グラフ上のボロノイゲームとその困難性

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Abstract. ボロノイゲームは競争施設配置問題をモデル化した二人ゲームである. これまで, ボロノイゲームは連続領域の上で定義され, 二つの特別な場合 (1 次元の場合と 1-ラウンドの場合) についてのみ良く調べられている. 本論文では, 離散的なボロノイゲームを導入する. つまり, 与えられたグラフ上でのボロノイゲームを考える. はじめに, 与えられたグラフが十分大きな完全 k 分木である場合の最良の戦略を示す. また, 一般のグラフ上の 1-ラウンド 離散ボロノイゲームに対する $\mathcal{NP}$-困難性を示す.

Key words: ボロノイゲーム, $\mathcal{NP}$-完全性.

Voronoi game on graphs and its complexity

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Abstract. The Voronoi game is a two-person game which is a model for a competitive facility location. The game is done on a continuous domain, and only two special cases (1-dimensional case and 1-round case) are well investigated. We introduce the *discrete* Voronoi game of which the game arena is given as a graph. We first show the best strategy when the game arena is a large complete k-ary tree. We also show that the discrete Voronoi game is $\mathcal{NP}$-hard on a given general graph, even in 1-round case.

Key words: Voronoi Game, $\mathcal{NP}$-completeness.

1 Introduction

The Voronoi game is an idealized model for a competitive facility location, which was proposed by Ahn, Cheng, Cheong, Golin, and Oostrum [1]. The Voronoi game is played on a bounded continuous arena by two players. Two players $\mathcal{W}$ (white) and $\mathcal{B}$ (black) put n points alternately, and the continuous field is subdivided according to the *nearest neighbor rule*. At the final step, the player who dominates larger area wins.

The Voronoi game is a natural game, but the general case seems to be very hard to analyze from the theoretical point of view. Hence, in [1], Ahn et al. investigated the case that the game field is a bounded 1-dimensional continuous domain. On the other hand, Cheong, Har-Peled, Linial, and Matoušek [2], and Fekete and Meijer [3] deal with a 2-dimensional case, but they restrict themselves to one-round game; first, $\mathcal{W}$ puts all n points, and next $\mathcal{B}$ puts all n points.

In this paper, we introduce *discrete* Voronoi game. Two players alternately occupy n vertices on a graph, which is a bounded discrete arena. (Hence the graph contains at least $2n$ vertices.) This restriction seems to be appropriate since real estates are already bounded in general, and we have to build shops in the bounded area. More precisely, the discrete Voronoi game is played on a given finite graph G, instead of a bounded continuous arena. Each vertex of G can be assigned to nearest vertices occupied by $\mathcal{W}$ or $\mathcal{B}$, according to the *nearest neighbor rule*. (Hence some vertex can be "tie" when it has the same distance from a vertex occupied by $\mathcal{W}$ and another vertex occupied by $\mathcal{B}$.) Finally, the player who dominates larger area (or a larger number of vertices) wins. We note that two players can tie in some cases.

We first consider the case that the graph G is a complete k-ary tree. A complete k-ary tree is a natural generalization of a path which is the discrete analogy of 1-dimensional continuous domain. We also mention that complete k-ary trees form very natural and nontrivial graph class. In [1], Ahn et al. showed that the second player $\mathcal{B}$ has an advantage on a 1-dimensional continuous domain. In contrast to the fact, we first show that the first player $\mathcal{W}$ has an advantage for the discrete Voronoi game on a complete k-ary tree, when the tree is sufficiently large (comparing to n and k). More precisely, we show that $\mathcal{W}$ has a winning strategy if (1) $2n \le k$, or (2) k is odd and the complete k-ary tree contains at least $4n^2$ vertices. On the other hand, when k is even and $2n > k$, two players tie if they do their best.

Next, we show the hardness results of the discrete Voronoi game. When we admit a general graph as a game arena, the discrete Voronoi game becomes intractable even in the strongly restricted case. We consider the following strongly restricted case; the game arena is an arbitrary graph, the first player $\mathcal{W}$ occupies just one vertex which is predetermined, the second player $\mathcal{B}$ occupies n vertices in any way. The decision problem for the strongly restricted discrete Voronoi game is defined as follows; the problem is to determine if $\mathcal{B}$ has a winning strategy for given graph G with the occupied vertex by $\mathcal{W}$. This restricted case seems to be advantageous for $\mathcal{B}$. However, the decision problem is $\mathcal{NP}$-complete. This result is also quite different from the previously known results in the 2-dimensional problem (i.e. $\mathcal{B}$ can always dominate the fraction $\frac{1}{2}+\varepsilon$ of the 2-dimensional domain) by Cheong et al. [2] and Fekete et al. [3].

2 Problem definitions

In this section, we formulate the discrete Voronoi game on a graph. Let denote a Voronoi game $VG(G,n)$, where G is the game arena, and the players play n rounds. Hereafter, the game arena intends an undirected and unweighted simple graph $G=(V,E)$ with $N=|V|$ vertices.

