricting our attention to $\Psi(A)$. This is exactly what we do in Schütte's method. We prove the following correspondence between $M^+ = \langle W_L/\sim, R^+, \models^+\rangle$ and $M_0 = \langle W_0, R_0, \models_0\rangle$.

[9] This is the filtration we introduced. For other kinds of filtration, consult e.g. [7].

Lemma 53

1. *if* $U \in W_L$*, then* $\Psi(A) \cap U \in W_0$.

2. *if* $V_0 \in W_0$*, then there exists* $V \in W_L$ *with* $V_0 = \Psi(A) \cap V$.

Proof.

1. Let $U \in W_L$ and suppose to the contrary that $\Psi(A) \cap U$ is inconsistent[10]. Then there are $A_1, \dots, A_k \in \Psi(A) \cap U$ and $B_1, \dots, B_l \in \Psi(A) \cap U^c$ such that $A_1, \dots, A_k \to B_1, \dots, B_l$ is provable. Since A_i's and B_j's are in U and U^c respectively, this contradicts the consistency of U.

2. Let $V_0 \in W_0$. Regarding V_0 as a subset of Φ, we can extend $\langle V_0, \Psi(A) - V_0 \rangle$ to $\langle V, V^c \rangle$. Clearly $V_0 \subseteq \Psi(A) \cap V$. Conversely, suppose to the contrary that $B \in \Psi(A) \cap V$ and $B \notin V_0$. Then B must be in $\Psi(A) - V_0$. Since $\Psi(A) - V_0 \subset V^c$, $B \in V^c$, so that $B \notin V$, which gives the desired contradiction.

❏

We now show that M^+ and M_0 are "isomorphic", with which we mean the existence of a p-morphism that is 1-1 and onto. Define a function θ from $W/{\sim}$ to W_0 with $\theta([U]) = \Psi(A) \cap U$. By the remark above, $[U]R^+[V]$ iff $(\Psi(A) \cap U)_\Box \subseteq (\Psi(A) \cap V)$, i.e., $\theta([U])R_0\theta([V])$. For the other condition, suppose that $\theta([U]) \subseteq T_0$. We want some $[V]$ with $\theta([V]) = T_0$. By the lemma 53 we have $T_0 = \Psi(A) \cap T$ for some $T \in W_L$. Then $\theta([T]) = T_0$ and $[U]R^+[T]$ using the remark above again. By lemma 53 θ is clearly onto. To prove the injectivity of the function, suppose $[U] \neq [V]$. Then there exists $C \in \Psi(A)$ such that $U \models C$ but $V \not\models C$, that is, $C \in (U - V) \cap \Psi(A) = (U \cap \Psi(A)) - (V \cap \Psi(A))$, whence C witnesses the difference of $\theta([U])$ and $\theta([V])$.

Next we turn to the world of algebras. An algebraic version of the lemma 53 is a straightforward translation:

Lemma 54

1. *If* U *is* A*-saturated, then* $B \cap U$ *is* B*-saturated.*

2. *If* U_0 *is* B*-saturated, then* $U_0 = U \cap B$ *for some* $U \in M_A$.

Proof. Just changing the terminology provides the proof.

1. If $\langle U, U^c \rangle$ is A-saturated and $\langle B \cap U, B \cap U^c \rangle$ is overflowing, there exist $u_1, \dots, u_k \in B \cap U$ and $c_1, \dots, c_m \in B \cap U^c$, for which
$$b_1 \wedge \cdots \wedge b_k \leqslant c_1 \wedge \cdots \wedge c_m.$$
Since b_i's and c_j's are in U and U^c, respectively, the above inequality contradicts the saturability of U.

2. Given B-saturated U_0, extend $\langle U_0, B - U_0 \rangle$ to A-saturated pair $\langle U, U^c$. We want $U_0 = U \cap B$. We have the forward inclusion by construction. To prove the converse, let $b \in U \cap B$ and suppose $b \notin U_0$. Then $b \in B - U_0 \subseteq U^c$, so that $b \notin U$. Contradiction.

❏

[10] $\langle \Psi(A) \cap U, \Psi(A) \cap U^c \rangle$ is inconsistent.

These lemmas seem to suggest the following[11]:

> If the largest filtration provides the FMP of a logic, then Schütte's method is applicable to the logic.

This seems to partially answer *when it is possible* to use Schütte's method. In employing the largest filtration, we essentially take back our generosity (of considering "global" maximally consistent sets) and localizing each maximally consistent set. If this is possible, then we can be thrift from the beginning and consider only maximally $\Psi(A)$-consistent sets, as suggested by the above lemmas.

Summing up, we may have the following series of implication.

> the largest filtration $\rightarrow$ Schütte's method $\rightarrow$ the FEP

But again, it is not at all clear about the applicability of the largest filtration. We need a more general and formal notion of *a logic admitting the largest filtration*. But for the time being, we have not fully investigated this direction, nor understood where to go. Some relevant information can be found pp.142ff of [7].

[11] The idea is suggested by Dr. Tadeusz Litak when the author presented the contents of this chapter at laboratory seminar.

Chapter 4

The FEP for Heyting algebras

The FEP for some intuitionistic algebras (including those enriched with a modality) is investigated in this chapter. Schütte's method was originally used for intuitionistic logic, so that the proof seems to work naturally for Heyting algebras. By our argument in the previous chapter, algebraizing it might be a promising way to the FEP. In fact, it does give a proof of the FEP. For more details on intuitionistic modal logics, see [27] and [12].

4.1 Heyting algebra

First we consider intuitionistic algebras without modality, i.e., Heyting algebras. In this section we are to rephrase what was said about modal logics/algebras as appropriate for Heyting algebras. Things are straightforward and we shall be as taciturn as possible. As for the presentation of preliminary materials, we deserve to be accused of violating copyright of [7]. Let us start from answering what is Heyting algebra.

4.1.1 Definition

We summarize semantics for intuitionistic logic. Again we introduce Kripke-style and algebraic semantics. First let us discuss Kripke semantics. An partially ordered set $\langle M, \leqslant\rangle$ is a Kripke frame for intuitionistic logic, called *intuitionistic frames*. A valuation V is a map from the set of propositional variables to $\mathcal{U}(M)$, where $\mathcal{U}(M)$ is the set of all upward closed (with respect to $\leqslant$) subsets of M. The relation $\models$ is defined in the similar way as for modal logic with a little modification on $\to$ and $\neg$.

$$\begin{aligned}
a \models p &\quad\Leftrightarrow\quad a \in V(p),\ \ p \text{ is a propositional variable}\\
a \models \alpha \wedge \beta &\quad\Leftrightarrow\quad a \models \alpha \text{ and } a \models \beta\\
a \models \alpha \vee \beta &\quad\Leftrightarrow\quad a \models \alpha \text{ or } a \models \beta\\
a \models \alpha \to \beta &\quad\Leftrightarrow\quad \text{for any } b \text{ with } a \leqslant b, b \not\models \alpha \text{ or } b \models \beta\\
a \models \neg\alpha &\quad\Leftrightarrow\quad \text{for any } b \text{ with } a \leqslant b, b \not\models \alpha
\end{aligned}$$

Intuitionistic logic is Kripke-complete as might be expected. We prove it later along with the FMP.

Theorem 55 *A formula is provable in* LJ *iff it is valid in all intuitionistic frames.*

The algebraic counterpart is *Heyting algebra*. Note we do not include $'$ in our language. We write a' for $a \to 0$.

Definition 56 A sextuple $\langle H, \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra if

- $\langle H, \wedge, \vee\rangle$ is a bounded lattice having 0 as the least element and 1 as the greatest.
- $a \wedge c \leqslant b \Leftrightarrow c \leqslant a \to b$ holds for any $a, b, c \in H$.

Heyting algebra is a simplified version of residuated lattice[1]. The simplification is $\cdot$ coincides with $\wedge$. Notion of validity in algebras is defined as before[2]. Lindenbaum-Tarski construction gives completeness:

Theorem 57 *A formula* A *is provable in* LJ *iff it is valid in all Heyting algebras.*

4.1.2 Frames and algebras

We show a way from Heyting algebras to frames and the way back. The notion of filter is the same in Heyting algebra.

Definition 58 Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ be a Heyting algebra. A set $F \subseteq A$ is a filter in $\mathbf{A}$ if

1. $1 \in F$;
2. if $x, y \in F$, then $x \wedge y \in F$;
3. if $x \in F$ and $x \leqslant y$, then $y \in F$.

We introduce three kinds of filter. A filter F in $\mathbf{A}$ is

- *prime* if it is proper and $x \vee y \in F$ implies $x \in F$ or $y \in F$;
- *maximal* if it is not contained in a proper filter in $\mathbf{A}$ other than itself;
- *ultrafilter* if for all $a \in A$, $a \in F$ or $a \to 0 \in F$.

In Boolean algebras, these three all coincide. In Heyting algebra, too, the first two notions coincide. Then we can travel from algebras to frames. Recall that we took all ultrafilters of a given algebra as the set of possible worlds. We here take all prime filters instead. Hence we may well call it *prime filter frame*[3].

Definition 59 Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ a finite Heyting algebra. Then $\langle \mathcal{F}(A), \subseteq\rangle$ is the *prime filter frame* $\mathbf{A}_+$ of $\mathbf{A}$, where $\mathcal{F}(A)$ is the set of all prime filters of $\mathbf{A}$.

Again we can come back to algebras. The difference from modal case is that we take the set of upward closed sets instead of powerset.

