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# Algebraic aspects of cut elimination

**Abstract.** We will give here a purely algebraic proof of the cut elimination theorem for various sequent systems. Our basic idea is to introduce mathematical structures, called *Gentzen structures*, for a given sequent system *without cut*, and then to show the completeness of the sequent system without cut with respect to the class of algebras for the sequent system with cut, by using the *quasi-completion* of these Gentzen structures. It is shown that the quasi-completion is a generalization of the MacNeille completion. Moreover, the finite model property is obtained for many cases, by modifying our completeness proof. This is an algebraic presentation of the proof of the finite model property discussed by Lafont [12] and Okada-Terui [17].

*Keywords*: Algebraic Gentzen systems, cut elimination, substructural logics, residuated lattices, finite model property

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## 1. Introduction

In this paper, we will give an algebraic proof of cut elimination for various sequent systems. Our method can be modified so as to prove the completeness theorem of tableau systems with respect to algebraic semantics. Our motivation of giving an algebraic proof of the cut elimination theorem is to clarify the meaning of cut elimination from an algebraic point of view, and to give a proof of cut elimination attractive to algebraists, avoiding heavy syntactic arguments which are used in the standard cut elimination procedure. Such a proof might be useful for algebraists, as cut elimination offers us a useful tool for proving decidability. Our goal is to clarify the algebraic aspects of cut elimination and its consequences.[1]

Our basic idea of algebraic proofs is to introduce mathematical structures, which we call *Gentzen structures*, for a given sequent system *without cut*, and then to show the completeness of the sequent system without cut with respect to the class of algebras for the logic determined by the sequent system with cut. In this completeness proof, we will use the *quasi-completion*

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[1]A part of our algebraic proof of the cut elimination theorem is announced also in the paper [22] by the third author.

of Gentzen structures, which is a generalization of the MacNeille completion, as shown at the end of Section 5. Our method introduced here is closely related to those discussed by Maehara [13] and independently by Okada [16], in which semi-algebraic proofs of cut elimination are given, and is also inspired by the paper [9].

We note that there exist already several ways of proving cut elimination by using semantical methods. For instance, in 1960 Schütte [24] introduced the notion of *semi valuations* (or, Schütte's valuations in [7]) to prove the cut elimination theorem for higher order classical sequent system in a semantical way. Also, to show completeness of tableau systems for some modal logics and intuitionistic logic, which is essentially equivalent to cut elimination for them, Fitting introduced *consistency properties* in [4]. But, proofs in these papers and also [1], except [13] and [16], are not of an algebraic character in our sense.

Our method works well for a wide variety of sequent systems of nonclassical logics, both in propositional and predicate cases, including Gentzen's systems **LK** and **LJ** in [5] for classical and intuitionistic logic, respectively. To explain our basic idea, we take first the sequent system $\mathbf{FL_{ew}}$ for intuitionistic logic without the contraction rule as an example, and give a proof of cut elimination for it. The name **FL** comes from *full Lambek calculus* (i.e., Lambek calculus with conjunction and disjunction) and the $_{\mathbf{ew}}$ refers to the presence of the *exchange* and *weakening* rule (presented in the next section). In Section 6, we will show how our method can be applied to some other sequent systems of nonclassical logics, including modal logics.

By a slight modification of our completeness proof, we show in Sections 7 and 8 the finite model property of some of nonclassical logics. This is an algebraic presentation of the proof of the finite model property, discussed by Lafont [12] and Okada-Terui [17]. The proof will show how the finiteness of proof-search procedures, which is purely of proof-theoretic character, is related to such an algebraic property as the finite model property.

## 2. Sequent calculi and cut elimination — preliminaries

As we mentioned in the above, one of our motivations for giving an algebraic proof of cut elimination is to present a proof which is attractive to algebraists. For this purpose, it might be helpful to give first a brief explanation of sequent calculi and their syntactic properties, assuming that some of our readers may not be familiar with proof theory. On the other hand, readers familiar with cut elimination may skip this section. For more information on cut elimination theorems for nonclassical logics, readers are referred to a survey article [20] by the third author, which is written from a syntactic point of view.

We use the symbols $\to$, $\wedge$ and $\vee$ for implication, conjunction and disjunction, respectively, and the *zero* 0 and *unit* 1 for logical constants. The negation $\neg\alpha$ of a formula $\alpha$ is defined to be an abbreviation of $\alpha \to 0$. Lowercase Greek letters $\alpha$, $\beta$, $\gamma$, $\delta$ etc. are used for formulas, and uppercase Greek letters $\Gamma$, $\Sigma$, $\Delta$ etc. for finite (possibly empty) sequences of formulas. A *sequent* is an expression of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ with $m \geq 0$, where $\alpha_i$'s are formulas and $\beta$ is either a formula or empty. An informal interpretation of this sequent is that $\beta$ *follows from assumptions* $\alpha_1, \ldots, \alpha_m$.

In the next section, we will start to give an algebraic proof of the cut elimination theorem. We take first the sequent calculus $\mathbf{FL_{ew}}$ as an example, which is obtained from Gentzen's sequent calculus **LJ** for intuitionistic logic, by deleting the contraction rule. Here, the calculus **LJ** (in a slightly modified form) consists of initial sequents and rules of inference which are given as follows. An initial sequent is a sequent of one of the following forms; 1) $\alpha \Rightarrow \alpha$, 2) $0 \Rightarrow$ , 3) $\Rightarrow 1$. Rules of inference are divided into two groups, i.e. the first consists of rules for logical connectives, and the second consists of structural rules. In the rules below, $\Gamma, \Delta$ denotes the concatenation of sequences, and an expression like $\alpha, \Gamma$ prepends the formula $\alpha$ to the sequence $\Gamma$. Rules for logical constants and for logical connectives $\to$, $\wedge$ and $\vee$ are given as follows:

$$\frac{\Gamma \Rightarrow \delta}{1, \Gamma \Rightarrow \delta}\ (1 \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \delta}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\ (\wedge 2 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta \quad \beta, \Gamma \Rightarrow \delta}{\alpha \vee \beta, \Gamma \Rightarrow \delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

Structural rules consist of the weakening rules, the contraction rule, the exchange rule, and the cut rule.

$$\frac{\Gamma \Rightarrow \delta}{\alpha, \Gamma \Rightarrow \delta}\ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\Rightarrow w) \qquad \frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \delta}{\Gamma, \alpha, \Sigma \Rightarrow \delta}\ (c \Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \delta}\ (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}\ (cut)$$

A sequent $\Gamma \Rightarrow \delta$ is *provable* in **LJ** if it can be obtained from initial sequents by applying rules of inference repeatedly. A figure which shows how a given sequent $\Gamma \Rightarrow \delta$ is obtained is called a *proof* of $\Gamma \Rightarrow \delta$. In general, there are many proofs of each provable sequent. A *cut-free* proof is a proof which contains no applications of the cut rule. Here are two examples of proofs in **LJ**, both of which are cut-free. The first one is a proof of $\alpha, \beta \Rightarrow \alpha \wedge \beta$.

$$\frac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\ (w \Rightarrow) \quad \dfrac{\beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \beta}\ (w \Rightarrow)}{\alpha, \beta \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

The second is a proof of $\alpha \to (\beta \to \gamma), \alpha \wedge \beta \Rightarrow \gamma$.

$$\frac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge \beta \Rightarrow \alpha}\ (\wedge 1 \Rightarrow) \quad \dfrac{\dfrac{\beta \Rightarrow \beta}{\alpha \wedge \beta \Rightarrow \beta} \quad \gamma \Rightarrow \gamma}{\beta \to \gamma, \alpha \wedge \beta \Rightarrow \gamma}\ (\to\Rightarrow)}{\alpha \to (\beta \to \gamma), \alpha \wedge \beta, \alpha \wedge \beta \Rightarrow \gamma}\ (\to\Rightarrow)}{\alpha \to (\beta \to \gamma), \alpha \wedge \beta \Rightarrow \gamma}\ (c \Rightarrow)$$

We can show easily the following.