For each round, the two players, $\mathcal{W}$ (white) and $\mathcal{B}$ (black), alternately occupy an *empty* vertex on the graph G ($\mathcal{W}$ always starts the game, as in Chess). The empty vertex is defined as a vertex which has not been occupied so far. This implies that $\mathcal{W}$ and $\mathcal{B}$ cannot occupy a same vertex simultaneously. Hence it is implicitly assumed that the game arena G contains at least $2n$ vertices.

Let W_i (resp. B_i) be a set of vertices occupied by player $\mathcal{W}$ (resp. $\mathcal{B}$) at the end of the i-th round. We define the distance $d(v,w)$ between two vertices v and w as the number of edges along the shortest path between them if such path exists, otherwise $d(v,w)=\infty$. Each vertex of G can be assigned to the nearest vertices occupied by $\mathcal{W}$ and $\mathcal{B}$, according to the *nearest neighbor rule*. So, we define a *dominance set* $\mathcal{V}(A,B)$ (or *Voronoi regions*) of a subset $A \subset V$ against a subset $B \subset V$, where $A \cap B = \emptyset$ as

$$\mathcal{V}(A,B) = \{u \in V \mid \min_{v \in A} d(u,v) < \min_{w \in B} d(u,w)\}.$$

The dominance sets $\mathcal{V}(W_i,B_i)$ and $\mathcal{V}(B_i,W_i)$ represent the sets of vertices dominated at the end of the i-th round by $\mathcal{W}$ and $\mathcal{B}$, respectively. Let $\mathcal{V}_\mathcal{W}$ and $\mathcal{V}_\mathcal{B}$ denote $\mathcal{V}(W_n,B_n)$ and $\mathcal{V}(B_n,W_n)$, respectively. Since some vertex can be "tie" when it has the same distance from a vertex occupied by $\mathcal{W}$ and another vertex occupied by $\mathcal{B}$, there may exist set N_i of *neutral* vertices, $N_i := \{u \in V \mid \min_{v\in W_i} d(u,v) = \min_{w \in B_i} d(u,w)\}$, which does not belong to both of $\mathcal{V}(W_i,B_i)$ and $\mathcal{V}(B_i,W_i)$.

Finally, the player who dominates larger number of vertices wins, in the discrete Voronoi game. More precisely, $\mathcal{W}$ *wins* if $|\mathcal{V}_\mathcal{W}| > |\mathcal{V}_\mathcal{B}|$, $\mathcal{B}$ wins (or $\mathcal{W}$ *loses*) if $|\mathcal{V}_\mathcal{W}| < |\mathcal{V}_\mathcal{B}|$, and *tie* otherwise, since the *outcome* for each player, $\mathcal{W}$ or $\mathcal{B}$, is the size of the dominance set $|\mathcal{V}_\mathcal{W}|$ or $|\mathcal{V}_\mathcal{B}|$. In our model, note that any vertices in N_n do not contribute to the outcomes $\mathcal{V}_\mathcal{W}$ and $\mathcal{V}_\mathcal{B}$ of both players (see Fig. 1).

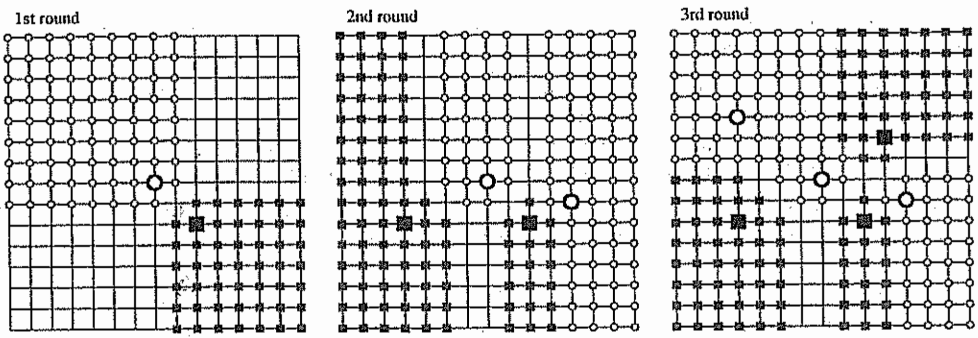

Fig. 1. Example for a discrete Voronoi game $VG(G,3)$, where G is the 15×15 grid graph; each bigger circle is a vertex occupied by $\mathcal{W}$, each smaller circle is an empty vertex dominated by $\mathcal{W}$, each bigger black square is a vertex occupied by $\mathcal{B}$, each smaller black square is an empty vertex dominated by $\mathcal{B}$, and the other are neutral vertices. In this example, the 2nd player $\mathcal{B}$ won by 108–96.