Definition 60 Let $F = \langle W, \leqslant\rangle$ be an intuitionistic frame. Define an algebra

$$F^+ = \langle \mathcal{H}(W), \cap, \cup, \to, \emptyset, W\rangle,$$

where $\mathcal{H}(W)$ is the set of upward closed (with respect to $\leqslant$) sets[4], $\cap$ and $\cup$ are the usual set-theoretic operations, and $\to$ is defined, for $X, Y \in \mathcal{H}(W)$, as

$$X \to Y = \{x \in W | \forall y (x \leqslant y \wedge y \in X \Rightarrow y \in Y)\}.$$

This algebra is called the *dual algebra* of F.

[1] In Heyting algebra, alias pseudo-Boolean algebra, the symbol $\to$ is also called *pseudo-complementation*.
[2] A slight change is that $\neg A$ as a formula is mapped to $v(A) \to 0$.
[3] This terminology is not standard.
[4] $\mathcal{H}$ is for *hereditary*.

Then we can prove a weaker version of Jónsson-Tarski's theorem for Heyting algebras. For proof, see theorem 7.30 of [7].

Theorem 61 *Let* $\mathbf{A}$ *be a finite Heyting algebra. The function* $r : A \to \mathcal{H}(\mathcal{F}(A))$ *that maps* a *to* $\{U \in \mathcal{F}(A) | a \in U\}$ *is an embedding from* $\mathbf{A}$ *into* $(\mathbf{A}_+)^+$.

4.1.3 Schütte's method, original version

Schütte's method was originally for intuitionistic logic in [36], as mentioned earlier. We briefly describe it following [28]. The argument is quite similar to that in modal logic.

Fix some formula A of intuitionistic logic and let $\Psi(A)$ be the set of all subformulae of A. We define $\langle U, V \rangle$is a maximally $\Psi(A)$-consistent in LJ if

$$A_1, \ldots, A_m \Rightarrow B_1 \vee \cdots \vee B_n$$

is not provable in LJ, where $U, V \subseteq \Psi(A)$, $U = \{A_1, \ldots, A_m\}$, and $V = \{B_1, \ldots, B_n\}$. Note that we have to juxtapose B_i's with $\vee$ between them because the right-hand side of intuitionistic sequent must be a singleton or empty. As remarked in the modal case, every consistent pair can be padded to maximally $\Psi(A)$-consistent one. This time we state formally:

Lemma 62 *If* $\langle U, V \rangle$ *is consistent, then a maximally* $\Psi(A)$*-consistent pair* $\langle U', V' \rangle$ *exists such that* $U \subseteq U'$ *and* $V \subseteq V'$.

Then parallel to the modal case, we define a intuitionistic model and have the following lemma.

$$\begin{array}{rcl} W_0 & = & \{U \subseteq \Psi(A) | \langle U, \Psi(A) - U \rangle \text{ is maximally } \Psi(A)\text{-consistent in LJ}\}. \\ UR_0V & \Leftrightarrow & U \subseteq V. \\ U \models_0 p & \Leftrightarrow & p \in U \text{ for each propositional variable } p. \end{array}$$

Lemma 63 *For* $U \in W_0$*, the following hold.*

1. *For* $A_i \in U$ *and* $B \in \Psi(A)$*, if* $A_1, \ldots, A_m \Rightarrow B$ *is provable in* LJ*, then* $B \in U$
2. *if* $A \wedge B \in \Psi(A)$, $A \wedge B \in U \Leftrightarrow A \in U$ *and* $B \in U$;
3. *if* $A \vee B \in \Psi(A)$, $A \vee B \in U \Leftrightarrow A \in U$ *or* $B \in U$;
4. *if* $A \to B \in \Psi(A)$, $A \to B \in U \Leftrightarrow$ *for all* $V \in W_0$ *with* URV, $A \notin U$ *or* $B \in U$;
5. *if* $\neg A \in \Psi(A)$, $\neg A \in U \Leftrightarrow$ *for any* $V \in W_0$ *with* URV, $A \notin U$;

This again entails, for any wff A,

$$U \models A \text{ iff } B \in U.$$

We give a proof of the FMP of intuitionistic logic with these.

Theorem 64 *A sequent* $\Gamma \Rightarrow D$ *is valid in all finite frames iff* $\Gamma \Rightarrow D$ *is provable in* LJ

Proof. Suppose $\Gamma \Rightarrow D$ is a sequent unprovable in LJ. Take $\bigwedge \Gamma \to D$ (or $\neg \bigwedge \Gamma$ if D is empty) as A in the above construction. Since A is not provable, so that $\langle \emptyset, \{A\} \rangle$ is consistent. Then lemma 62 provides $U \in W_0$ with $A \notin U$, i.e., $U \not\models A$. The constructed model falsify A. ❑

4.1.4 The FEP

A proof of the finite embeddability property of intuitionistic algebra due to McKinsey and Tarski[23, 22] is well-known[5]. Here we give another proof of the FEP for intuitionistic algebra.

Let B a finite partial subalgebra of an intuitionistic algebra A and M_B the set all saturated subsets of B as before. The following analog of the corollary 46 holds.

Lemma 65 *For any* $U \in M_B$,

1. *if* $a_1, \dots, a_m \in U$ *and* $b \in B$, $a_1 \wedge \cdots \wedge a_m \leqslant b$ *implies* $b \in U$.
2. *if* $a \wedge b \in B$, $a \wedge b \in U$ *iff* $a \in U$ *and* $b \in U$.
3. *if* $a \vee b \in B$, $a \vee b \in U$ *iff* $a \in U$ *or* $b \in U$.
4. *if* $a \to b \in B$, $a \to b \in U$ *iff for any* $V \in M_B$, $U \subseteq V$ *implies either* $a \notin V$ *or* $b \in V$.

This time we have $\to$ as primitive, the case of which is added. We provide proof only the case of $\to$.

Proof. ($\Rightarrow$) Let $a \to b \in U$. Suppose $U \subseteq V$ and $a \in V$. Then $b \in V$ because $a, a \to b \in V$ and $a \wedge a \to b \leqslant b$. Thus $a \notin V$ or $b \in V$.($\Leftarrow$) Assume $a \to b \notin U$. Then $\langle U \cup \{a\}, \{b\} \rangle$ is saturable. For if not, there exist $a_i \in U$ such that

$$a \wedge a_1 \wedge \cdots \wedge a_k \leqslant b,$$

which implies

$$a_1 \wedge \cdots \wedge a_k \leqslant a \to b.$$

But this implies $a \to b \in U$, contradicting our assumption. Then saturating $\langle U \cup \{a\}, \{b\} \rangle$ to get $\langle U^+, V^+ \rangle$ gives $U^+ \in M_B$ for which $U \subseteq U^+$, $a \in U^+$, and $b \notin U^+$. ❏

Thus we got an intuitionistic frame and want an algebra. We consider the set $\mathcal{H}(M_B)$ of all the upward-closed subsets of M_B, that is,

$$\mathcal{H}(M_B) = \{X \subseteq M_B | U \in X \text{ and } U \subseteq V \text{ imply } V \in X\}$$

We embed B into $\mathcal{H}(M_B)$ with the same function h above.

Recall the definition of $\to$ in $\mathcal{H}(M_B)$:

$$X \to Y = \{U \in M_B | \text{ for all } V \in M_B, \text{ if } U \subseteq V, \text{ then } V \notin X \text{ or } V \in Y\}.$$

We define U' as $U \to \emptyset$.

Recall that $h(b) = \{U | b \in U \in M_B\}$. Joins and meets are preserved as before. For $\to$, we have

$$\begin{aligned} U \in h(a \to b) \quad &\Leftrightarrow \quad \text{for all } V \in M_B, U \subseteq V \text{ implies } a \notin V \text{ or } b \in V. \\ &\Leftrightarrow \quad \text{for all } V \in M_B, U \subseteq V \text{ implies } V \notin h(a) \text{ or } V \in h(b) \\ &\Leftrightarrow \quad U \in h(a) \to h(b). \end{aligned}$$

Thus we have:

Theorem 66 *The variety of Heyting algebras has the FEP. Hence its universal theory is decidable.*

[5] see also [4], where a more general result is given.

4.2 Enriching with a box

The subject of this section is *intuitionistic modal logics*. We introduce some modal logics following [27]. We apply the method used in the preceding chapter and prove the FEP for some of these logics. Unfortunately, the proof seems not to work for all the logics under consideration. In intuitionistic modal logics, $\Box$ and $\Diamond$ are not dual, i.e., one is not definable from the other. Here we consider only logics with $\Box$.

We first introduce the syntax of intuitionistic modal logics in Hilbert-style formulation. In [27], Gentzen-style formulation is also provided. Then we introduce algebraic and Kripke-style semantics as before and proceed to prove the FEP.

4.2.1 Syntax

Let P be a Hilbert-style formulation of intuitionistic propositional logic. The intuitionistic modal logic L_0 is P with the following additional axioms and the inference rule:
Axioms:

- $\Box p \to p$
- $\Box p \to \Box\Box p$
- $\Box(p \to q) \to (\Box p \to \Box q)$

Inference rule:

Necessitation given φ, derive $\Box\varphi$.

Clearly $S4 = L_0 + (p \vee \neg p)$. Furthermore we consider decorating L_0 with the axioms below.

A_1 $\neg\Box p \to \Box\neg\Box p$

A_2 $(\Box p \to \Box q) \to \Box(\Box p \to \Box q)$

A_3 $\Box(\Box p \vee q) \to (\Box p \vee \Box q)$

A_4 $\Box p \vee \Box\neg\Box p$

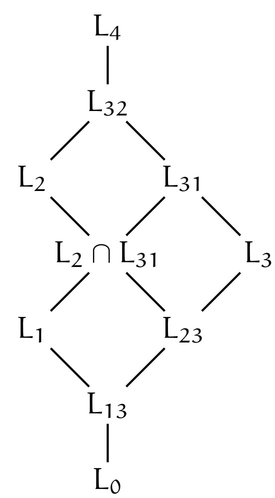

Figure 6. A lattice of intuitionistic modal logics

The extensions of L_0 with A_i is denoted by L_i for $i = 1, 2, 3, 4$. Moreover the extensions of L_3 with A_i is denoted by L_{3i} for $i = 1, 2$. Observe that adding A_i to $S4$ is $S5$ for any i.The relationship among them are depicted in 6, which is proved in [27].