PROPOSITION 2.1. *The following three conditions are mutually equivalent:*
*1.* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ *is provable in* **LJ**,
*2.* $\Rightarrow \alpha_1 \to (\alpha_2 \to (\ldots(\alpha_m \to \beta)\ldots))$ *is provable in* **LJ**,
*3.* $\alpha_1 \wedge \ldots \wedge \alpha_m \Rightarrow \beta$ *is provable in* **LJ**.

The equivalence of 1 and 2 is shown without using structural rules other than the cut rule, while both the weakening rule and the contraction rule are necessary to show the equivalence of 1 and 3, which one can see in the above examples of proofs. Cut elimination for **LJ** is expressed as follows.

PROPOSITION 2.2. *If a sequent* $\Gamma \Rightarrow \delta$ *is provable in* **LJ** *then it is provable in* **LJ** *without using the cut rule.*

The sequent calculus $\mathbf{FL_{ew}}$ is obtained from $\mathbf{LJ}$ by simply deleting the contraction rule. As noted above, we cannot show the equivalence of 1 and 3 in Proposition 2.1 for $\mathbf{FL_{ew}}$. In other words, commas in sequents of $\mathbf{FL_{ew}}$ should not be interpreted as conjunctions. So it is often convenient to introduce a logical connective $\cdot$, called the *fusion*, which represents commas explicitly in sequents of a sequent calculus which lacks the contraction rule or the weakening rule. This is in fact possible without affecting the provability of any sequent containing no $\cdot$, if we take rules for $\cdot$ given below.

$$\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \cdot \beta, \Gamma \Rightarrow \delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

Hereafter, by $\mathbf{FL_{ew}}$ we mean the sequent calculus with these rules for $\cdot$. Similarly to $\mathbf{LJ}$, we have the following propositions.

PROPOSITION 2.3. *The following three conditions are mutually equivalent:*
*1.* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ *is provable in* $\mathbf{FL_{ew}}$*,*
*2.* $\Rightarrow \alpha_1 \to (\alpha_2 \to (\ldots(\alpha_m \to \beta)\ldots))$ *is provable in* $\mathbf{FL_{ew}}$*,*
*3.* $\alpha_1 \cdot \ldots \cdot \alpha_m \Rightarrow \beta$ *is provable in* $\mathbf{FL_{ew}}$*.*

PROPOSITION 2.4. *If a sequent* $\Gamma \Rightarrow \delta$ *is provable in* $\mathbf{FL_{ew}}$ *then it is provable in* $\mathbf{FL_{ew}}$ *without using the cut rule.*

A syntactic proof of cut elimination for $\mathbf{FL_{ew}}$ is given in [23]. Many important syntactic properties, including the interpolation theorem and decidability of both propositional and predicate logics $\mathbf{FL_{ew}}$ are obtained from cut elimination [23], [10]. For more information on cut elimination and decidability of substructural logics, see [20] and [19].

Let $\mathbf{FL_{e}}$ (and $\mathbf{FL_{ec}}$) be the sequent calculus obtained from $\mathbf{LJ}$ by deleting both weakening and contraction rules (only weakening rules, respectively). (See e.g. [18] for the detailed definition.) Similar propositions as above hold for both $\mathbf{FL_{e}}$ and $\mathbf{FL_{ec}}$. They are known to be the intuitionistic linear logic without exponentials and the intuitionistic relevant logic without the distributive law, respectively, and their cut elimination theorems are proved by J.-Y. Girard [6] and essentially by R.K. Meyer [14], respectively.

We conclude this section with some additional remarks on the cut elimination theorem. In his paper [5], G. Gentzen introduced sequent calculi $\mathbf{LK}$ for classical logic and $\mathbf{LJ}$ for intuitionistic logic, and proved the cut elimination theorem for them. To show this, he gave a procedure for eliminating each application of the cut rule in a given proof of a given sequent $\Gamma \Rightarrow \delta$, and showed by using double induction that a cut-free proof of $\Gamma \Rightarrow \delta$ can be eventually obtained by repeated applications of this procedure. It should be noted here that the cut rule can never be obtained by combining applications of other rules of inference in a uniform way. That is, the cut rule is *not derivable* in the system obtained from **LJ** by deleting the cut rule. This implies that in eliminating each application of the cut rule, we must replace it depending on how it appears.

## 3. Commutative residuated lattices

We now introduce algebraic structures for $\mathbf{FL_{ew}}$. An algebra $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \to, 1\rangle$ is a *commutative residuated lattice* if it satisfies the following:

1. $\langle P, \wedge, \vee\rangle$ is a lattice,
2. $\langle P, \cdot, 1\rangle$ is a commutative monoid with the unit element 1,
3. $a \cdot b \leq c$ iff $a \leq (b \to c)$, for any $a, b, c \in P$.

By abuse of symbols, we use the same symbols for logical connectives and corresponding algebraic operations. The third condition in the above definition is called the law of residuation between the monoid operation $\cdot$ and the *residual* $\to$. In our paper, we assume that commutative residuated lattices under consideration are always *bounded*, that is, any of them has both a greatest element $\top$ and a least element $\bot$. In addition, we fix an element 0, called the zero element, of each commutative residuated lattice $\mathbf{P}$, to define a unary operation $\neg$ on it by $\neg x = x \to 0$ for any $x$. It determines an interpretation of the negation in $\mathbf{P}$ as shown below. Therefore, it will be natural from a logical standpoint to define a commutative residuated lattice to be an algebra $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \to, 0, 1\rangle$. For more information on residuated lattices, see e.g. [9] and [11].

Any commutative residuated lattice determines an interpretation of each formula, and in fact commutative residuated lattices serve as algebraic structures for substructural logics, as noted in Proposition 3.1 below. Since there is a one-to-one correspondence between the set of all formulas and the set of all terms in the language for residuated lattices, we will use the same symbols for logical connectives (and constants) in formulas and corresponding operations (and constants, respectively) in algebraic terms. In other words, terms will be regarded as synonyms of formulas, distinguished only by context. We use the letters $p, q, r, s, t$ etc. for terms.