3 Discrete Voronoi game on a complete k-ary tree

In this section, we consider the case that the game arena G is a complete k-ary tree, which is a rooted tree whose inner vertices have exactly k children, and all leaves are in a same level, or the highest level.

Firstly, we show a simple observation for Voronoi games $VG(T,n)$ which are satisfied $2n \le k$. In this game of a few rounds, $\mathcal{W}$ occupies the root of T with his first move, and then $\mathcal{W}$ can dominate at least $\frac{N-1}{k}n+1$ vertices. Since $\mathcal{B}$ dominate at most $\frac{N-1}{k}n$ vertices, $\mathcal{W}$ wins. More precisely, we show the following algorithm as $\mathcal{W}$'s winning strategy.

Algorithm 1: Simple strategy

Stage I: ($\mathcal{W}$'s fist move) $\mathcal{W}$ occupies the root of T;
Stage II: $\mathcal{W}$ occupies the empty children of the root for his remaining rounds;

In the strategy of Algorithm 1, $\mathcal{W}$ alternately pretends to occupy the empty children of root, though $\mathcal{W}$ may occupy any vertex. This strategy is obviously well-defined and winning strategy for $\mathcal{W}$, whenever the game arena T is satisfied $2n \le k$.

Proposition 1. *Let $VG(G,n)$ be the discrete Voronoi game such that G is a complete k-ary tree with $2n \le k$. Then the first player $\mathcal{W}$ always wins.*

We next turn to more general case. We call a k-ary tree an odd (resp. even) if k odd (resp. even). Let T be a complete k-ary tree as a game arena, N be the number of vertices of T, and H be the height of T. Note that $N = \frac{k^{H+1}-1}{k-1}$ and $H \sim \log_k N^\star$. For this game, we show the following theorem.

Theorem 1. *In the discrete Voronoi game $VG(G,n)$ where G is a complete k-ary tree such that $N \ge 4n^2$, the first player $\mathcal{W}$ always wins if G is odd k-ary tree, otherwise the game ends in tie when the players do their best.*

In section 3.1, we first show winning strategy for the first player $\mathcal{W}$ when k is odd and the complete k-ary tree contains at least $4n^2$ vertices. In idea of any winning strategy, it is necessary to deliberate the relation between the number of children k and the game round n. Indeed, $\mathcal{W}$ chooses one of two strategies according to the relation between k and n. We next consider the even k-ary tree in section 3.2, which completes the proof of Theorem 1.

3.1 Discrete Voronoi game on a large complete odd k-ary tree

We generalize the simple strategy to Voronoi games $VG(T,n)$ on a large complete k-ary tree, where $2n > k$ and k is odd ($k \ge 3$). We define that a level h is *keylevel* if the number k^h of vertices satisfies $n \le k^h < 2n$, and a vertex v is a *key-vertex* if v is in the keylevel. Let T_i denote the number of vertices in the subtree rooted at a vertex in level i (i.e., $T_0 = N$, $T_i = kT_{i+1}+1$). Let $\{V_1^h, V_2^h, \ldots, V_{k^{h-1}}^h\}$ be a family of vertices in the keylevel h such that set V_i^h consists of k vertices which have the same parent for each i.

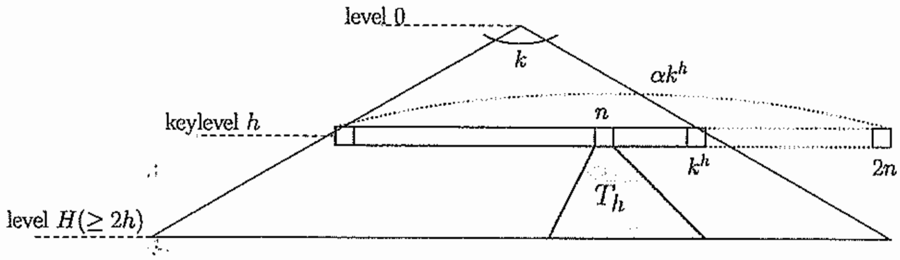

Fig. 2. The notations on the game arena T.

As mentioned above, a winning strategy is sensitive for the relation between k, h, and n.So, we firstly introduce a magic number $\alpha = \frac{2n}{k^h}$, $1 < \alpha < k$ (see Fig. 2). We note that since k is odd, we have neither $\alpha = 1$ nor $\alpha = k$. By assumption, we have that the game arena T is sufficiently large such that the subtrees rooted at level h contain sufficient vertices comparing to the number of vertices between level 0 and level h. More precisely, by assumption $N \ge 4n^2$, we have $H \ge 2h$ and $N \ge \frac{4n^2}{\alpha^2}$. We define $\gamma := H - 2h$, and hence $\gamma \ge 0$.

The winning strategy for $\mathcal{W}$ chooses one of two strategies according to the condition whether the magic number α is greater than $1 + \frac{2}{k} - \frac{1}{k-1} + \frac{1}{k^{h+\gamma}(k-1)}$ or not. The strategy is shown in Algorithm 2.

