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Kripke completeness of some distributive substructural logics

By TOMOYUKI SUZUKI

A thesis submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
Graduate Program in Information Science

Written under the direction of
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March, 2007

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Contents

1 Introduction

Substructural logics are obtained by deleting some or all of structural rules from formal systems LK, LJ introduced by Gentzen. Through researching substructural logics, we can consider the relation between logical properties and structural rules. Moreover, substructural logics are including well researched logics, like many-valued logics, fuzzy logics, relevance logics, etc. A semantics for substructural logics is usually based on algebras and a lot of logical properties are proved by algebraic methods (see [15] or [16]).

One of main reasons we introduce algebraic semantics is the Lindenbaum-Tarski technique, yielding almost immediate soundness and completeness results, although some authors are not satisfied with this kind of completeness (e.g. [21]). On the other hand, relational semantics introduced by Kripke are recently subject of intensive research because of their intuitive character and connection with applicative structures like automata or transition systems in computer science. They are particularly popular in modal logic and intuitionistic logic (see [2], [4] or [13]).

Although it may seem these two types of semantics have nothing in common, Stone's representation theorem provides a bridge between algebraic semantics and relational semantics. For example, it is known that relational completeness results for *canonical* modal logics can be immediately proved using Stone's duality.

In recent years, several relational semantics for substructural or other logics were introduced [8], [10], [11] or [12]. These results were mostly based on Priestley duality. On the other hand, relevance logics which form a subclass of substructural logics possess a relational semantics, called Routley-Meyer semantics. Urquhart studied the duality between relevance algebras and Routley-Meyer semantics in [20]. In addition, in [18] and [19], a relational semantics for relavance modal logics is defined and Sahlqvist theorem is also proved.

But, there are relatively few results for distributive substructural logics. Distinct points of our approach are as follows:

- Since our relational semantics based not on Pristley's duality but Stone's one, our relational semantics consist of just one underlying set and just one ternary relation.
- Moreover, the single ternary relation provides an interpretation for almost all connectives, that is, $\vee$, $\wedge$, $\circ$, $\backslash$ and $/$.
- Because of its simplicity (one set & one relation), our semantics resembles Kripke semantics for modal logics, which allows for easier transfer of methods and techniques from the well-developed metatheory of those systems.

2 Logic

This chapter introduces basic concepts and terminology of logics.

2.1 Basic sequent calculus

Formulas are built from propositional variables (in this paper, countably many propositional variables are only considered), special constants 1, 0, $\top$ and $\bot$, and logical connectives. In this paper, $p, q, r, \ldots$ are used for propositional variables (Φ: the set of all propositional variables), $\phi, \psi, \chi, \ldots$ formulas, φ a formula which may be empty, and $\Gamma, \Sigma, \Delta, \Xi$ sequences or sets of formulas which may be empty. Moreover, as logical connectives, $\vee$, $\wedge$, $\rightarrow$, $\neg$, $\circ$, $\backslash$, $/$, $\Box$ and $\Diamond$ are used, where $\vee$, $\wedge$, $\rightarrow$ and $\neg$ are considered as "or", "and", "implication" and "not", repsectively, and $\Box$ and $\Diamond$ are "modal operators". We define two types of formulas as follows.

Definition 2.1 (Formula)
A formula ϕ is a *FL formula*, if

$$\phi ::= p \mid 1 \mid 0 \mid \top \mid \bot \mid \psi \vee \chi \mid \psi \wedge \chi \mid \psi \circ \chi \mid \psi \backslash \chi \mid \chi / \psi.$$

A formula ϕ is a *modal formula*, if

$$\phi ::= p \mid \top \mid \bot \mid \psi \vee \chi \mid \psi \wedge \chi \mid \psi \rightarrow \chi \mid \neg\psi \mid \Box\psi \mid \Diamond\psi.$$

We denote $\mathrm{Frm}_{\mathrm{FL}}(\Phi)$ ($\mathrm{Frm}_{\Box}(\Phi)$) as the set of all FL formulas (modal formulas).

In later section, we often call them just formulas, and $\mathrm{Frm}(\Phi)$ denotes the set of all formulas. In this paper, as logical (proof) system, sequent calculi introduced by Gentzen are mainly used. Sequent calculi usually calculate a type of objects, called sequents, instead of formulas. Therefore, we define sequents as follows.

Definition 2.2 (Sequent)
For any sequent Γ (not set) of formulas and a formula φ, an expression $\Gamma \Rightarrow \varphi$ is a *sequent*.

The basic sequent calculus in this paper, known as Full Lambek calculus (denoted by FL), is defined by the following.

Definition 2.3 (Full Lambek calculus)

Initial sequents:

$$\phi \Rightarrow \phi \qquad \Gamma \Rightarrow \top \qquad \Gamma, \bot, \Sigma \Rightarrow \varphi \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

Cut rule:

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \phi, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi}\ (\mathrm{cut})$$

Rules for logical connectives:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi}\ (1\text{ w}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0\text{ w})$$

$$\frac{\Gamma, \phi, \Delta \Rightarrow \varphi \qquad \Gamma, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \vee \psi, \Delta \Rightarrow \varphi}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \phi, \Delta \Rightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \circ \psi, \Delta \Rightarrow \varphi}\ (\circ \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \phi \circ \psi}\ (\Rightarrow \circ)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Xi, \psi, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \phi\backslash\psi, \Delta \Rightarrow \varphi}\ (\backslash \Rightarrow) \qquad \frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi\backslash\psi}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Xi, \psi, \Delta \Rightarrow \varphi}{\Xi, \psi/\phi, \Gamma, \Delta \Rightarrow \varphi}\ (/ \Rightarrow) \qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \psi/\phi}\ (\Rightarrow /)$$

Sequent calculi consist of initial sequents and inference rules. Initial sequents are starting points of calculations. In other words, we have to start any calculation with this type of sequents. Inference rules express what kind of inferences are permitted in a system. Each inference rule has one or two sequents above the line, called the upper sequents, and has one sequent below the line, called the lower sequent. Then, inference rules permit to derive the lower sequent, if all upper sequents have already been obtained.

We define a sequent $\Gamma \Rightarrow \varphi$ to be *provable* in a sequent calculus S (denoted by $\vdash_{\mathrm{S}} \Gamma \Rightarrow \varphi$, if not, denoted by $\nvdash_{\mathrm{S}} \Gamma \Rightarrow \varphi$) if $\Gamma \Rightarrow \varphi$ can be derived in the system. In other words, in the system, a diagram, called a proof, can be drawn. For example, see the three diagrams below.

$$\dfrac{\dfrac{\phi \Rightarrow \phi \qquad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \phi \circ \psi}\ (\Rightarrow \circ)}{\psi \Rightarrow \phi\backslash(\phi \circ \psi)}\ (\Rightarrow \backslash) \qquad \dfrac{\dfrac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \psi \circ \psi}\ (\Rightarrow \circ)}{\psi \Rightarrow \phi\backslash(\psi \circ \psi)}\ (\Rightarrow \backslash) \qquad \dfrac{\dfrac{\phi \Rightarrow \phi \qquad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \phi \circ \psi}\ (\Rightarrow \circ)}{\phi \Rightarrow \psi\backslash(\phi \circ \psi)}$$

The left diagram is a proof in FL, but the other two diagrams are not proofs in FL, because the middle diagram does not start with initial sequents and, in the right diagram, the last inference rule is not permitted in FL. For example, a sequent $(\phi \wedge \psi) \vee (\phi \wedge \chi) \Rightarrow \phi \wedge (\psi \vee \chi)$ is provable in FL, because the following diagram is a proof of $(\phi \wedge \psi) \vee (\phi \wedge \chi) \Rightarrow \phi \wedge (\psi \vee \chi)$.

$$\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \wedge \psi \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \quad \dfrac{\dfrac{\psi \Rightarrow \psi}{\phi \wedge \psi \Rightarrow \psi}\,(\wedge 2 \Rightarrow)}{\phi \wedge \psi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 1)}{\phi \wedge \psi \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\Rightarrow \wedge) \quad \dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \wedge \chi \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \quad \dfrac{\dfrac{\chi \Rightarrow \chi}{\phi \wedge \chi \Rightarrow \chi}\,(\wedge 2 \Rightarrow)}{\phi \wedge \chi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 2)}{\phi \wedge \chi \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\Rightarrow \wedge)}{(\phi \wedge \psi) \vee (\phi \wedge \chi) \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\vee \Rightarrow)$$

In the above diagram, every top sequent is an initial sequent of FL, and each inference rule is defined in FL. On the other hand, we can show $\nvdash_{\mathrm{FL}} \phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$. For any formula ϕ, if $\Rightarrow \phi$ is provable in a system S, ϕ is *provable* in it (denoted by $\vdash_{\mathrm{S}} \phi$, if not , denoted by $\nvdash_{\mathrm{S}} \phi$.).

Other basic sequent calculi are obtained from FL, by adding some or all of the following rules, called structural rules.

Definition 2.4 (Structural rules)
Structural rules:

$$\dfrac{\Gamma, \phi, \phi, \Delta \Rightarrow \varphi}{\Gamma, \phi, \Delta \Rightarrow \varphi}\,(\mathrm{c} \Rightarrow) \qquad \dfrac{\Gamma, \phi, \psi, \Delta \Rightarrow \varphi}{\Gamma, \psi, \phi, \Delta \Rightarrow \varphi}\,(\mathrm{e} \Rightarrow)$$

$$\dfrac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \phi, \Delta \Rightarrow \varphi}\,(\mathrm{w} \Rightarrow) \qquad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \phi}\,(\Rightarrow \mathrm{w})$$

The rules (c $\Rightarrow$), (e $\Rightarrow$), (w $\Rightarrow$) and ($\Rightarrow$ w) are called *the contraction rule*, *the exchange rule*, *the left weakening rule* and *the right weakening rule*, respectively. Sometimes, we call both the left and the right weakening rule just *the weakening rules*. If FL has some or all of structural rules, the rules defined in the system are denoted by the initials as indexes. For example, FL_w denotes the sequent calculus FL having the weakening rules. Besides, it is known that each structural rule can be replaced by the following initial sequents.

(c $\Rightarrow$) : $\phi \Rightarrow \phi \circ \phi$.

(e $\Rightarrow$) : $\phi \circ \psi \Rightarrow \psi \circ \phi$.

(w $\Rightarrow$) : $\phi \Rightarrow 1$.

($\Rightarrow$ w) : $0 \Rightarrow \phi$.

We note that the intuitionistic sequent calculus LJ, introduced by Gentzen, is FL_{cew}, since we can consider $\circ$ as $\wedge$. Besides, FL_{cw} can prove exactly the same sequents which are provable in FL_{cew}, because the exchange rule can be derived in FL_{cw}. See below.

$$\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\psi, \phi \Rightarrow \phi}\,(\mathrm{w} \Rightarrow) \quad \dfrac{\psi \Rightarrow \psi}{\psi, \phi \Rightarrow \psi}\,(\mathrm{w} \Rightarrow)}{\psi, \phi \Rightarrow \phi \wedge \psi}\,(\Rightarrow \wedge) \quad \dfrac{\dfrac{\dfrac{\Gamma, \phi, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \wedge \psi, \psi, \Delta \Rightarrow \varphi}\,(\wedge 1 \Rightarrow)}{\Gamma, \phi \wedge \psi, \phi \wedge \psi, \Delta \Rightarrow \varphi}\,(\wedge 2 \Rightarrow)}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi}\,(\mathrm{c} \Rightarrow)}{\Gamma, \psi, \phi, \Delta \Rightarrow \varphi}\,(\mathrm{cut})$$

Sometimes, we use sequents of the form $\Gamma \Rightarrow \varphi$ with a set Γ of formulas, when, for example, we introduce a sequent system for intuitionistic logic. In this formulation, we do not need to use both the exchange and contraction rules. On the other hand, our purpose of the present thesis is to clarify roles of each structural rule explicitly. Therefore, we will not take such a formulation here. For example, although, as we mentioned before, $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is not provable in FL, but it is provable in LJ, as the following proof shows.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \psi \Rightarrow \phi}\,(\mathrm{w} \Rightarrow) \quad \dfrac{\psi \Rightarrow \psi}{\phi, \psi \Rightarrow \psi}\,(\mathrm{w} \Rightarrow)}{\phi, \psi \Rightarrow \phi \wedge \psi}\,(\Rightarrow \wedge)}{\phi, \psi \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\Rightarrow \vee 1) \quad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \chi \Rightarrow \phi}\,(\mathrm{w} \Rightarrow) \quad \dfrac{\chi \Rightarrow \chi}{\phi, \chi \Rightarrow \chi}\,(\mathrm{w} \Rightarrow)}{\phi, \chi \Rightarrow \phi \wedge \chi}\,(\Rightarrow \wedge)}{\phi, \chi \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\Rightarrow \vee 2)}{\phi, \psi \vee \chi \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\vee \Rightarrow)}{\phi \wedge (\psi \vee \chi), \psi \vee \chi \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\wedge 1 \Rightarrow)}{\phi \wedge (\psi \vee \chi), \phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\wedge 2 \Rightarrow)}{\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\mathrm{c} \Rightarrow)
$$

The contraction rule and the left weakening rule are essentially used in this proof. Like this, adding structural rules may increase the number of provable sequents.

The exchange rule (e $\Rightarrow$) removes the difference between $\backslash$ and $/$, since we can derive the inference rules for $/$, by using the rules for $\backslash$ and the exchange rule, as shown below.

$$
\dfrac{\dfrac{\Gamma \Rightarrow \phi \qquad \Xi, \psi, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \phi\backslash\psi, \Delta \Rightarrow \varphi}\,(\backslash \Rightarrow)}{\Xi, \phi\backslash\psi, \Gamma, \Delta \Rightarrow \varphi}\,(\mathrm{e} \Rightarrow)^{*}
\qquad\qquad
\dfrac{\dfrac{\Gamma, \phi \Rightarrow \psi}{\phi, \Gamma \Rightarrow \psi}\,(\mathrm{e} \Rightarrow)^{*}}{\Gamma \Rightarrow \phi\backslash\psi}\,(\Rightarrow \backslash)
$$

and the converse, too.

$$
\dfrac{\dfrac{\Gamma \Rightarrow \phi \qquad \Xi, \psi, \Delta \Rightarrow \varphi}{\Xi, \psi/\phi, \Gamma, \Delta \Rightarrow \varphi}\,(/ \Rightarrow)}{\Xi, \Gamma, \psi/\phi, \Delta \Rightarrow \varphi}\,(\mathrm{e} \Rightarrow)^{*}
\qquad\qquad
\dfrac{\dfrac{\phi, \Gamma \Rightarrow \psi}{\Gamma, \phi \Rightarrow \psi}\,(\mathrm{e} \Rightarrow)^{*}}{\Gamma \Rightarrow \psi/\phi}\,(\Rightarrow /)
$$

So, as the abbreviation of both $\backslash$ and $/$, $\rightarrow$ is used, for formulas, inference rules, etc. Besides, we define $\neg\phi$ as an abbreviation of $\phi \rightarrow 0$. Then, the following inference rules can be used for the sequent calculi with the exchange rule. (Of course, they are the abbreviations of the original inference rules.)

$$
\dfrac{\Gamma \Rightarrow \phi \qquad \Xi, \psi, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \phi \rightarrow \psi, \Delta \Rightarrow \varphi}\,(\rightarrow\Rightarrow)
\qquad\qquad
\dfrac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \rightarrow \psi}\,(\Rightarrow\rightarrow)
$$

$$
\dfrac{\Gamma \Rightarrow \phi}{\neg\phi, \Gamma \Rightarrow}\,(\neg \Rightarrow)
\qquad\qquad
\dfrac{\Gamma, \phi \Rightarrow}{\Gamma \Rightarrow \neg\phi}\,(\Rightarrow \neg)
$$

It is known that the left (right) weakening rule remove the difference between 1 (0) and $\top$ ($\bot$). It is also known that the contraction and the left weakening remove the difference between $\circ$ and $\wedge$. Therefore, in FL_{cw}, we can consider that a formula are defined by

$$
\phi ::= p \mid \top \mid \bot \mid \psi \vee \chi \mid \psi \wedge \chi \mid \psi \rightarrow \chi \mid \neg\psi.
$$

By adding some initial sequents or inference rules to FL, we obtain *extensions of FL*. For example, it gives us well known classical sequent calculus LK, to add initial sequents $\neg\neg\phi \Rightarrow \phi$, called as the involutivity, to LJ. Of course, all basic sequent calculi are extensions of FL. In addition, we define some other extensions of FL which are used later.

Definition 2.5 (Distributive sequent calculus)
The distributive sequent calculus DFL is obtained by adding initial sequents (Distributivity) to FL.

(Distributivity) : $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$.

Definition 2.6 (Normal modal sequent calculus)
The normal modal sequent calculus K is obtained by adding a inference rule ($\Box$) to LK.

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ (\Box)$$

Here, a sequence $\Box\Gamma$ denotes $\Box\phi_1, \Box\phi_2, \ldots, \Box\phi_n$, if $\Gamma = \phi_1, \phi_2, \ldots, \phi_n$.

2.2 Substructural logic

To define a logic, we define the following.

Definition 2.7 (Uniform substitution)
Let ρ be a function from Φ to $\mathrm{Frm}(\Phi)$. ρ can be inductively extended from $\mathrm{Frm}(\Phi)$ to $\mathrm{Frm}(\Phi)$ as follows.

1. $\rho(1) := 1$.
2. $\rho(0) := 0$.
3. $\rho(\top) := \top$.
4. $\rho(\bot) := \bot$.
5. $\rho(\phi \vee \psi) := \rho(\phi) \vee \rho(\psi)$.
6. $\rho(\phi \wedge \psi) := \rho(\phi) \wedge \rho(\psi)$.
7. $\rho(\phi \circ \psi) := \rho(\phi) \circ \rho(\psi)$.
8. $\rho(\phi \backslash \psi) := \rho(\phi) \backslash \rho(\psi)$.
9. $\rho(\psi / \phi) := \rho(\psi) / \rho(\phi)$.
10. $\rho(\phi \to \psi) := \rho(\phi) \to \rho(\psi)$.
11. $\rho(\neg\phi) := \neg\rho(\phi)$.

12. $\rho(\Box\phi) := \Box\rho(\phi)$.

13. $\rho(\Diamond\phi) := \Diamond\rho(\phi)$.

For any formula ϕ, we define $\rho(\phi)$ as a *uniform substitution instance of ϕ under ρ.*

Then, a set Σ of formulas is *closed under uniform substitution*, if $\rho(\phi) \in \Sigma$ for any formula $\phi \in \Sigma$.

We define here a logic as a set of formulas.

Definition 2.8 (Logic)
A set Γ of FL formulas is a *substructural logic* over FL, if Γ satisfies the following (see also [15]).

1. Γ contains all FL formulas which are provable in FL.
2. If $\phi \in \Gamma$ and $\phi\backslash\psi \in \Gamma$, then $\psi \in \Gamma$.
3. If $\phi, \psi \in \Gamma$, then $\phi \wedge \psi \in \Gamma$.
4. If $\phi \in \Gamma$, then $\varphi\backslash(\phi \circ \varphi), (\varphi \circ \phi)/\varphi \in \Gamma$, for an arbitrary formula φ.
5. Γ is closed under uniform substitution.

A set Γ of modal formulas is a *normal modal logic*, if Γ satisfies the following.

1. Γ contains all modal formulas which are provable in K.
2. If $\phi \in \Gamma$ and $\phi \to \psi \in \Gamma$, then $\psi \in \Gamma$.
3. Γ is closed under uniform substitution.
4. If $\phi \in \Gamma$, then $\Box\phi \in \Gamma$.

As long as possible, we call a substructural logic and a normal modal logic just a *logic.*

Based on our sequent calculi, we can show the following theorem.

Theorem 2.9
Given a sequent calculus S which is an extension of FL (K), the set ***S*** *of formulas which are provable in S is a logic. In other words, the set* ***S*** *is closed under all of the conditions above.*

Proof
We show only substructural logics here, since we can prove analogously about normal modal logics. We give here a proof of the conditions 2, 3 and 4, because 1 and 5 are obvious.