Let $p$ and $q$ be terms, and let $\mathbf{P}$ be a commutative residuated lattice. A *valuation* $f$ on $\mathbf{P}$ is any mapping from the set of term variables to $\mathbf{P}$. As usual, $f$ can be extended to a mapping from the set of terms to $\mathbf{P}$. We write

$\mathbf{P} \models p \leq q$, whenever $f(p) \leq f(q)$ holds in $\mathbf{P}$ for any valuation $f$. We say that a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is *valid* in a commutative residuated lattice $\mathbf{P}$ if $\mathbf{P} \models (\alpha_1 \cdot \ldots \cdot \alpha_m) \leq \beta$. Here we are identifying formulas and terms. We take 1 for $\alpha_1 \cdot \ldots \cdot \alpha_m$ when $m = 0$, and an empty formula $\beta$ is considered as the term 0.

A commutative residuated lattice is *integral* if the unit 1 is equal to the greatest element $\top$ and 0 is equal to $\bot$. It is *increasing idempotent* if $a \leq a \cdot a$ for each $a$. It is easy to see that a commutative residuated lattice $\mathbf{P}$ is both integral and increasing idempotent if and only if $a \cdot b = a \wedge b$ for all $a, b \in P$. Thus, any integral, increasing idempotent commutative residuated lattice is a Heyting algebra, and vice versa. In fact, integrality and increasing idempotency correspond to the weakening rule and contraction rule of sequent calculi, respectively, and in particular, integral commutative residuated lattices are algebraic structures for $\mathbf{FL_{ew}}$, as the following proposition shows.

PROPOSITION 3.1. *The propositional logic* $\mathbf{FL_{ew}}$ *is complete with respect to the class of integral commutative residuated lattices. More precisely, for all formulas* $\alpha_1, \ldots, \alpha_m, \beta$ *a sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ *is provable in* $\mathbf{FL_{ew}}$ *if and only if* $\mathbf{P} \models (\alpha_1 \cdot \ldots \cdot \alpha_m) \leq \beta$ *for any integral commutative residuated lattice* $\mathbf{P}$.

The only-if part is proved by showing that each rule of $\mathbf{FL_{ew}}$ preserves the validity. For instance, in case of ($\Rightarrow\rightarrow$) we need to show that if $\mathbf{P} \models (r \cdot p) \leq q$ then $\mathbf{P} \models r \leq (p \rightarrow q)$. But, this follows from the fact that $a \cdot b \leq c$ implies $a \leq (b \rightarrow c)$, for any $a, b, c \in P$. The if-part is shown by using standard argument, making use of free integral commutative residuated lattice, i.e. the *Lindenbaum algebra* of $\mathbf{FL_{ew}}$. Note that similar completeness results holds for $\mathbf{FL_{e}}$ (and $\mathbf{FL_{ec}}$) with respect to the class of commutative residuated lattices (and increasing idempotent commutative residuated lattices, respectively).

We now present a standard way of constructing a residuated lattice from a commutative monoid. Suppose that a commutative monoid $\mathbf{M} = \langle M, \cdot, 1 \rangle$ is given. A unary function $C$ on $\wp(M)$ is a *closure operator* if for all $X, Y \in \wp(M)$,

1. $X \subseteq C(X)$,
2. $CC(X) \subseteq C(X)$,
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,

4. $C(X) * C(Y) \subseteq C(X * Y)$.

Here, $*$ is defined by $W * Z = \{wz : w \in W, z \in Z\}$ for $W, Z \in \wp(M)$. A subset $X$ of $M$ is $C$-closed if $C(X) = X$. Let $C(\wp(M))$ denote the set of all $C$-closed subsets. Define operations $\cup_C$, $*_C$ and $\Rightarrow$ on $C(\wp(M))$ as follows. For all $C$-closed sets $X$ and $Y$:

- $X \cup_C Y = C(X \cup Y)$,
- $X *_C Y = C(X * Y)$,
- $X \Rightarrow Y = \{z : \{z\} * X \subseteq Y\}$.

Then we have the following result (see e.g. [18] and also Lemma 7.1 of [9]). Here, a residuated lattice is said to be *complete* if the lattice reduct is a complete lattice.

LEMMA 3.2. *The algebra* $\mathbf{C_M} = \langle C(\wp(M)), \cap, \cup_C, *_C, \Rightarrow, O, C(\{1\})\rangle$ *forms a complete, commutative bounded residuated lattice with the lattice order* $\subseteq$ *which has greatest element* $M$*, least element* $C(\emptyset)$*, and unit element* $C(\{1\})$*, where* $O$ *is an arbitrary* $C$*-closed subset of* $M$*.*

We suppose now that a commutative monoid $\mathbf{M}$ is moreover partially ordered by an order $\leq$ satisfying

1. for any $x, y, z \in M$, $x \leq y$ implies $x \cdot z \leq y \cdot z$, and
2. the unit element 1 is the greatest element.

For each subset $X$ of $M$, define $X^{\rightarrow}$ and $X^{\leftarrow}$ to be the set of all upper bounds and of all lower bounds of $X$, respectively. More precisely, $X^{\rightarrow} = \{u \in M : x \leq u \text{ for any } x \in X\}$, and $X^{\leftarrow} = \{v \in M : v \leq x \text{ for any } x \in X\}$. Define a mapping $D$ on $\wp(M)$ by $DX = (X^{\rightarrow})^{\leftarrow}$. Then, we can show that $D$ is a closure operator in the above sense, which satisfies $D(\{1\}) = M$. Therefore, by using Lemma 3.2, $\mathbf{D_M}$ forms an integral commutative residuated lattice, when we take $C(\emptyset)$ for $O$. Moreover, by Theorem 5 in [21] we have the following.

PROPOSITION 3.3. *The mapping* $h : M \to D(\wp(M))$ *defined by* $h(u) = \{x \in M : x \leq u\}$ *is a complete embedding, i.e. an order isomorphism which preserves all products and all existing residuals,* (*infinite*) *joins and meets in* $M$*.*

This complete, integral commutative residuated lattice $\mathbf{D_M}$ is called the *MacNeille completion* of a commutative partially ordered monoid $\mathbf{M}$. For more information, see e.g. [21].

## 4. Gentzen structures for the sequent calculus $\mathbf{FL_{ew}}$

As shown in the previous section, algebraic structures for the sequent calculus $\mathbf{FL_{ew}}$ are integral commutative residuated lattices. In this section we will introduce structures for $\mathbf{FL_{ew}}$ *without cut*, which we call *Gentzen structures* for $\mathbf{FL_{ew}}$.

Proposition 2.3 says that commas in sequents can be interpreted as fusions in $\mathbf{FL_{ew}}$. But, to derive the the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ from the sequent $\alpha_1 \cdot \ldots \cdot \alpha_m \Rightarrow \beta$, we need the cut rule, in general. Thus, in $\mathbf{FL_{ew}}$ without cut it becomes necessary to interpret commas as they stand.