* In this paper, we denote by $f(x) \sim g(x)$ when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Algorithm 2: Keylevel strategy for $\mathcal{W}$

if $\alpha > 1 + \frac{2}{k} - \frac{1}{k-1} + \frac{1}{k^{h+\gamma}(k-1)}$ **then**

  Stage (a)-I:

    $\mathcal{W}$ occupies an empty key-vertex so that at least one vertex is occupied in each V_i^h;

    `(Stage (a)-I ends after the last key-vertex is occupied by either` $\mathcal{W}$ `or` $\mathcal{B}$`. Note that the game may finish in Stage (a)-I.)`

  end

  Stage (a)-II:

    $\mathcal{W}$ occupies an empty vertex which is a child of the vertex v, such that v is occupied by $\mathcal{B}$, and v has the minimum level greater than or equal to h;

    `(`$\mathcal{W}$ `dominates as much vertices as possible from` $\mathcal{B}$`.)`

  end

else

  Stage (b)-I:

    $\mathcal{W}$ occupies an empty vertex in level $h-1$;

    `(Stage (b)-I ends when such empty vertices are not exists.)`

  end

  Stage (b)-II:

    $\mathcal{W}$ occupies an empty key-vertex whose parent is not occupied by $\mathcal{W}$;

    `(Stage (b)-II ends when such empty key-vertices are not exist.)`

  end

  Stage (b)-III:

    if *there exists an empty vertex* v *in level* $h+1$ *such that the parent of* v *is occupied by* $\mathcal{B}$ **then** $\mathcal{W}$ occupies v;

    else $\mathcal{W}$ occupies an empty key-vertex in level $h+1$ whose parent is occupied by $\mathcal{W}$;

  end

end

Lemma 1. *The keylevel strategy is well-defined in a discrete Voronoi game $VG(T,n)$, where T is a sufficient large complete k-ary tree so that $N \geq 4n^2$.*

Proof. By assumption, there exists the keylevel h.

In the Stage (a)-I, if $\mathcal{B}$ occupied a key-vertex in V_i^h and $\mathcal{W}$ has not occupied any vertex in V_i^h, $\mathcal{W}$ occupies an empty key-vertex in V_i^h rather than occupies the other empty key-vertices. This implies that $\mathcal{W}$ can occupy at least one key-vertex in each $V_i^h, i = 1,2,\ldots,k^{h-1}$. Since the situation $\mathcal{W}$ follows the Stage (a)-II is happened when $\mathcal{B}$ occupies at least one key-vertex, there exists such a children. If $\mathcal{W}$ follows the case (b), then this is obviously well-defined. So, the keylevel strategy is well-defined. □

Lemma 2. *The keylevel strategy is a winning strategy for $\mathcal{W}$ in a discrete Voronoi game $VG(T,n)$, where T is a sufficient large complete odd k-ary tree so that $N \geq 4n^2$.*

Proof. We first argue that $\mathcal{W}$ follows the case (a), or $\alpha > 1 + \frac{2}{k} - \frac{1}{k-1} + \frac{1}{k^{h+\gamma}(k-1)}$. When the game ends in the Stage (a)-I (i.e., $\mathcal{B}$ never occupies any key-vertices, or does not occupy so many key-vertices), the best strategy of $\mathcal{B}$ follows, occupying all vertices in level $h-1$ for the first k^{h-1} rounds, and then occupying a child of key-vertex dominated by $\mathcal{W}$ to dominate as much vertices as possible with his remaining moves. In fact, the winner dominates more leaves than that of the opposite. So, it is not so significant to occupy the vertices in a level strictly greater than $h+1$, and strictly less than $h-1$.

Now, we estimate their outcomes $|\mathcal{V}_{\mathcal{W}}|$ and $|\mathcal{V}_{\mathcal{B}}|$. Firstly, $\mathcal{W}$ dominates nT_h vertices and $\mathcal{B}$ dominates $(k^h - n)T_h + \frac{k^h-1}{k-1}$ vertices. Since $\mathcal{B}$ dominates the subtrees of $\mathcal{W}$ with his remaining $n - k^{h-1}$ vertices,

$$|\mathcal{V}_{\mathcal{W}}| = nT_h - (n - k^{h-1})\, T_{h+1},$$

$$|\mathcal{V}_{\mathcal{B}}| \leq (k^h - n)\, T_h + (n - k^{h-1})\, T_{h+1} + \frac{k^h - 1}{k-1}.$$

Since $2n = \alpha k^h$ and $\alpha > 1 + \frac{2}{k} - \frac{1}{k-1} + \frac{1}{k^{h+\gamma}(k-1)}$,

$$
\begin{aligned}
|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| &\geq nT_h - 2(n - k^{h-1})\, T_{h+1} - (k^h - n)\, T_h - \frac{k^h - 1}{k-1} \\
&> (k^{h+1}\alpha + 2k^{h-1} - k^h\alpha - k^{h+1})T_{h+1} - \frac{k^h - 1}{k-1} \\
&\geq \frac{1}{k^\gamma}\, T_{h+1} - \frac{k^h - 1}{k-1}. \qquad (1)
\end{aligned}
$$