2. Assume $\phi \in \mathbf{S}$ and $\phi\backslash\psi \in \mathbf{S}$. Then, we can draw a proof as follows.

$$\dfrac{\Rightarrow \phi \qquad \dfrac{\Rightarrow \phi\backslash\psi \qquad \dfrac{\phi \Rightarrow \phi \qquad \psi \Rightarrow \psi}{\phi, \phi\backslash\psi \Rightarrow \psi}\,(\backslash \Rightarrow)}{\phi \Rightarrow \psi}\,(\text{cut})}{\Rightarrow \psi}\,(\text{cut})$$

   Therefore, $\psi \in \mathbf{S}$.

3. Assume $\phi \in \mathbf{S}$ and $\psi \in \mathbf{S}$. Then,

$$\dfrac{\Rightarrow \phi \qquad \Rightarrow \psi}{\Rightarrow \phi \wedge \psi}\,(\Rightarrow \wedge)$$

   Therefore, $\phi \wedge \psi \in \mathbf{S}$.

4. Suppose that φ is an arbitrary formula and $\phi \in \mathbf{S}$. Then, we can prove $\varphi\backslash(\phi \circ \varphi)$ and $(\varphi \circ \phi)/\varphi$, as follows.

$$\dfrac{\dfrac{\Rightarrow \phi \qquad \varphi \Rightarrow \varphi}{\varphi \Rightarrow \phi \circ \varphi}\,(\Rightarrow \circ)}{\Rightarrow \varphi\backslash(\phi \circ \varphi)}\,(\Rightarrow \backslash) \qquad\qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \Rightarrow \phi}{\varphi \Rightarrow \varphi \circ \phi}\,(\Rightarrow \circ)}{\Rightarrow (\varphi \circ \phi)/\varphi}\,(\Rightarrow /)$$

   Therefore, $\varphi\backslash(\phi \circ \varphi) \in \mathbf{S}$ and $(\varphi \circ \phi)/\varphi \in \mathbf{S}$. (Q.E.D)

Thus, we have immediately the following corollary.

Corollary 2.10
The sets of formulas which are provable in FL, FL_c, FL_e, FL_w, FL_{ce}, FL_{ew}, FL_{cew} and (K) are substructural (modal) logics.

In this paper, to distinguish logics from sequent calculi, logics are denoted by boldface letters, like $\mathbf{FL}_{ew}$.

Sometimes, we say "a sequent is in a logic $\mathbf{S}$". But, this sentence makes sense with the following two facts (see [15]).

Fact 1 The following conditions are equivalent in any extension of FL.

- $\vdash_{\text{S}} \phi \Rightarrow \psi$.
- $\vdash_{\text{S}} \phi\backslash\psi$.
- $\vdash_{\text{S}} \psi/\phi$.

Fact 2 For any formula ϕ and a sequent calculus S which is an extension of FL, $\vdash_{\text{S}} \phi$ if and only if $\vdash_{\text{S}} 1 \Rightarrow \phi$.

To define our logics, we define the following.

Definition 2.11 (Extension)
Given a logic **L**, a logic **L'** is an *extension* of **L**, if **L** is a subset of **L'** (**L** $\subseteq$ **L'**).

For example, $\mathbf{FL}_{cew}$ is an extension of **FL**, because of the definition of logics. However, **FL** is not an extension of $\mathbf{FL}_{cew}$, since, as we saw before, $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is provable in FL_{cew} but it is not provable in FL.

Another example is the following. Obviously, $\mathbf{FL}_{cew}$ is an extension of $\mathbf{FL}_{cw}$. On the other hand, $\mathbf{FL}_{cw}$ is also an extension of $\mathbf{FL}_{cew}$, because, as we saw before, the exchange rule can be derived in FL_{cw}. Therefore, $\mathbf{FL}_{cew}$ is equal to $\mathbf{FL}_{cw}$, as a logic.

Definition 2.12 (Axiomatization)
A substructural (normal modal) logic **L** is *axiomatized over a logic* ***S*** by a set Σ of FL (modal) formulas, if **L** is the smallest extension of **S** containing Σ.

For example, we can say that **LK** is axiomatized over **LJ** by $\{\neg\neg p \to p\}$.

Next, we define the distributive substructural logic, as an extension of **FL**.

Definition 2.13 (The distributive substructural logic DFL)
The distributive substructural logic ***DFL*** is the set of all provable formulas in the distributive sequent calculus DFL. In other words, **DFL** is axiomatized over **FL** by the distributivity.

A logic **L** is a *DFL logic*, if **L** is an extension of **DFL**. Besides, **DFL**, $\mathbf{DFL}_c$, $\mathbf{DFL}_e$, $\mathbf{DFL}_w$, $\mathbf{DFL}_{ce}$, $\mathbf{DFL}_{ew}$ and $\mathbf{DFL}_{cew}$ are called *basic*.

The distributive law seems a strong assumption, but DFL logics include many of nonclassical logics like relevance logics, many-valued logics and fuzzy logics.

We define the normal modal logic.

Definition 2.14 (The normal modal logic)
The set of all provable modal formulas in K is *the normal modal logic* ***K***.

It is known that **K** is the smallest normal modal logic. So, every extension of **K** is called *a normal modal logic.*

3 Algebra and algebraic semantics

In this chapter, we introduce algebraic concepts and terminology, and algebraic semantics for logics.

3.1 Algebraic preliminaries

We introduce some basic mathematical concepts (see [1], [3] or [5]).

Definition 3.1 (Lattice)
A structure $\mathfrak{A} = \langle A, \vee, \wedge \rangle$ is a *lattice*, if $\vee$ and $\wedge$, called join and meets, are binary operations on A satisfying the following equations.

1. (Idempotency): $a \vee a = a$, $a \wedge a = a$.
2. (Commutativity): $a \vee b = b \vee a$, $a \wedge b = b \wedge a$.
3. (Associativity): $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
4. (Absorption): $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.

A lattice $\mathfrak{A}$ is called *distributive*, if $\mathfrak{A}$ also satisfies one of the following equations.

5. (Distributivity): $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
6. (Distributivity'): $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$.

In fact, we can show that either of the above two conditions (5 and 6) derives another.

It is well known that a partial order $\leq$ is naturally introduced in every lattice, by defining $\leq$ as follows.

$$a \leq b \iff a \wedge b = a$$

In any lattice, $a \wedge b = a$ is equivalent to $a \vee b = b$, because $b = b \wedge (a \vee b) = (a \vee b) \wedge b = a \vee b$ and $a = a \vee (a \wedge b) = a \wedge b$. Therefore, $a \leq b$ can be also defined by $a \vee b = b$.

If a lattice $\mathfrak{A}$ has a maximum element $\top$ and a minimum element $\bot$, then it is called *bounded.*

Next, we will define some basic notions for lattices.

Definition 3.2 (Filter)
Given a lattice $\mathfrak{A} = \langle A, \vee, \wedge \rangle$, a non-empty subset F of A is a *filter* over $\mathfrak{A}$, if it satisfies, for each $a, b \in A$,

$$a \in F \text{ and } b \in F \iff a \wedge b \in F.$$

Besides, if F also satisfies the following condition and $F \neq A$, it is called *prime.* For each $a, b \in A$,

$$a \in F \text{ or } b \in F \iff a \vee b \in F$$

Given an element $a \in \mathfrak{A}$, the set $\uparrow a := \{b \in A \mid a \leq b\}$ is always a filter, which is called *the principal filter generated by* a.

Definition 3.3 (Ideal)
Given a lattice $\mathfrak{A} = \langle A, \vee, \wedge \rangle$, a non-empty subset I of A is a *ideal* over $\mathfrak{A}$, if it satisfies, for each $a, b \in A$,

$$a \in I \text{ and } b \in I \iff a \vee b \in I.$$

Moreover, given an element $a \in \mathfrak{A}$, the set $\downarrow a := \{b \in A \mid b \leq a\}$ is always a ideal, which is called *the principal ideal generated by* a.

We can prove the following proposition.

Proposition 3.4
Given a non-empty subset S of A, the set $F_S := \{a \in A \mid s_1 \wedge \cdots \wedge s_n \leq a \text{ for some } s_1, \ldots, s_n \in S\}$ is the smallest filter containing S.

Proof
It is clear that F_S is non-empty.

Suppose that $f_1, f_2 \in F_S$. Then, there exist $s_1, \ldots, s_m \in S$ and $t_1, \ldots, t_n \in S$ such that $s_1 \wedge \cdots \wedge s_m \leq f_1$ and $t_1 \wedge \cdots \wedge t_n \leq f_2$ hold. So, $s_1 \wedge \cdots \wedge s_m \wedge t_1 \wedge \cdots \wedge t_n \leq f_1 \wedge f_2$ holds. Therefore, $f_1 \wedge f_2 \in F_S$.

Suppose that $f_1 \wedge f_2 \in F_S$. Then, by the definition, $f_1 \in F_S$ and $f_2 \in F_S$ obviously hold, since $f_1 \wedge f_2 \leq f_1, f_2$. (Q.E.D)

This filter F_S is called the *filter generated by* S. As a corollary, we can show the following.

Corollary 3.5
Given a filter F and an element a, the filter F^a generated by $F \cup \{a\}$ is represented as follows.

$$F^a := \{b \in A \mid a \wedge f \leq b \text{ for some } f \in F\}$$

Since Zorn's lemma is effectively used in later, we introduce the lemma here (see [6]).

Lemma 3.6 (Zorn)
Given a partially ordered set P, if every chain in P has an upper bound in P, then P has a maximal element in P.

Definition 3.7 (Monoid)
A structure $\mathfrak{A} = \langle A, \circ, 1 \rangle$ is a *monoid*, if it satisfies

1. (Associativity): For any $a, b, c \in A$, $a \circ (b \circ c) = (a \circ b) \circ c$.

2. (Identity): For any $a \in A$, $a \circ 1 = 1 \circ a = a$.

Definition 3.8 (Residuated lattice)
A tuple $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1\rangle$ is a *residuated lattice*, if

1. $\langle A, \vee, \wedge\rangle$ is a lattice.
2. $\langle A, \circ, 1\rangle$ is a monoid.
3. $\mathfrak{A}$ satisfies the residuation law.
   (Residuation law): For each $a, b, c \in A$, $a \circ b \leq c \iff b \leq a\backslash c \iff a \leq c/b$.

If $\circ$ is commutative, $a\backslash b$ is equivalent to b/a. In this case, we use the symbol $\to$ instead of $\backslash$ and $/$. So, the residuation law is

(Residuation law'):
For each $a, b, c \in A$, $a \circ b \leq c \iff b \leq a \to c \iff a \leq b \to c$.

Since algebraic semantics for logics are defined on some classes of residuated lattices, we here introduce some classes of residuated lattice (see [10]).

Definition 3.9 (FL-algebra)
A structure $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot\rangle$ is a *FL-algebra*, if $\langle A, \vee, \wedge, \circ, \backslash, /, 1, \top, \bot\rangle$ is a bounded residuated lattice and 0 is an element of A.

We denote the class of all FL-algebras as $\mathfrak{C}^{\mathrm{FL}}$.

Definition 3.10 (DFL-algebra)
A structure $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot\rangle$ is a *DFL-algebra*, if $\mathfrak{A}$ is a FL-algebra and satisfies the distributivity.

We denote the class of all DFL-algebras as $\mathfrak{C}^{\mathrm{DFL}}$. To define algebras for modal logics, we define the following.

Definition 3.11 (Boolean algebra)
A structure $\mathfrak{A} = \langle A, \vee, \wedge, \neg, \top, \bot\rangle$ is a *Boolean algebra*, if $\langle A, \vee, \wedge, \top, \bot\rangle$ is a bounded distributive lattice and $\mathfrak{A}$ satisfies the following conditions.

1. (Complementation): $a \wedge (\neg a) = \bot$, $a \vee (\neg a) = \top$,
2. (Boundedness): $a \vee \bot = a$, $a \vee \top = \top$,

We denote the class of all Boolean algebras as $\mathfrak{C}^{\mathrm{B}}$

Definition 3.12 (Modal algebra)
A structure $\mathfrak{A} = \langle A, \vee, \wedge, \neg, \Box, \top, \bot\rangle$ is a *modal algebra*, if $\langle A, \vee, \wedge, \neg, \top, \bot\rangle$ is a Boolean algebra and $\Box$ a unary operation on A satisfying the following conditions.

1. (Meet preservation): $\Box(a \wedge b) = \Box a \wedge \Box b$.
2. (Top preservation): $\Box\top = \top$.

We denote the class of all modal algebras as $\mathfrak{C}^{\Box}$.

Definition 3.13 (Homomorphism)
Given two algebras $\mathfrak{A} = \langle A, f_1^A, \ldots, f_n^A \rangle$ and $\mathfrak{B} = \langle B, f_1^B, \ldots, f_n^B \rangle$, a function h from $\mathfrak{A}$ to $\mathfrak{B}$ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$, if h satisfies, for any i and any elements $a_1, \ldots, a_m \in A$,

$$h(f_i^A(a_1, \ldots, a_m)) = f_i^B(h(a_1), \ldots, h(a_m)).$$

If there exists a surjective homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then we say that $\mathfrak{B}$ is a *homomorphic image of* $\mathfrak{A}$. Beside, if there exists a bijective (injective) homomorphism, called *isomorphism (embedding)* from $\mathfrak{A}$ to $\mathfrak{B}$, then we say that $\mathfrak{A}$ is *isomorphic (embeddable)* to $\mathfrak{B}$ (denoted by $\mathfrak{A} \cong \mathfrak{B}$).

3.2 Algebraic semantics and soundness

In this section, we define algebraic semantics for logics.

An assignment f on an algebra $\mathfrak{A}$ is a function from Φ to $\mathfrak{A}$. We can inductively extend f to a function from $\mathrm{Frm}(\Phi)$ to $\mathfrak{A}$ as follows.

For FL formulas:

- $f(1) := 1$.
- $f(0) := 0$.
- $f(\top) := \top$.
- $f(\bot) := \bot$.
- $f(\phi \vee \psi) := f(\phi) \vee f(\psi)$.
- $f(\phi \wedge \psi) := f(\phi) \wedge f(\psi)$.
- $f(\phi \circ \psi) := f(\phi) \circ f(\psi)$.
- $f(\phi \backslash \psi) := f(\phi) \backslash f(\psi)$.
- $f(\psi / \phi) := f(\psi) / f(\phi)$.

For modal formulas:

- $f(\top) := \top$.
- $f(\bot) := \bot$.
- $f(\phi \vee \psi) := f(\phi) \vee f(\psi)$.
- $f(\phi \wedge \psi) := f(\phi) \wedge f(\psi)$.

- $f(\phi \to \psi) := f(\phi) \to f(\psi)$.
- $f(\neg\phi) := \neg f(\phi)$.
- $f(\Box\phi) := \Box f(\phi)$.
- $f(\Diamond\phi) := \neg(\Box(\neg f(\phi)))$.

A formula ϕ is *true* in $\mathfrak{A}$ under f (denoted by $\mathfrak{A}, f \models \phi$) is defined by $1 \leq f(\phi)$. If not, we denote $\mathfrak{A}, f \not\models \phi$. For an arbitrary assignment f, if $\mathfrak{A}, f \models \phi$, then ϕ is *valid* on $\mathfrak{A}$ (denoted by $\mathfrak{A} \models \phi$). A set Γ of formulas is valid on $\mathfrak{A}$ (denoted by $\mathfrak{A} \models \Gamma$), if $\mathfrak{A} \models \phi$, for any $\phi \in \Gamma$. On a class $\mathfrak{C}$ of algebras a formula ϕ is valid (denoted by $\mathfrak{C} \models \phi$), if $\mathfrak{A} \models \phi$, for any $\mathfrak{A} \in \mathfrak{C}$.

It is easy to check that a sequent $\Gamma \Rightarrow \varphi$ is *true* in an algebra $\mathfrak{A}$ under f if and only if $f(\Gamma^\circ) \leq f(\varphi)$, where Γ° is the formula $\phi_1 \circ \cdots \circ \phi_n$ if $\Gamma = \phi_1, \ldots, \phi_n$, and the left- (right-) hand side of a sequent is 1 (0), if it is empty.

We note that, for extensions of **LJ** (or **K**), Γ° is $\Gamma^\wedge$, because $\circ$ is equal to $\wedge$.

The following correspondence lemma can be proved (see [15]).

Lemma 3.14
Each structural rule corresponds to the following DFL-algebra conditions, respectively.

(c ⇒) : $a \leq a \circ a$.

(e ⇒) : $a \circ b = b \circ a$.

(w ⇒) : $a \leq 1$ for any $a \in A$.

(⇒ w) : $0 \leq a$ for any $a \in A$.

Then, we can show the soundness theorem for **DFL** and **K**.

Theorem 3.15 (Soundness)
*Every formula in **DFL** (**K**) is valid on the class $\mathfrak{C}^{DFL}$ ($\mathfrak{C}^{\Box}$).*

Proof
Since $\mathfrak{C}^B \subseteq \mathfrak{C}^{\mathrm{DFL}}$, we firstly prove the soundness theorem for **DFL**. It suffices to check both every initial sequent is valid and, in each of cut rule and rules for logical connectives of DFL, if the upper sequents are valid, then the lower sequent is also valid, on any DFL-algebra.

We check here the following. Let $\mathfrak{A}$ be an arbitrary DFL-algebra.

$\phi \Rightarrow \phi$: By definition.

$\Gamma \Rightarrow \top$: Since $\top$ is the maximal element in $\mathfrak{A}$, $f(\Gamma^\circ) \leq f(\top)$ for an arbitrary assignment f on $\mathfrak{A}$. Therefore, $\Gamma \Rightarrow \top$ is valid on $\mathfrak{A}$.

$\Gamma, \bot, \Sigma \Rightarrow \varphi$: Since $\bot$ is the minimal element in $\mathfrak{A}$, $f(\bot) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Sigma^\circ))$ for an arbitrary assignment f on $\mathfrak{A}$. By the residuation law, $f(\Gamma^\circ) \circ f(\bot) \circ f(\Sigma^\circ) \leq f(\varphi)$. Therefore, $\Gamma, \bot, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

$\Rightarrow 1$: By definition.

$0 \Rightarrow$: By definition.

$\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$: For an arbitrary assignment f on $\mathfrak{A}$, by the distributivity, $f(\phi \wedge (\psi \vee \chi)) = f(\phi) \wedge (f(\psi) \vee f(\chi)) \leq (f(\phi) \wedge f(\psi)) \vee (f(\phi) \wedge f(\chi)) = f((\phi \wedge \psi) \vee (\phi \wedge \chi))$. Therefore, $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is valid on $\mathfrak{A}$.

(cut) : Suppose that $\Gamma \Rightarrow \phi$ and $\Sigma, \phi, \Xi \Rightarrow \varphi$ are valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the first assumption, $f(\Gamma^\circ) \leq f(\phi)$. By the second assumption and the residuation law, $f(\phi) \leq f(\Sigma^\circ)\backslash(f(\varphi)/f(\Xi^\circ))$. So, $f(\Gamma^\circ) \leq f(\Sigma^\circ)\backslash(f(\varphi)/f(\Xi^\circ))$. By the residuation law, $f(\Sigma \circ \Gamma \circ \Xi) \leq f(\varphi)$. Therefore, $\Sigma, \Gamma, \Xi \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

(1 w) : Suppose that $\Gamma, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ) \circ f(\Delta^\circ) \leq f(\varphi)$. Since $1 (= f(1))$ is the identity element in $\mathfrak{A}$, $f(\Gamma^\circ) \circ f(1) \circ f(\Delta^\circ) \leq f(\varphi)$. Therefore, $\Gamma, 1, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

(0 w) : By definition.

($\vee \Rightarrow$) : Suppose that $\Gamma, \phi, \Delta \Rightarrow \varphi$ and $\Gamma, \psi, \Delta \Rightarrow \varphi$ are valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumptions and the residuation law, $f(\phi) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$ and $f(\psi) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. So, $f(\phi) \vee f(\psi) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. By the residuation law, $f(\Gamma^\circ \circ (\phi \vee \psi) \circ \Delta^\circ) \leq f(\varphi)$. Therefore, $\Gamma, \phi \vee \psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

($\Rightarrow \vee 1$) : Suppose that $\Gamma \Rightarrow \phi$ is valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ) \leq f(\phi) \leq f(\phi) \vee f(\psi)$. Therefore, $\Gamma \Rightarrow \phi \vee \psi$ is valid on $\mathfrak{A}$.

($\Rightarrow \vee 2$) : Suppose that $\Gamma \Rightarrow \psi$ is valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ) \leq f(\psi) \leq f(\phi) \vee f(\psi)$. Therefore, $\Gamma \Rightarrow \phi \vee \psi$ is valid on $\mathfrak{A}$.