Now, for a given nonempty set $Q$, let $Q^*$ be the set of all (finite, possibly empty) multisets whose elements are in $Q$. The empty multiset is denoted by $\varepsilon$ in the following. For members $x$ and $y$ of $Q^*$, $xy$ denotes the multiset union of $x$ and $y$. The multiset consisting of elements $a_1, \ldots, a_m \in Q$ is denoted by $\langle a_1, \ldots, a_m \rangle$. Sometimes, we identify an element $c \in Q$ with the singleton multiset $\langle c \rangle$ when no confusions will occur. Thus, for example, when $y$ is a singleton multiset $\langle c \rangle$, $xy$ is written as $xc$. Obviously, $Q^*$ forms a commutative monoid with respect to multiset union whose unit is $\varepsilon$. In the following, letters $x, y, z, u, v$ etc. are used for expressing members of $Q^*$, and letters $a, b, c, d$ for elements of $Q \cup \{\varepsilon\}$.

A *Gentzen structure* for $\mathbf{FL_{ew}}$ is a structure $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0_Q, 1_Q \rangle$ such that $0_Q, 1_Q \in Q$, $\wedge, \vee, \cdot, \rightarrow$ are binary operations on $Q$, and $\preceq$ is a subset of $Q^* \times (Q \cup \{\varepsilon\})$, which satisfies the following conditions:

- $a \preceq a$,
- $0_Q \preceq c$,
- $\varepsilon \preceq 1_Q$,
- $x \preceq c$ implies $dx \preceq c$,
- $x \preceq a$ and $by \preceq c$ imply $(a \rightarrow b)xy \preceq c$,
- $ax \preceq b$ implies $x \preceq a \rightarrow b$,
- $ax \preceq c$ and $bx \preceq c$ imply $(a \vee b)x \preceq c$,
- $x \preceq a$ implies $x \preceq a \vee b$,
- $x \preceq b$ implies $x \preceq a \vee b$,
- $ax \preceq c$ implies $(a \wedge b)x \preceq c$,

- $bx \preceq c$ implies $(a \wedge b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \wedge b$,
- $abx \preceq c$ implies $(a \cdot b)x \preceq c$,
- $x \preceq a$ and $y \preceq b$ imply $xy \preceq a \cdot b$.

Each of these conditions corresponds to either an instance of an initial sequent or an instance of a rule of inference, if we replace $\preceq$ by $\Rightarrow$ and elements of $Q$ by formulas. Conversely, each rule of inference except the exchange rule and the cut rule is represented by one of these conditions. While the exchange rule is incorporated implicitly into the definition of $Q^*$, no condition in the above represents the cut rule. Sometimes we will omit the subscript $Q$ of $0_Q$ and $1_Q$ when confusion is unlikely.

As for residuated lattices, a valuations on a Gentzen structure $\mathbf{Q}$ is defined as a homomorphism from the algebra of terms to the algebra reduct $\langle Q, \wedge, \vee, \cdot, \rightarrow, 0_Q, 1_Q\rangle$. A sequent $s_1, \ldots, s_m \Rightarrow t$ is said to hold in $\mathbf{Q}$ for $\mathbf{FL_{ew}}$, ($\mathbf{Q} \models s_1, \ldots, s_m \Rightarrow t$, in symbols) if $\langle h(s_1), \ldots, h(s_m)\rangle \preceq h(t)$ holds for any valuation $h$ on $\mathbf{Q}$. (Here, we assume that $h(t) = \varepsilon$ when $t$ is empty.) It is obvious that if a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in $\mathbf{FL_{ew}}$ without using the cut rule then $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ holds in every Gentzen structure for $\mathbf{FL_{ew}}$. The converse of this implication can be shown also by taking the *free* Gentzen structure for $\mathbf{FL_{ew}}$. As shown below, the proof goes essentially the same as, but is much simpler than, the standard proof of completeness of $\mathbf{FL_{ew}}$ with respect to the class of integral commutative residuated lattices, using the Lindenbaum algebra. Let $Q^+$ be the set of all terms (in the language for residuated lattices). Obviously, both constants 0 and 1 belong to $Q^+$, and operations $\wedge, \vee, \cdot, \rightarrow$ are defined on the set $Q^+$ in a trivial way. We define the relation $\preceq^+$ as follows:

> $\langle \alpha_1, \ldots, \alpha_m\rangle \preceq^+ \beta$ holds if and only if the corresponding sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in $\mathbf{FL_{ew}}$ without using the cut rule, for $m > 0$. Also, $\varepsilon \preceq^+ \beta$ holds if and only if the sequent $\Rightarrow \beta$ is provable in $\mathbf{FL_{ew}}$ without using the cut rule.

Recall here that the correspondence between formulas and terms is bijective. The structure $\mathbf{Q}^+$ thus obtained becomes a Gentzen structure for $\mathbf{FL_{ew}}$ with the property that if a sequent is not provable in $\mathbf{FL_{ew}}$ without cut, then the corresponding sequent does not hold under the trivial valuation, i.e. the valuation $f$ satisfying that $f(w) = w$ for any term variable $w$. Thus, we have the following.

LEMMA 4.1. *A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in* $\mathbf{FL_{ew}}$ *without using the cut rule if and only if $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ holds in every Gentzen structure for* $\mathbf{FL_{ew}}$.

Next we show that every integral commutative residuated lattice can be regarded as a particular Gentzen structure for $\mathbf{FL_{ew}}$. First, suppose that an integral commutative residuated lattice $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is given. Let $\leq$ be the lattice order of $\mathbf{P}$. Define a subset $\preceq$ of $P^* \times (P \cup \{\varepsilon\})$ by the condition that $\langle a_1, \ldots, a_m\rangle \preceq c$ holds if and only if $(a_1 \cdot \ldots \cdot a_m) \leq c$ holds in $\mathbf{P}$, when $c \in P$. (Let $(a_1 \cdot \ldots \cdot a_m) = 1$ when $m = 0$. Also, define $\langle a_1, \ldots, a_m\rangle \preceq \varepsilon$ if and only if $(a_1 \cdot \ldots \cdot a_m) \leq 0$ holds in $\mathbf{P}$.) Then $\mathbf{P}' = \langle P, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ becomes a Gentzen structure for $\mathbf{FL_{ew}}$. Moreover, the following *strong transitivity* holds in $\mathbf{P}'$:

$$x \preceq a \text{ and } ay \preceq c \text{ imply } xy \preceq c.$$

Conversely, suppose that a Gentzen structure $\mathbf{Q}$ for $\mathbf{FL_{ew}}$ with a strongly transitive relation $\preceq$ is given. Let $\preceq_0$ be the restriction of $\preceq$ to $Q \times Q$. We note here that $\preceq$ is strongly transitive if and only if both of the following hold:

1. the relation $\preceq_0$ is transitive,
2. $\langle a_1, \ldots, a_m\rangle \preceq c$ if and only if $(a_1 \cdot \ldots \cdot a_m) \preceq_0 c$.

Moreover, in $\mathbf{Q}$ we have that $ax \preceq b$ if and only if $x \preceq a \rightarrow b$. To see this, it is enough to show the if-part since the only-if part holds always. From $a \preceq a$ and $b \preceq b$, $(a \rightarrow b) \cdot a \preceq b$ follows. So, if $x \preceq a \rightarrow b$ then we can show that $ax = xa \preceq (a \rightarrow b) \cdot a$. Then, by the strong transitivity of $\preceq$, we have $ax \preceq b$. This gives the following result.