By the definition of γ with $\gamma = H - 2h$,

$$
\begin{aligned}
\frac{1}{k^\gamma}\, T_{h+1} - \frac{k^h - 1}{k-1} &= \frac{1}{k^\gamma}(kT_{h+2} + 1) - \frac{k^h - 1}{k-1} = \frac{1}{k^\gamma}\left(k^2 T_{h+3} + k + 1\right) - \frac{k^h - 1}{k-1} \\
&= \frac{1}{k^\gamma}\left(k^i\, T_{h+1+i} + \sum_{j=0}^{i-1} k^j\right) - \frac{k^h - 1}{k-1}, \quad (i = 1, 2, \ldots, H - h - 1) \\
&= \frac{1}{k^\gamma}\,\frac{k^{H-h} - 1}{k-1} - \frac{k^h - 1}{k-1} = \frac{1}{k^\gamma}\,\frac{k^{(2h+\gamma)-h} - 1}{k-1} - \frac{k^h - 1}{k-1} \\
&= \frac{1}{k-1}\left(1 - \frac{1}{k^\gamma}\right) > 0.
\end{aligned}
$$

Next, we consider the case that $\mathcal{W}$ follows Stage (a)-II. At level greater than h, there are three types of $\mathcal{B}$'s occupation (see Fig. 3). In cases (2) and (3) of Fig. 3, $\mathcal{B}$ has no profits. Therefore, when $\mathcal{B}$ uses his best strategy, we

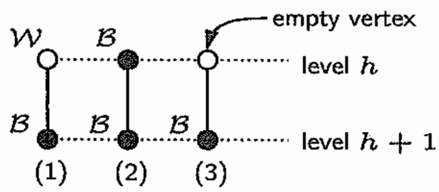

Fig. 3. $\mathcal{B}$'s occupations at the level greater than h.

can assume that $\mathcal{B}$ only occupies vertices under $\mathcal{W}$'s vertices. This implies that $\mathcal{B}$ tries to perform the similar strategy of $\mathcal{W}$, that is to occupy much key-vertices. More precisely, $\mathcal{B}$ chooses his move from following ways at every round:

- $\mathcal{B}$ occupies an empty key-vertex, or
- occupies a vertex v in level $h+1$, where the parent of v is a key-vertex of $\mathcal{W}$, or
- occupies a vertex w in level $h+1$, where the parent of w is a key-vertex of $\mathcal{B}$.

This implies that almost all key-vertices are occupied by either $\mathcal{W}$ or $\mathcal{B}$, and then the subtree of T consisted by the vertices in level 0 through $h-1$ is negligible small so that these vertices cannot have much effect on outcomes of $\mathcal{W}$ and $\mathcal{B}$. It is not significant to the occupation of these vertices for both players.

Let x_i (resp. y_i) be the number of vertices occupied by $\mathcal{W}$ (resp. $\mathcal{B}$) in level i. Let y_i^+ (resp. y_i^-) be the number of vertices occupied by $\mathcal{B}$ in higher (resp. lower) than or equal to level i.

When Stage (a)-I ends, $\mathcal{W}$ has x_h key-vertices and $\mathcal{B}$ has y_h key-vertices. Note that $x_h + y_h \leq k^h$ and $y_h < \lceil \frac{k^h}{2} \rceil \leq x_h < n$. x_{h+1} is the number of vertices occupied in Stage (a)-II. Let y'_{h+1} be the number of occupations used to dominate vertices of $\mathcal{W}$'s dominance set by $\mathcal{B}$ in level $h+1$, and y''_{h+1} be $y_{h+1} - y'_{h+1}$. (see Fig. 4). Note that $x_h - y_h \geq y'_{h+1} - x_{h+1}$ (it has equality if $y''_{h+1} + y^-_{h-1} + y^+_{h+2} = 0$.) Now, we estimate their outcomes. Since $\mathcal{W}$ can dominate at least $x_h T_h + (x_{h+1} - y'_{h+1})T_{h+1}$ vertices, and $\mathcal{W}$ dominates $y_h T_h + (y'_{h+1} - x_{h+1})T_{h+1}$ vertices, the difference between the outcomes of $\mathcal{W}$ and $\mathcal{B}$ is

$$
\begin{aligned}
|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| &= x_h T_h + (x_{h+1} - y'_{h+1})T_{h+1} - y_h T_h - (y'_{h+1} - x_{h+1})T_{h+1} \\
&\geq \left(k(x_h - y_h) - 2(y'_{h+1} - x_{h+1})\right)T_{h+1} > T_{h+1} > 0.
\end{aligned}
$$

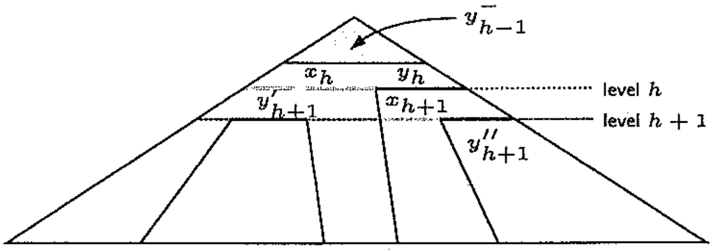

Fig. 4. The notations in the case (a) of keylevel strategy.