($\wedge 1 \Rightarrow$) : Suppose that $\Gamma, \phi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ \circ \phi \circ \Delta^\circ) \leq f(\varphi)$. By the residuation law, $f(\phi) \wedge f(\psi) \leq f(\phi) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. So, by the residuation law, $f(\Gamma^\circ \circ (\phi \wedge \psi) \circ \Delta^\circ) \leq f(\varphi)$. Therefore, $\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

($\wedge 2 \Rightarrow$) : Suppose that $\Gamma, \psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ \circ \psi \circ \Delta^\circ) \leq f(\varphi)$. By the residuation law,

$f(\phi) \wedge f(\psi) \leq f(\psi) \leq f(\Gamma^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. So, by the residuation law, $f(\Gamma^\circ \circ (\phi \wedge \psi) \circ \Delta^\circ) \leq f(\varphi)$. Therefore, $\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

$(\Rightarrow \wedge)$: Suppose that $\Gamma \Rightarrow \phi$ and $\Gamma \Rightarrow \psi$ are valid on $\mathfrak{A}$. For an arbitrary assignment f on $\mathfrak{A}$, by the assumptions, $f(\Gamma^\circ) \leq f(\phi)$ and $f(\Gamma^\circ) \leq f(\psi)$. So, $f(\Gamma^\circ) \leq f(\phi) \wedge f(\psi)$. Therefore, $\Gamma \Rightarrow \phi \wedge \psi$ is valid on $\mathfrak{A}$.

$(\circ \Rightarrow)$: By definition.

$(\Rightarrow \circ)$: By the monotonicity of $\circ$.

$(\backslash \Rightarrow)$: Assume that $\Gamma \Rightarrow \phi$ and $\Xi, \psi, \Delta \Rightarrow \varphi$ are valid. For an arbitrary assignment f on $\mathfrak{A}$, by the first assumption and the monotonicity of $\circ$, $f(\Gamma^\circ) \circ f(\phi\backslash\psi) \leq f(\phi) \circ f(\phi\backslash\psi) \leq f(\psi)$. By the second assumption and the residuation law, $f(\psi) \leq f(\Xi^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. So, by the residuation law, $f(\Xi^\circ) \circ f(\Gamma^\circ) \circ f(\phi\backslash\psi) \circ f(\Delta^\circ) \leq f(\varphi)$. Therefore, $\Xi, \Gamma, \phi\backslash\psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

$(\Rightarrow \backslash)$: Assume that $\phi, \Gamma \Rightarrow \psi$ is valid. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\phi) \circ f(\Gamma^\circ) \leq f(\psi)$. By the residuation law, $f(\Gamma^\circ) \leq f(\phi)\backslash f(\psi)$. Therefore, $\Gamma \Rightarrow \phi\backslash\psi$ is valid on $\mathfrak{A}$.

$(/ \Rightarrow)$: Assume that $\Gamma \Rightarrow \phi$ and $\Xi, \psi, \Delta \Rightarrow \varphi$ are valid. For an arbitrary assignment f on $\mathfrak{A}$, by the first assumption and the monotonicity of $\circ$, $f(\psi/\phi) \circ f(\Gamma^\circ) \leq f(\psi/\phi) \circ f(\phi) \leq f(\psi)$. By the second assumption and the residuation law, $f(\psi) \leq f(\Xi^\circ)\backslash(f(\varphi)/f(\Delta^\circ))$. So, by the residuation law, $f(\Xi^\circ) \circ f(\psi/\phi) \circ f(\Gamma^\circ) \circ f(\Delta^\circ) \leq f(\varphi)$. Therefore, $\Xi, \psi/\phi, \Gamma, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

$(\Rightarrow /)$: Assume that $\Gamma, \phi \Rightarrow \psi$ is valid. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\circ) \circ f(\phi) \leq f(\psi)$. By the residuation law, $f(\Gamma^\circ) \leq f(\psi)/f(\phi)$. Therefore, $\Gamma \Rightarrow \psi/\phi$ is valid on $\mathfrak{A}$.

Thus, $\mathfrak{C}^{\mathrm{DFL}} \models$ **DFL**. Thanks to Lemma 3.14 (We note that any DFL-algebra satisfying all the conditions in Lemma 3.14 validates **LJ**.), it suffices to check the inference rule $(\Box)$ on the class $\mathfrak{C}^{\Box}$. Let $\mathfrak{A}$ be an arbitrary modal algebra.

$(\Box)$: Assume that $\Gamma \Rightarrow \phi$ is valid. For an arbitrary assignment f on $\mathfrak{A}$, by the assumption, $f(\Gamma^\wedge) \leq f(\phi)$. By the definition of $\leq$, $f(\Gamma^\wedge) \wedge f(\phi) = f(\Gamma^\wedge)$. By the meet preservation, $f((\Box\Gamma)^\wedge) \wedge f(\Box\phi) = \Box(f(\Gamma^\wedge) \wedge f(\phi)) = \Box f(\Gamma^\wedge) = f((\Box\Gamma)^\wedge)$. So, $f((\Box\Gamma)^\wedge) \leq f(\Box\phi)$. Therefore, $\Box\Gamma \Rightarrow \Box\phi$ is valid on $\mathfrak{A}$. (Q.E.D)

Given a logic **L**, an algebra $\mathfrak{A}$ is a L-algebra, if $\mathfrak{A} \models$ **L**. The set of all L-algebras is denoted by $\mathfrak{C}^{\mathrm{L}}$. Because of Theorem 3.15, the name (DFL-algebra) makes sense.

3.3 Algebraic completeness via Lindenbaum-Tarski algebra

In the next section, we will show the algebraic completeness theorem for DFL (normal modal) logics. To prove this theorem, we define some notions here. In this section, **L** denotes any extension of **DFL** (**K**).

Let Frm(Φ) be the set of all formulas based on Φ. A relation $\equiv_{\mathrm{L}}$ on Frm(Φ) can be defined as follows in **L**.

$$\phi \equiv_{\mathrm{L}} \psi \iff \phi\backslash\psi \wedge \psi\backslash\phi \text{ (or, } (\phi \to \psi) \wedge (\psi \to \psi)) \in \mathbf{L}.$$

Then, we can easily check this relation $\equiv_{\mathrm{L}}$ is a equivalence relation (the equivalence class of ϕ is denoted by $[\phi]$).

We can make the quotient set $\mathrm{Frm}(\Phi)/\equiv_{\mathrm{L}}$ under this relation. Then, we define the following operations on it.

- $[\phi] \vee [\psi] := [\phi \vee \psi]$.
- $[\phi] \wedge [\psi] := [\phi \wedge \psi]$.
- $[\phi] \circ [\psi] := [\phi \circ \psi]$.
- $[\phi]\backslash[\psi] := [\phi\backslash\psi]$.
- $[\psi]/[\phi] := [\psi/\phi]$.
- $[\phi] \to [\psi] := [\phi \to \psi]$.
- $\neg[\phi] := [\neg\phi]$.
- $\Box[\phi] := [\Box\phi]$.

Moreover, we can check that these operations are well defined. Then, we define Lindenbaum-Tarski algebra of **L**.

Definition 3.16 (Lindenbaum-Tarski algebra of L)
Given a set Φ of propositional variables, the *Lindenbaum-Tarski algebra* $\mathfrak{L}(\Phi)$ *of* ***L*** is the tuples $\mathfrak{L}(\Phi) = \langle \mathrm{Frm}(\Phi)/\equiv_{\mathrm{L}}, \vee, \wedge, \circ, \backslash, /, [1], [0], [\top], [\bot] \rangle$ for DFL logics, and $\mathfrak{L}(\Phi) = \langle \mathrm{Frm}(\Phi)/\equiv_{\mathrm{L}}, \vee, \wedge, \neg, \Box, [\top], [\bot] \rangle$ for normal modal logics.

In the Lindenbaum-Tarski algebra of **L**, we can introduce the order $\leq$ as follows.

Lemma 3.17
In the Lindenbaum-Tarski algebra, $[\phi] \leq [\psi]$ if and only if $\phi\backslash\psi$ (or, $\phi \to \psi$) is in **L**. In sequent calculi, this condition is equivalent to $\phi \Rightarrow \psi \in \mathbf{L}$.

Proof
Since $[\phi] \leq [\psi]$ is the abbreviation of $[\phi] \wedge [\psi] = [\phi]$, $\phi \wedge \psi \Rightarrow \phi$ and $\phi \Rightarrow \phi \wedge \psi$ are in **L**. If $\phi \Rightarrow \psi$ is in **L**, we can prove as follows.

$$\frac{\phi \Rightarrow \phi}{\phi \wedge \psi \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \qquad\qquad \frac{\phi \Rightarrow \phi \qquad \phi \Rightarrow \psi}{\phi \Rightarrow \phi \wedge \psi}\,(\Rightarrow \wedge)$$

If $[\phi] \leq [\psi]$, then see below.

$$\frac{\phi \Rightarrow \phi \wedge \psi \qquad \dfrac{\psi \Rightarrow \psi}{\phi \wedge \psi \Rightarrow \psi}\,(\wedge 2 \Rightarrow)}{\phi \Rightarrow \psi}\,(\text{cut})$$

So, $[\phi] \leq [\psi]$ if and only if $\phi \Rightarrow \psi$ is in **L**. (Q.E.D)

Then, we show the following proposition.

Proposition 3.18
The Lindenbaum-Tarski algebra $\mathfrak{L}(\Phi)$ of **L** is a L-algebra.

Proof
We prove first that $\langle \mathrm{Frm}(\Phi)/\equiv_{\mathbf{L}}, \vee, \wedge, [\top], [\bot]\rangle$ is a bounded distributive lattice as follows.

Idempotency : $\phi \vee \phi \Rightarrow \phi$ and $\phi \Rightarrow \phi \vee \phi$ are provable.

$$\frac{\phi \Rightarrow \phi \qquad \phi \Rightarrow \phi}{\phi \vee \phi \Rightarrow \phi}\,(\vee \Rightarrow) \qquad\qquad \frac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee \phi}\,(\Rightarrow \vee 1)$$

Therefore, $[\phi] \vee [\phi] = [\phi]$. $[\phi] \wedge [\phi] = [\phi]$ can be analogously proved.

Commutativity : $\phi \vee \psi \Rightarrow \psi \vee \phi$ and $\psi \vee \phi \Rightarrow \phi \vee \psi$ are provable.

$$\frac{\dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \psi \vee \phi}\,(\Rightarrow \vee 2) \qquad \dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \psi \vee \phi}\,(\Rightarrow \vee 1)}{\phi \vee \psi \Rightarrow \psi \vee \phi}\,(\vee \Rightarrow) \qquad \frac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \phi \vee \psi}\,(\Rightarrow \vee 2) \qquad \dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee \psi}\,(\Rightarrow \vee 1)}{\psi \vee \phi \Rightarrow \phi \vee \psi}\,(\vee \Rightarrow)$$

Therefore, $[\phi] \vee [\psi] = [\psi] \vee [\phi]$. $[\phi] \wedge [\psi] = [\psi] \wedge [\phi]$ can be analogously proved.

Associativity : $\phi \vee (\psi \vee \chi) \Rightarrow (\phi \vee \psi) \vee \chi$ and $(\phi \vee \psi) \vee \chi \Rightarrow \phi \vee (\psi \vee \chi)$ are provable.

$$\frac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee \psi}\,(\Rightarrow \vee 1)}{\phi \Rightarrow (\phi \vee \psi) \vee \chi}\,(\Rightarrow \vee 1) \qquad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \phi \vee \psi}\,(\Rightarrow \vee 2)}{\psi \Rightarrow (\phi \vee \psi) \vee \chi}\,(\Rightarrow \vee 1) \qquad \dfrac{\chi \Rightarrow \chi}{\chi \Rightarrow (\phi \vee \psi) \vee \chi}\,(\Rightarrow \vee 2)}{\psi \vee \chi \Rightarrow (\phi \vee \psi) \vee \chi}\,(\vee \Rightarrow)}{\phi \vee (\psi \vee \chi) \Rightarrow (\phi \vee \psi) \vee \chi}\,(\vee \Rightarrow)$$

$$\frac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee (\psi \vee \chi)}\,(\Rightarrow \vee 1) \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 1)}{\psi \Rightarrow \phi \vee (\psi \vee \chi)}\,(\Rightarrow \vee 2)}{\phi \vee \psi \Rightarrow \phi \vee (\psi \vee \chi)}\,(\vee \Rightarrow) \qquad \dfrac{\dfrac{\chi \Rightarrow \chi}{\chi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 2)}{\chi \Rightarrow \phi \vee (\psi \vee \chi)}\,(\Rightarrow \vee 2)}{(\phi \vee \psi) \vee \chi \Rightarrow \phi \vee (\psi \vee \chi)}\,(\vee \Rightarrow)$$

Therefore, $[\phi] \vee ([\psi] \vee [\chi]) = ([\phi] \vee [\psi]) \vee [\chi]$. $[\phi] \wedge ([\psi] \wedge [\chi]) = ([\phi] \wedge [\psi]) \wedge [\chi]$ can be analogously proved.

Absorption : $\phi \wedge (\phi \vee \psi) \Rightarrow \phi$ and $\phi \Rightarrow \phi \wedge (\phi \vee \psi)$ are provable.

$$\dfrac{\phi \Rightarrow \phi}{\phi \wedge (\phi \vee \psi) \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \qquad \dfrac{\phi \Rightarrow \phi \quad \dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee \psi}\,(\Rightarrow \vee 1)}{\phi \Rightarrow \phi \wedge (\phi \vee \psi)}\,(\Rightarrow \wedge)$$

Therefore, $[\phi] \wedge ([\phi] \vee [\psi]) = [\phi]$. $[\phi] \vee ([\phi] \wedge [\psi]) = [\phi]$ can be analogously proved.

Distributivity : $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is an initial sequent and $(\phi \wedge \psi) \vee (\phi \wedge \chi) \Rightarrow \phi \wedge (\psi \vee \chi)$ are provable.

$$\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \wedge \psi \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \quad \dfrac{\dfrac{\psi \Rightarrow \psi}{\phi \wedge \psi \Rightarrow \psi}\,(\wedge 2 \Rightarrow)}{\phi \wedge \psi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 1)}{\phi \wedge \psi \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\Rightarrow \wedge) \quad \dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \wedge \chi \Rightarrow \phi}\,(\wedge 1 \Rightarrow) \quad \dfrac{\dfrac{\chi \Rightarrow \chi}{\phi \wedge \chi \Rightarrow \chi}\,(\wedge 2 \Rightarrow)}{\phi \wedge \chi \Rightarrow \psi \vee \chi}\,(\Rightarrow \vee 2)}{\phi \wedge \chi \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\Rightarrow \wedge)}{(\phi \wedge \psi) \vee (\phi \wedge \chi) \Rightarrow \phi \wedge (\psi \vee \chi)}\,(\vee \Rightarrow)$$

Therefore, $[\phi] \wedge ([\psi] \vee [\chi]) = ([\phi] \wedge [\psi]) \vee ([\phi] \wedge [\chi])$.

$\top$-, $\bot$-Boundedness : $\phi \Rightarrow \top$ and $\bot \Rightarrow \phi$ are initial sequents. Therefore, $[\phi] \leq [\top]$ and $[\bot] \leq [\phi]$.

Next, we prove that $\langle \mathrm{Frm}(\Phi)/\equiv, \vee, \wedge, \circ, \backslash, /, [1] \rangle$ is a residuated lattice as follows.

Associativity : We need to prove $[\phi] \circ ([\psi] \circ [\chi]) = ([\phi] \circ [\psi]) \circ [\chi]$. This is that $\phi \circ (\psi \circ \chi) \Rightarrow (\phi \circ \psi) \circ \chi$ and $(\phi \circ \psi) \circ \chi \Rightarrow \phi \circ (\psi \circ \chi)$ are provable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \phi \circ \psi}\,(\Rightarrow \circ) \quad \chi \Rightarrow \chi}{\phi, \psi, \chi \Rightarrow (\phi \circ \psi) \circ \chi}\,(\Rightarrow \circ)}{\phi, \psi \circ \chi \Rightarrow (\phi \circ \psi) \circ \chi}\,(\circ \Rightarrow)}{\phi \circ (\psi \circ \chi) \Rightarrow (\phi \circ \psi) \circ \chi}\,(\circ \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \dfrac{\psi \Rightarrow \psi \quad \chi \Rightarrow \chi}{\psi, \chi \Rightarrow \psi \circ \chi}\,(\Rightarrow \circ)}{\phi, \psi, \chi \Rightarrow \phi \circ (\psi \circ \chi)}\,(\Rightarrow \circ)}{\phi \circ \psi, \chi \Rightarrow \phi \circ (\psi \circ \chi)}\,(\circ \Rightarrow)}{(\phi \circ \psi) \circ \chi \Rightarrow \phi \circ (\psi \circ \chi)}\,(\circ \Rightarrow)$$

Therefore, $[\phi] \circ ([\psi] \circ [\chi]) = ([\phi] \circ [\psi]) \circ [\chi]$.

Identity : We need to prove $[\phi] \circ [1] = [\phi]$ and $[1] \circ [\phi] = [\phi]$. This is that $\phi \circ 1 \Rightarrow \phi$, $\phi \Rightarrow \phi \circ 1$, $1 \circ \phi \Rightarrow \phi$ and $\phi \Rightarrow 1 \circ \phi$ are provable.

$$\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, 1 \Rightarrow \phi}\,(1\ \mathrm{w})}{\phi \circ 1 \Rightarrow \phi}\,(\circ \Rightarrow) \qquad \dfrac{\phi \Rightarrow \phi \quad \Rightarrow 1}{\phi \Rightarrow \phi \circ 1}\,(\Rightarrow \circ) \qquad \dfrac{\dfrac{\phi \Rightarrow \phi}{1, \phi \Rightarrow \phi}\,(1\ \mathrm{w})}{1 \circ \phi \Rightarrow \phi}\,(\circ \Rightarrow) \qquad \dfrac{\Rightarrow 1 \quad \phi \Rightarrow \phi}{\phi \Rightarrow 1 \circ \phi}\,(\Rightarrow \circ)$$

Therefore, $[\phi] \circ [1] = [\phi]$ and $[1] \circ [\phi] = [\phi]$ hold.

Residuation law : It is suffices to check the following.

$$[\phi] \circ [\psi] \leq [\chi] \Longrightarrow [\psi] \leq [\phi] \backslash [\chi] \Longrightarrow [\phi] \leq [\chi]/[\psi] \Longrightarrow [\phi] \circ [\psi] \leq [\chi]$$

Assume $[\phi] \circ [\psi] \leq [\chi]$.

$$\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \qquad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \phi \circ \psi}\,(\Rightarrow \circ) \qquad \phi \circ \psi \Rightarrow \chi}{\phi, \psi \Rightarrow \chi}\,(\text{cut})}{\psi \Rightarrow \phi \backslash \chi}\,(\Rightarrow \backslash)$$

Therefore, $[\phi] \circ [\psi] \leq [\chi] \Longrightarrow [\psi] \leq [\phi] \backslash [\chi]$.

Assume $\psi \Rightarrow \phi \backslash \chi$.

$$\dfrac{\dfrac{\psi \Rightarrow \phi \backslash \chi \qquad \dfrac{\phi \Rightarrow \phi \qquad \psi \Rightarrow \psi}{\phi, \phi \backslash \psi \Rightarrow \psi}\,(\backslash \Rightarrow)}{\phi, \psi \Rightarrow \chi}\,(\text{cut})}{\phi \Rightarrow \chi / \psi}\,(\Rightarrow /)$$

Therefore, $[\psi] \leq [\phi] \backslash [\chi] \Longrightarrow [\phi] \leq [\chi]/[\psi]$.

Assume $\phi \Rightarrow \chi/\psi$.

$$\dfrac{\dfrac{\phi \Rightarrow \chi/\psi \qquad \dfrac{\chi \Rightarrow \chi \qquad \psi \Rightarrow \psi}{\chi/\psi, \psi \Rightarrow \chi}\,(/ \Rightarrow)}{\phi, \psi \Rightarrow \chi}\,(\text{cut})}{\phi \circ \psi \Rightarrow \chi}\,(\circ \Rightarrow)$$

Therefore, $[\phi] \leq [\chi]/[\psi] \Longrightarrow [\phi] \circ [\psi] \leq [\chi]$.