4.2.2 Semantics

Again we introduce algebraic and Kripke-style semantics. First comes algebraic ones. An *modal Heyting algebra*[6] is a pair $\langle H, \Box\rangle$, where H is a Heyting algebra with the top element 1 and $\Box$ is a unary operation on it satisfying:

- $\Box(a \wedge b) = \Box a \wedge \Box b$,

[6]This term sounds somewhat odd but is used to keep uniformity with *Heyting algebra*. Below we stick to the more usual term *intuitionistic modal algebra*. This has yet another pseudonym *topological pseudo-Boolean algebra* in [27].

- $\Box a \leqslant a$,
- $\Box\Box a = \Box a$,
- $\Box 1 = 1$.

An element a of H is called *open* if $\Box a = a$. We can characterize L_j's for $j = 1, 2, 3, 31, 32, 4$ as follows.

0) Any intuitionistic modal algebra is of type 0

1) An intuitionistic modal algebra is of type 1 if the complement of any open element in it is also open.

2) An intuitionistic modal algebra is of type 2 if the set of all open elements in it constitutes a sub-(Heyting) algebra of H.

3) An intuitionistic modal algebra is of type 3 if it satisfies the inequality

$$\Box(\Box a \vee b) \leqslant \Box a \vee \Box b$$

3i) An intuitionistic modal algebra is of type $3i$ ($i = 1, 2$) if it is both of type 3 and of type i.

4) An intuitionistic modal algebra is of type 4 if it is of type 2 and the subalgebra constituted by the set of all open elements is a Boolean algebra.

Lindenbaum-Tarski construction gives:

Theorem 67 *A formula is provable in L_j iff it is valid in all intuitionistic modal algebras of type j, for $j = 0, 1, 2, 3, 31, 32, 4$.*

Next we consider Kripke-type semantics[7]. A triple $\langle M, \leqslant, R\rangle$ is an intuitionistic Kripke frame if

- M is a nonempty set with a partial order $\leqslant$,
- R is a reflexive and transitive relation on M such that $x \leqslant y$ implies xRy for each $x, y \in M$.

The truth definition is as follows

$$\begin{array}{rcl} a \models p & \Leftrightarrow & b \in V(p) \text{ for any } b \text{ s.t.} a \leqslant b, \ \ p \text{ is a propositional variable} \\ a \models \alpha \wedge \beta & \Leftrightarrow & a \models \alpha \text{ and } a \models \beta \\ a \models \alpha \vee \beta & \Leftrightarrow & a \models \alpha \text{ or } a \models \beta \\ a \models \alpha \to \beta & \Leftrightarrow & \text{for any } b \text{ with } a \leqslant b, b \not\models \alpha \text{ or } b \models \beta \\ a \models \neg\alpha & \Leftrightarrow & \text{for any } b \text{ with } a \leqslant b, b \not\models \alpha \\ a \models \Box\alpha & \Leftrightarrow & \text{for any } b \text{ with } aRb, b \models \alpha \end{array}$$

We define the types for frames as for algebras:

0) Any frame is of type 0.

[7] called I models in [27]

1) An intuitionistic Kripke frame $\langle M, \leqslant, R\rangle$ is of type 1 if for every $x, y \in M$, xRy implies the existence of $y' \in M$ with $x \leqslant y'$ and yRy'.

2) An intuitionistic Kripke frame $\langle M, \leqslant, R\rangle$ is of type 2 if for every $x, y \in M$, xRy implies the existence of $y' \in M$ with $x \leqslant y'$ and $y \sim y'$.

3) An intuitionistic Kripke frame $\langle M, \leqslant, R\rangle$ is of type 3 if for every $x, y \in M$, xRy implies the existence of $x' \in M$ with $x \sim x'$ and $x' \sim y$.

3j) An intuitionistic Kripke frame is of type $3j$ if it is both of type 3 and of type j for $j = 1, 2$

4) An intuitionistic Kripke frame is of type 4 if R is symmetric.

Not surprisingly, intuitionistic modal logics are complete with respect to the frames of appropriate type. See [27] for proof.

Theorem 68 *A formula is provable in L_j iff it is valid in any intuitionistic modal algebra of type j, for $j = 0, 1, 2, 3, 31, 32, 4$.*

Similar construction gives a way connecting between frames and algebras. See theorems 3.4, 3.5 of [27].

Theorem 69 *Let $\langle W, \leqslant, R\rangle$ be an intuitionistic Kripke frame of type j. Then define an algebra $\langle \mathcal{H}(W), \cap, \cup, \to, \emptyset, W, \Box\rangle$, where $\cap$ and $\cup$ are set-theoretic operations and*

- *$\mathcal{H}(W)$ is the set of all those subsets of W that are upward closed with respect to $\leqslant$;*
- *$U \to V = \{x \in W | \text{for any } y \text{ with } x \leqslant y, y \notin U \text{ or } y \in V\}$;*
- *$\Box U = \{a \in W | \text{for any } y \text{ with } aRb, b \in U\}$.*

This algebra is an intuitionistic modal algebra of type j.

Theorem 70 *Let $\langle \mathbf{H}, \Box\rangle$ be an intuitionistic modal algebra. Then $\langle \mathcal{F}(H), \subseteq, R\rangle$, where $\subseteq$ is usual inclusion and*

- *$\mathcal{F}(H)$ is the set of all prime filters in $\mathbf{H}$*
- *URV is defined to be $U^{\Box} \subseteq V^{\Box}$, where $U^{\Box} = \{\Box a | \Box a \in U\}$.*

4.3 The FEP

Now we prove the FEP for L_0 and L_4. For the others, the method of the previous chapter seems not to work.

Theorem 71 *The variety of all intuitionistic modal algebras has the FEP, whence its universal theory is decidable.*

Proof. As before, given a finite partial subalgebra $\mathbf{B}$ of an algebra $\mathbf{A}$, let M_B be the set of all saturated sets partially ordered by $\subseteq$. We define R as

$$URV \Leftrightarrow U^{\Box} \subseteq V^{\Box},$$

where $U^{\Box} = \{\Box a | \Box a \in U\}$. Then $\langle M_B, \subseteq, R\rangle$ is an intuitionistic Kripke frame. We have to check corollary 46 holds for $\Box$. That is, we have to check, for any $\Box a \in B$

$$\Box a \in U \Leftrightarrow \text{ for any } V \in M_B, URV \text{ implies } a \in V.$$

From left-to-right, suppose $\Box a \in U$, i.e., $\Box a \in U^{\Box}$. Then by definition of R, $\Box a \in V^{\Box} \subseteq V$. Since V is upward closed and $\Box a \leqslant a$, $a \in V$. For the converse, assume $\Box a \notin U$. Note $\langle U^{\Box}, \{a\}\rangle$ is saturable; otherwise there are $\Box u_1, \ldots, \Box u_k \in U^{\Box}$ such that

$$\Box u_1 \wedge \cdots \wedge \Box u_k \leqslant a,$$

from which we can infer using 4 and monotone-increasingness of $\Box$,

$$\Box u_1 \wedge \cdots \wedge \Box u_k \leqslant \Box a.$$

This implies $\Box a \in U$ to the contrary to our assumption. Thus we can extend $\langle U^{\Box}, \{a\}\rangle$ to get the desired U' with $U^{\Box} \subseteq (U')^{\Box}$ and $a \in U'$. ❑

Changing the partial order and the binary relation after [27], we can prove the FEP for L_4.

Theorem 72 *The variety of intuitionistic modal algebras of type 4 has the FEP. Hence its universal theory is decidable.*

Proof. Let M_B as above. Define relations R and $\preccurlyeq$ on M_B by

$$URV \Leftrightarrow U^{\Box} = V^{\Box},$$

and

$$U \preccurlyeq V \Leftrightarrow URV \text{ and } U \subseteq V.$$

Then evidently R is symmetric and $U \preccurlyeq V$ implies URV, so that $\langle M_B, \preccurlyeq, R\rangle$ is an intuitionistic Kripke frame of type 4. Again, the last thing is to confirm is that the change does not affect corollary 46:

$$\Box a \in U \Leftrightarrow \text{ for any } V \in M_B, URV \text{ implies } a \in V.$$

The left-to-right is the same as in the preceding proof. For the reverse direction, Suppose $\Box a \notin U$. We show $\langle U^{\Box}, (B \setminus U)^{\Box} \cup \{a\}$ is saturable. Evidently the saturation of the pair gives U' such that URU' and $a \notin U'$. Assume to the contrary that there are $\Box u_1, \ldots, \Box u_k \in U$ and $\Box w_1, \ldots, \Box w_l \in (B \setminus U)^{\Box}$ s.t.

$$\Box u_1 \wedge \cdots \wedge \Box u_k \leqslant \Box w_1 \vee \cdots \vee \Box w_l \vee a,$$

from which we can derive with 4 and monotonicity of $\Box$

$$\Box u_1 \wedge \cdots \wedge \Box u_k \leqslant \Box(\Box w_1 \vee \cdots \vee \Box w_l \vee a).$$

Recall that we can use 3, i.e., $\Box(\Box a \vee b) \leqslant \Box a \vee \Box b$. Repeated application of 3 gives

$$\begin{aligned}
\Box(\Box w_1 \vee (\Box w_2 \vee \cdots \vee \Box w_l \vee a)) &\leqslant \Box w_1 \vee \Box(\Box w_2 \vee \cdots \vee \Box w_l \vee a) \\
&\leqslant \Box w_1 \vee \Box w_2 \vee \Box(\Box w_3 \vee \cdots \vee \Box w_l \vee a) \\
&\vdots \\
&\leqslant \Box w_1 \vee \Box w_2 \vee \cdots \vee \Box w_l \vee \Box a.
\end{aligned}$$

But this means $\Box a \in U$ together with the inequality above and assumption on w_i's. This gives the desired contradiction and completes the proof. ❑

Chapter 5

The FEP for modal residuated lattices

So far we have been considering logics with full structural rules. In this chapter we turn our attention to logics obtained by enriching a substructural logic FL_{ew} with $\Box$, which was introduced in [31][1]. We prove the FEP for *modal residuated residuated lattices*, an algebraic counterpart of $\mathsf{S4}$-like modal substructural logic. The contents of this chapter was presented at the 39th MLG meeting [2].