LEMMA 4.2. *Let $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ be any Gentzen structure for* $\mathbf{FL_{ew}}$ *with a strongly transitive $\preceq$. If the restriction $\preceq_0$ of $\preceq$ to $Q \times Q$ is moreover antisymmetric and therefore a partial order then $\mathbf{Q}_0 = \langle Q, \wedge, \vee, \rightarrow, 0, 1\rangle$ is an integral commutative residuated lattice with the lattice order $\preceq_0$.*

The assumption that $\preceq_0$ is antisymmetric is not essential. For, if $\preceq_0$ is both reflexive and transitive, by using the congruence relation $\sim$ determined by $\preceq_0$ we can introduce a quotient algebra, in which the relation $\preceq_0$ congruent modulo $\sim$ becomes a partial order. On the other hand, we cannot take a quotient structure of a Gentzen structure in general, since it lacks transitivity.

In conclusion, we can roughly say that any Gentzen structure with a strongly transitive relation can be identified with an integral commutative residuated lattice, and vice versa.

## 5. Quasi-completions and cut elimination

Results in the last part of the previous section tell us that each integral commutative residuated lattice can be regarded as a particular Gentzen structure for $\mathbf{FL_{ew}}$. Therefore, for all terms $s_1, \ldots, s_m$ and $t$, if a sequent $s_1, \ldots, s_m \Rightarrow t$ holds in any Gentzen structure for $\mathbf{FL_{ew}}$ then the corresponding inequality $(s_1 \cdot \ldots \cdot s_m) \leq t$ holds in any integral commutative residuated lattice. Our Theorem 5.1, proved in the present section, says that the converse is also true, and turns out to be equivalent to cut elimination (see Lemma 5.5). This is the algebraic content of cut elimination for $\mathbf{FL_{ew}}$, and leads directly to the main result in Theorem 5.6.

THEOREM 5.1. *For all terms $s_1, \ldots, s_m$ and $t$, if $\mathbf{P} \models (s_1 \cdot \ldots \cdot s_m) \leq t$ for any integral commutative residuated lattice $\mathbf{P}$, then $\mathbf{Q} \models s_1, \ldots, s_m \Rightarrow t$ for any Gentzen structure $\mathbf{Q}$ for $\mathbf{FL_{ew}}$.*

We devote most of this section to proving the above result. Taking the contraposition, suppose that $s_1, \ldots, s_m \Rightarrow t$ fails to hold in a Gentzen structure $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ under a valuation $f$, i.e. $\langle f(s_1), \ldots, f(s_m)\rangle \preceq f(t)$ does not hold in $\mathbf{Q}$. Our goal is to construct an integral commutative residuated lattice $\mathbf{P}$ in which $(s_1 \cdot \ldots \cdot s_m) \leq t$ does not hold.

Since $Q^*$ is a commutative monoid, we have that $\mathbf{C_{Q^*}}$ is a complete commutative bounded residuated lattice for any closure operator $C$ on $\wp(Q^*)$, by Lemma 3.2. We will introduce a particular closure operator $C$ on $\wp(Q^*)$ in the following. First, for each $x \in Q^*$ and each $a \in Q \cup \{\varepsilon\}$, define

$$[x; a] = \{w \in Q^* : wx \preceq a\}.$$

The set $\mathcal{B}$ of all subsets of $A$ of the above form determines a *closed base* (in topological sense). Note that $Q^*$ belongs to $\mathcal{B}$ since $Q^* = [\varepsilon; 1]$. We call $\mathcal{B}$, the closed base determined by $\mathbf{Q}$ or simply the closed base determined by $\preceq$. By using $\mathcal{B}$ we can define a closure operation $C$ (in the usual sense) on $\wp(Q^*)$ as follows: For each subset $X$ of $Q^*$,

$$C(X) = \bigcap\{[x; a] : X \subseteq [x; a] \text{ for } x \in Q^* \text{ and } a \in Q \cup \{\varepsilon\}\}.$$

LEMMA 5.2. *The map $C$ is a closure operator (in our sense), which satisfies $C(\{\varepsilon\}) = Q^*$ and $C(\{0\}) = C(\emptyset)$.*

**Proof**. We show first that $C(X) * C(Y) \subseteq C(X * Y)$. Take any $w \in C(X) * C(Y)$. By the definition of $*$, there are $u \in C(X)$ and $v \in C(Y)$ such that $w = uv$. Suppose that $X * Y \subseteq [z; c]$ for $z \in Q^*$ and $c \in Q \cup \{\varepsilon\}$.

Then for any $x \in X$ and any $y \in Y$, $xyz \preceq c$. This implies that $X \subseteq [yz; c]$. Then, since $u \in C(X)$ by our assumption, $u \in [yz; c]$, i.e. $yuz = uyz \preceq c$ for any $y \in Y$. Thus, $Y \subseteq [uz; c]$. Since $v \in C(Y)$, it follows that $v \in [uz; c]$ and hence $uvz = vuz \preceq c$. Therefore, we have shown that $w = uv \in [z; c]$ whenever $X * Y \subseteq [z; c]$. Thus we have $w \in C(X * Y)$.

Next, we show that $C(\{\varepsilon\}) = Q^*$. Suppose that $\{\varepsilon\} \subseteq [z; c]$. This means that $z = \varepsilon z \preceq c$. Then by the fourth condition of Gentzen structures (i.e. the weakening rule), we have $wz \preceq c$, i.e. $w \in [z; c]$ for any $w \in Q^*$. Thus, $Q^* \subseteq C(\{\varepsilon\})$. Similarly we can show that $C(\{0\}) = C(\emptyset)$.

By Lemmas 5.2 and 3.2, $\mathbf{C_{Q^*}}$ is a complete, integral commutative residuated lattice with identity element $C(\{\varepsilon\})$, by taking $C(\{0\})$ as the zero element. This residuated lattice $\mathbf{C_{Q^*}}$, which is determined uniquely by a given Gentzen structure $\mathbf{Q}$ for $\mathbf{FL_{ew}}$, is called the *quasi-completion* of $\mathbf{Q}$.

It would be nice if we get an embedding from $\mathbf{Q}$ to $\mathbf{C_{Q^*}}$ like the one in Proposition 3.3. But we cannot expect that much, as $\mathbf{Q}$ has only a weak mathematical structure. Still we can prove the following theorem that confirms the existence of a *quasi-embedding* from $\mathbf{Q}$ to $\mathbf{C_{Q^*}}$, which will be shown to be sufficient for our purpose.

Let us define a map $k : Q \to C(\wp(Q^*))$ by $k(a) = [a]$, where $[a] = [\varepsilon; a]$, i.e. $[a] = \{w \in Q^* : w \preceq a\}$. Then we can show the following. This can be proved essentially in the same way as the proof of Lemma 7.3 in [9] (see also Maehara [13] and Okada [16]).