Thus, the Lindenbaum-Tarski algebra of **L** is at least a DFL-algebra. Next, if **L** is a normal modal logic, then we can show the following.

Complementation : $\bot \Rightarrow \phi \wedge \neg\phi$ is an initial sequent and $\phi \wedge \neg\phi \Rightarrow \bot$ is provable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \neg\phi \Rightarrow}\,(\neg \Rightarrow)}{\phi \wedge \neg\phi, \neg\phi \Rightarrow}\,(\wedge 1 \Rightarrow)}{\phi \wedge \neg\phi, \phi \wedge \neg\phi \Rightarrow}\,(\wedge 2 \Rightarrow)}{\phi \wedge \neg\phi \Rightarrow}\,(\text{c} \Rightarrow)}{\phi \wedge \neg\phi \Rightarrow \bot}\,(\bot\ \text{w})$$

Therefore, $[\phi] \wedge [\neg\phi] = [\bot]$. $[\phi] \vee [\neg\phi] = [\top]$ can be analogously proved.

Boundedness : $\phi \vee \bot \Rightarrow \phi$ and $\phi \Rightarrow \phi \vee \bot$ are provable, as follows.

$$\dfrac{\phi \Rightarrow \phi \qquad \bot \Rightarrow \phi}{\phi \vee \bot \Rightarrow \phi}\,(\vee \Rightarrow) \qquad\qquad \dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \vee \bot}\,(\Rightarrow \vee 1)$$

Therefore, $[\phi] \vee [\bot] = [\phi]$. $[\phi] \vee [\top] = [\top]$ can be analogously proved.

Thus, $\langle \text{Frm}(\Phi)/\equiv_{\mathbf{L}}, \vee, \wedge, \neg, [\top], [\bot] \rangle$ is a Boolean algebra. Next, we prove the following conditions of $\Box$.

Meet preservation : $\Box(\phi\wedge\psi) \Rightarrow \Box\phi\wedge\Box\psi$ and $\Box\phi\wedge\Box\psi \Rightarrow \Box(\phi\wedge\psi)$ are provable.

$$\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi \wedge \psi \Rightarrow \phi}\,(\wedge 1 \Rightarrow)}{\Box(\phi \wedge \psi) \Rightarrow \Box\phi}\,(\Box) \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\phi \wedge \psi \Rightarrow \psi}\,(\wedge 2 \Rightarrow)}{\Box(\phi \wedge \psi) \Rightarrow \Box\psi}\,(\Box)}{\Box(\phi \wedge \psi) \Rightarrow \Box\phi \wedge \Box\psi}\,(\Rightarrow \wedge)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \psi \Rightarrow \phi}\,(\mathrm{w} \Rightarrow) \qquad \dfrac{\psi \Rightarrow \psi}{\phi, \psi \Rightarrow \psi}\,(\mathrm{w} \Rightarrow)}{\phi, \psi \Rightarrow \phi \wedge \psi}\,(\Rightarrow \wedge)}{\Box\phi, \Box\psi \Rightarrow \Box(\phi \wedge \psi)}\,(\Box)}{\Box\phi \wedge \Box\psi, \Box\psi \Rightarrow \Box(\phi \wedge \psi)}\,(\wedge 1 \Rightarrow)}{\Box\phi \wedge \Box\psi, \Box\phi \wedge \Box\psi \Rightarrow \Box(\phi \wedge \psi)}\,(\wedge 2 \Rightarrow)}{\Box\phi \wedge \Box\psi \Rightarrow \Box(\phi \wedge \psi)}\,(\mathrm{c} \Rightarrow)$$

Therefore, $\Box([\phi] \wedge [\psi]) = (\Box[\phi] \wedge \Box[\psi])$.

Top preservation : $\Box\top \Rightarrow \top$ is an initial sequent and $\top \Rightarrow \Box\top$ is provable.

$$\dfrac{\dfrac{\Rightarrow \top}{\Rightarrow \Box\top}\,(\Box)}{\top \Rightarrow \Box\top}\,(\mathrm{w} \Rightarrow)$$

Therefore, $\Box[\top] = [\top]$.

Thus, the Lindenbaum-Tarski algebra of **L** is a modal algebra. Therefore, we have already proved that the Lindenbaum-Tarski algebra of **DFL** (**K**) is a DFL- (K-) algebra.

It remains to show the following. Given an extension **L** of **DFL** (**K**), every foruma ϕ in **L** is valid on the Lindenbaum-Tarski algebra $\mathfrak{L}(\Phi)$ of **L**.

Since every logic is closed under uniform substitution, if ϕ is in **L**, any uniform substitution instance is also in **L**. In other words, any uniform substitution instance is provable.

Let f be an arbitrary assignment on $\mathfrak{L}(\Phi)$. For any propositional variable $p \in \Phi$, we can take a formula $\rho(p)$ in the equivalent class $f(p)$ as its representative. That is, $f(p) = [\rho(p)]$. Then, we can view ρ as a function from Φ to $\mathrm{Frm}(\Phi)$. In other words, ρ is a uniform substitution.

If ϕ is in **L**, the uniform substitution instance $\rho(\phi)$ is also in **L**. For any assignment f on $\mathfrak{L}(\Phi)$, if we take the representation $\rho(p)$ for any propositional variable $p \in \Phi$,

$$f(\phi) = [\rho(\phi)]$$

Here, $\rho(\phi) \in \mathbf{L}$. Therefore, $[1] \leq [\rho(\phi)]$. (Q.E.D)

Here, we define a special assignment $f_{\mathfrak{L}}$ for the Lindenbaum-Tarski algebra $\mathfrak{L}(\Phi)$ of **L** as follows.

$$f_{\mathfrak{L}}(p) := [p] \text{ for any propositional variable } p \in \Phi$$

Besides, the assignment can be inductively extended, as usual. Then, we can check that $f(\phi) = [\phi]$ for any formula $\phi \in \mathrm{Frm}(\Phi)$. Finally, we can prove the completeness theorem of any DFL (normal modal) logic.

Theorem 3.19 (Algebraic completeness theorem)
Given any DFL (normal modal) logic $\boldsymbol{L}$*, if a formula* ϕ *is valid on the class* $\mathfrak{C}^{\boldsymbol{L}}$ *of all L-algebras, then* ϕ *is in* $\boldsymbol{L}$*.*

Proof
Assume $\phi \notin \mathbf{L}$ (i.e. ϕ is not provable in L). Then, $1 \Rightarrow \phi$ is not provable. From Lemma 3.17, in the Lindenbaum algebra $\mathfrak{L}(\Phi)$ of $\mathbf{L}$, $[1] \leq [\phi]$ does not hold. So, $\mathfrak{L}(\Phi), f_{\mathfrak{L}} \not\models \phi$. By Proposition 3.18, $\mathfrak{L}(\Phi)$ is a L-algebra. Therefore, ϕ is not valid on the class $\mathfrak{C}^{\mathrm{L}}$. (Q.E.D)

4 Relational semantics and canonicity

In this chapter, we introduce relational semantics for DFL (normal modal) logics and define some concepts and terminology about them.

4.1 Relational semantics

We define the relational semantics for our logics.

Definition 4.1 (Kripke model)
A tuple $\mathfrak{M} = \langle W, R_\Box, V\rangle$ is a *Kripke model*, if W is a non-empty set, $R_\Box$ is a binary relation on W, and V is a function from Φ to W, called a *valuation*. The tuple $\mathfrak{F} = \langle W, R\rangle$ is called a *Kripke frame*. Besides, $\mathfrak{C}_\Box$ denotes the class of all Kripke frames.

Definition 4.2 (DFL-model)
A tuple $\mathfrak{F} = \langle W, O, N, R_\circ\rangle$ is a *DFL-frame*, if W is a non-empty set, O a non-empty subset of W, N a subset of W and $R_\circ$ a ternary relation on W, that satisfy the following conditions.

1. ($R_\circ$-reflexivity):
   For any $w \in W$, there exist o and o' in O such that $R_\circ(w, o, w)$ and $R_\circ(w, w, o')$.

2. ($R_\circ$-transitivity):
   For any $w, v, u, w', v', u' \in W$,
   if $R_\circ(w, v, u)$ and $w \leq w'$ and $v' \leq v$ and $u' \leq u$, then $R_\circ(w', v', u')$.

3. ($R_\circ$-associativity):
   For any $w, v, u, s \in W$, there exists $x \in W$ such that $R_\circ(w, x, s)$ and $R_\circ(x, v, u)$, if and only if there exists $y \in W$ such that $R_\circ(w, v, y)$ and $R_\circ(y, u, s)$.

4. O is closed under $\leq$.
   (That is, for each $w, v \in W$, if $w \leq v$ and $w \in O$, then $v \in O$.)

5. N is closed under $\leq$.

Here, $w \leq v$ is defined by the condition that there exists $o \in O$ such that $R_\circ(v, o, w)$ or $R_\circ(v, w, o)$. A subset W' of W is $R_\circ$*-upward closed*, if $w \leq v$ and $w \in W'$ imply $v \in W'$ for any $w, v \in W$. Thus, the above definition of DFL-frames says that O and N must be $R_\circ$-upward closed. Besides, $\mathfrak{C}_{\mathrm{DFL}}$ denotes the class of all DFL-frames.

The set of all $R_\circ$-upward closed sets over W is denoted by $Up(W)$. Moreover, given a DFL-frame $\mathfrak{F} = \langle W, O, N, R_\circ, V\rangle$, a tuple $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is a *DFL-model*, where V is a function from Φ to $Up(W)$, called a *valuation*.

Although we defined several relational semantics, we call just frame, model or valuation as long as possible.

Given a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle = \langle W, \dots, V \rangle$, an element $w \in W$ (sometimes, we say w is an element of $\mathfrak{F}$ or $\mathfrak{M}$) and a formula ϕ, we define a relation $\mathfrak{M}, w \Vdash \phi$ as follows. (If $\mathfrak{M}, w \Vdash \phi$ does not hold, then we denote $\mathfrak{M}, w \nVdash \phi$.)

For FL formulas:

- $\mathfrak{M}, w \Vdash p \iff w \in V(p)$, for an arbitrary propositional variable $p \in \Phi$.
- $\mathfrak{M}, w \Vdash 1 \iff w \in O$.
- $\mathfrak{M}, w \Vdash 0 \iff w \in N$.
- $\mathfrak{M}, w \Vdash \top$ always holds.
- $\mathfrak{M}, w \nVdash \bot$ always holds.
- $\mathfrak{M}, w \Vdash \phi \vee \psi \iff \mathfrak{M}, w \Vdash \phi$ or $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \phi \wedge \psi \iff \mathfrak{M}, w \Vdash \phi$ and $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \phi \circ \psi \iff$
  there exist $v, u \in W$ such that $R_\circ(w, v, u)$, $\mathfrak{M}, v \Vdash \phi$ and $\mathfrak{M}, u \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \phi \backslash \psi \iff$
  for any $v, u \in W$, if $R_\circ(u, v, w)$ and $\mathfrak{M}, v \Vdash \phi$, then $\mathfrak{M}, u \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \psi / \phi \iff$
  for any $v, u \in W$, if $R_\circ(u, w, v)$ and $\mathfrak{M}, v \Vdash \phi$, then $\mathfrak{M}, u \Vdash \psi$.

For modal formulas:

- $\mathfrak{M}, w \Vdash p \iff w \in V(p)$, for an arbitrary propositional variable $p \in \Phi$.
- $\mathfrak{M}, w \Vdash \top$ always holds.
- $\mathfrak{M}, w \nVdash \bot$ always holds.
- $\mathfrak{M}, w \Vdash \phi \vee \psi \iff \mathfrak{M}, w \Vdash \phi$ or $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \phi \wedge \psi \iff \mathfrak{M}, w \Vdash \phi$ and $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \phi \to \psi \iff$ if $\mathfrak{M}, w \Vdash \phi$, then $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \neg\phi \iff \mathfrak{M}, w \nVdash \phi$.
- $\mathfrak{M}, w \Vdash \Box\phi \iff$ for any $v \in W$, if $R_\Box(w, v)$, then $\mathfrak{M}, v \Vdash \phi$.
- $\mathfrak{M}, w \Vdash \Diamond\phi \iff$ there exists $v \in W$ such that $R_\Box(w, v)$ and $\mathfrak{M}, v \Vdash \phi$.

We might want to call $\mathfrak{M}, w \Vdash \phi$ that a fomula ϕ is true at w in $\mathfrak{F}$ under V, at first. However, for FL formulas, this relation $\mathfrak{M}, w \Vdash \phi$ is, in fact, not enough. With this relation $\Vdash$, we define *truth*, *global truth* or *validity* as follows. (As we can see later, in normal modal logics, $\mathfrak{M}, w \Vdash \phi$ is completely equivalent to the following definition.)

- A formula ϕ is *true* at w in $\mathfrak{F}$ under V ($\mathfrak{F}, V, w \Vdash_t \phi$), if and only if $\mathfrak{F}, V, w \Vdash \phi$ and $w \in O$.
- A formula ϕ is *globally true* in $\mathfrak{F}$ under V ($\mathfrak{F}, V \Vdash_t \phi$), if and only if $\mathfrak{F}, V, w \Vdash \phi$, for all $w \in O$.
- A formula ϕ is *valid* on $\mathfrak{F}$ ($\mathfrak{F} \Vdash_t \phi$), if and only if $\mathfrak{F}, V \Vdash_t \phi$, for any valuation V.

Given a set Σ of formulas and a class $\mathfrak{C}$ of frames, Σ is valid on $\mathfrak{C}$ (denoted by $\mathfrak{C} \Vdash_t \Sigma$), if and only if every formula in Σ is valid on any frame in $\mathfrak{C}$.

Besides, by the above definition of truth, global truth and validity, we can consider that a sequent is *true*, *globally true* and *valid* as follows.

- A sequent $\Gamma \Rightarrow \varphi$ is true at w in $\mathfrak{F}$ under V (denoted by $\mathfrak{F}, V, w \Vdash_t \Gamma \Rightarrow \varphi$), if and only if $\mathfrak{F}, V, w \Vdash \Gamma^\circ$ implies $\mathfrak{F}, V, w \Vdash \varphi$.
- A sequent $\Gamma \Rightarrow \varphi$ is globally true in $\mathfrak{F}$ under V (denoted by $\mathfrak{F}, V \Vdash_t \Gamma \Rightarrow \varphi$), if and only if $\Gamma \Rightarrow \varphi$ is true for any $w \in \mathfrak{F}$.
- A sequent $\Gamma \Rightarrow \varphi$ is valid on $\mathfrak{F}$ (denoted by $\mathfrak{F} \Vdash_t \Gamma \Rightarrow \varphi$), if and only if $\Gamma \Rightarrow \varphi$ is globally true for any valuation V.

Note that global truth and validity correspond to algebraic truth and validity, respectively. Moreover, the following proposition for DFL-models holds.

Proposition 4.3
Let $\mathfrak{M} = (\mathfrak{F}, V)$ be any DFL-model, w, w' arbitrary elements in $\mathfrak{M}$ and ϕ an arbitrary formula, if $\mathfrak{M}, w \Vdash \phi$ and $w \leq w'$, then $\mathfrak{M}, w' \Vdash \phi$ holds. In other words, the relation $\Vdash$ is $R_\circ$-upward closed. Therefore, truth, global truth and validity also are $R_\circ$-upward closed.

Proof
Induction on ϕ.

- If $\mathfrak{M}, w \Vdash p$, then $\mathfrak{M}, w' \Vdash p$, because $V(p)$ is $R_\circ$-upward closed.
- If $\mathfrak{M}, w \Vdash 1$, then $\mathfrak{M}, w' \Vdash 1$, because O is $R_\circ$-upward closed.
- If $\mathfrak{M}, w \Vdash 0$, then $\mathfrak{M}, w' \Vdash 0$, because N is $R_\circ$-upward closed.
- If $\mathfrak{M}, w \Vdash \psi \vee \chi$, then $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. By the induction hypothesis, $\mathfrak{M}, w' \Vdash \psi$ or $\mathfrak{M}, w' \Vdash \chi$. So, $\mathfrak{M}, w' \Vdash \psi \vee \chi$.

- If $\mathfrak{M}, w \Vdash \psi \wedge \chi$, then $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$. By the induction hypothesis, $\mathfrak{M}, w' \Vdash \psi$ and $\mathfrak{M}, w' \Vdash \chi$. So, $\mathfrak{M}, w' \Vdash \psi \wedge \chi$.
- If $\mathfrak{M}, w \Vdash \psi \circ \chi$, then these exist $v, u \in \mathfrak{M}$ such that $R_\circ(w, v, u)$, $\mathfrak{M}, v \Vdash \psi$ and $\mathfrak{M}, u \Vdash \chi$. Moreover, by means of $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(w', v, u)$ holds. Therefore, $\mathfrak{M}, w' \Vdash \psi \circ \chi$.
- If $\mathfrak{M}, w' \nVdash \psi \backslash \chi$, then there exist $v, u \in \mathfrak{M}$ such that $R_\circ(u, v, w')$, $\mathfrak{M}, v \Vdash \psi$ but $\mathfrak{M}, u \nVdash \chi$. Moreover, by means of $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(u, v, w)$ holds. Therefore, $\mathfrak{M}, w \nVdash \psi \backslash \chi$.
- If $\mathfrak{M}, w' \nVdash \chi / \psi$, then there exist $v, u \in \mathfrak{M}$ such that $R_\circ(u, w', v)$, $\mathfrak{M}, v \Vdash \psi$ but $\mathfrak{M}, u \nVdash \chi$. Moreover, by means of $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(u, w, v)$ holds. Therefore, $\mathfrak{M}, w \nVdash \chi / \psi$.

Since O is $R_\circ$-upward closed, truth, global truth and validity are also $R_\circ$-upward closed. (Q.E.D)

On relational semantics, we say that an axiom (initial sequents or an inference rule) *corresponds* to a frame condition, if any logic **L** containing the axiom if and only if any frame satisfying the frame condition validates **L**. Next, we prove the following correspondence lemma for DFL-frames.

Lemma 4.4
Each structural rule corresponds to the following DFL-frame conditions, respectively.

(c $\Rightarrow$) : For any $w, v \in W$, if $w \leq v$, then $R_\circ(v, w, w)$.
Or, equivalently on DFL-frame, for any $w \in W, R_\circ(w, w, w)$.

(e $\Rightarrow$) : For any $w, v, u \in W$, if $R_\circ(w, v, u)$, then $R_\circ(w, u, v)$.

(w $\Rightarrow$) : For any $w \in W$, $w \in O$. (That is, $O = W$)

($\Rightarrow$ w) : $N = \emptyset$.

Proof
(c $\Rightarrow$) : Suppose that $\Gamma, \phi, \phi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$ satisfying (c $\Rightarrow$) DFL-frame condition. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Gamma^\circ \circ \phi \circ \Delta^\circ$, then there exist $v, u, x, s \in \mathfrak{M}$ such that $R_\circ(w, x, s)$, $R_\circ(x, v, u)$, $\mathfrak{M}, v \Vdash \Gamma^\circ$, $\mathfrak{M}, u \Vdash \phi$ and $\mathfrak{M}, s \Vdash \Delta^\circ$. By $R_\circ$-reflexivity, $u \leq u$. So, by (c $\Rightarrow$) DFL-frame condition, $R_\circ(u, u, u)$ holds. Then, $\mathfrak{M}, u \Vdash \phi \circ \phi$. Then, $\mathfrak{M}, w \Vdash \Gamma^\circ \circ \phi \circ \phi \circ \Delta^\circ$. From the first assumption, $\mathfrak{M}, w \Vdash \varphi$. Therefore, $\Gamma, \phi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{A}$.

Let $\mathfrak{M} = \langle \{w, v, u\}, R_\circ, \{w, v\}, \emptyset, V \rangle$ be a DFL-model, which is not satisfying, $R \circ (u, u, u)$.

$$R_\circ = \{(w, w, w), (v, v, v), (u, w, u), (u, u, v)\}$$

$$V(p) = \{u\}$$

In this model, $\mathfrak{M}, u \Vdash p$ and $\mathfrak{M}, u \nVdash p \circ p$. So, although $p \circ p \Rightarrow p \circ p$ is valid, $p \Rightarrow p \circ p$ is not true. Therefore, (c $\Rightarrow$) is not valid.