We cannot define Kripke models for substructural logics, so that filtration simply fails us: we have nothing to filtrate. We employ a fairly different construction by Blok and van Alten[4, 5][2]. We recall two relevant theorems from [4].

Theorem 73 [Blok-van Alten] *The variety of residuated lattice has the FEP.*

This is proved below along with our result. Due to the following result, we have to consider FL_{ew} as the base logic when concerned about the FEP.

Theorem 74 [Blok-van Alten] *The variety corresponding to* FL_e[3] *does not have the FEP.*

We first introduce our main subject of modal substructural logic and modal residuated lattice. Then we prove the FEP of the variety of modal residuated lattices. Lastly we mention some known results in linear logic and their possible relevance to us.

5.1 A modal substructural logic

First we define an $\mathsf{S4}$-like modal logic based on FL_{ew} in the obvious way. We define the $\mathsf{S4}_{\mathsf{FL}_{ew}}$ as FL_{ew} augmented with $\Box$ and the following axioms:

- $\Box\neg\bot$,
- $\Box\alpha \cdot \Box\beta \to \Box(\alpha \cdot \beta)$,
- $\Box\alpha \to \alpha$,
- $\Box\alpha \to \Box\Box\alpha$.

[1] Modal logics over FL is investigated there.
[2] This is algebraic translation of construction given by Okada and Terui[26].
[3] called *intuitionistic linear algebra* in [4].

Further we add the following rule of inference: From $\alpha \to \beta$ infer $\Box\alpha \to \Box\beta$.

We define the algebraic semantics obviously as follows.

Definition 75 A pair $\langle \mathbf{M}, \Box \rangle$ is a *modal residuated lattice* if $\mathbf{M}$ is a residuated lattice and $\Box$ satisfies the following:

- $1 \leqslant \Box 1$,
- $\Box x \cdot \Box y \leqslant \Box(x \cdot y)$,
- $\Box x \leqslant x$,
- $\Box x \leqslant \Box\Box x$
- if $x \leqslant y$, then $\Box x \leqslant \Box y$.

5.2 The FEP

5.2.1 Basic class

A natural extension (for the modal operator) of the method used by Blok and van Alten gives the FEP of the corresponding variety of modal residuated lattices. We repeat the necessary construction from Blok and van Alten[4] for completeness' sake. We do not wish to ascribe each lemma or proposition to Blok and van Alten as should be done because it is rather cumbersome.

Let $\mathbf{B}$ be a finite partial subalgebra of a modal residuated lattice $\mathbf{A}$. We define M to be the (base set of) submonoid generated by B, i.e., we set $M = \{b_1^{n_1}, \ldots, b_k^{n_k} : n_i < \omega\}$, where $B = \{b_1, \ldots b_k\}$.

We construct the underlying set. For each $a \in M$ and $b \in B$, define

$$(a \rightsquigarrow b] = \{c \in M : ac \leqslant b\} = \{c \in M : c \leqslant a \to b\}$$

Each $(a \rightsquigarrow b]$ is downward closed, for if $d \leqslant c \in (a \rightsquigarrow b]$ then $ad \leqslant ac$ and $ac \leqslant b$, so that $ad \leqslant b$, hence $d \in (a \rightsquigarrow b]$. We write $(1 \rightsquigarrow a]$ as $(a]$.

Now set

$$\overline{D} = \{(a \rightsquigarrow b] : a \in M \text{ and } b \in B\} (\subseteq \mathcal{P}(M))$$

and then define D as

$$D = \{\cap\mathcal{X} : \mathcal{X} \subseteq \overline{D}\} (\subseteq \mathcal{P}(M))$$

Each element of D is a downward closed subset of M and D is closed under intersection. Note that $M \in D$.

Let us define the closure operator[4] C on $\mathcal{P}(M)$ associated with $\overline{D}$; define C as follows:

$$C(X) = \cap\{Y \in \overline{D} : X \subseteq Y\}$$

Note that $C(X)$ is the smallest element of D that contains X. Thus the following proposition holds.

Proposition 76 *If* $X \subseteq M$ *and* $Y \in D$, *then* $X \subseteq Y$ *implies* $C(X) \subseteq Y$.

[4] See below for the definition and properties.

Next we define an algebra whose underlying set is D. For $X, Y \subseteq M$ and $a \in M$, put $XY = \{ab : a \in X \text{ and } b \in Y\}$ and $Xa = X\{a\}$.

Let us define an algebra $\mathbf{D} = \langle D, \wedge^{\mathbf{D}}, \vee^{\mathbf{D}}, \cdot^{\mathbf{D}}, \rightarrow^{\mathbf{D}}, 0^{\mathbf{D}}, 1^{\mathbf{D}}, \Box^{\mathbf{D}} \rangle$. For $X, Y \subseteq D$, $X_i \in D$ $(i \in I)$ define[5]

$$
\begin{aligned}
\bigwedge_{i\in I}^{\mathbf{D}} X_i &= \bigcap_{i\in I} X_i \\
\bigvee_{i\in I}^{\mathbf{D}} X_i &= C(\bigcup_{i\in I} Y) \\
X \cdot^{\mathbf{D}} Y &= C(XY) \\
X \rightarrow^{\mathbf{D}} Y &= \{a \in M : Xa \subseteq Y\} \\
0^{\mathbf{D}} &= \bigcap \overline{D} \\
1^{\mathbf{D}} &= M \\
\Box^{\mathbf{D}} X &= C(\{a \in X | a = \Box a\})
\end{aligned}
$$

Now we need to prove three things:

1. $\mathbf{D}$ is a modal residuated lattice;
2. we can embed $\mathbf{B}$ into $\mathbf{D}$;
3. $\mathbf{D}$ is finite.

As a first step to show that $\mathbf{D}$ really is a modal residuated lattice, we must show that each operation on D is well-defined, that is, D is closed under all operations. It is clearly closed under the operations $\wedge^{\mathbf{D}}$, $\vee^{\mathbf{D}}$, $\cdot^{\mathbf{D}}$, $\Box^{\mathbf{D}}$ and contains both $0^{\mathbf{D}}$ and $1^{\mathbf{D}}$. Thus we must show closure under $\rightarrow^{\mathbf{D}}$. For the moment we concentrate on modality-free part of $\mathbf{D}$ and show that it is a residuated lattice, and then go on to show that $D^{\mathbf{D}}$ is the legitimate modality, so that $\mathbf{D}$ as a whole is a modal residuated lattice.

First we show the following lemma:

Lemma 77 *For* $X \subseteq M$ *and* $Y_i \subseteq M$, *where* $i \in I$, $X \rightarrow^{\mathbf{D}} \bigcap_{i\in I} Y_i = \bigcap_{i\in I} X \rightarrow^{\mathbf{D}} Y_i$

PROOF.

$$
\begin{aligned}
a \in X \rightarrow^{\mathbf{D}} \bigcap_{i\in I} Y_i &\iff Xa \subseteq \bigcap_{i\in I} Y_i \\
&\iff Xa \subseteq Y_i \text{ for all } i \in I \\
&\iff a \in X \rightarrow^{\mathbf{D}} Y_i \text{ for all } i \in I \\
&\iff a \in \bigcap_{i\in I} X \rightarrow^{\mathbf{D}} Y_i
\end{aligned}
$$

❑

Using this we have the desired closure under $\rightarrow^{\mathbf{D}}$:

Lemma 78 *If* $X \subseteq M$ *and* $Y \in D$, *then* $X \rightarrow^{\mathbf{D}} Y \in D$

[5] The definition of modality follows [1].

PROOF. Since $Y \in D$, we can write $Y = \bigcap(a_i \rightsquigarrow b_i]$, where $\{(a_i \rightsquigarrow b_i]\}_{i \in I} \subseteq \overline{D}$. By the preceding lemma, $X \rightarrow^{\mathbf{D}} \bigcap(a_i \rightsquigarrow b_i] = \bigcap(X \rightarrow^{\mathbf{D}} (a_i \rightsquigarrow b_i])$, so that if $X \rightarrow^{\mathbf{D}} (a_i \rightsquigarrow b_i] \in D$, then $X \rightarrow^{\mathbf{D}} \bigcap(a_i \rightsquigarrow b_i] \in D$. Then

$$\begin{aligned} c \in X \rightarrow^{\mathbf{D}} (a_i \rightsquigarrow b_i] &\iff Xc \subseteq (a_i \rightsquigarrow b_i] \\ &\iff xc \in (a_i \rightsquigarrow b_i] \text{ for all } x \in X \\ &\iff xca_i \leqslant b_i \text{ for all } x \in X \\ &\iff c \leqslant xa_i \rightarrow b_i \text{ for all } x \in X \\ &\iff c \in \bigcap_{x \in X} (xa_i \rightsquigarrow b_i], \end{aligned}$$

so that $X \rightarrow^{\mathbf{D}} (a_i \rightsquigarrow b_i] = \bigcap_{x \in X}(xa_i \rightsquigarrow b_i]$, which implies $X \rightarrow^{\mathbf{D}} (a_i \rightsquigarrow b_i] \in D$ as desired. ❑

As another preliminary step, We show that C satisfies the following conditions and hence is a *nucleus operator*[6].