$\mathcal{W}$ can dominates at least T_{h+1} vertices more than that of $\mathcal{B}$, which is more vertices dominated by $\mathcal{B}$ using y_0 vertices between level 0 and h. So, $\mathcal{W}$ wins when $\alpha > 1+\frac{2}{k}-\frac{1}{k-1}+\frac{1}{k^{h+\gamma}(k-1)}$.

We next argue that $\mathcal{W}$ follows the case (b), or $\alpha \le 1+\frac{2}{k}-\frac{1}{k-1}+\frac{1}{k^{h+\gamma}(k-1)}$. When $x_{h-1}=k^{h-1}$, the best strategy for $\mathcal{B}$ is to occupied as much key-vertex as possible. So, the differences of outcomes are estimated as follow;

$$\begin{aligned}|\mathcal{V}_{\mathcal{W}}|-|\mathcal{V}_{\mathcal{B}}| &= (k^h-2n)\,T_h+2(n-k^{h-1})\,T_{h+1}+\frac{k^h-1}{k-1}\\ &\ge (k^{h+1}-2k^{h-1}-k^h(k-1)\alpha)T_{h+1}+2\cdot\frac{k^h-1}{k-1}\\ &\ge 2\cdot\frac{k^h-1}{k-1}-\frac{1}{k^\gamma}T_{h+1} = 2\frac{k^h-1}{k-1}-\frac{1}{k^\gamma}\frac{k^{h+\gamma}-1}{k-1}=\frac{1}{k-1}\left(k^h-2+\frac{1}{k^\gamma}\right)\\ &> 0.\end{aligned}$$

Finally, we consider the case of $\alpha < 1+\frac{2}{k}-\frac{1}{k-1}+\frac{1}{k^{h+\gamma}(k-1)}$ and $x_{h-1}<k^{h-1}$ (or $x_{h-1}+y_{h-1}=k^{h-1}$). In this case, the similar arguments in which $\mathcal{W}$ follows Stage (a)-II can be applied. Each x_{h-1}, x_h, and x_{h+1} is the number of vertices occupied in Stage (b)-I, (b)-II, and (b)-III, respectively. As mentioned above, y^-_{h-2} and y^+_{h+2} should be 0 to maximize his outcome $|\mathcal{V}_{\mathcal{B}}|$. Let y'_h be the number of key-vertices occupied by $\mathcal{B}$ whose parent is occupied by $\mathcal{W}$, and $y''_h = y_h - y'_h$. Fig. 5 shows these notations. If $\mathcal{W}$ does not follows Stage (b)-III, then $\mathcal{W}$ wins since $x_{h-1}-y_{h-1}\ge y'_h - x_h$

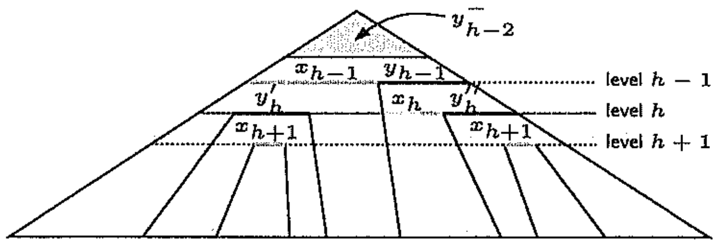

Fig. 5. The notations in the case (b) of keylevel strategy.

and $k(x_{h-1}-y_{h-1})-2(x_h-y'_h)>0$. If $\mathcal{W}$ follows Stage (b)-III, then we have $y_{h-1}+y'_h+y''_h\le n$, $x_h+y''_h=y_{h-1}$, and $x_{h-1}>\frac{1}{2}k^{h-1}>y_{h-1}$ by the keylevel strategy. We can estimate the outcome of $\mathcal{W}$ as follows;

$$\begin{aligned}|\mathcal{V}_{\mathcal{W}}|-|\mathcal{V}_{\mathcal{B}}| &= x_{h-1}T_{h-1}+(x_h-2y'_h-y''_h)T_h+2x_{h+1}T_{h+1}\\ &> kx_{h-1}+x_h-2y'_h-y''_h\\ &\ge k^h+2(k^{h-1}-x_{h-1})-\alpha k^h \ge \frac{k^{h-1}}{k-1}-\frac{1}{k^\gamma(k-1)}\\ &> 0.\end{aligned}$$