(e $\Rightarrow$) : Suppose that $\Gamma, \phi, \psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$ satisfying (e $\Rightarrow$) DFL-frame condition. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Gamma^{\circ} \circ \psi \circ \phi \circ \Delta^{\circ}$, then there exist $v, u, x, s \in \mathfrak{M}$ such that $R_{\circ}(w, x, s)$, $R_{\circ}(x, v, u)$, $\mathfrak{M}, v \Vdash \Gamma^{\circ}$, $\mathfrak{M}, u \Vdash \psi \circ \phi$ and $\mathfrak{M}, s \Vdash \Delta^{\circ}$. Moreover, from $\mathfrak{M}, u \Vdash \psi \circ \phi$, there exist $y, z \in \mathfrak{M}$ such that $R_{\circ}(u, y, z)$, $\mathfrak{M}, y \Vdash \psi$ and $\mathfrak{M}, z \Vdash \phi$. By (e $\Rightarrow$) DFL-frame condition, $R_{\circ}(u, z, y)$ holds. Also, $\mathfrak{M}, u \Vdash \phi \circ \psi$ holds. Then, $\mathfrak{M}, w \Vdash \Gamma^{\circ} \circ \phi \circ \psi \circ \Delta^{\circ}$. By the assumption, $\mathfrak{M}, w \Vdash \varphi$. Therefore, $\Gamma, \psi, \phi, \Delta \Rightarrow \varphi$ is valid.

Let $\mathfrak{M} = \langle \{w, v, u\}, R_{\circ}, \{w, v\}, \emptyset, V \rangle$ be a DFL-model, which is not satisfying $R_[circ(u, w, u)$ implies $R_{\circ}(u, u, w)$.

$$R_{\circ} = \{(w, w, w), (v, v, v), (u, w, u), (u, u, v)\}$$

$$V(p) = \{w\}$$

$$V(q) = \{u\}$$

In this model, $\mathfrak{M}, u \Vdash p \circ q$ and $\mathfrak{M}, u \nVdash q \circ p$. So, although $q \circ p \Rightarrow q \circ p$ is valid, $p \circ q \Rightarrow q \circ p$ is not true. Therefore, (e $\Rightarrow$) is not valid.

(w $\Rightarrow$) : Suppose that $\Gamma, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$ satisfying (w $\Rightarrow$) DFL-frame condition. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Gamma^{\circ} \circ \phi \circ \Delta^{\circ}$, then there exist $v, u, x, s \in \mathfrak{M}$ such that $R_{\circ}(w, x, s)$, $R_{\circ}(x, v, u)$, $\mathfrak{M}, v \Vdash \Gamma^{\circ}$, $\mathfrak{M}, u \Vdash \phi$ and $\mathfrak{M}, s \Vdash \Delta^{\circ}$. By (w $\Rightarrow$) DFL-frame condition, $u \in O$. So, $v \leq x$ holds. Then, by $R_{\circ}$-transitivity and $R_{\circ}$-reflexivity, $R_{\circ}(w, v, s)$ holds. Therefore, $\mathfrak{M}, w \Vdash \Gamma^{\circ} \circ \Delta^{\circ}$. By the assumption, $\mathfrak{M}, w \Vdash \varphi$. Therefore, $\Gamma, \phi, \Delta \Rightarrow \varphi$ is valid.

Let $\mathfrak{M} = \langle \{w, v\}, R_{\circ}, \{w\}, \emptyset, V \rangle$ be a DFL-model, which is not satisfying $v \in O$.

$$R_{\circ} = \{(w, w, w), (v, v, w), (v, w, v)\}$$

$$V(p) = \{v\}$$

In this modes, $\mathfrak{M}, v \Vdash p$ and $\mathfrak{M}, v \nVdash 1$. So, although, $\Rightarrow 1$ is valid, $p \Rightarrow 1$ is not true. Therefore, (w $\Rightarrow$) is not valid.

($\Rightarrow$ w) : Suppose that $\Gamma \Rightarrow$ is valid on $\mathfrak{F}$ satisfying ($\Rightarrow$ w) DFL-frame condition. For an arbitrary valuation V , there is no element $w \in \mathfrak{F}$ such that $\mathfrak{F}, V, w \Vdash \Gamma^{\circ}$, by the assumption. Therefore, $\Gamma \Rightarrow \varphi$ is valid.

Let $\mathfrak{M} = \langle \{w, v\}, R_{\circ}, \{w\}, \{v\}, V \rangle$ be a DFL-model, which is not satisfying $w \notin N$.

$$R_{\circ} = \{(w, w, w), (v, w, v), (v, v, w)\}$$

$$V(p) = \{w\}$$

In this model, $\mathfrak{M}, v \Vdash 0$ and $\mathfrak{M}, v \nVdash p$. So, although, $0 \Rightarrow$ is valid, $0 \Rightarrow p$ is not true. Therefore, ($\Rightarrow$ w) is not valid. (Q.E.D)

4.2 Soundness

We prove the soundness of **DFL** and **K**.

Theorem 4.5 (Soundness)
*Every formula in **DFL** and **K** is valid on the classes $\mathfrak{C}_{DFL}$ and $\mathfrak{C}_{\Box}$, respectively.*

Proof
As in the case of algebraic semantics, we firstly prove the soundness theorem for **DFL**, because $\mathfrak{C}_{\Box}$ is a class of some DFL-frames with a binary relation.

It suffices to check both every initial sequent is valid and, in each of cut rule and rules for logical connectives of DFL, the lower sequent is valid, if the upper sequents are valid. Let $\mathfrak{F}$ be an arbitrary DFL-frame.

$\phi \Rightarrow \phi$: By definition.

$\Gamma \Rightarrow \top$: This is immediate, because $\mathfrak{F} \Vdash \top$ always holds.

$\Gamma, \bot, \Sigma \Rightarrow \varphi$: This is immediate, because $\mathfrak{F} \nVdash \bot$ always holds.

$\Rightarrow 1$: By definition.

$0 \Rightarrow$: By definition.

$\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$: For an arbitrary valuation V on $\mathfrak{F}$ and an arbitrary $w \in \mathfrak{F}$, assume $\mathfrak{M} = (\mathfrak{F}, V), w \Vdash \phi \wedge (\psi \vee \chi)$. Then, $\mathfrak{M}, w \Vdash \phi$ and either $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. Either $\mathfrak{M}, w \Vdash \phi$ and $\mathfrak{M}, w \Vdash \psi$, or $\mathfrak{M}, w \Vdash \phi$ and $\mathfrak{M}, w \Vdash \chi$. So, $\mathfrak{M}, w \Vdash (\phi \wedge \psi) \vee (\phi \wedge \chi)$. Therefore, $\phi \wedge (\psi \vee \chi) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ is valid on $\mathfrak{F}$.

(cut) : Suppose that $\Gamma \Rightarrow \phi$ and $\Sigma, \phi, \Xi \Rightarrow \varphi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Sigma^\circ \circ \Gamma^\circ \circ \Xi^\circ$, then there exist $v, u, s, x \in \mathfrak{M}$ such that $R_\circ(w, x, s)$, $R_\circ(x, v, u)$, $\mathfrak{M}, v \Vdash \Sigma^\circ$, $\mathfrak{M}, u \Vdash \Gamma^\circ$, and $\mathfrak{M}, s \Vdash \Xi^\circ$. By the first assumption, $\mathfrak{M}, u \Vdash \phi$. So, $\mathfrak{M}, w \Vdash \Sigma^\circ \circ \phi \circ \Xi^\circ$. Then, by the second assumption, $\mathfrak{M}, w \Vdash \varphi$. Therefore, $\Sigma, \Gamma, \Xi \Rightarrow \varphi$ is valid on $\mathfrak{F}$.

(1 w) : Suppose that $\Gamma, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Gamma^\circ \circ 1 \circ \Delta^\circ$, then there exist $v, u, s, x \in \mathfrak{M}$ such that $R_\circ(w, x, s)$, $R_\circ(x, v, u)$, $\mathfrak{M}, v \Vdash \Gamma^\circ$, $\mathfrak{M}, u \Vdash 1$ and $\mathfrak{M}, s \Vdash \Delta^\circ$. Since $u \in O$, $v \leq x$ holds. From Proposition 4.3, $\mathfrak{M}, x \Vdash \Gamma^\circ$ holds. So, $\mathfrak{M}, w \Vdash \Gamma^\circ \circ \Delta^\circ$. By the assumption, $\mathfrak{M}, w \Vdash \varphi$. Therefore, $\Gamma, 1, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$.

(0 w) : By definition.

$(\vee\Rightarrow)$: Suppose that $\Gamma,\phi,\Delta\Rightarrow\varphi$ and $\Gamma,\psi,\Delta\Rightarrow\varphi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}\circ(\phi\vee\psi)\circ\Delta^{\circ}$, then there exist v,u,x,s such that $R_{\circ}(w,x,s)$, $R_{\circ}(x,v,u)$, $\mathfrak{M},v\Vdash\Gamma^{\circ}$, $\mathfrak{M},u\Vdash\phi\vee\psi$ and $\mathfrak{M},s\Vdash\Delta^{\circ}$. If $\mathfrak{M},u\Vdash\phi$, then, from the first assumption, $\mathfrak{M},w\Vdash\varphi$ holds. Otherwise, $\mathfrak{M},u\Vdash\psi$, then, from the second assumption $\mathfrak{M},w\Vdash\varphi$ holds. Afterall, $\mathfrak{M},w\Vdash\varphi$. Therefore, $\Gamma,\phi\vee\psi,\Delta\Rightarrow\varphi$ is valid on $\mathfrak{F}$.

$(\Rightarrow\vee 1)$: Suppose that $\Gamma\Rightarrow\phi$ is valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}$, then, from the assumption, $\mathfrak{M},w\Vdash\phi$ holds. So, $\mathfrak{M},w\Vdash\phi\vee\psi$. Therefore, $\Gamma\Rightarrow\phi\vee\psi$ is valid on $\mathfrak{F}$.

$(\Rightarrow\vee 2)$: Suppose that $\Gamma\Rightarrow\psi$ is valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}$, then, from the assumption, $\mathfrak{M},w\Vdash\psi$ holds. So, $\mathfrak{M},w\Vdash\phi\vee\psi$. Therefore, $\Gamma\Rightarrow\phi\vee\psi$ is valid on $\mathfrak{F}$.

$(\wedge 1\Rightarrow)$: Suppose that $\Gamma,\phi,\Delta\Rightarrow\varphi$ is valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}\circ(\phi\wedge\psi)\circ\Delta^{\circ}$, then there exist $v,u,x,s\in\mathfrak{M}$ such that $R_{\circ}(w,x,s)$, $R_{\circ}(x,v,u)$, $\mathfrak{M},v\Vdash\Gamma^{\circ}$, $\mathfrak{M},u\Vdash\phi\wedge\psi$ and $\mathfrak{M},s\Vdash\Delta^{\circ}$. Then, $\mathfrak{M},u\Vdash\phi$ holds. So, from the assumption, $\mathfrak{M},w\Vdash\varphi$ holds. Therefore, $\Gamma,\phi\wedge\psi,\Delta\Rightarrow\varphi$ is valid on $\mathfrak{F}$.

$(\wedge 2\Rightarrow)$: Suppose that $\Gamma,\psi,\Delta\Rightarrow\varphi$ is valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}\circ(\phi\wedge\psi)\circ\Delta^{\circ}$, then there exist $v,u,x,s\in\mathfrak{M}$ such that $R_{\circ}(w,x,s)$, $R_{\circ}(x,v,u)$, $\mathfrak{M},v\Vdash\Gamma^{\circ}$, $\mathfrak{M},u\Vdash\phi\wedge\psi$ and $\mathfrak{M},s\Vdash\Delta^{\circ}$. Then, $\mathfrak{M},u\Vdash\psi$ holds. So, from the assumption, $\mathfrak{M},w\Vdash\varphi$ holds. Therefore, $\Gamma,\phi\wedge\psi,\Delta\Rightarrow\varphi$ is valid on $\mathfrak{F}$.

$(\Rightarrow\wedge)$: Suppose that $\Gamma\Rightarrow\phi$ and $\Gamma\Rightarrow\psi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}$, then, by the first assumption, $\mathfrak{M},w\Vdash\phi$, and, by the second assumption, $\mathfrak{M},w\Vdash\psi$ hold. So, $\mathfrak{M},w\Vdash\phi\wedge\psi$ holds. Therefore, $\Gamma\Rightarrow\phi\wedge\psi$ is valid on $\mathfrak{F}$.

$(\circ\Rightarrow)$: By definition.

$(\Rightarrow\circ)$: Suppose that $\Gamma\Rightarrow\phi$ and $\Delta\Rightarrow\psi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Gamma^{\circ}\circ\Delta^{\circ}$, then there exist $v,u\in\mathfrak{M}$ such that $R_{\circ}(w,v,u)$, $\mathfrak{M},v\Vdash\Gamma^{\circ}$ and $\mathfrak{M},u\Vdash\Delta^{\circ}$. From the first assumption, $\mathfrak{M},v\Vdash\phi$, and, from the second assumption, $\mathfrak{M},u\Vdash\psi$ hold. So, $\mathfrak{M},w\Vdash\phi\circ\psi$. Therefore, $\Gamma,\Delta\Rightarrow\phi\circ\psi$ is valid on $\mathfrak{F}$.

$(\backslash\Rightarrow)$: Suppose that $\Gamma\Rightarrow\phi$ and $\Xi,\psi,\Delta\Rightarrow\varphi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w\in\mathfrak{F}$, if $\mathfrak{M}(=\mathfrak{F},V),w\Vdash\Xi^{\circ}\circ\Gamma^{\circ}\circ(\phi\backslash\psi)\circ\Delta^{\circ}$, then there exist $v,u,x,s\in\mathfrak{M}$ such that $R_{\circ}(w,x,s)$, $R_{\circ}(x,v,u)$, $\mathfrak{M},v\Vdash\Xi^{\circ}$,

$\mathfrak{M}, u \Vdash \Gamma^\circ \circ (\phi\backslash\psi)$ and $\mathfrak{M}, s \Vdash \Delta^\circ$. Moreover, from $\mathfrak{M}, u \Vdash \Gamma^\circ \circ (\phi\backslash\psi)$, there exist $y, z \in \mathfrak{M}$ such that $R_\circ(u, y, z)$, $\mathfrak{M}, y \Vdash \Gamma^\circ$ and $\mathfrak{M}, z \Vdash \phi\backslash\psi$. By the first assumption, $\mathfrak{M}, y \Vdash \phi$ holds. So, $\mathfrak{M}, u \Vdash \psi$ holds. By the second assumption, $\mathfrak{M}, w \Vdash \varphi$ holds. Therefore, $\Xi, \Gamma, \phi\backslash\psi, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$.

$(\Rightarrow \backslash)$: Assume that $\Gamma \Rightarrow \phi\backslash\psi$ is not valid on $\mathfrak{F}$. Then, there exist a valuation V' and an element $w' \in \mathfrak{F}$ such that $\mathfrak{M}(= \mathfrak{F}, V'), w' \Vdash \Gamma^\circ$ but $\mathfrak{M}, w' \nVdash \phi\backslash\psi$. So, there exist $v, u \in \mathfrak{M}$ such that $R_\circ(u, v, w')$, $\mathfrak{M}, v \Vdash \phi$ but $\mathfrak{M}, u \nVdash \psi$. On the other hand, since $R_\circ(u, v, w')$, $\mathfrak{M}, v \Vdash \phi$ and $\mathfrak{M}, w' \Vdash \Gamma^\circ$ hold, then $\mathfrak{M}, u \Vdash \phi \circ \Gamma^\circ$. So, $\mathfrak{M}, w' \nVdash \phi, \Gamma \Rightarrow \psi$. Therefore, $\phi, \Gamma \Rightarrow \psi$ is not valid on $\mathfrak{F}$.

$(/ \Rightarrow)$: Suppose that $\Gamma \Rightarrow \phi$ and $\Xi, \psi, \Delta \Rightarrow \varphi$ are valid on $\mathfrak{F}$. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash \Xi^\circ \circ (\psi/\phi) \circ \Gamma^\circ \circ \Delta^\circ$, then there exist $v, u, x, s \in \mathfrak{M}$ such that $R_\circ(w, x, s)$, $R_\circ(x, v, u)$, $\mathfrak{M}, v \Vdash \Xi^\circ$, $\mathfrak{M}, u \Vdash (\psi/\phi) \circ \Gamma^\circ$ and $\mathfrak{M}, s \Vdash \Delta^\circ$. Moreover, from $\mathfrak{M}, u \Vdash (\psi/\phi) \circ \Gamma^\circ$, there exist $y, z \in \mathfrak{M}$ such that $R_\circ(u, y, z)$, $\mathfrak{M}, y \Vdash \psi/\phi$ and $\mathfrak{M}, z \Vdash \Gamma^\circ$. By the first assumption, $\mathfrak{M}, z \Vdash \phi$ holds. So, $\mathfrak{M}, u \Vdash \psi$ holds. By the second assumption, $\mathfrak{M}, w \Vdash \varphi$ holds.Therefore, $\Xi, \psi/\phi, \Gamma, \Delta \Rightarrow \varphi$ is valid on $\mathfrak{F}$.

$(\Rightarrow /)$: Assume that $\Gamma \Rightarrow \psi/\phi$ is not valid on $\mathfrak{F}$. Then, there exist a valuation V' and an element $w' \in \mathfrak{F}$ such that $\mathfrak{M}(= \mathfrak{F}, V'), w' \Vdash \Gamma^\circ$ but $\mathfrak{M}, w' \nVdash \psi/\phi$. So, there exist $v, u \in \mathfrak{M}$ such that $R_\circ(u, w', v)$, $\mathfrak{M}, v \Vdash \phi$ but $\mathfrak{M}, u \nVdash \psi$. On the other hand, since $R_\circ(u, w', v)$, $\mathfrak{M}, v \Vdash \phi$ and $\mathfrak{M}, w' \Vdash \Gamma^\circ$ hold, then $\mathfrak{M}, u \Vdash \Gamma^\circ \circ \phi$. So, $\mathfrak{M}, w' \nVdash \Gamma, \phi \Rightarrow \psi$. Therefore, $\Gamma, \phi \Rightarrow \psi$ is not valid on $\mathfrak{F}$.

Thus, $\mathfrak{C}_{\mathrm{DFL}} \Vdash_t \mathbf{DFL}$. We only check the inference rule $(\Box)$ on $\mathfrak{C}_\Box$ here. Let $\mathfrak{F}$ be an arbitrary Kripke frame.

$(\Box)$: Suppose that $\Gamma \Rightarrow \phi$ is valid. For an arbitrary valuation V and an arbitrary $w \in \mathfrak{F}$, if $\mathfrak{M}(= \mathfrak{F}, V), w \Vdash (\Box\Gamma)^\wedge$, by the Meet preservation, $\mathfrak{M}, w \Vdash \Box(\Gamma^\wedge)$. So, for any $v \in \mathfrak{F}$, if $R_\Box(w, v)$, then $\mathfrak{M}, v \Vdash \Gamma^\wedge$. By the assumption, $\mathfrak{M}, v \Vdash \phi$. Then, $\mathfrak{M}, w \Vdash \Box\phi$. Therefore, $\Box\Gamma \Rightarrow \Box\phi$ is valid on $\mathfrak{F}$. (Q.E.D)

Given a logic $\mathbf{L}$, a frame $\mathfrak{F}$ is a *L-frame*, if $\mathfrak{F} \Vdash_t \mathbf{L}$. The set of all L-frames is denoted by $\mathfrak{C}_{\mathrm{L}}$. By Theorem 4.5, the name (DFL-frame) makes sense.

As a corollary, we can show the following.

Theorem 4.6
Any DFL (normal modal) logic $\boldsymbol{L}$ is sound for the class $\mathfrak{C}_L$ of L-frames.

4.3 Kripke completeness via canonicity

Now, we will show Kripke completeness of some DFL (normal modal) logics. As we saw before, thanks to the Lindenbaum-Tarski algebra, any extension $\mathbf{L}$ of $\mathbf{DFL}$ and $\mathbf{K}$ is algebraic complete with respect to the class $\mathfrak{C}^{\mathrm{L}}$. However, on a relational semantics, we do not know the way we can detect every Kripke complete logic, yet.

It is also known that there exist some non-Kripke complete logics in normal modal logics (see e.g. chap 4.4 in [2], [4] or [9]). But, in normal modal logics, there exists a way to check some Kripke complete logics automatically. So, we take the same strategy for DFL logics.