THEOREM 5.3. *Suppose that $a, b \in Q$ and that $U$ and $V$ are arbitrary $C$-closed subsets of $Q^*$ such that $a \in U \subseteq k(a)$ and $b \in V \subseteq k(b)$. Then for each $\star \in \{\wedge, \vee, \cdot, \to\}$, $a \star b \in U \star_C V \subseteq k(a \star b)$, where $\star_C$ denotes $\cap, \cup_C, *_C$ and $\Rightarrow$, respectively. Thus, in particular $a \star b \in k(a) \star_C k(b) \subseteq k(a \star b)$.*

**Proof**. We note first that the following is a necessary and sufficient condition for a given subset $W$ of $Q^*$ to be $C$-closed: for any $x \in Q^*$,

> $x \in W$ whenever $W \subseteq [z; c]$ implies $x \in [z; c]$ for any $z \in Q^*$ and $c \in Q \cup \{\varepsilon\}$.

We will give here a proof of our theorem when $\star_C$ is either $\cup_C$ or $\Rightarrow$. First let $\star_C$ be $\cup_C$. To show that $a \vee b \in U \cup_C V$, suppose that $U \cup V \subseteq [z; c]$ for $z \in Q^*$ and $c \in Q \cup \{\varepsilon\}$. Since $a \in U$ and $b \in V$, $a, b \in [z; c]$. That is, both $az \preceq c$ and $bz \preceq c$ hold. From this $(a \vee b)z \preceq c$ follows. Hence $a \vee b \in [z; c]$. Therefore, $a \vee b \in C(U \cup V) = U \cup_C V$. Next we show that $U \cup_C V \subseteq k(a \vee b)$. For this, it is enough to show that $U \cup V \subseteq k(a \vee b)$

since $k(a \vee b)$ is $C$-closed. By our assumption $U \subseteq k(a)$ holds, and also by a condition of Gentzen structures $k(a) \subseteq k(a \vee b)$ holds. Therefore, $U \subseteq k(a \vee b)$. Similarly, $V \subseteq k(a \vee b)$ holds.

Next suppose that $\star_C$ is $\Rightarrow$. We show that $a \to b \in U \Rightarrow V$, i.e. $\{a \to b\} * U \subseteq V$. For this, it is enough to show that $\{a \to b\} * k(a) \subseteq V$, since $U \subseteq k(a)$ by our assumption. Take any element $w \in k(a)$ and any $[z; c]$ such that $V \subseteq [z; c]$. Then, $w \preceq a$, and $b$ must belong to $[z; c]$, i.e. $bz \preceq c$ holds since $b$ is an element of a $C$-closed set $V$. Therefore, $(a \to b)wz \preceq c$ and hence $(a \to b)w \in [z; c]$. This implies $(a \to b)w \in V$. Thus, we have $\{a \to b\} * k(a) \subseteq V$. We show next that $U \Rightarrow V \subseteq k(a \to b)$. Take any $w \in U \Rightarrow V$. Then, $\{w\} * U \subseteq V \subseteq k(b)$. Since $a \in U$, this implies $aw = wa \preceq b$ and hence $w \preceq a \to b$. Thus, $w \in k(a \to b)$.

A map such as $k$, which has the properties described in the above theorem, is called a *quasi-embedding*. As shown later, the notion of quasi-embedding can be regarded as a generalization of complete embedding.

Recall that we assumed $\langle f(s_1), \ldots, f(s_m)\rangle \preceq f(t)$ does not hold in our Gentzen structure $\mathbf{Q}$ where $f$ is a valuation on $Q$. Now we show that the corresponding inequality $(s_1 \cdot \ldots \cdot s_m) \leq t$ does not hold in the quasi-completion $\mathbf{C_{Q^*}}$ of $\mathbf{Q}$. We define a valuation $g$ on $\mathbf{C_{Q^*}}$ by $g(q) = k(f(q))$ for each propositional variable $q$, where $k$ is the quasi-embedding. Then, we can prove the following by induction on the length of the term $r$. Note that for any $a \in Q$, $a \in C(\{a\}) \subseteq [a]$, and that $f(0) = 0, f(1) = 1, g(0) = C(\{0\})$ and $g(1) = C(\{1\})$. Thus, the lemma below holds for propositional variables and logical constants. For the induction step, we can use Theorem 5.3.

LEMMA 5.4. *For any term $r$, $f(r) \in g(r) \subseteq k(f(r))$.*

Now suppose to the contrary that $(s_1 \cdot \ldots \cdot s_m) \leq t$ holds in $\mathbf{C_{Q^*}}$. Then, $(g(s_1) *_C \ldots *_C g(s_m)) \subseteq g(t)$ must hold in particular. Since $f(s_i) \in g(s_i)$ for each $i$ by Lemma 5.4, $\langle f(s_1), \ldots, f(s_m)\rangle \in (g(s_1) * \ldots * g(s_m))$ and hence $\langle f(s_1), \ldots, f(s_m)\rangle \in g(t) \subseteq k(f(t))$ hold. (Recall that the monoid operation on $Q^*$ is the multiset union.) But this implies that $\langle f(s_1), \ldots, f(s_m)\rangle \preceq f(t)$. This is a contradiction. Thus, $(s_1 \cdot \ldots \cdot s_m) \leq t$ does not hold in $\mathbf{C_{Q^*}}$. This concludes the proof of Theorem 5.1.

The following lemma is an immediate consequence of Proposition 3.1 and Lemma 4.1.

LEMMA 5.5. *The statement of Theorem 5.1 is equivalent to the statement that cut elimination holds for* $\mathbf{FL_{ew}}$.

Hence we have shown our main result.

THEOREM 5.6. *Cut elimination holds for* $\mathbf{FL_{ew}}$. *In other words, the sequent system* $\mathbf{FL_{ew}}$ *without the cut rule is complete with respect to the class of all integral commutative residuated lattices.*

In algebraic terms, our theorem says that for all terms $s_1, \ldots, s_m$ and $t$, $(s_1 \cdot \ldots \cdot s_m) \leq t$ holds in all integral commutative residuated lattices if and only if the relation $s_1, \ldots, s_m \Rightarrow t$ can be derived by using only conditions described in the definition of Gentzen structures for $\mathbf{FL_{ew}}$ (with all $\preceq$ replaced by $\Rightarrow$).

The cut elimination theorem says that the cut rule is admissible in the system obtained from $\mathbf{FL_{ew}}$ by deleting the cut rule. In other words, if both $s_1, \ldots, s_m \Rightarrow t_0$ and $t_0, t_1, \ldots, t_n \Rightarrow r$ hold in any Gentzen structure for $\mathbf{FL_{ew}}$, then $s_1, \ldots, s_m, t_1, \ldots, t_n \Rightarrow r$ holds also in any Gentzen structure for $\mathbf{FL_{ew}}$. This should be distinguished from the fact that the strong transitivity condition, i.e. $x \preceq a$ and $ay \preceq c$ imply $xy \preceq c$, does not always hold in a Gentzen structure for $\mathbf{FL_{ew}}$, which is equivalent to the non-derivability of the cut rule in $\mathbf{FL_{ew}}$, as discussed in the previous section.