Therefore, the first player $\mathcal{W}$ wins when he follows case (b) in the keylevel strategy. This completes the proof of Lemma 2. □

3.2 Discrete Voronoi game on a large complete even k-ary tree

We consider the case that the game arena T is a large complete even k-ary tree. We assume that the game $VG(T,n)$ is sufficed $k > 2n$, since $\mathcal{W}$ always wins if $k \le 2n$ as mentioned above. Moreover, we assume that game arena T contains at least $4n^2$ vertices. Hence the first player $\mathcal{W}$ always loses if he occupies the root of T, since the second player $\mathcal{B}$ can use the keylevel strategy of $\mathcal{W}$ and $\mathcal{W}$ cannot drive $\mathcal{B}$ in disadvantage.

In fact, since T is an even k-ary tree, $\mathcal{B}$ can take the symmetric moves of $\mathcal{W}$ if $\mathcal{W}$ does not occupy the root. Therefore, $\mathcal{B}$ never loses. However, we can show that $\mathcal{W}$ also never loses if he follows the keylevel strategy.

If $\mathcal{B}$ has a winning strategy, then the strategy must not be the symmetric strategy of $\mathcal{W}$. However, such a strategy does not exist, since $\mathcal{W}$ can occupy at least half of vertices on the important level, although the important level is varied by the condition $\alpha > 1 + \frac{2}{k} - \frac{1}{k-1} + \frac{1}{k^{h+\gamma}(k-1)}$. This implies that $\mathcal{W}$ can dominate at least half vertices of T if he follows the keylevel strategy. Therefore, if both players do their best, then the game always ends in tie.

4 $\mathcal{NP}$-hardness for general graphs

In this section, we show that the discrete Voronoi game is intractable on general graphs even if we restrict ourselves to the one-round case. To show this, we consider the following special case:

Problem 1:

Input: A graph $G = (V,E)$, a vertex $u \in V$, and n.
Output: Determine whether $\mathcal{B}$ has the winning strategy on G by n occupations after just one occupation of u by $\mathcal{W}$.

That is, $\mathcal{W}$ first occupies u, and never occupy any more, and $\mathcal{B}$ can occupy n vertices in any way. Then we have the following Theorem:

Theorem 2. *Problem 1 is $\mathcal{NP}$-complete.*

Proof. It is clear Problem 1 is in $\mathcal{NP}$. Hence we prove the completeness by showing the polynomial time reduction from a restricted 3SAT such that each variable appears at most three times in a given formula [5, Proposition 9.3]. Let F be a given formula with the set W of variables $\{x_1,x_2,\ldots,x_n\}$ and the set C of clauses $\{c_1,c_2,\ldots,c_m\}$, where $n = |W|$ and $m = |C|$. Each clause contains at most 3 literals, and each variable appears at most 3 times. Hence we have $3n \ge m$.

Now we show a construction of G. Let $W^+ := \{x_i^+ \mid x_i \in W\}$, $W^- := \{x_i^- \mid x_i \in W\}$, $Y := \{y_i^j \mid i \in \{1,2,\ldots,n\}, j \in 1,2,3\}$, $Z := \{z_i^j \mid i \in \{1,2,\ldots,n\}, j \in 1,2,3\}$, $C' := \{c'_1,c'_2,\ldots,c'_m\}$, $D := \{d_1,d_2,\ldots,d_{2n-2}\}$. Then the set of vertices of G is defined by $V := \{u\} \cup W^+ \cup W^- \cup Y \cup Z \cup C \cup C' \cup D$. The set of edges E is defined by the union of the following edges; $\{\{u,z\} \mid z \in Z\}$, $\{\{y_i^j,z_i^j\} \mid y_i^j \in Y, z_i^j \in Z \text{ with } 1 \le i \le n, 1 \le j \le 3\}$, $\{\{x_i^+,y_i^j\} \mid x_i^+ \in W^+, y_i^j \in Y \text{ with } 1 \le i \le n, 1 \le j \le 3\}$, $\{\{x_i^-,y_i^j\} \mid x_i^- \in W^+, y_i^j \in Y \text{ with } 1 \le i \le n, 1 \le j \le 3\}$, $\{\{x_i^+,c_j\} \mid x_i^+ \in W^+, c_j \in C$ if c_j contains literal $x_i\}$, $\{\{x_i^-,c_j\} \mid x_i^- \in W^-, c_j \in C$ if c_j contains literal $\bar{x}_i\}$, $\{\{c_j,c'_j\} \mid c_j \in C, c'_j \in C' \text{ with } 1 \le j \le m\}$, $\{\{c'_j,u\} \mid c'_j \in C' \text{ with } 1 \le j \le m\}$, and $\{\{u,d_i\} \mid d_i \in D \text{ with } 1 \le i \le 2n-2\}$.