Definition 4.7 (Kripke complete)
A logic $\mathbf{L}$ is *Kripke complete with respect to the class* $\mathfrak{C}_L$ *of L-frames*, if $\mathfrak{C}_\mathrm{L} \Vdash_t \mathbf{L}$. Sometimes, we say just *Kripke complete*.

To define canonical models, we introduce some terminology.

Definition 4.8 (L-set)
Given a logic $\mathbf{L}$, a set Σ of formulas is a ***L**-set*, if Σ satisfying the following.

1. If $\phi \in \Sigma$ and $\psi \in \Sigma$, then $\phi \wedge \psi \in \Sigma$.

2. If $\phi \in \Sigma$ and $\phi \backslash \psi \in \mathbf{L}$, then $\psi \in \Sigma$.

Σ is *consistent*, if Σ also satisfies the following.

3. $\bot \notin \Sigma$.

Σ is *prime*, if Σ also satisfies the following.

4. If $\phi \vee \psi \in \Sigma$, then $\phi \in \Sigma$ or $\psi \in \Sigma$.

We know that, given a logic $\mathbf{L}$, the set $\mathrm{Con}(\mathrm{Frm}(\Phi))$ of all $\mathbf{L}$-consistent sets is a partially ordered set under the set inclusion. Therefore, we can define the following.

Definition 4.9 (Maximal consistent set)
Given a logic $\mathbf{L}$, a set Σ of formulas is *maximal*, if Σ is a maximal element in the set $\mathrm{Con}(\mathrm{Frm}(\Phi))$ of all $\mathbf{L}$-consistent sets.

It is known that, for any normal modal logic $\mathbf{L}$, every $\mathbf{L}$-prime consistent set is maximal.

We define the canonical model here.

Definition 4.10 (Canonical model)
Given a DFL logic $\mathbf{L}$, the tuple $\mathfrak{M}^\mathbf{L} = \langle W^\mathbf{L}, O^\mathbf{L}, N^\mathbf{L}, R^\mathbf{L}_\circ, V^\mathbf{L}\rangle$ is *the canonical model of **L***, where

1. $W^\mathbf{L}$ is the set of all $\mathbf{L}$-prime consistent sets,

2. $O^\mathbf{L}$ is the set of all $\mathbf{L}$-prime consistent sets containing 1,

3. $N^\mathbf{L}$ is the set of all $\mathbf{L}$-prime consistent sets containing 0,

4. $R^\mathbf{L}_\circ(\Sigma_1, \Sigma_2, \Sigma_3) \iff$
   for any formulas ϕ, ψ, χ, if $\psi \in \Sigma_2$, $\chi \in \Sigma_3$ and $(\psi \circ \chi)\backslash\phi \in \mathbf{L}$, then $\phi \in \Sigma_1$,

5. $V^{\mathbf{L}}(p) := \{\Sigma \mid p \in \Sigma\}$, for any propositional variable $p \in \Phi$.

The condition 4 can be considered as $R^{\mathbf{L}}_{\circ}(\Sigma_1, \Sigma_2, \Sigma_3) \iff \Sigma_2 \circ \Sigma_3 \subseteq \Sigma_1$, where

$$\Sigma_2 \circ \Sigma_3 := \{\phi \mid \text{there exist } \psi \in \Sigma_2, \chi \in \Sigma_3 \text{ and } (\psi \circ \chi)\backslash\phi \in \mathbf{L}\}.$$

Given a normal modal logic $\mathbf{L}$, the tuple $\mathfrak{M}^{\mathbf{L}} = \langle W^{\mathbf{L}}, R^{\mathbf{L}}_{\Box}, V^{\mathbf{L}} \rangle$ is *the canonical model of* $\boldsymbol{L}$, where

1. $W^{\mathbf{L}}$ is the set of all $\mathbf{L}$-maximal consistent sets,
2. $R^{\mathbf{L}}_{\Box}(\Sigma_1, \Sigma_2) \iff \{\phi \mid \Box\phi \in \Sigma_1\} \subseteq \Sigma_2$,
3. $V^{\mathbf{L}}(p) := \{\Sigma \mid p \in \Sigma\}$, for any propositional variable $p \in \Phi$.

The tuple $\mathfrak{F}^{\mathbf{L}}$ deleting $V^{\mathbf{L}}$ from the canonical model $\mathfrak{M}^{\mathbf{L}}$ of $\mathbf{L}$ is called *the canonical frame of* $\boldsymbol{L}$.

A logic $\mathbf{L}$ is *canonical*, if $\mathfrak{F}^{\mathbf{L}} \in \mathfrak{C}_{\mathrm{L}}$. We claim that every canonical logic is Kripke complete. To show this, we prove the following lemmas.

Lemma 4.11
Given a logic $\mathbf{L}$, for any $\Sigma_1 \in W^{\mathbf{L}}$, there exist $\Sigma_2, \Sigma_3 \in W^{\mathbf{L}}$ (Σ_2) such that $R^{\mathbf{L}}_{\circ}(\Sigma_1, \Sigma_2, \Sigma_3)$, $\phi \in \Sigma_2$ and $\psi \in \Sigma_3$ ($R^{\mathbf{L}}_{\Box}(\Sigma_1, \Sigma_2)$ and $\phi \in \Sigma_2$), if $\phi \circ \psi \in \Sigma_1$ ($\Diamond\phi \in \Sigma_1$).

Proof
We prove $R^{\mathbf{L}}_{\circ}$ here, since $R^{\mathbf{L}}_{\Box}$ can be analogously proved. If $\phi_1 \circ \phi_2 \in \Sigma_1$, we can define two sets of formulas as follows.

$$\Gamma_1 := \{\phi \mid \phi_1\backslash\phi \in \mathbf{L}\}$$

$$\Gamma_2 := \{\phi \mid \phi_2\backslash\phi \in \mathbf{L}\}$$

Then, Γ_1 and Γ_2 are $\mathbf{L}$-consistent sets satisfying $\Gamma_1 \circ \Gamma_2 \subseteq \Sigma_1$, $\phi_1 \in \Gamma_1$ and $\phi_2 \in \Gamma_2$, because, if $\bot \in \Gamma_1(\Gamma_2)$, $\bot \in \Sigma_1$, a contradiction. We want to show that there exists a $\mathbf{L}$-prime consistent set Σ_2 such that $\Sigma_2 \circ \Gamma_2 \subseteq \Sigma_1$, $\Gamma_1 \subseteq \Sigma_2$. From now on, we construct Σ_2.

We define $\mathcal{F} := \{\Gamma \in \mathrm{Con}(\mathrm{Frm}(\Phi)) \mid \Gamma_1 \subseteq \Gamma \text{ and } \Gamma \circ \Gamma_2 \subseteq \Sigma_1\}$. $\mathcal{F}$ is non-empty and inductive. Therefore, by Lemma 3.6, there exists a maximal element Σ_2 of $\mathcal{F}$. We claim Σ_2 is prime. Assume $\psi_1 \notin \Sigma_2$, $\psi_2 \notin \Sigma_2$ but $\psi_1 \vee \psi_2 \in \Sigma_2$. Since Σ_2 is a maximal element of $\mathcal{F}$, the smallest $\mathbf{L}$-set containing ψ_1 and Σ_2 is not in $\mathcal{F}$. Therefore, there exist $\psi'_1, \psi'_2 \in \Sigma_2, \chi_1, \chi_2 \in \Gamma_3, \theta_1, \theta_2 \notin \Sigma_1$ such that $((\psi_1 \wedge \psi'_1) \circ \chi_1)\backslash\theta_1 \in \mathbf{L}$ and $((\psi_2 \wedge \psi'_2) \circ \chi_2)\backslash\theta_2 \in \mathbf{L}$. Then ,$(((\psi_1 \vee \psi_2) \wedge \psi'_1 \wedge \psi'_2) \circ (\chi_1 \wedge \chi_2))\backslash(\theta_1 \vee \theta_2) \in \mathbf{L}$, a contradiction. The same strategy can be useful for the other side. (Q.E.D)

Lemma 4.12
Given a logic $\mathbf{L}$, for any formula ϕ, $\mathfrak{M}, \Sigma \Vdash \phi \iff \phi \in \Sigma$.

Proof
Thanks to Lemma 4.11, this is immediate by induction on ϕ. (Q.E.D)

Lemma 4.13
Given a logic $\mathbf{L}$, if $\phi \notin \mathbf{L}$, then there exists a $\mathbf{L}$-prime consistent set Σ such that $\mathbf{L} \subseteq \Sigma$ and $\phi \notin \Sigma$.

Proof
We define the class $\mathcal{F} := \{\Gamma \in \mathrm{Con}(\mathrm{Frm}(\Phi)) \mid \mathbf{L} \subseteq \Gamma \text{ and } \phi \notin \Gamma\}$. Since $\mathbf{L} \in \mathcal{F}$, $\mathcal{F}$ is non-empty. Then, since $\mathcal{F}$ is an inductive set, by Lemma 3.6, there exists a maximal element Σ in $\mathcal{F}$.

We want to show that Σ is prime. Assume that $\psi_a \notin \Sigma$ and $\psi_b \notin \Sigma$ but $\phi_a \vee \psi_b \in \Sigma$. Since Σ is maximal, for the smallest $\mathbf{L}$-consistent set Σ_a (Σ_b) containing $\Sigma \cup \{\psi_a\}$ ($\Sigma \cup \{\psi_b\}$), Σ_a (Σ_b) is not in $\mathcal{F}$. So, there exists $\chi_a \in \Sigma$ (χ_b) such that $(\psi_a \wedge \chi_a)\backslash\phi$ ($(\psi_b \wedge \chi_b)\backslash\phi$) is in $\mathbf{L}$. Then, we can prove $(\psi_a \vee \psi_b)\backslash\phi$, a contradiction. (Q.E.D)

Then, we can prove the following theorem.

Theorem 4.14 (Kripke completeness via canonicity)
Every canonical logic $\mathbf{L}$ is Kripke complete.

Proof
Assume that $\phi \notin \mathbf{L}$. By Lemma 4.13, there exists a $\mathbf{L}$-prime consistent set Σ such that $\mathbf{L} \subseteq \Sigma$ and $\phi \notin \Sigma$. Then, by Lemma 4.12, $\mathfrak{M}^{\mathbf{L}}, \Sigma \not\Vdash \phi$. Now, since $\mathbf{L}$ is canonical, $\mathfrak{F}^{\mathbf{L}} \in \mathfrak{C}_{\mathbf{L}}$. Therefore, $\mathfrak{C}_{\mathbf{L}} \not\Vdash_t \phi$. (Q.E.D)

Next, we show Kripke completeness of $\mathbf{K}$ and $\mathbf{DFL}$. To prove Proposition 4.17, we prove the following lemmas.

Lemma 4.15
Given two $\mathbf{L}$-sets Σ_1 and Σ_2, $\Sigma_1 \circ \Sigma_2$ is again a $\mathbf{L}$-set.

Proof
We show that $\Sigma_1 \circ \Sigma_2$ satisfies the conditions of $\mathbf{L}$-sets.

If $\phi_1 \in \Sigma_1 \circ \Sigma_2$ and $\phi_2 \in \Sigma_1 \circ \Sigma_2$, then there exist $\psi_1, \psi_2 \in \Sigma_1, \chi_1, \chi_2 \in \Sigma_2$ such that $(\psi_1 \circ \chi_1)\backslash\phi_1, (\psi_2 \circ \chi_2)\backslash\phi_2 \in \mathbf{L}$. Then, $((\psi_1 \wedge \psi_2) \circ (\chi_1 \wedge \chi_2))\backslash(\phi_1 \wedge \phi_2)$. Therefore, $\phi_1 \wedge \phi_2 \in \Sigma_1 \circ \Sigma_2$.

If $\phi \in \Sigma_1 \circ \Sigma_2$ and $\phi \Rightarrow \psi \in \mathbf{L}$, then obviously $\psi \in \Sigma_1 \circ \Sigma_2$. (Q.E.D)

Lemma 4.16
For any $\mathbf{L}$-consistent sets Σ_2, Σ_3 and a $\mathbf{L}$-prime consistent set Σ_1, if $\Sigma_2 \circ \Sigma_3 \subseteq \Sigma_1$, there exist $\mathbf{L}$-prime consistent sets Σ_2' and Σ_3' such that $R(\Sigma_1, \Sigma_2', \Sigma_3')$, $\Sigma_2 \subseteq \Sigma_2'$ and Σ_3'.

Proof
Assume $\Sigma_2 \circ \Sigma_3 \subseteq \Sigma_1$. We find Σ_2' firstly. Let $\mathcal{F}$ be the class of all **L**-consisntent sets Δ satisfies $\Sigma_2 \subseteq \Delta$ and $\Delta \circ \Sigma_3 \subseteq \Sigma_1$. Since $\mathcal{F}$ is an inductive set, by Lemma 3.6, there exists a maximal element $\Sigma_2' \in \mathcal{F}$.

We want to show that Σ_2' is prime. Assume $\phi_1 \notin \Sigma_2'$, $\phi_2 \notin \Sigma_2'$ but $\phi_1 \vee \phi_2 \in \Sigma'$. Then, in the smallest **L**-consistent set Σ_{2a}' (Σ_{2b}') satisfying $\Sigma_2' \cup \{\phi_1\} \subseteq \Sigma_{2a}'$ ($\Sigma_2' \cup \{\phi_2\} \subseteq \Sigma_{2b}'$), there exist $\chi_1, \chi_2 \in \Sigma_1$, $\psi_1, \psi_2 \in \Sigma_2'$, $\theta_1, \theta_2 \in \Sigma_3$ such that $((\phi_1 \wedge \psi_1) \circ \theta_1) \backslash \chi_1$ and $((\phi_2 \wedge \psi_2) \circ \theta_2) \backslash \chi_2$, because Σ_2' is maximal of $\mathcal{F}$. Then, by the distributivity, $(((\phi_1 \vee \phi_2) \wedge \psi_1 \wedge \psi_2) \circ (\theta_1 \wedge \theta_2)) \backslash (\chi_1 \vee \chi_2)$, a contradiction. For Σ_3, we can take the same strategy. (Q.E.D)

Then, we show the following.

Proposition 4.17
DFL and **K** are canonical.

Proof
We need to show that $\mathfrak{F}^{\mathbf{DFL}}$ ($\mathfrak{F}^{\mathbf{K}}$) is a DFL- (Kripke) frame. But, it is obvious that $\mathfrak{F}^{\mathbf{K}}$ is Kripke frame. So, we here show that $\mathfrak{F}^{\mathbf{DFL}}$ is a DFL-frame. In other words, $\mathfrak{F}^{\mathbf{DFL}}$ satisfies the conditions for DFL-frames.

$R_\circ^{\mathbf{DFL}}$**-reflexivity** : For any **L**-prime consistent set Σ, $\Sigma \circ \mathbf{L} \subseteq \Sigma$. By Lemma 4.16, there exists a **L**-prime consistent set Γ such that $1 \in \mathbf{L} \subseteq \Gamma$ and $R_\circ^{\mathbf{DFL}}(\Sigma, \Gamma, \Sigma)$. The converse is analogous.

$R_\circ^{\mathbf{DFL}}$**-transitivity** : Assume that $R_\circ^{\mathbf{DFL}}(\Sigma_1, \Sigma_2, \Sigma_3)$, $\Sigma_1 \subseteq \Gamma_1$, $\Gamma_2 \subseteq \Sigma_2$ and $\Gamma_3 \subseteq \Sigma_3$. For any formula ϕ, $\phi \in \Gamma_2 \circ \Gamma_3 \subseteq \Sigma_2 \circ \Sigma_3 \subseteq \Sigma_1 \subseteq \Gamma_1$. Therefore, $R_\circ^{\mathbf{DFL}}(\Gamma_1, \Gamma_2, \Gamma_3)$.

$R_\circ^{\mathbf{DFL}}$**-associativity** : Assume $R_\circ^{\mathbf{DFL}}(\Sigma_1, \Sigma_2, \Sigma_3)$ and $R_\circ^{\mathbf{DFL}}(\Sigma_2, \Sigma_4, \Sigma_5)$. Then, by the associativity of $\circ$, $\Sigma_4 \circ \Sigma_5 \circ \Sigma_3 \subseteq \Sigma_1$. By Lemma 4.16, there exists a **DFL**-prime consistent set Γ such that $R_\circ^{\mathbf{DFL}}(\Sigma_1, \Sigma_4, \Gamma)$, $R_\circ^{\mathbf{DFL}}(\Gamma, \Sigma_5, \Sigma_3)$. The converse is analogous.

$O^{\mathbf{DFL}}$ **is closed under** $\subseteq$: If $\Sigma_1 \subseteq \Sigma_2$ and $\Sigma_1 \in O^{\mathbf{DFL}}$, $1 \in \Sigma_1 \subseteq \Sigma_2$. Therefore, $\Sigma_2 \in O^{\mathbf{DFL}}$.

$N^{\mathbf{DFL}}$ **is closed under** $\subseteq$: If $\Sigma_1 \subseteq \Sigma_2$ and $\Sigma_1 \in N^{\mathbf{DFL}}$, $0 \in \Sigma_1 \subseteq \Sigma_2$. Therefore, $\Sigma_2 \in N^{\mathbf{DFL}}$. (Q.E.D)

Then, we can prove the following, as a corollary of Theorem 4.14.

Theorem 4.18 (Kripke completeness of DFL and K)
DFL *and* ***K*** *are Kripke complete.*

Proof
This is immediate from Proposition 4.17 and Theorem 4.14. (Q.E.D)

Next, we prove Kripke completeness of all basic DFL logics.

Theorem 4.19 (Kripke completeness of all basic DFL logics)
Any basic DFL logic is Kripke complete.

Proof
We need to check every canonical model satisfying each rule satisfies the correspondence DFL-frame condition.

(c $\Rightarrow$) : If **L** satisfies (c $\Rightarrow$), then $\phi\backslash(\phi \circ \phi) \in \mathbf{L}$ for any formula ϕ. Therefore, if $\phi \in \Sigma$, then $\phi \circ \phi \in \Sigma$.

(e $\Rightarrow$) : If **L** satisfies (e $\Rightarrow$), then $(\phi \circ \psi)\backslash(\psi \circ \phi) \in \mathbf{L}$ for any formula ϕ and ψ. Therefore, if $\phi \circ \psi \in \Sigma$, then $\psi \circ \phi \in \Sigma$.

(w $\Rightarrow$) : If **L** satisfies (w $\Rightarrow$), then $\phi\backslash 1 \in \mathbf{L}$ for any formula ϕ. Therefore, $1 \in \Sigma$ for any Σ.

($\Rightarrow$ w) : If **L** satisfies ($\Rightarrow$ w), then $0\backslash\phi \in \mathbf{L}$ for any formula ϕ. Therefore, there is no Σ such that $0 \in \Sigma$.

Therefore, every canonical model of **L** satisfying each rule is in $\mathfrak{C}_{\mathrm{L}}$. That is, **L** is canonical. (Q.E.D)

5 Algebraic semantics and Relational semantics via Stone's duality

In the previous chapter, we defined a relational semantics for DFL logics and proved Kripke completeness of basic DFL logics and **K**. Here, we introduce Stone's duality on which our relational semantics for DFL logics are based.(see [8], [10], [12], [14], [18], [19] or [20]).

5.1 Stone's duality

We introduce Stone's duality here.

Definition 5.1 (Prime filter frame)
Given a bounded distributive lattice $\mathfrak{A} = \langle A, \vee, \wedge, \top, \bot \rangle$, a tuple $\mathfrak{A}_- = \langle Pf(A), \subseteq \rangle$ is *the prime filter frame of* $\mathfrak{A}$, where $Pf(A)$ is the set of all prime filters over A and $\subseteq$ the set inclusion.

Definition 5.2 (Set algebra)
Given a poset $\mathfrak{F} = \langle W, \leq \rangle$, the tuple $\mathfrak{F}^- = \langle Up(W), \cup, \cap, W, \emptyset \rangle$ is *the set algebra of* $\mathfrak{F}$, where

1. $Up(W)$ is the set of all upward closed sets over W,
2. $\cup$ ($\cap$) is the set union (intersection).

Then, we can introduce the following theorem.