An actual example witnessing the non-derivability of cut can be constructed as follows. Take a sequent system $L$ obtained from cut-free $\mathbf{FL_{ew}}$ by adding two axioms:

$$p \Rightarrow q \quad \text{and} \quad q \Rightarrow r \qquad \text{for distinct variables } p, q, r.$$

Obviously, $p \Rightarrow r$ is not provable in $L$ (because every formula in the upper sequent of each rule of cut-free $\mathbf{FL_{ew}}$ appears in the lower sequent as a subformula of some formula). Note that the derivability relation of $L$ determines a Gentzen structure for $\mathbf{FL_{ew}}$.

It should be remarked also that only properties of the monoid operation of $Q^*$ and of the relation $\preceq$ determine the structure of the integral commutative residuated lattice $\mathbf{C_{Q^*}}$. In other words, the structure is not affected by properties of any algebraic operation or constant, related to *logical connectives* and *logical constants*. We think that this fact is regarded as an intrinsic algebraic feature of substructural logics. Also, as we can see in the proof of Theorem 5.3, we use only conditions concerned with a given operation $\star$ when proving our theorem for $\star$. From these observation, we can derive an algebraic proof of the next theorem.

Let $\Phi$ be any nonempty subset of the set $\{\wedge, \vee, \cdot, \rightarrow, 0, 1\}$ of all algebraic operations and constants. We say that a term $t$ is a $\Phi$-*term* if it consists only of symbols in $\Phi$ and variables. Also, a $\Phi$-*Gentzen structure* for $\mathbf{FL_{ew}}$ is a structure defined similarly as a usual Gentzen structure for $\mathbf{FL_{ew}}$, but by restricting the structure and conditions to those related only to members of $\Phi$. In logical terms, $\Phi$-Gentzen structures for $\mathbf{FL_{ew}}$ are precisely the Gentzen structures for the $\Phi$-fragment of $\mathbf{FL_{ew}}$. Now, we have the following theorem on the conservativity of each fragment of $\mathbf{FL_{ew}}$. The theorem is usually proved syntactically as a consequence of the *subformula property* of $\mathbf{FL_{ew}}$, which in turn is one of the most important consequences of cut elimination of $\mathbf{FL_{ew}}$.

THEOREM 5.7. *Let $\Phi$ be any nonempty subset of $\{\wedge, \vee, \cdot, \rightarrow, 0, 1\}$. For $\Phi$-terms $s$ and $t$, the following three conditions are mutually equivalent:*
*1. $\mathbf{P} \models s \leq t$ for any integral commutative residuated lattice $\mathbf{P}$,*
*2. $\mathbf{Q} \models s \Rightarrow t$ for any $\Phi$-Gentzen structure $\mathbf{Q}$ for $\mathbf{FL_{ew}}$,*
*3. $\mathbf{P}' \models s \leq t$ for any $\Phi$-reduct $\mathbf{P}'$ of integral commutative residuated lattices.*

In the rest of this section, we show how quasi-completions are related to the MacNeille completions. As mentioned in the previous section, each integral commutative residuated lattice $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ with the lattice order $\leq$ can be identified with a Gentzen structure $\mathbf{P}'$ if we define the relation $\preceq$ for $\mathbf{P}'$ by

$$\langle a_1, \ldots, a_m\rangle \preceq c \text{ holds if and only if } (a_1 \cdot \ldots \cdot a_m) \leq c \text{ holds.}$$

For each $z = \langle a_1, \ldots, a_m\rangle$ in $P^*$, let $\tilde{z}$ denote an element $a_1 \cdot \ldots \cdot a_m$ in $P$. (Define $\tilde{z} = 1$ when $z = \varepsilon$.) The above relation enables us to identify each $z$ of $P^*$ with an element $\tilde{z}$ of $P$, and $\preceq$ with $\leq$. Under this identification, each set of the form $[x; a]$ is regarded as a subset $\{d \in P : d \cdot \tilde{x} \leq a\}$, or equivalently $\{d \in P : d \leq \tilde{x} \rightarrow a\}$. Since any element in $P$ can be expressed by an element of the form $\tilde{x} \rightarrow a$ for some $x \in P^*$ and some $a \in P$, we can assume that our closed base $\mathcal{B}$ consists of a set of the form $[c]$ for $c \in P$, where $[c] = \{d \in P : d \leq c\}$. Now let us define an operation $C$ on $\wp(P)$, instead of $\wp(P^*)$, by

$$C(X) = \bigcap\{[c] : X \subseteq [c] \text{ for an element } c \in P\} \quad \text{for each subset } X \text{ of } P.$$

Then it is easily seen that

$$z \in C(X) \text{ if and only if } c \in X^{\rightarrow} \text{ implies } z \leq c \text{ for any c,}$$

where $X^{\rightarrow}$ denotes the set of all upper bounds of $X$, and hence $C(X) = (X^{\rightarrow})^{\leftarrow}$, where $Y^{\leftarrow}$ denotes the set of all lower bounds of $Y$. It means that $C(X)$ is equal to $D(X)$, discussed in Section 3, and therefore the quasi-completion $\mathbf{C}_{(\mathbf{P}')^*}$ of $\mathbf{P}'$ is isomorphic to the MacNeille completion of $\mathbf{P}$.

Recall that the quasi-embedding $k : P' \to C(\wp((P')^*))$ is defined by $k(a) = [\varepsilon; a] = \{w \in Q^* : w \preceq a\}$, and $a \star b \in k(a) \star_C k(b) \subseteq k(a \star b)$ by Theorem 5.3. If $\preceq$ is strongly transitive, $a \star b \in k(a) \star_C k(b)$ implies that $k(a \star b) \subseteq k(a) \star_C k(b)$. Hence $k(a \star b) = k(a) \star_C k(b)$ holds. Since $k$ is injective, this means that $k$ is an "isomorphism", and in fact this $k$ is identified with the complete embedding of an integral commutative residuated lattice $\mathbf{P}$ into its MacNeille completion described in Proposition 3.3. Thus we have the following.

Corollary 5.8. *The quasi-completion of any integral commutative residuated lattice* $\mathbf{Q}$ *is isomorphic to its MacNeille completion, and the quasi-embedding of* $\mathbf{Q}$ *to its quasi-completion is a complete embedding, when* $\mathbf{Q}$ *is considered as a Gentzen structure with a strongly transitive relation.*

We end this section with some brief remarks about the possibility of replacing Gentzen structures with bonafide first-order structures, so that $\preceq$ is a binary relation on $Q$, rather than a relation from $Q^*$ to $Q$. Such an approach is indeed possible, and permits Gentzen structures to be defined by a short list of universal horn sentences, where the sequence constructor (comma) is replaced by fusion. A notion of *algebraic Gentzen proof* can now be formulated as a restriction of the standard notion of quasi-equational proof, and derivations in this system are somewhat shorter since fusion-elimination steps are omitted. From an algebraic standpoint this provides an even tighter connection between proof theory and universal algebra. But our present approach also has some advantages. For example the semantics of (non-first-order) Gentzen structures exactly capture the provability relation for sequents of standard Gentzen systems (with or without cut), and the presence of comma-separated sequences allows some distinctions to be made that cannot be expressed by the first-order language.