1. $X \subseteq C(X)$;
2. $C(X) = C(C(X))$;
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$;
4. $C(X)C(Y) \subseteq C(XY)$.

It is easy to see the first and third condition hold. The second condition is also easily seen to follow from proposition 76. We are left to prove the fourth condition, which is stated as the following lemma:

Lemma 79 *For all* $X, Y \subseteq M$, $C(X)C(Y) \subseteq C(XY)$

PROOF. First recall that if $X \subseteq M$ and $Y \in D$, then $X \subseteq Y$ implies $C(X) \subseteq Y$.

For $X, Y, Z \subseteq M$, by the definition of $\rightarrow^{\mathbf{D}}$, we have

$$\begin{aligned} XY \subseteq Z &\iff Xy \subseteq Z \text{ for all } y \in Y \\ &\iff y \in X \rightarrow^{\mathbf{D}} Z \text{ for all } y \in Y \\ &\iff Y \subseteq X \rightarrow^{\mathbf{D}} Z. \end{aligned}$$

Then

$$\begin{aligned} XY \subseteq C(XY) &\iff Y \subseteq X \rightarrow^{\mathbf{D}} C(XY) \\ &\iff C(Y) \subseteq X \rightarrow^{\mathbf{D}} C(XY) \\ &\iff XC(Y) \subseteq C(XY) \\ &\iff C(Y)X \subseteq C(XY) \\ &\iff X \subseteq C(Y) \rightarrow^{\mathbf{D}} C(XY) \\ &\iff C(X) \subseteq C(Y) \rightarrow^{\mathbf{D}} C(XY) \\ &\iff C(X)C(Y) \subseteq C(XY). \end{aligned}$$

❑

Now we can show that the modality-free part is a residuated lattice in the following two lemmas.

[6]This term is from [34]. The first three conditions are the requirements for an operation to be a *closure operator*.

Lemma 80 *The algebra* $\langle D, \cdot^{\mathbf{D}}, 1^{\mathbf{D}} \rangle$ *is a commutative monoid.*

Proof. The commutativity of $\cdot^{\mathbf{D}}$ immediately follows from that of the original monoid operation in $\mathbf{A}$. To show the associativity of $\cdot^{\mathbf{D}}$, let $X, Y, Z \in D$. Then $(X \cdot^{\mathbf{D}} Y) \cdot^{\mathbf{D}} Z = C(C(XY)Z) = C(C(XY)C(Z)) \subseteq C(C((XY)Z)) = C((XY)Z)$. As C is a closure operator, $C((XY)Z) \subseteq C(C(XY)C(Z))$, whence $(X \to^{\mathbf{D}} Y) \to^{\mathbf{D}} Z) = C((XY)Z)$. Similarly $X \to^{\mathbf{D}} (Y \to^{\mathbf{D}} Z) = C(X(YZ))$. The associativity of $\cdot^{A}$ implies $(XY)Z = X(YZ)$, so that $\cdot^{\mathbf{D}}$ is associative.

Next let us see that $1^{\mathbf{D}} = M$ is the identity for $\mathbf{D}$. Suppose X is in D. Since X is downward closed, $X \supseteq XM$, and conversely $X \subseteq XM$ for 1 is in M. Then $X = C(X) = C(XM) = X \cdot^{\mathbf{D}} M$, which shows $1^{\mathbf{D}}$ is the identity for $\cdot^{\mathbf{D}}$. ❏

Lemma 81 *The structure* $\langle D, \cdot^{\mathbf{D}}, \to^{\mathbf{D}}, 0^{\mathbf{D}}, 1^{\mathbf{D}}, \subseteq \rangle$ *is a residuated lattice . Moreover the partial order* $\subseteq$ *is a complete lattice order with lattice meet* $\wedge^{\mathbf{D}}$*, lattice join* $\vee^{\mathbf{D}}$*, largest element* $1^{\mathbf{D}}$*, and smallest element* $0^{\mathbf{D}}$*.*

Proof. We have only to observe that the residuation law holds.

$$\begin{aligned} X \cdot^{\mathbf{D}} Y \subseteq Z &\iff C(XY) \subseteq Z \\ &\iff XY \subseteq Z \text{ by proposition 76 and } XY \subseteq C(XY) \\ &\iff Y \subseteq X \to^{\mathbf{D}} Z \text{ by the previous argument.} \end{aligned}$$

It is clear that $1^{\mathbf{D}}$ and $0^{\mathbf{D}}$ are the largest and greatest element, respectively. Lastly, we prove that $A \wedge^{\mathbf{D}} B = B$ iff $B \subseteq A$. Suppose $B \subseteq A$. Then $A \wedge^{\mathbf{D}} B = C(A \cap B) = C(B) = B$, where the last equality uses the fact that $B \in D$. Conversely, assume $A \wedge^{\mathbf{D}} B = C(A \cap B) = B$. Then $B = C(A \cap B) \subseteq C(A) = A$. ❏

We turn to the modality and show that $\Box^{\mathbf{D}} X$ indeed satisfies the requirement: it is S4-like. We omit the superscript $\mathbf{D}$ for brevity.

Lemma 82 $\mathbf{D}$ *is a modal residuated lattice.*

Proof. With the above lemmas at hand, we only deal with $\Box^{\mathbf{D}}$.

First we clearly have $\Box X \subseteq \Box Y$ if $X \subseteq Y$.

Next we show $\Box M = M$. Trivially $\Box^{\mathbf{D}} M \subseteq M$. For the converse, note $\{a \in M | a = \Box a\}$ contains 1 since $\Box 1 = 1$. Then the downward closure of it contains all the elements of M.

We next show $\Box X \cdot \Box Y \subseteq \Box(X \cdot Y)$. We write X° for $\{a \in X | a = \Box a\}$.

$$\begin{aligned} \Box X \cdot \Box Y &= C(C(X^{\circ})C(Y^{\circ})) \\ &\subseteq C(C(X^{\circ}Y^{\circ})) \\ &= C(X^{\circ}Y^{\circ}) \\ &= C(\{ab | a \in X \text{ and } a = \Box a \text{ and } b \in Y \& b = \Box b\}) \\ &\subseteq C(\{ab \in C(XY) | ab = \Box(ab)\}) \\ &\subseteq C(\{c \in C(XY) | c = \Box c\}) = \Box(X \cdot Y) \end{aligned}$$

The transition from 4th to 5th line holds because $\{a \cdot b | a \in X \& a = \Box a \text{ and } b \in Y \text{ and } b = \Box b\} \subseteq \{ab \in C(XY) | ab = \Box(ab)\}$. In fact, consider ab in the former set. Then

$$ab = \Box a \Box b \leqslant \Box(ab) \leqslant ab,$$

hence ab is in the latter set.

We turn to prove $\Box X \subseteq X$ for any $X \in D$. It is easy to see $X^\circ \subseteq X$. Then proposition 76 provides the inclusion.

Lastly we want $\Box X \subseteq \Box\Box X$. Let $x \in X^\circ$. Then $x \in C(X^\circ)$ and $x = \Box x$. Thus we have

$$X^\circ \subseteq \{x \in C(X^\circ) | x = \Box x\},$$

from which we can infer by the property of closure operator

$$\Box X = C(X^\circ) \subseteq C(\{x \in C(X^\circ) | x = \Box x\}) = \Box\Box X,$$

as desired. ❏

Next we want an embedding from $\mathbf{B}$ into $\mathbf{D}$. Exactly the same embedding $h : a \mapsto (a] = \{b \in M | b \leqslant a\}$[7] as in [4] is used.

We show this mapping preserves every operation. Again the modality-free part is a reproduction from [4].

Lemma 83 *The map $h : B \to D$ which sends a to $(a]$ is an embedding of $\mathbf{B}$ into $\mathbf{D}$. Furthermore, h preserves all meets and join that exists in B. In particular, if 0 is the least element of $\mathbf{A}$ and $0 \in B$, then $h(0) = 0^{\mathbf{D}}$.*

Proof. Let $a, b \in B$. Recall that all we have to show is that h preserves all *existing* operations.

1. Suppose $a \to b \in B$. We need to show that $h(a \to b) = h(a) \to^{\mathbf{D}} h(b)$; that is, $\{c \in M : c \leqslant a \to b\} = \{c \in M : (a]c \subseteq (b]\}$, which can be shown as follows:

$$\begin{aligned} c \leqslant a \to b &\iff ac \leqslant b \\ &\iff ac \in (b] \\ &\iff (a]c \subseteq (b] \\ &\iff c \in (a] \to^{\mathbf{D}} (b]. \end{aligned}$$

   The implication from the second line to the third holds for if $dc \in (a]c$, then $dc \leqslant ac \in (b]$, and so $dc \in (b]$.

2. As for monoid operation $\cdot$, we want to show $h(ab) = (ab] = (a] \cdot^{\mathbf{D}} (b] = C((a](b])$. Since $ab \in (a](b] \subseteq C((a](b])$ and $C((a](b])$ is downward closed, $(ab] \subseteq C((a](b])$.