Theorem 5.3 (Stone)
Every bounded distributive lattice $\mathfrak{A}$ is embeddable into $(\mathfrak{A}_-)^-$, with the following embedding h from $\mathfrak{A}$ to $(\mathfrak{A}_-)^-$.

$$h(a) := \{F \in Pf(A) \mid a \in F\}$$

We have separately considered algebraic semantics and relational semantics so far. But, in modal logics, it is known that Theorem 5.3 represents a connection between algebraic semantics and relational semantics. Since we define DFL-frames based on this viewpoint, we will consider, in parallel with some results in modal logics, the connection between DFL-algebras and DFL-frames.

5.2 Dual algebra

As in the case of set algebras, we introduce dual algebras.

Definition 5.4 (Dual DFL-algebra)
Given a DFL-frame $\mathfrak{F} = \langle W, O, N, R_\circ \rangle$, $\mathfrak{F}^+ = \langle Up(W), \cup, \cap, *, \backslash, /, O, N, W, \emptyset \rangle$ is the *dual DFL-algebra* of $\mathfrak{F}$, where

1. $Up(W)$ is the set of all $R_\circ$-upward closed subsets of W,
2. $\cup$ ($\cap$) is the set theoretical union (intersection),
3. for any $X, Y \in Up(W)$,
   $X * Y := \{w \in W \mid \text{for some } v, u \in W, R_\circ(w, v, u), v \in X \text{ and } u \in Y\}$,
4. for any $X, Y \in Up(W)$,
   $X \searrow Y := \{w \in W \mid \text{for all } v, u \in W, \text{if } R_\circ(u, v, w) \text{ and } v \in X, \text{ then } u \in Y\}$,
5. for any $X, Y \in Up(W)$,
   $Y \swarrow X := \{w \in W \mid \text{for all } v, u \in W, \text{if } R_\circ(u, w, v) \text{ and } v \in X, \text{ then } u \in Y\}$.

Although Definition 5.4 gives us the definition of dual DFL-algebras, we need to make sure that the definition is well defined. The following lemma shows this.

Lemma 5.5
Given a DFL-frame $\mathfrak{F}$, in the dual DFL-algebra $\mathfrak{F}^+$, $X \cup Y$, $X \cap Y$, $X * Y$ $X \searrow Y$ and $Y \swarrow X$ are $R_\circ$-upward closed, for any $X, Y \in Up(W)$.

Therefore, the above definition is well defined.

Proof
For any $X, Y \in Up(W)$,

$X \cup Y$:
Assume $w \leq w'$ and $w \in X \cup Y$. Then, $w \in X$ or $w \in Y$. Here, both X and Y are $R_\circ$-upward closed. Therefore, $w' \in X$ or $w' \in Y$. So, $w' \in X \cup Y$.

$X \cap Y$:
Assume $w \leq w'$ and $w \in X \cap Y$. Then, $w \in X$ and $w \in Y$. Here, both X and Y are $R_\circ$-upward closed. Therefore, $w' \in X$ and $w' \in Y$. So, $w' \in X \cap Y$.

$X * Y$:
Assume $w \leq w'$ and $w \in X * Y$. Then, by the definition of $*$, there exist $v, u \in W$ such that $R_\circ(w, v, u)$, $v \in X$ and $u \in Y$. Here, by $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(w', v, u)$ holds. Therefore, $w' \in X * Y$.

$X \searrow Y$:
Assume $w \leq w'$ and $w' \notin X \searrow Y$. Then, by the definition of $\searrow$, there exist $v, u \in W$ such that $R_\circ(u, v, w')$, $v \in X$ but $u \notin Y$. Here, by $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(u, v, w)$ holds. Therefore, $w \notin X \searrow Y$.

$Y \swarrow X$:
Assume $w \leq w'$ and $w' \notin Y \swarrow X$. Then, by the definition of $\swarrow$, there exist $v, u \in W$ such that $R_\circ(u, w', v)$, $v \in X$ but $u \notin Y$. Here, by $R_\circ$-transitivity and $R_\circ$-reflexivity, $R_\circ(u, w, v)$ holds. Therefore, $w \notin Y \swarrow X$. (Q.E.D)

Then, we prove the following.

Theorem 5.6
Given any DFL-frame $\mathfrak{F}$, the dual DFL-algebra $\mathfrak{F}^+$ is a DFL-algebra.

Proof
It suffices to show the following conditions. First, we show that$\langle Up(W), \cup, \cap, W, \emptyset\rangle$ is a bounded distributive lattice, and $\langle Up(W), *, O\rangle$ is a monoid. Then, $\mathfrak{F}^+$ satisfies the residuation law.

$\langle Up(W), \cup, \cap, W, \emptyset\rangle$ is a bounded distributive lattice :

By Lemma 5.5, $Up(W)$ is closed under $\cup$ and $\cap$. Beside, by the definition of W ($\emptyset$), W ($\emptyset$) is the maximum (minimum) element of $Up(W)$. Then, $\langle Up(W), \cup, \cap, W, \emptyset\rangle$ is a bounded lattice.

It is suffices to prove $X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z)$, for the distributivity. For any $w \in X \cap (Y \cup Z)$, $w \in X$ and, either $w \in Y$ or $w \in Z$, hold.

If $w \notin Y$, then $w \in X \cap Z$. It derives $w \in (X \cap Y) \cup (X \cap Z)$. Otherwise $w \notin Z$, then $w \in X \cap Y$. It also derives $w \in (X \cap Y) \cup (X \cap Z)$. So, $w \in (X \cap Y) \cup (X \cap Z)$.

Therefore, $\langle Up(W), \cup, \cap, W, \emptyset\rangle$ is a bounded distributive lattice.

$\langle Up(W), *, O\rangle$ is a monoid :

Associativity : Assume $w \in X * (Y * Z)$. Then, there exist $v, u, s, t \in W$ such that $R_\circ(w, v, u)$, $R_\circ(u, s, t)$, $v \in X$, $s \in Y$ and $t \in Z$. By means of $R_\circ$-associativity, there exists $w' \in W$ such that $R_\circ(w, w', t)$ and $R_\circ(w', v, s)$. So, $w \in (X*Y)*Z$. Therefore, $X*(Y*Z) \subseteq (X*Y)*Z$ holds. The converse is analogous.

$\langle Up(W), *, O\rangle$ satisfies the associativity.

Identity : Assume $w \in X * O$. Then, there exist $v, o \in W$ such that $R_\circ(w, v, o)$, $v \in X$ and $o \in O$. This leads $v \leq w$. Now, X is $R_\circ$-upward closed. So, $w \in X$. Therefore, $X * O \subseteq X$.

Assume $w \in X$. By $R_\circ$-reflexivity, there exists $o \in W$ such that $R_\circ(w, w, o)$ and $o \in O$. So, $w \in X * O$. Therefore, $X \subseteq X * O$.

$O * X = X$ can be analogously proved.

$\mathfrak{F}^+$ satisfies the residuation law :

For any $X, Y, Z \in Up(W)$, the residuation law is proved along the following way.

$$X * Y \leq Z \Longrightarrow Y \leq X \backslash Y \Longrightarrow X \leq Z / Y \Longrightarrow X * Y \leq Z$$

First implication. Assume $X * Y \leq Z$, $w \in Y$, $v \in X$ and $R_\circ(u, v, w)$. From the latter three assumptions, $u \in X * Y$ holds. Besides, by the first assumption, $u \in Z$ holds. Therefore, $X * Y \leq Z \Longrightarrow Y \leq X \backslash Z$.

Second implication. Assume $Y \leq X \searrow Z$, $w \in X$, $v \in Y$ and $R_\circ(u, w, v)$. The latter three assumptions mean $v \in Y$, $w \in X$ and $R_\circ(u, w, v)$. Besides, by the first assumption, $u \in Z$ holds. Therefore, $Y \leq X \searrow Z \Longrightarrow X \leq Z \swarrow Y$.

The last implication. Assume $X \leq Z \swarrow Y$ and $w \in X * Y$. $w \in X * Y$ means there exist $v, u \in W$ such that $R_\circ(w, v, u)$, $v \in X$ and $u \in Y$. From the first assumption, $w \in Z$ holds. Therefore, $X \leq Z \swarrow Y \Longrightarrow X * Y \leq Z$. (Q.E.D)

Based on this duality, we prove the following proposition for validity.

Proposition 5.7
For any formula ϕ, ϕ is valid on a DFL-frame $\mathfrak{F}$, if and only if ϕ is valid on the dual DFL-algebra $\mathfrak{F}^+$.

Proof
Given a DFL-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, the valuation V can be inductively extended as follows.

- $V(1) = O$.
- $V(0) = N$.
- $w \in V(\phi \vee \psi)$ if and only if $w \in V(\phi)$ or $w \in V(\psi)$.
- $w \in V(\phi \wedge \psi)$ if and only if $w \in V(\phi)$ and $w \in V(\psi)$.
- $w \in V(\phi \circ \psi)$ if and only if
  there exist $v, u \in W$ such that $R_\circ(w, v, u)$, $v \in V(\phi)$ and $u \in V(\psi)$.
- $w \in V(\phi \backslash \psi)$ if and only if
  for any $v, u \in W$, if $R_\circ(u, v, w)$ and $v \in V(\phi)$, then $u \in V(\psi)$.
- $w \in V(\psi / \phi)$ if and only if
  for any $v, u \in W$, if $R_\circ(u, w, v)$ and $v \in V(\phi)$, then $u \in V(\psi)$.

From this point of view, any valuation on a DFL-frame itself is an assignment on the dual DFL-algebra. (Q.E.D)

Definition 5.8 (Dual modal algebra)
Given a Kripke frame $\mathfrak{F} = \langle W, R_\Box \rangle$, the tuple $\mathfrak{F}^+ = \langle \wp(W), \cup, \cap, -, \Box, W, \emptyset \rangle$ is *the dual modal algebra of* $\mathfrak{F}$, where

1. $\wp(W)$ is the power set of W,
2. $\cup$ is the set union,
3. $\cap$ is the set intersection,
4. $-$ is the complement operation,

5. for any $X \in \wp(W)$,
   $\Box X := \{w \in W \mid \text{for all } w' \in W, \text{ if } R_\Box(w, w'), \text{ then } w' \in X\}$.

Then, we prove the following.

Theorem 5.9
Given any Kripke frame $\mathfrak{F}$, the dual modal algebra $\mathfrak{F}^+$ is a modal algebra.

Proof
We check the following.

$\langle \wp(W), \cup, \cap, -, W, \emptyset \rangle$ is a Boolean algebra : Thanks to Theorem 5.6, we need to check (Complementation) and (Boundedness). But, both are obvious.

(Meet preservation) : Let X, Y be arbitrary elements of $\wp(W)$. If $w \in \Box X \cap \Box Y$, then $w \in \Box X$ and $w \in \Box Y$. For an arbitrary element $v \in W$, if $R_\Box(w, v)$, then $v \in X$ and $v \in Y$. So, $w \in \Box(X \cap Y)$.

Assume $w \notin \Box X \cap \Box Y$. Then, there exists an element $v \in W$ such that $R_\Box(w, v)$ and either $v \notin X$ or $v \notin Y$. Therefore, $w \notin \Box(X \cap Y)$.

(Top preservation) : This is obvious. (Q.E.D)

Then, we can show the following propositions along with Proposition 5.7.

Proposition 5.10
For any formula ϕ, ϕ is valid on a Kripke frame $\mathfrak{F}$, if and only if ϕ is valid on the dual modal algebra $\mathfrak{F}^+$.

5.3 Dual frame

Next, we define the duality like prime filter frames.

Definition 5.11 (Dual DFL-frame)
Given a DFL-algebra $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot \rangle$, the tuple $\mathfrak{A}_+ = \langle Pf(A), Pf_1(A), Pf_0(A), R_\circ \rangle$ is the *dual DFL-frame of* $\mathfrak{A}$, where

1. $Pf(A)$ is the set of all prime filters over $\mathfrak{A}$,
2. $Pf_1(A)$ is the set of all prime filters containing 1,
3. $Pf_0(A)$ is the set of all prime filters containing 0,
4. for any $F_1, F_2, F_3 \in Pf(A)$,
   $R_\circ(F_1, F_2, F_3)$ iff $b \circ c \leq a$, $b \in F_2$ and $c \in F_3$ imply $a \in F_1$, for all $a, b, c \in A$.

The last condition can be also written by $R_\circ(F_1, F_2, F_3) \iff F_2 \circ F_3 \subseteq F_1$, if we define a binary operation $\circ$ on the set of filters of A, as follows.

$$F_2 \circ F_3 := \{a \in A \mid \text{for some } b, c \in A, b \circ c \leq a,\ b \in F_2 \text{ and } c \in F_3\}$$

Given an assignment f on a DFL-algebra $\mathfrak{A}$, the *dual valuation* V_f on the dual DFL-frame $\mathfrak{A}_+$ is defined below.

$$V_f(p) := \{F \in Pf(A) \mid f(p) \in F\}$$

To prove Theorem 5.15, we need to show several lemmas.

Lemma 5.12
Given arbitrary filters F and F', $F \circ F'$ is again a filter.

Proof
Since F and F' are filters, each filter has at least one element. So, $F \circ F'$ is non-empty.

Assume $f_1, f_2 \in F \circ F'$. Then, there exist $a, b \in F$ and $a', b' \in F'$ such that $a \circ a' \leq f_1$ and $b \circ b' \leq f_2$. By means of the monotonicity of $\circ$,

$$(a \wedge b) \circ (a' \wedge b') \leq f_1 \wedge f_2$$

$f_1 \wedge f_2 \in F \circ F'$ holds, because $a \wedge b \in F$ and $a' \wedge b' \in F'$.

The other direction is rather trivial. If $f_1 \wedge f_2 \in F \circ F'$, then there exist $a \in F$ and $a' \in F'$ such that $a \circ a' \leq f_1 \wedge f_2$. So, $a \circ a' \leq f_1 \in F \circ F'$ and $a \circ a' \leq f_2 \in F \circ F'$.

Therefore, $F \circ F'$ is a filter. (Q.E.D)

The following lemma, which is called *Squeeze lemma* in [7], is often used in relevant logics (see also [18] or [20]).

Lemma 5.13 (Squeeze lemma)
Let F_2, F_3 be arbitrary filters and P_1 a prime filter. If $F_2 \circ F_3 \subseteq P_1$ holds, there exist two prime filters P_2 and P_3 such that $F_2 \subseteq P_2$, $F_3 \subseteq P_3$ and $P_2 \circ P_3 \subseteq P_1$.

Proof
Let $\mathcal{F}$ be the class of filters F satisfying $F_2 \subseteq F$ and $F \circ F_3 \subseteq P_1$. Then, $\mathcal{F}$ is non-empty, since $F_2 \in \mathcal{F}$. Besides, $\mathcal{F}$ is a poset with respect to the set inclusion $\subseteq$. Then, $\mathcal{F}$ is inductive, because the infinite union of it is an element of $\mathcal{F}$, for every chain in $\mathcal{F}$. By Lemma 3.6, $\mathcal{F}$ has a maximal element F_{max}.

We claim that F_{max} is prime. Assume that $a \notin F_{max}$, $b \notin F_{max}$ but $a \vee b \in F_{max}$. Then, let F^a_{max} be the smallest filter containing F_{max} and a. Since F_{max} is a maximal element of $\mathcal{F}$, $F^a_{max} \circ F_3 \not\subseteq P_1$ holds. It means that there exist $f_1 \in F_{max}$, $g_1 \in F_3$ and $h_1 \notin P_1$ such that $(f_1 \wedge a) \circ g_1 \leq h_1$.

Similarly, let F^b_{max} be the smallest filter containing F_{max} and b. Then, there exist $f_2 \in F_{max}$, $g_2 \in F_3$ and $h_2 \notin P_1$ such that $(f_2 \wedge b) \circ g_2 \leq h_2$. By the monotonicity of $\circ$, $((f_1 \wedge f_2 \wedge a) \vee (f_1 \wedge f_2 \wedge b)) \circ (g_1 \wedge g_2) \leq h_1 \vee h_2$ holds. From the distributivity,

$(f_1 \wedge f_2 \wedge (a \vee b)) \circ (g_1 \wedge g_2) \leq h_1 \vee h_2$. Therefore, $h_1 \vee h_2 \in P_1$. This contradicts that P_1 is prime.

Therefore, there exists a prime filter P_2 such that $P_2 \circ F_3 \subseteq P_1$ and $F_2 \subseteq P_2$. Along with the same argument, we can prove the existence of a prime filter P_3 such that $P_2 \circ P_3 \subseteq P_1$ and $F_3 \subseteq P_3$. (Q.E.D)

Lemma 5.14 (Prime filter theorem)
Given a filter F, if $a \notin F$ for an element a, then there exists a prime filter P such that $F \subseteq P$ and $a \notin P$.

Proof
Let $\downarrow a$ be the principal ideal generated by a. Then, F and $\downarrow a$ are disjoint, because if there exists $f \in F \cap \downarrow a$ then $a \in F$.

Here, let $\mathcal{F}$ be the class of filters F' satisfying $F \subseteq F'$ and $a \notin F'$. Then, $\mathcal{F}$ is non-empty, because $F \in \mathcal{F}$. Besides, $\mathcal{F}$ is a poset with respect to the set inclusion $\subseteq$. Then, $\mathcal{F}$ is inductive, because the infinite union of it is an element of $\mathcal{F}$, for every chain in $\mathcal{F}$. By Lemma 3.6, $\mathcal{F}$ has a maximal element F_{max}.

Then, all we have to show is that F_{max} is prime. Assume that $b \notin F_{max}$, $c \notin F_{max}$ but $b \vee c \in F_{max}$. Then, let F^b_{max} be the smallest filter containing F_{max} and b, and F^c_{max} the smallest filter containing F_{max} and c. Since F_{max} is a maximal element of $\mathcal{F}$, there exist $f_1, f_2 \in F_{max}$ such that $f_1 \wedge f_2 \wedge b \leq a$ and $f_1 \wedge f_2 \wedge c \leq a$. Then, $(f_1 \wedge f_2 \wedge b) \vee (f_1 \wedge f_2 \wedge c) \leq a \vee a = a$ holds. By means of the distributivity, $f_1 \wedge f_2 \wedge (b \vee c) \leq a$. Therefore, $a \in F_{max}$. It contradicts $a \notin F_{max}$. (Q.E.D)

The above lemmas are very helpful in the following proof.

Theorem 5.15
Given any DFL-algebra $\mathfrak{A}$, the dual DFL-frame $\mathfrak{A}_+$ is a DFL-frame.

Proof
We show that $\mathfrak{A}_+ = \langle Pf(A), Pf_1(A), Pf_0(A), R_\circ \rangle$ satisfies all conditions for DFL-frames.

($R_\circ$-reflexivity) :

For any prime filter F, both $F \circ \uparrow 1 \subseteq F$ and $\uparrow 1 \circ F \subseteq F$ hold, where $\uparrow 1$ is the principal filter generated by 1. By the prime filter theorem (Theorem 5.14), there exist prime filters P_1 and P_2 such that $F \circ P_1 \subseteq F$, $1 \in P_1$, $P_2 \circ F \subseteq F$ and $1 \in P_2$.

Therefore, for all prime filter F, there exist prime filters P_1 and P_2 containing 1 such that $R_\circ(F, F, P_1)$ and $R_\circ(F, P_2, F)$.

($R_\circ$-transitivity) :

Suppose $R_\circ(F_1, F_2, F_3)$, $F_1 \leq F'_1$, $F'_2 \leq F_2$ and $F'_3 \leq F_3$. Generally, in the dual DFL-frame, $F \leq F'$ means that every element of F is included in F'.

So, if $b \in F'_2$, $c \in F'_3$ and $b \circ c \leq a$, then $b \in F_2$, $c \in F_3$ and $b \circ c \leq a$. By the first assumption, $a \in F_1$. So, $a \in F'_1$. Therefore, $R_\circ(F'_1, F'_2, F'_3)$ holds.

($R_\circ$-associativity) :

Assume $R_\circ(F_1, X, F_4)$ and $R_\circ(X, F_2, F_3)$. Then, $(F_2 \circ F_3) \circ F_4 \subseteq F_1$. By the associativity of $\circ$, $(F_2 \circ F_3) \circ F_4 = F_2 \circ (F_3 \circ F_4)$. Here, from Lemma 5.12, $F_3 \circ F_4$ is also a filter satisfying $F_2 \circ (F_3 \circ F_4) \subseteq F_1$.

