Polynomial time algorithms for graph reconstruction problem

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Many problems handled by computers are modeled in graph structures. Graph theory and algorithm theory are important to solve these problems efficiently. "Graph Isomorphism" to determine whether two given graphs are isomorphic is one of the most basic problems in graph theory and algorithm theory. However, whether or not the algorithm that solves this problem efficiently exists has been open for several decades.

"Graph reconstruction conjecture" is another open problem which has also been open for several decades. Let G^* be a graph consisting of n vertices $v_1, v_2, ..., v_n$. For $i = 1, 2, ..., n$, let G_i be the graph obtained by removing v_i from G^*. We call the set of $G_i (i = 1, ..., n)$ *Deck* of G^*. "Graph reconstruction conjecture" is the following conjecture. Given n graphs consisting of $n - 1$ vertices with no labels, there exist at most one graph of n vertices whose *Deck* is the given graphs. "Graph reconstruction conjecture" is expected to be true for any n greater than one. However, "Graph reconstruction conjecture" has not been proven except for very restricted cases.

On the other hand, it is known that there is a deep relation between "Graph reconstruction conjecture" and "Graph Isomorphism." Studying algorithms to reconstruct a graph may help the analysis of complexity of

"Graph Isomorphism." It may also help us to find new graph classes in which "Graph reconstruction conjecture" holds.

In this paper, we propose algorithms to reconstruct a graph G^* from given *Deck*. More accurately, when there are graphs whose *Deck* are the input, the algorithm outputs one of such graphs. If there is no such graph, the algorithm outputs "No." Notice that we do not mention about the uniquness of G^* in this paper. "Graph reconstruction problem" is defined as follows. The input is n graphs consisting of $n-1$ vertices. If G^* exists, the algorithm outputs one of G^*. And if G^* doesn't exist, the algorithm outputs "No."

A naive algorithm that solves "Graph reconstruction problem" exists. The naive algorithm is as follows. First we chose a graph G_1 arbitrarily from *Deck*. Then we insert a vertex to G_1. Let the resulting graph G^+. There are 2^{n-1} ways to insert a vertex to G_1, and thus we construct 2^{n-1} G^+s. If there is G^+ such that its *Deck* is identical to the input, output the G^+ as G^*. However, this algorithm constructs exponentially many (in the input size) candidates of G^*. Moreover it is necessary to excute graph isomorphisms exponentially many times in the input size. We study polynomial time algorithms to reconstruct graph G^* from given *Deck* under the assumption that G^* belongs to some graph class.

If an algorithm to construct graph G^* from *Deck* in polynomial time exists for G^* in some graph class, the proof of "graph reconstruction conjecture" for the graph class may be a bit easier than that for other classes. And, it can be expected that some knowledges for the "Graph reconstruction conjecture" will be obtained through the analysis of such an algorithm. "Graph reconstruction conjecture" gives only the static and rough insight. We propose a layered structure of the difficulty of the "Graph reconstruction conjecture" by giving a new measure of time complexity.

In this paper, we treat graph classes which are hereditary and with whose any two element graphs "Graph Isomorphism" can be solved in the polynomial time, where graph class C is hereditary means that for any graph $G \in C$, a graph obtained from G by removing a vertex also belongs to C. Interval graphs, cographs and distance-hereditary graphs are such graph classes.

Let graph class C be hereditary and with whose any two element graphs

"Graph Isomorphism" solved in polynomial time. We proved that the graph reconstruction problem for the unconnected graph G^* that belongs to C can be reduced to that for the connected graph that belongs to C.

Next, as the concrete example, we study graph reconstruction algorithms for proper interval graphs and quasi threshold graphs. The following proper inclusion of graph classes are known;

proper interval graphs $\subset$ *interval graphs*

quasi threshold graphs $\subset$ *cographs* $\subset$ *distance-hereditary graphs*

These graph classes are hereditary, and graph isomorphism can be determined in polynomial time. In this paper, we propose an $O(n^3m)$ time graph reconstruction algorithm for proper interval graphs and an $O(nm)$ time graph reconstruction algorithm for quasi threshold graphs.