   Conversely, suppose $cd \in (a](b]$. Then $c \leqslant a$ and $d \leqslant b$, and so $cd \leqslant ab$. Hence $cd \in (ab]$ and so $(a](b] \subseteq (ab]$. Since $(a](b] \subseteq M$ and $(ab] \in D$, $C((a](b]) \subseteq (ab]$. Thus $(ab] = C((a](b]) = (a] \cdot^{\mathbf{D}} (b]$.

3. If $1 \in B$, then $(1] = M = 1^{\mathbf{D}}$ by definition.

4. For meet. Let $a_i \in A$ for $i \in I$ and $\bigwedge^{\mathbf{D}}_{i \in I} a_i$ exists in A and it is in B. Then

$$\begin{aligned} x \in (\textstyle\bigwedge_{i\in I} a_i] &\iff x \leqslant a_i \text{ for all } i \in I \\ &\iff x \in (a_i] \text{ for all } i \in I \\ &\iff x \in \textstyle\bigcap_{i\in I}(a_i] = C(\bigcap_{i\in I}(a_i]) = \bigwedge^{\mathbf{D}}_{i\in I}(a_i] \text{ since } \bigcap_{i\in I}(a_i] \in D, \end{aligned}$$

   as desired.

[7] Caution: this time $a \in B$ and $b \in M$ as opposed to the usage in the definition of $(a \rightsquigarrow b]$.

5. To prove h preserves join, suppose $a_i \in B$ for each $i \in I$, $\bigvee_{i\in I} a_i$ exists in A, and $\bigvee_{i\in I} a_i \in B$. We need to show $(\bigvee_{i\in I} a_i] = \bigvee^{\mathbf{D}}_{i\in I}(a_i] = C(\cup_{i\in I}(a_i])$. Since $a_i \leqslant \bigvee a_i$, $(a_i] \subseteq (\bigvee a_i]$, so that $\cup(a_i] \subseteq (\bigvee a_i]$. It then follows $C(\cup(a_i]) \subseteq (\bigvee a_i]$.

   For the converse, let $C(\cup(a_i]) = \cap\{(c \rightsquigarrow d] \in \overline{D} : \cup(a_i] \subseteq (c \rightsquigarrow d]\} = \cap\mathcal{X}$. Take any $(c \rightsquigarrow d] \in \mathcal{X}$. As $a_i \in \cup(a_i] \subseteq (c \rightsquigarrow d]$ for all $i \in I$, $a_i c \leqslant d$, so that $\bigvee(a_i c) \leqslant d$. By lemma 24, $c(\bigvee a_i) \leqslant d$, which means $\bigvee a_i \in (c \rightsquigarrow d]$. Thus $\bigvee a_i \in \bigcap\mathcal{X} = C(\cup(a_i])$.

6. Suppose $\mathbf{A}$ has the smallest element 0 and $0 \in B$. We want $h(0) = (0] = 0^{\mathbf{D}}$. Each set of the form $(c \rightsquigarrow d] \in \overline{D}$ is downward closed, so $0 \in (c \rightsquigarrow d]$, and so $0 \in \cap\overline{D}$. Thus $(0] = \{0\} \subseteq \cap\overline{D}$.

   For the reverse inclusion, note $(0]$ is in $\overline{D}$, so that $\cap\overline{D} \subseteq (0]$. Thus $h(0) = (0] = \cap\overline{D} = 0^{\mathbf{D}}$.

7. We want to show, for $\Box a \in B$, $(\Box a] = \Box^{\mathbf{D}}(a]$. By definition $\Box^{\mathbf{D}}(a] = C(\{b \in (a] | b = \Box b\}) = C(\{b \leqslant a | b = \Box b\})$. We show $(\Box a] = C(\{b \leqslant a | b = \Box b\})$. To prove the right-to-left inclusion, take $b \leqslant a$ with $b = \Box b$. By monotonicity of the original $\Box$, we have $b = \Box b \leqslant \Box a$. This shows $\{b \leqslant a | b = \Box b\} \subseteq (\Box a]$. Since $(\Box a]$ is in D, $C(\{b \leqslant a | b = \Box b\}) \subseteq (\Box a]$.

   For the left-to-right inclusion, take an arbitrary $(c \rightsquigarrow d] \in \overline{D}$ that includes $\{b \leqslant a | b = \Box b\}$. We want $(\Box a] \subseteq (c \rightsquigarrow d]$. Note that $\Box a = \Box\Box a$ and that $\Box a \leqslant a$, whence[8] $\Box a$ is in $\{b \leqslant a | b = \Box b\}$. Therefore $(\Box a] \subseteq (c \rightsquigarrow d]$ because $(c \rightsquigarrow d]$ is downward closed. Thus $(\Box a] \subseteq C(\{b \leqslant a | b = \Box b\})$.

❏

Now we got the algebra $\mathbf{D}$ into which we can embed $\mathbf{B}$. The last thing we have to do is to prove the finiteness of the constructed algebra. Since we start the generated submonoid M, the proof is exactly the same as in [4]. We reproduce the proof here[9].

Lemma 84 *If* $\mathbf{B}$ *is finite, the* $\mathbf{D}$ *is also finite.*

Proof. Let $\mathbf{F}(k)$ be the free commutative monoid on k generators $\{x_1, \ldots, x_k\}$. The elements of $\mathbf{F}(k)$ can be regarded as the product $x_1^{n_1} \cdot \cdots \cdot x_k^{n_k}$, where $n_i < \omega$ and $x_j^0 = 1$. Define a relation $\leqslant^{\mathbf{F}(k)}$ on $\mathbf{F}(k)$ by setting $x_1^{n_1} \cdot \cdots \cdot x_k^{n_k} \leqslant^{\mathbf{F}(k)} x_1^{m_1} \cdot \cdots \cdot x_k^{m_k} \iff n_i \geqslant m_i$ for all $i \in \{1, \ldots, k\}$. The relation $\leqslant^{\mathbf{F}(k)}$ is a partial order on $F(k)$. Further $\mathbf{F}(k)$ is residuated[10], where

$$x_1^{m_1} \cdot \cdots \cdot x_k^{m_k} \rightarrow^{\mathbf{F}(k)} x_1^{l_1} \cdot \cdots \cdot x_k^{l_k} = x_1^{l \dot{-} m_1} \cdot \cdots \cdot x_k^{l \dot{-} m_k}$$

[8] More importantly $\Box a \in M$ because we consider $\Box a$ that exits in B.

[9] We summarized preliminary facts about well-quasi-order in appendix A.

[10] Note that this is not a residuated lattice since it has no lattice operation (though it can be obviously defined). Here follows the proof that $\dot{-}$ is the residual of $\cdot$.

Proof.

$$\begin{aligned} & x_1^{l_1} \cdot \cdots \cdot x_k^{l_k} \cdot^{\mathbf{F}(k)} x_1^{m_1} \cdot \cdots \cdot x_k^{m_k} \leqslant x_1^{n_1} \cdot \cdots \cdot x_k^{n_k} \\ \Leftrightarrow \quad & m_i + l_i \geqslant n_i \text{ for all } i \in \{1, \ldots, k\} \\ \Leftrightarrow \quad & l_i \geqslant n_i \dot{-} m_i \\ \Leftrightarrow \quad & x_1^{l_1} \cdot \cdots \cdot x_k^{l_k} \leqslant x_1^{m_1} \cdot \cdots \cdot x_k^{m_k} \rightarrow^{\mathbf{F}(k)} x_1^{n_1} \cdot \cdots \cdot x_k^{n_k} \end{aligned}$$

❏

Here $\dot{-}$ is defined on $\mathbb{N}$ as

$$m \dot{-} l = \begin{cases} m - l & \text{if } l \leqslant m \\ 0 & \text{otherwise} \end{cases}$$

Recall that a partially ordered set is well-quasi-ordered if it is well-founded and contains no infinite antichains. The direct product of two well-quasi-ordered sets is again well-quasi-ordered (see the appendix A). Since the partially ordered set of natural numbers $\langle N, \leqslant \rangle$ is well-ordered, so *a fortiori* well-quasi-ordered, so is $\langle N, \leqslant \rangle^k$, for any $k < \omega$; In particular it contains no infinite antichain.

If we write $\mathbb{N}$ for the structure $\langle N, +, \dot{-}, 0, \leqslant \rangle$, then the algebra $\mathbf{F}(k)$ is dually isomorphic to $\mathbb{N}^k$ and so dually well-quasi-ordered.

Let $B = \{b_1, \ldots, b_k\}$. The map which takes x_i to b_i can be extended naturally to a map $h : F(k) \to A$ that preserves the monoid operation[11]. Recall that M is the submonoid generated by B, so that the image of h is M.

For each $X \subseteq F(k)$, the set Max$X$ of maximal elements of X is an antichain. Since the order on $\mathbf{F}(k)$ is a dual well-quasi-order, Max$X$ *is finite*. For each $b \in B$ let $\text{Crit}(b) = \text{Max}h^{-1}((b])$, which is finite. Consider $Z = \cup_{b \in B} \cup_{z \in \text{Crit}(b)} [z)$. Since $[z)$ is finite for any $z \in F(k)$, and so are B and Crit(b), it follows that Z is finite.