So, by the Squeeze lemma (Lemma 5.13), there exists a prime filter Y such that $R_\circ(F_1, F_2, Y)$ and $R_\circ(Y, F_3, F_4)$. The converse is analogous.

($R_\circ$-upward closed property of $Pf_1(A)$) :

Assume $F_1 \in Pf_1(A)$ and $F_1 \leq F_2$. From the definition of $Pf_1(A)$, $1 \in F_1$ holds. Then, for each element of F_1 is included in F_2, by $F_1 \leq F_2$. So, $1 \in F_2$. Therefore, $F_2 \in Pf_1(A)$.

($R_\circ$-upward closed property of $Pf_0(A)$) :

Assume $F_1 \in Pf_0(A)$ and $F_1 \leq F_2$. From the definition of $Pf_0(A)$, $0 \in F_1$ holds. Then, for each element of F_1 is included in F_2, by $F_1 \leq F_2$. So, $0 \in F_2$. Therefore, $F_2 \in Pf_0(A)$. (Q.E.D)

Based on this duality, we prove the following proposition for validity.

Proposition 5.16
For any formula ϕ, ϕ is valid on a DFL-algebra $\mathfrak{A}$, if it is valid on the dual DFL-frame $\mathfrak{A}_+$.

Proof
Assume a formula ϕ is not valid on a DFL-algebra $\mathfrak{A}$. Then, there exists a assignment f satisfying $\mathfrak{A}, f \not\models \phi$, which means $1 \not\leq f(\phi)$. Now, let $\uparrow 1$ be the principal filter generated by 1. From $1 \not\leq f(\phi)$, $f(\phi) \notin \uparrow 1$ holds.

By the prime filter theorem (Lemma 5.14), there exists a prime filter P such that $\uparrow 1 \subseteq P$ and $f(\phi) \notin P$. So, on the dual DFL-frame, $\mathfrak{A}_+, V_f, P \not\Vdash \phi$ holds. Besides, $P \in Pf_1(A)$. Therefore, $\mathfrak{A}_+ \not\Vdash_t \phi$. (Q.E.D)

However, it is known that the converse does not generally hold.

Definition 5.17 (Dual Kripke frame)
Given a modal algebra $\mathfrak{A} = \langle A, \vee, \wedge, \neg, \Box, \top, \bot \rangle$, the tuple $\mathfrak{A}_+ = \langle Pf(A), R_\Box \rangle$ is the dual Kripke frame of $\mathfrak{A}$, where

1. $Pf(A)$ is the set of all prime filters over $\mathfrak{A}$,
2. for any $F_1, F_2 \in Pf(A)$, $R_\Box(F_1, F_2)$ if and only if $\{a \mid \Box a \in F_1\} \subseteq F_2$.

Given an assignment f on a modal algebra $\mathfrak{A}$, the *dual valuation* V_f on the dual Kripke frame $\mathfrak{A}_+$ is defined below.

$$V_f(p) := \{F \in Pf(A) \mid f(p) \in F\}$$

Then, we can show the following propositions along with Proposition 5.16.

Proposition 5.18
For any formula ϕ, ϕ is valid on a modal algebra $\mathfrak{A}$, if ϕ is valid on the dual Kripke frame $\mathfrak{A}_+$.

5.4 Canonicity via canonical extension

Through Stone's duality, we introduce the following.

Definition 5.19 (Canonical extension)
Given an algebra $\mathfrak{A}$, a bidual algebra $(\mathfrak{A}_+)^+$ is *the canonical extension of* $\mathfrak{A}$.

To state something about the connection between models (frames), we define a morphism.

Definition 5.20 (Bounded morphism)
Given two models $\mathfrak{M} = \langle W, O, N, R_\circ, V\rangle, \mathfrak{M}' = \langle W', O', N', R'_\circ, V'\rangle$, a function h from W to W' is a *bounded morphism*, if it satisfies the following.

1. For any $w \in W$ and $p \in \Phi$, $w \in V(p)$ if and only if $h(w) \in V'(p)$.
2. For any $w_0, \ldots, w_n \in W$, if $R_\circ(w_0, \ldots, w_n)$, then $R'_\circ(h(w_0), \ldots, h(w_n))$.
3. If $R'_\circ(h(w), v'_1, \ldots, v'_n)$, there exists $v_1, \ldots, v_n \in W$ such that, for all i ($1 \leq i \leq n$), $h(v_i) = v'_i$ and $R_\circ(w, v_1, \ldots, v_n)$.

In addition, h is called *isomorphism*, if h is bijecive. If there exists a isomorphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then $\mathfrak{M}$ is *isomorphic* to $\mathfrak{M}'$ (denoted by $\mathfrak{M} \cong \mathfrak{M}'$).

We introduce a relation between canonicity and canonical extension here.

Theorem 5.21
Given a logic $\mathbf{L}$, the dual frame $\mathfrak{L}(\Phi)_+$ of the Lindenbaum-Tarski algebra $\mathfrak{L}(\Phi)$ of $\mathbf{L}$ is isomorphic to the canonical frame $\mathfrak{F}^{\mathbf{L}}$ of $\mathbf{L}$.

Proof
Let h be a homomorphism from $\mathfrak{F}^{\mathbf{L}} = \langle W^{\mathbf{L}}, O^{\mathbf{L}}, N^{\mathbf{L}}, R^{\mathbf{L}}_\circ\rangle$ to $\mathfrak{L}(\Phi)_+ = \langle W_{\mathbf{L}}, O_{\mathbf{L}}, N_{\mathbf{L}}, R_{\mathbf{L}}$ defined by the following.

$$h(\Sigma) := \{[\phi] \mid \phi \in \Sigma\}$$

Then, we need to check h is a well defined isomorphism.

Well defined : Given a $\mathbf{L}$-prime consistent set Σ, $h(\Sigma) = \{[\phi] \mid \phi \in \Sigma\}$ is a prime filter over $\mathfrak{L}(\Phi)$. It is obvious, because every point in $\mathfrak{L}(\Phi)$ is closed under $\equiv_{\mathrm{L}}$.

Homomorphism : $h(W^{\mathbf{L}}) \subseteq W_{\mathbf{L}}$, $h(O^{\mathbf{L}}) \subseteq O_{\mathbf{L}}$ and $h(N^{\mathbf{L}}) \subseteq N_{\mathbf{L}}$ are rather obvious. Assume $R^{\mathbf{L}}_\circ(\Sigma_1, \Sigma_2, \Sigma_3)$. If $[\psi] \in h(\Sigma_2)$, $[\chi] \in h(\Sigma_3)$ and $(\psi \circ \chi)\backslash\phi \in \mathbf{L}$, then, by the assumption, $\phi \in \Sigma_1$. That is, $[\phi] \in h(\Sigma_1)$. Therefore, $R_{\mathbf{L}}(h(\Sigma_1), h(\Sigma_2), h(\Sigma_3))$. The converse is analogous.

Injective : Assume $\Sigma_1 \neq \Sigma_2$. There exists a formula ϕ such that $\phi \in \Sigma_1$ but $\phi \notin \Sigma_2$. If $h(\Sigma_1) = h(\Sigma_2)$, there exists $\psi \in \Sigma_2$ such that $[\phi] = [\psi]$. Then, $\psi\backslash\phi \in \mathbf{L}$ and $\psi \in \Sigma_2$, a contradiction.

Surjective : Let F be an arbitrary prime filter over $\mathfrak{L}(\Phi)$. We want to show that $\Sigma' := \{\phi \mid [\psi] \in F\}$ is a **L**-prime consistent set. $\perp \notin \Sigma'$ is obvious. If $\phi \in \Sigma'$ and $\psi \in \Sigma'$, then $[\phi], [\psi] \in F \iff [\phi] \wedge [\psi] \in F$. Therefore, $\phi \wedge \psi \in \Sigma'$. If $\phi \in \Sigma'$ and $\phi \backslash \psi \in \mathbf{L}$, then $[\phi] \in F$ and $[\phi] \leq [\psi]$. So, $[\psi] \in F$. Therefore, $\psi \in \Sigma'$. If $\phi \vee \psi \in \Sigma'$, then $[\phi \vee \psi] = [\phi] \vee [\psi] \in F$. So, $[\phi] \in F$ or $[\psi] \in F$. Therefore, $\phi \in \Sigma'$ or $\psi \in \Sigma'$. (Q.E.D)

Then, we can prove the following.

Theorem 5.22
For any logic $\boldsymbol{L}$, $\boldsymbol{L}$ is canonical, if $\mathfrak{C}^{\boldsymbol{L}}$ is closed under canonical extension.

Proof
If $\mathfrak{C}^{\mathbf{L}}$ is closed under canonical extension, the bidual algebra $(\mathfrak{L}(\Phi)_+)^+$ of the Lindenbaum-Tarski algebra $\mathfrak{L}(\Phi)$ of **L** is also in $\mathfrak{C}^{\mathbf{L}}$. By Propositions 5.7 or 5.10, $\mathfrak{L}(\Phi)_+ \in \mathfrak{C}_{\mathbf{L}}$. Here, by Theorem 5.21, **L** is canonical. (Q.E.D)

We mention here that the converse direction of Theorem 5.22 for countably many propositional variables is still an open problem.
We can prove a main theorem.

Theorem 5.23
Every canonical logic is Kripke complete.

Proof
We will show two things. First, we show that, if the class $\mathfrak{C}^{\mathbf{L}}$ is closed under canonical extension, the dual DFL-frames validates every formula in **L**. Then, we prove that **L** is Kripke complete.

If the class $\mathfrak{C}^{\mathbf{L}}$ is closed under canonical extension, then, for any L-algebra $\mathfrak{A}$, the bidual algebra $(\mathfrak{A}_+)^+$ is again a L-algebra. So, by the definition of L-algebras, $(\mathfrak{A}_+)^+ \models \phi$, for any formula $\phi \in \mathbf{L}$. From Proposition 5.7, we conclude $\mathfrak{A}_+ \models \phi$.

Next, if ϕ is not in **L**, then, by Theorem 3.19, $\mathfrak{L}(\Phi) \not\models \phi$. By Proposition 3.18, $\mathfrak{L}(\Phi) \in \mathfrak{C}^{\mathrm{L}}$. Besides, by Proposition 5.16, $\mathfrak{L}(\Phi)_+ \not\Vdash_t \phi$. Since $\mathfrak{L}(\Phi) \in \mathfrak{C}^{\mathbf{L}}$, $\mathfrak{L}(\Phi)_+$ is an element of the class $\mathfrak{C}_{\mathbf{L}}$. Therefore, $\mathfrak{C}_{\mathbf{L}} \not\Vdash_t \phi$. (Q.E.D)

We can also prove Kripke completeness of basic DFL logics via canonical extension.

Theorem 5.24 (Kripke completeness theorem for basic DFL logics)
*Each basic DFL logics (**DFL**, $\boldsymbol{DFL}_c$, $\boldsymbol{DFL}_e$, $\boldsymbol{DFL}_w$, $\boldsymbol{DFL}_{ce}$, $\boldsymbol{DFL}_{ew}$ and $\boldsymbol{DFL}_{cew}$) is Kripke complete.*

Proof
Thanks to Proposition 5.7 and Theorem 5.23, we only check that, if any DFL-algebra $\mathfrak{A}$ satisfies the condition which validates each structural rule, then the dual DFL-frame $\mathfrak{A}_+$ is satisfying the correspondence condition.

(c ⇒) : Assume that $a \leq a \circ a$ for any $a \in A$ and $F_1 \subseteq F_2$. If $b, c \in F_1$, then $b \wedge c \in F_1$. Here, by the first assumption, $b \wedge c \leq (b \wedge c) \circ (b \wedge c)$. Then, by the monotonicity of $\circ$, $b \wedge c \leq (b \wedge c) \circ (b \wedge c) \leq b \circ c$. So, by the second assumption, $b \circ c \in F_2$. Therefore, $F_1 \circ F_1 \subseteq F_2$.

(e ⇒) : Assume that $a \circ b = b \circ a$ for any $a, b \in A$ and $F_2 \circ F_3 \subseteq F_1$. If $a \in F_2$ and $b \in F_3$, then, by the second assumption, $a \circ b \in F_1$. Therefore, by the first assumption, $b \circ a \in F_1$.

(w ⇒) : Assume that $a \leq 1$ for any $a \in A$. If $F \in Pf(A)$, then F is non-empty. So, there exists $a \in F$. By the assumption, $1 \in F$. Therefore, $F \in Pf_1(A)$.

(⇒ w) : Assume that $0 \leq a$ for any $a \in A$. If $F \in Pf(A)$, then $0 \notin F$, because, if $0 \in F$, then $F = A$. It contradicts $F \in Pf(A)$. Therefore, $Pf_0(A) = \emptyset$. (Q.E.D)

5.5 Canonicity via general frame

To research canonicity via canonical extension, as in the case of modal logics, we define general frames for DFL logics.

Definition 5.25 (General DFL-frame)
A tuple $\mathfrak{G} = \langle \mathfrak{F}, P \rangle$ is a *general DFL-frame*, if $\mathfrak{F}$ is a DFL-frame and P is a subset of $Up(W)$ satisfying the following.

1. $O, N, W, \emptyset \in P$.
2. P is closed under $\cup$, $\cap$, $*$, $\searrow$ and $\swarrow$ (these operations are in Definition 5.4).

We also define an *admissible valuation* V on a general DFL-frame $\mathfrak{G} = \langle \mathfrak{F}, P \rangle$, as a function from Φ to P.

Moreover, we can see DFL-frames as a special case of general DFL-frame which P is the set of all $R_\circ$-upward closed sets.

Then, we introduce the duality.

Definition 5.26 (Dual general DFL-frame)
Given a DFL-algebra $\mathfrak{A} = \langle A, \vee, \wedge, \circ, \backslash, /, 1, 0, \top, \bot \rangle$, a tuple $\mathfrak{A}_* = \langle \mathfrak{A}_+, \hat{A} \rangle$ is the *dual general DFL-frame*, if $\mathfrak{A}_+$ is the dual DFL-frame and $\hat{A}$ is defined by the following.

$$\hat{A} := \{\hat{a} \subseteq Pf(A) \mid a \in A\}$$
$$\hat{a} := \{F \in Pf(A) \mid a \in F\}$$

Definition 5.27 (Dual DFL-algebra)
Given a DFL-frame $\mathfrak{G} = \langle \mathfrak{F}, P \rangle$, a tuple $\mathfrak{G}^* = \langle P, \cup, \cap, *, \searrow, \swarrow, O, N, W, \emptyset \rangle$ is the *dual DFL-algebra*.

To consider canonicity, we here prove the following propositions.

Proposition 5.28
For any formula ϕ, ϕ is valid on a DFL-algebra $\mathfrak{A}$, if and only if ϕ is valid on the dual general DFL-frame $\mathfrak{A}_*$.

Proof
We need firstly to prove that every admissible valuation V on the dual general DFL-frame $\mathfrak{A}_*$ is a dual valuation of an assignment on $\mathfrak{A}$.

Let V be an arbitrary admissible valuation on $\mathfrak{A}_*$. If $V(p) := \hat{a}$ for any propositional variable $p \in \Phi$, then $\hat{a}$ is an element of $\hat{A}$. So, by the definition of $\hat{A}$, there exists an element $a \in A$ such that $\hat{a} = \{F \in Pf(A) \mid a \in F\}$. Therefore, we can define an assignment f_V corresponding to V as $f_V(p) = a$. Then, V is also the dual valuation of f_V, conversely.

Assume that ϕ is not valid on a DFL-algebra $\mathfrak{A}$. Then, there exists an assignment f such that $1 \not\leq f(\phi)$. By the dual valuation $V_f(1) \not\subseteq V_f(\phi)$, $\mathfrak{A}_*, V_f \not\Vdash_t \phi$. So, $\mathfrak{A}_* \not\Vdash_t \phi$.

Assume that ϕ is not valid on a dual general DFL-frame $\mathfrak{A}_*$. Then, there exists an admissible valuation V such that $\mathfrak{A}_*, V \not\Vdash_t \phi$. By the above assignment f_V, $1 \not\leq f_V(\phi)$. So, $\mathfrak{A} \not\models \phi$. (Q.E.D)

Proposition 5.29
For any formula ϕ, ϕ is valid on a general DFL-frame $\mathfrak{G}$, if and only if ϕ is valid on the dual DFL-algebra $\mathfrak{G}^*$.

Proof
Since P is both the range of admissible valuations and the underlying set of the dual algebra, we can take the same strategy with Proposition 5.7. (Q.E.D)

Although these propositions are like Proposition 5.7 and Proposition 5.16, there are some difference. We, especially, note that "only if" part of Proposition 5.16.

These propositions guarantee that any class of algebras is closed under this restricted canonical extension. That is, for any algebra $\mathfrak{A}$, the bidual algebra $(\mathfrak{A}_*)^*$ is also in the same class of algebras.

Again, this does not mean that any class of algebras is closed under canonical extension. However, with the following definitions, we can discuss more concretely.

Definition 5.30 (Descriptive DFL-frame)
Given a general DFL-frame $\mathfrak{G}$, it is *descriptive*, if $\mathfrak{G} \cong (\mathfrak{G}_*)^*$.

Definition 5.31 ($\mathfrak{D}$-persistency)
For any formula ϕ, the formula ϕ is *$\mathfrak{D}$-persistent*, if ϕ satisfies the following.

$$\text{For any descriptive DFL-frame } \mathfrak{G} = \langle \mathfrak{F}, P \rangle, \text{ if } \mathfrak{G} \Vdash \phi, \text{ then } \mathfrak{F} \Vdash \phi.$$

Then, we can prove the following.

Proposition 5.32
For any DFL logic $\mathbf{L}$, the class $\mathfrak{C}^{\mathbf{L}}$ of L-algebras is closed under canonical extension, if $\mathbf{L}$ is axiomatized only by $\mathfrak{D}$-persistent formulas.

Proof
We need to check that every $\mathfrak{D}$-persistent formula ϕ in $\mathbf{L}$ is valid on the dual DFL-frame.

Let $\mathfrak{A}$ be a L-algebra. Then, for any formula ϕ in $\mathbf{L}$, $\mathfrak{A} \models \phi$. By Proposition 5.28, $\mathfrak{A}_* \Vdash_t \phi$. Here, since ϕ is $\mathfrak{D}$-persistent, $\mathfrak{A}_+ \Vdash_t \phi$. (Q.E.D)

By Theorem 5.23 and Proposition 5.32, we can conclude that every DFL logic $\mathbf{L}$ is Kripke complete with respect to the class $\mathfrak{C}_{\mathbf{L}}$ of L-frames if $\mathbf{L}$ is axiomatized only by $\mathfrak{D}$-persistent formulas. Moreover, we can show the following.

Theorem 5.33
Every logic $\boldsymbol{L}$ is complete with respect to the class of all descriptive L-frames.

6 Conclusion

This paper has defined DFL-frames and general DFL-frames. The main results we obtained can be summed up as follows:

- For all basic extensions of DFL, we identified corresponding frame conditions and proved completeness results.
- We extended Stone duality to duality between DFL algebras and DFL frames.
- Finally, we have obtained general completeness result: every DFL logic is complete with respect to a class of descriptive frames.

A natural question is how our semantics is related to existing semantics for distributive substructural logics, in particular intuitionistic Kripke frames and Routley-Meyer semantics. Kripke frames for **LJ** are partially ordered sets. It is easy to see that in DFL_{cew}-frames, $R_\circ(w, v, u) \iff v \leq w$, where $\leq$ is defined as in Definition 4.2. Hence, Kripke frames for **LJ** and DFL_{cew} frames in the sense of present paper are in fact equivalent. As for Routley-Meyer semantics, a technical problem is posed by the fact that the interpretation of negation in these semantics is provided by unary function called *the Routley star*. However, it is possible to relate our treatment of negation to that of relevant logicians. This subject is outside the scope of the present thesis, but we have already obtained results in this direction which are currently prepared for publication.

In the course of our research, we identified the following open problems.

- Can we adapt the filtration technique for DFL logics? That would lead to straightforward finite model property and decidability results.
- What types of FL formulas are $\mathfrak{D}$-persistent? In particular, is there and equivalent of Sahlqvist theorem for DFL logics?
- Which classes of DFL frames are definable? Is it possible to develop rich correspondence theory for DFL logics? Our main goal here is a variant of Goldblatt-Thomason theorem.
- Characterize along the lines of [17] and [22] those general frames which are duals of subdirectly irreducible algebras.
- Finally, can we extend this semantics to modal logics based on DFL?

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