An example of the reduction for the formula $F = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$ is depicted in Fig. 6: Small white and black circles are the vertices in Z and Y, respectively, large black circles are the vertices in $W^+ \cup W^-$, black and white rectangles are the vertices in C and C', respectively, two white large diamonds are the same vertex u, and small diamonds are the vertices in D. It is easy to see that G contains $10n + 2m - 1$ vertices, and hence the reduction can be done in polynomial time.

Now we show that F is satisfiable if and only if $\mathcal{B}$ has a winning strategy. We first observe that for $\mathcal{B}$, occupying the vertices in $W^+ \cup W^-$ gives more outcome than occupying the vertices in $Y \cup Z \cup C \cup C'$. More precisely, occupying either x_i^+ or x_i^- for each i with $1 \le i \le n$, $\mathcal{B}$ dominates all vertices in $W^+ \cup W^- \cup Y$, and it is easy to see that any other ways archive less outcome. Therefore, we can assume that $\mathcal{B}$ occupies one of x_i^+ and x_i^- for each i with $1 \le i \le n$.

When there is an assignment $(a_1,a_2,\ldots,a_n)$ that satisfies F, $\mathcal{B}$ can also dominates all vertices in C by occupying x_i^+ if $a_i = 1$, and occupying x_i^- if $a_i = 0$. Hence, $\mathcal{B}$ dominates $5n + m$ vertices in the case, and then $\mathcal{W}$ dominates all vertices in Z, C' and D, that is, $\mathcal{W}$ dominates $1 + 3n + m + 2n - 2 = 5n + m - 1$ vertices. Therefore, $\mathcal{B}$ wins if F is satisfiable.

On the other hand, if F is unsatisfiable, $\mathcal{B}$ can dominate at most $5n + m - 1$ vertices. In the case, the vertex in C corresponding to the unsatisfied clause is dominated by u. Thus $\mathcal{W}$ dominates at least $5n + m$ vertices, and hence $\mathcal{W}$ wins if F is unsatisfiable.

Therefore, Problem 1 is $\mathcal{NP}$-complete. □

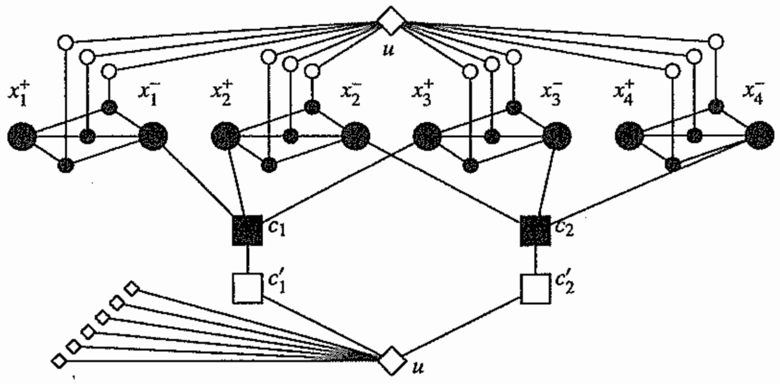

Fig. 6. Reduction from $F = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$

Next we show that the discrete Voronoi game is $\mathcal{NP}$-hard even in the one-round case. More precisely, we show the $\mathcal{NP}$-completeness of the following problem:

Problem 2:

Input: A graph $G = (V, E)$, a vertex set $S \subseteq V$ with $n := |S|$.

Output: Determine whether $\mathcal{B}$ has the winning strategy on G by n occupations after n occupations of the vertices in S by $\mathcal{W}$.

Corollary 1. *Problem 2 is $\mathcal{NP}$-complete.*

Proof. We use the same reduction in the proof of Theorem 2. Let S be the set that contains u and $(n-1)$ vertices in D. Then we immediately have $\mathcal{NP}$-completeness of Problem 2. □

Corollary 2. *The (n-round) discrete Voronoi game on a general graph is $\mathcal{NP}$-hard.*

5 Concluding Remarks and Further Researches

We give winning strategies for the first player $\mathcal{W}$ on the discrete Voronoi game $VG(T, n)$, where T is a large complete k-ary tree with odd k. It seems that $\mathcal{W}$ has an advantage even if the complete k-ary tree is not large, which is a future work.

In our strategy, it is essential that each subtree of the same depth has the same size. Therefore, considering general trees is the next problem. The basic case is easy: When $n = 1$, the discrete Voronoi game on a tree is essentially equivalent to find a *median* vertex of a tree. The deletion of a median vertex partitions the tree so that no component contains more than $n/2$ of the original n vertices. It is well known that a tree has either one or two median vertices, which can be found in linear time (see, e.g. [4]). In the former case, $\mathcal{W}$ wins by occupying the median vertex. In the later case, two players tie. This algorithm corresponds to our Algorithm 1.

We also show that the discrete Voronoi game is intractable on a general graph even if we restrict to the one-round case. We conjecture that the discrete Voronoi game on a general graph is $PSPACE$-complete in n-round case.

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