Let $a \in M$ and $b \in B$. We show $h^{-1}((a \rightsquigarrow b]) = (Y]$ for some $Y \subseteq Z$. Suppose $h(y) = a$ for $y \in F(k)$. Such y exists as $a \in M$. Then

$$\begin{aligned} x \in h^{-1}((a \rightsquigarrow b]) &\iff h(x) \in (a \rightsquigarrow b] \\ &\iff ah(x) \leqslant b \\ &\iff h(y)h(x) \leqslant b \text{ as } h(y) = a \\ &\iff h(yx) \leqslant b \text{ as } h \text{ preseve } \cdot \\ &\iff h(yx) \in (b] \\ &\iff yx \in h^{-1}((b]) \\ &\iff yx \leqslant^{\mathbf{F}(k)} z \text{ for some } z \in \text{Crit}(b) = \text{Max}h^{-1}((b]) \\ &\iff x \leqslant^{\mathbf{F}(k)} y \to^{\mathbf{F}(k)} z \text{ for some } z \in \text{Crit}(b) \end{aligned}$$

Note that $y \to^{\mathbf{F}(k)} z \geqslant z$, which implies $[y \to^{\mathbf{F}(k)} z) \subseteq [z)$. This in turn implies the set $Y = \{y \to^{\mathbf{F}(k)} z : z \in \text{Crit}(b)\}$ is a subset of Z[12]. By the argument above, $h^{-1}((a \rightsquigarrow b]) = (Y]$. Since Z is finite and h is surjective this shows there can be only finitely many distinct sets of the form $(a \rightsquigarrow b]$[13]. Thus $\overline{D}$ is finite, and so is D, whence finally $\mathbf{D}$ is a finite algebra.

❑

At last we can conclude:

Theorem 85 *The variety of modal residuated lattices has the FEP. Hence its universal theory is decidable.*

5.2.2 Open questions

- What about K- or KT-like modal substructural logics? That is, do the varieties of $K_{FL_{ew}}$- and $KT_{FL_{ew}}$-algebras have the FEP?

[11] just put $h(x_1^{n_1} \cdot \cdots \cdot x_k^{n_k}) = b_1^{n_1} \cdot \cdots \cdot b_k^{n_k}$

[12] if $y \to^{\mathbf{F}(k)} z \in [z)$

[13] Some more details. Consider the list $h((Y_1]), \ldots, h((Y_{2^{|Z|}}])$. By the argument above, we have some $(Y]$ for each $(a \rightsquigarrow b]$ with $h^{-1}((a \rightsquigarrow b]) = (Y]$, which means all the sets of the form $(a \rightsquigarrow b]$ must appear in the list. Then the number of sets of the form $(a \rightsquigarrow b]$ is finite since the list is finite.

- In another paper by Blok and van Alten[5], the FEP for FL_w-algebras is proved. Then can we prove the FEP for the algebras FL_w with $\mathsf{S4}$-like modality?

5.3 Notes

We said in the introduction that modal substructural logics are rarely on serious investigation. This is far from precise because linear logic[16], which is a family of substructural logics, is originally endowed with modalities called *exponentials*.

In this section we discuss some results of Terui[38] that are related to ours in this chapter. We introduce or restate some linear logics and comments on the construction in the paper. Since correspondences between linear logics and modal substructural logics are not exact[14], our discussion might be a little informal.

The base modality-free part of *intuitionistic linear logic*, ILL, is essentially FL_e. Several modalities are the part of linear logic. We confine ourselves to !, which roughly is our $\Box$. The following is the axioms for !.

Functricity	$A \multimap B$ implies $!A \multimap !B$;
Monoidalness1	$!A \otimes !B \multimap !(A \otimes B)$;
Monoidalness2	$!1$;
Dereliction	$!A \multimap A$;
Digging	$!A \multimap !!A$;
Weakening	$!A \multimap 1$;
Contraction	$!A \multimap !A \otimes !A$.

The operations $\otimes$ and $\multimap$ corresponds to our $\cdot$ and $\to$, respectively. The constant 1 is the top element. With these in mind, Functricity and Monoidalness are easily seen to be the requirement for a normal modality. Dereliction is T and Digging is 4. Weakening is irrelevant for us thinking modal logics over FL_{ew}.

The crucial difference is Contraction. In algebraic terms, this is $\Box a \leqslant (\Box a)^2$. In presence of this, we have $dist_\Box$: $\Box(a \cdot b) = \Box a \Box b$ as follows.

$$\Box ab \leqslant (\Box ab)^2 \leqslant \Box a \Box b.$$

Hence all linear logic can be seen as modal substructural logic with $dist_\Box$[15].

Now rough correspondence is summarized in table 2[16].

Among others, Terui proves the FMP of ILAL or $\mathsf{K}_{\mathsf{FL}_{ew}} + dist_\Box$ in our terminology. As noted earlier, Blok and van Alten's construction is an algebraic rephrasing of syntactic construction[17] by Okada and Terui[26]. The corresponding construct to our **D** is *phase semantics*. Why cannot we algebraically rephrase the proof of the FMP for ILAL to prove the FEP of

[14] To say the least, the languages are quite different. The author is not sure if they are the same logics although clearly they are essentially equivalent.

[15] The converse seems not to hold. Hence Contraction is stronger that $dist_\Box$, if we want to be precise. By the way, $\Box a = \Box a \Box a$ is equivalent to $\Box a \Box b = \Box(a \wedge b)$, which is called *Exponential Isomorphism* in [38].

[16] The expanded acronyms: ILL=intuitionistic linear logic; IMALL=the multiplicative-additive fragment of intuitionistic linear logics; IMAAL=the multiplicative-additive fragment of intuitionistic affine logic; IAL=intuitionistic affine logic; ILLL=intuitionistic light linear logic; ILAL=intuitionistic light affine logic. Here follow the rules of thumb for remembering these appalling explosion of logics. When weakening is allowed, *linear* is replaced with *affine*. A logic is *light* if T(Dereliction) and 4(Digging) are dropped. The adjective *multiplicative-additive* means modality-free.

[17] Is it a coincidence that our proof of the FEP in chapter 3 and 4 is also rephrasing syntactic proof of Schütte?

Linear logic	**Modal substructural logic**
IMALL	FL_e
ILL	$\mathsf{S4}_{\mathsf{FL}_e} + dist_\Box$
IMAAL	FL_{ew}
IAL	$\mathsf{S4}_{\mathsf{FL}_{ew}} + dist_\Box$
ILLL	$\mathsf{K}_{\mathsf{FL}_e} + dist_\Box$
ILAL	$\mathsf{K}_{\mathsf{FL}_{ew}} + dist_\Box$

Table 2. A family of linear logics

$\mathsf{K}_{\mathsf{FL}_{ew}} + dist_\Box$? The reason is somewhat obscure, but one obvious obstacle is that Terui's proof "wraps" the argument one further step. That is, instead of proving the completeness to finite phase semantics, Terui uses a notion of *generalized* phase semantics and prove the FMP with respect to it. Moreover, in the definition of phase semantics for ILAL, treatment of modality is different from ours. A *light affine phase structure* is defined to be a quadruple $\langle M, C, f, h\rangle$[18], where (roughly)

- M and C can be seen our M and C in the construction;
- f is a function from M to $\{X \in M | X \leqslant X \cdot X\}$;
- f is used in order to define $!$. $!X = C(f(X))$;
- h is a function for defining another modality and irrelevant to us.

In a word, our modality is fixed in some way while it is not in phase semantics. How relevant this is to us is not clear. However, a proper algebraization of Terui's result could be a first step toward our open questions in the preceding section.

[18]See definition 7.32 there.

Appendix A

Notes on well-quasi-order

In this appendix we collect some definition and fact, all of which revolves around *well-quasi-order*. Our treatment is due to [13, 40]. The goal is to prove that $\mathbb{N}^k$ is well-quasi-ordered with the order used in [4], which plays a crucial role in the proof of the finiteness of the constructed algebra $\mathbf{D}$. First we define quasi-order:

Definition 86 A binary relation R on a set A is a *preorder* or *quasi-order* if it is reflexive and transitive.

We usually write $\leqslant$ for an order and also write $a \leqslant b$ as. With $\leqslant$ fixed, we further define its strict version $<$ as

$$a < b \text{ iff } a \leqslant b \text{ and } b \nleqslant a.$$

We say that a and b are *incomparable*, denoted $a|b$ in symbols, if $a \nleqslant b$ and $b \nleqslant a$.

Definition 87 Given a set A and a preorder $\leqslant$ on it, a sequence $\langle a_i \rangle$ of elements[1] is an *infinite descending chain* when $a_i > a_{i+1}$ for all $i \geqslant 0$. A sequence $\langle a_i \rangle$ is an *infinite antichain* if $a_i | a_j$ for all $0 \leqslant i < j$.

Now we can define the notion of well-quasi-order, which is mysteriously ubiquitous in mathematics.

Definition 88 A preorder $\leqslant$ on A is a *well-quasi-order* if A has no infinite descending chain nor infinite antichain with respect to $\leqslant$.

We abbreviate *well-quasi-order* to wqo. Before we go on to the main theorem, we show the following characterizations of wqo.

Lemma 89 *The following are equivalent.*

1. $\leqslant$ *is a wqo;*
2. *For any sequence* $\langle a_i \rangle$ *there are natural numbers* i, j *with* $i < j$ *such that* $a_i \leqslant a_j$[2]*;*
3. *Any sequence* $\langle a_i \rangle$ *has an infinite subsequence*[3] $\langle a'_i \rangle$ *such that* $a'_i \leqslant a'_{i+1}$ *for all* $i \geqslant 0$.

[1] Formally a sequence of elements of A is a function from $\mathbb{N}$ to A.

[2] Such a sequence is also called a *good* sequence.

[3] Formal definition: A sequence $\langle a'_i \rangle$ is a *infinite subsequence* of a sequence $\langle a_i \rangle$ if there is a strictly increasing function f from $\mathbb{N}$ to $\mathbb{N}$ such that $a'_i = a_{f(i)}$ for all $i \geqslant 0$. It is also written as $a_{f(i)}$.

Proof. Clearly $3 \to 2 \to 1$. We show $1 \to 3$. Suppose $\leqslant$ is a wqo and consider a sequence $\langle a_i \rangle$. Then for each pair $\langle i, j \rangle \in \mathbb{N}^2$ with $i < j$, exactly one of $a_i > a_j$, $a_i \leqslant a_j$ and $a_i | a_j$ holds. Note that one of these holds for infinitely many pairs because the number of pairs is infinite. By our assumption the sequence contains no infinite antichain and no descending chain. This means neither $a_i > a_j$ nor $a_i | a_j$ holds infinitely many times. Thus $a_i \leqslant a_j$ must occur for infinitely many pairs. Then we can construct an infinite nondecreasing subsequence from those pairs[4]. ❏

Given two preorders $\leqslant_A$ on A and $\leqslant_B$ on B, we define a preorder $\leqslant$ on $A \times B$ with

$$\langle a, b \rangle \leqslant \langle a', b' \rangle. \text{ iff } a \leqslant_A a' \text{ and } b \leqslant_B b'$$

We next show that this definition preserves well-quasi-orderedness.

Theorem 90 *If $\leqslant_A$ and $\leqslant_B$ are wqo's respectively on A and on B, then $\leqslant$ as defined above is a wqo on $A \times B$.*

Proof. Let $\langle \langle a_i, b_i \rangle \rangle$ be a sequence in $A \times B$. Since $\langle a_i \rangle$ is a sequence in A, we have an infinite subsequence $\langle a_{f(i)} \rangle$ with $a_{f(i)} \leqslant_A a_{f(i+1)}$ for all $i \geqslant 0$. Then we turn our attention to the sequence $\langle b_{f(i)} \rangle$, i.e., the subsequence of $\langle b_i \rangle$ induced by f. By the preceding lemma, we have some $i, j \in \mathbb{N}$ with $f(i) < f(j)$ such that $b_{f(i)} \leqslant_B b_{f(j)}$. Then we have $\langle a_{f(i)}, b_{f(i)} \rangle \leqslant \langle a_{f(j)}, b_{f(j)} \rangle$, so that $\leqslant$ is a wqo once again by the preceding lemma. ❏

An easy induction provides our goal:

Corollary 91 *Let $\leqslant$ be a wqo on $\mathbb{N}$. Then the order $\leqslant_n$ induced by $\leqslant$ on $\mathbb{N}^k$ is also a wqo.*

Note The names associated with above theorem are Higman[17], Kruskal[19], and Nash-Williams[25][5].

[4] This proof implicitly appeals to Ramsey's theorem. It says that $f : [\mathbb{N}]^2 \to \{|, \leqslant, >\}$ is monochromatic.

[5] A web crawling says: Higman is a mathematician at Oxford and is now retired. Kruskal is a computer scientist at Bell Laboratory and (seemingly) actively works on something around multidimensional scaling. Nash-Williams is a British mathematician who was at Reading in the end of his career. He died some five years ago. In [13], the above corollary is ascribed to Dickson[10], who is an American mathematician at Chicago around the first half of the 20th century.

Appendix B

Notes on literature

We give some comments on the literature.

For algebraic preliminaries, knowing the definitions, which takes off a lot of scare invoked by those jargon, would almost always suffice in the author's opinion. An appendix "Algebraic toolkit" in [3] is valuable for that purpose. For more information on universal algebra, see [6], which the author has never read it seriously.

Although decidability is a main motivation for this thesis, it is not necessary to know what is all about, as mentioned in the text. Again the toolkit in [3] would be helpful. For a detailed introduction, see [37]. On history around the origin of models of computation, [14] is an interesting survey. Important papers of Turing are collected in [9], with readable introductions by the editor to each paper. Decidability in relation to the FMP is discussed in [3].

If you can read Japanese, [28] would be what you should read first. Most of what is necessary to read this thesis is covered. Among others, the presentation of Schütte's method is largely due to this book. It also deals with intuitionistic logic and Heyting algebra.

Intuitionistic modal logics we discussed are from [27]. More comprehensive discussion from a modern viewpoint can be found in [12] and references therein.

For substructural logics and its algebraic semantics, consult [29, 30]. The former[29] is a detailed account of the lattice of logics above FL_{ew}. The latter contains proof theoretical discussion of various substructural logics and a quick survey of their relationships with residuated lattices. Novices might want to read [29] first. Modal substructural logics are introduced in [31].

The deep subject of well-quasi-order is readably summarized in the survey[13] although it concerns proof theory. Textbook presentation is in [40].

Bibliography

[1] van Alten, C. J., The finite model property for knotted extensions of propositional linear logic. *the Journal of Symbolic Logic*, 70(1), 84–98, 2005.

[2] Amano, S. J., The finite embeddability property of the variety of modal residuated lattices, in *Proceedings of the 39th MLG meeting*, Gamagori, Japan, December 2005.

[3] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Sience*, Cambridge University Press, 2001.

[4] Blok, W. J. and C. J. van Alten, The finite embeddability property for residuated lattices, pocrims and BCK-algebras. *Algebra Universalis*, 48, 253–271, 2002.

[5] Blok, W. J. and C. J. van Alten, On the finite embeddability property for residuated ordered groupoid. *Transactions of the American Mathematical Society*, 357, 4141–4157, 2005.

[6] Burris, S. and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, 2000.

[7] Chagrov, A. and M. Zakharyaschev, *Modal Logic*, volume 35 of *Oxford Logic Guides*, Claredon Press, 1997.

[8] Church, A., Review of Turing 1936-37. *the Journal of Symbolic Logic*, 2, 42–43, 1937.

[9] Copeland, B. J., ed., *the Essential Turing*, Oxford University Press, 2004.

[10] Dickson, L. E., Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35, 413–426, 1913.

[11] Došen, K. and P. Schroeder-Heister, eds., *Substructural Logics*, Oxford University Press, 1993.

[12] Gabbay, D. M., A. Kurucz, F. Wolter, and M. Zakharyaschev, *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*, Elsevier, 2003.

[13] Gallier, J. H., What's so special about Kruskal's theorem and the ordinal Γ_0?: A survey of some results in proof theory. *Annals of Pure and Applied Logic*, 53, 199–260, 1991.

[14] Gandy, R., The confluence of ideas in 1936, in R. Herken, ed., *the Universal Turing Machine: A Half-Century Survey*, pp. 51–102, Springer-Verlag, 1995.

[15] Gentzen, G., Untersuchungen über das logische schliessen I. II. *Mathematische Zeitschrift*, 39, 176–210;405–431, 1934.

[16] Girard, J.-Y., Linear logic. *Theoretical Computer Science*, 50, 1–102, 1987.

[17] Higman, G., Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 3(2), 326–336, 1952.

[18] Kleene, S. C., Recursive predicates and quantifiers. *Transactions of the American Mathematical Society*, 53, 41–73, 1943.

[19] Kruskal, J. B., Well-quasi-ordering, the tree theorem, and Vázsonyi's conjecture. *Transactions of the American Mathematical Society*, 95, 210–225, 1960.

[20] Lambek, J., The mathematics of sentence structure. *American Mathematical Monthly*, 12, 166–178, 1958.

[21] McKinsey, J. C. C., A solution of the decision problem for the Lewis systems S2 and S4. *the Journal of Symbolic Logic*, 6, 117–134, 1941.

[22] McKinsey, J. C. C. and A. Tarski, The algebra of topology. *Annals of Mathematics*, 45, 141–191, 1944.

[23] McKinsey, J. C. C. and A. Tarski, On closed elements in closure algebras. *Annals of Mathematics*, 47, 122–162, 1946.

[24] Mostowski, A., On definable sets of positive integers. *Fundamenta Mathematicae*, 34, 81–112, 1947.

[25] Nash-Williams, C. S. J. A., On well-quasi-ordering finite trees. *Proceedings of the Cambridge Philosophical Society*, 59, 833–835, 1963.

[26] Okada, M. and K. Terui, The finite model property for various fragments of intuitionistic linear logic. *the Journal of Symbolic Logic*, 64, 790–802, 1999.

[27] Ono, H., On some intuitionistic modal logics. *Publications of the Research Institute of Mathematical Science, Kyoto University*, 13, 687–722, 1977.

[28] Ono, H., *Jouhokagaku ni okeru Ronnri (Logic in Information Science; In Japanese)*, Nihon Hyronsha, 1994.

[29] Ono, H., Logics without contraction rule and residuated lattices I, 2001, unpublished.

[30] Ono, H., Substructural logics and reresiduated lattices–an Introduction, in V. F. Hendricks and J. Malinowski, eds., *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pp. 193–228, Kluwer Academic Publishers, 2003.

[31] Ono, H., Modalities in substructural logics – a preliminary report, September 2005, Kurt Gödel Society Lecture, Technical Unvierysity of Vienna.

[32] Ono, H. and Y. Komori, Logics without the contraction rule. *the Journal of Symbolic Logic*, 50, 169–201, 1985.

[33] Post, E. L., Formal reductions of the general combinatorial decision problem. *American Journal of Mathematics*, 65, 197–215, 1943.

[34] Rosenthal, K. I., *Quantales and Their Applications*, volume 234 of *Pitman Research Notes in Mathematics*, Longman, 1990.

[35] Sato, M., A study of Kripke-type models for some modal logic by Gentzen's sequential method. *Publications of the Research Institute of Mathematical Science, Kyoto University*, 13, 381–468, 1977.

[36] Schütte, K., *Vollständige Systeme modaler und intuitionistischer Logik*, volume 42 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, 1968.

[37] Sipser, M., *Introduction to the Theory of Computation*, PWS Publishing Company, 1997.

[38] Terui, K., *Light Logic and Polynomial Time Computation*, PhD thesis, Keio University, 2002.

[39] Turing, A. M., On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society, 2nd series*, 42, 230–265, 1936-7.

[40] Welcher, W., *Universal Algebra for Computer Scientists*, Springer-Verlag, 1992.