## 6. Cut elimination for other systems

Our algebraic proof of cut elimination works for various sequent systems. For example, an outline of an algebraic proof of cut elimination of Gentzen's sequent system **LJ** for intuitionistic logic is given in [22]. It is not hard to modify our method to apply it to intuitionistic substructural logics like $\mathbf{FL_e}$

and $\mathbf{FL_{ec}}$. In this section, we explain briefly how to extend our method to other sequent systems and tableau systems.

First, we show that our method is naturally extended to sequent systems for predicate logics, taking the sequent system $\mathbf{QFL_{ew}}$ which is a natural predicate extension of $\mathbf{FL_{ew}}$. We note that in [13] Maehara gave a semi-algebraic proof of cut elimination for predicate systems $\mathbf{LK}$ and $\mathbf{LJ}$. We will give here an outline of the algebraic proof of cut elimination for $\mathbf{QFL_{ew}}$. The sequent system $\mathbf{QFL_{ew}}$ is obtained from $\mathbf{FL_{ew}}$ by adding the following rules for quantifiers $\forall$ and $\exists$:

$$\frac{\alpha[t/v], \Gamma \Rightarrow \delta}{\forall v\alpha, \Gamma \Rightarrow \delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[z/v]}{\Gamma \Rightarrow \forall v\alpha}\ (\Rightarrow \forall)$$

$$\frac{\alpha[z/v], \Gamma \Rightarrow \delta}{\exists v\alpha, \Gamma \Rightarrow \delta}\ (\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[t/v]}{\Gamma \Rightarrow \exists v\alpha.}\ (\Rightarrow \exists)$$

Here, $t$ is a term (in the first-order language), $v$ and $z$ are individual variables, and $\alpha[z/v]$ ($\alpha[t/v]$) are the formula obtained from $\alpha$ by replacing all free occurrences of $v$ in $\alpha$ by $z$ (by $t$, respectively). Moreover, in applications of $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ $z$ should not occur as a free variable in the lower sequent.

In algebras, quantifiers are interpreted as infinite meets and joins. Let $\mathbf{P}$ be a complete integral commutative residuated lattice and let $D$ be a nonempty subset of $\mathbf{P}$, called the *individual domain*. Moreover, suppose that $\alpha$ is a first-order formula which contains no free individual variable other than $v$. Then, for any valuation $f$, $\forall v\alpha$ and $\exists v\alpha$ are defined as follows:

$$f(\forall v\alpha) = \bigwedge\{f(\alpha[\hat{i}/v]) : i \in D\} \text{ and } f(\exists v\alpha) = \bigvee\{f(\alpha[\hat{i}/v]) : i \in D\},$$

where $\hat{i}$ denotes the name of an element $i \in D$. We call such a pair $\langle \mathbf{P}, D\rangle$, an *algebraic frame* for $\mathbf{QFL_{ew}}$. Then we can show the completeness of $\mathbf{QFL_{ew}}$ with respect to the class of all algebraic frames for $\mathbf{QFL_{ew}}$ (see e.g. [18]).

Gentzen structures for $\mathbf{QFL_{ew}}$ are defined in the same way as those for $\mathbf{FL_{ew}}$. A Gentzen structure for $\mathbf{QFL_{ew}}$ consists of a Gentzen structure $\mathbf{Q}$ for $\mathbf{FL_{ew}}$ and a nonempty set $D$. Suppose that $\bigwedge\{a_i : i \in D\}$ and $\bigvee\{a_i : i \in D\}$ are elements of $Q$, which are defined for some subsets $\{a_i : i \in D\}$ of $Q$. (In the following, we write $\bigwedge a_i$ and $\bigvee a_i$, instead of $\bigwedge\{a_i : i \in D\}$ and $\bigvee\{a_i : i \in D\}$, respectively.) When they are defined, $\mathbf{Q}$ satisfies in addition the following conditions:

- $a_j x \preceq c$ for some $j \in D$ implies $(\bigwedge a_i)x \preceq c$,
- $x \preceq a_j$ for all $j \in D$ implies $x \preceq \bigwedge a_i$,
- $a_j x \preceq c$ for all $j \in D$ implies $(\bigvee a_i)x \preceq c$,
- $x \preceq a_j$ for some $j \in D$ implies $x \preceq \bigvee a_i$.

Then, similarly to Theorem 5.3, we have the following, where $\bigcup_C U_i$ means $C(\bigcup U_i)$.

LEMMA 6.1. *Suppose that for all $j \in D$, $a_j \in Q$ and $U_j$ is a $C$-closed subset of $Q^*$ such that $a_j \in U_j \subseteq k(a_j)$. Under these assumptions, if $\bigwedge a_i$ exists then $\bigwedge a_i \in \bigcap U_i \subseteq k(\bigwedge a_i)$, and also if $\bigvee a_i$ exists then $\bigvee a_i \in \bigcup_C U_i \subseteq k(\bigvee a_i)$.*

**Proof**. We give here a proof of the second part. Suppose that $\bigcup_C U_i \subseteq [z; c]$. Then, $a_j \in U_j \subseteq [z; c]$ and hence $a_j z \preceq c$ for all $j \in D$. Thus, $(\bigvee a_i)z \preceq c$. That is, $\bigvee a_i \in [z; c]$. Since $\bigcup_C U_i$ is $C$-closed, $\bigvee a_i \in \bigcup_C U_i$. Next, suppose that $x \in \bigcup_C U_i$. Then, $x \in U_j \subseteq k(a_j)$ for some $j$. That is, $x \preceq a_j$. Hence $x \preceq \bigvee a_i$. Therefore, $\bigcup_C U_i \subseteq k(\bigvee a_i)$.

By using Lemma 6.1, we can show Lemma 5.4 for the present case. Note here that we need to take a Gentzen structure **Q** for $\mathbf{QFL_{ew}}$ and a valuation $f$ with the property that every quantifier in the sequent under discussion is interpreted in it. To do so, it is enough to take the free Gentzen structure of $\mathbf{QFL_{ew}}$ for **Q**, and the *canonical mapping* for $f$. (For the details, consult [13].) Thus we have the following.

THEOREM 6.2. *Cut elimination holds for the sequent system $\mathbf{QFL_{ew}}$, which is the predicate extension of $\mathbf{FL_{ew}}$.*

Next, we discuss sequent systems with sequents of the form $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are sequences of formulas. A typical example is Gentzen's sequent system **LK** for classical logic. For **LK**, a semi-algebraic proof of cut elimination is given in the paper [13] by Maehara. Also the first author of the present paper explored in his thesis [2], an algebraic proof of cut elimination of sequent systems of this kind for substructural logics, i.e. sequent systems of *classical* substructural logics, based on a draft of an algebraic proof of cut elimination for $\mathbf{FL_{ew}}$ by the third author. In the following, by taking the sequent system $\mathbf{CFL_{ew}}$ as an example, we will explain how to modify our proof of cut elimination for these sequent systems. Note that a syntactic proof of the cut elimination theorem for $\mathbf{CFL_{ew}}$ is given by Grishin in [8].

The sequent system $\mathbf{CFL_{ew}}$ is obtained from $\mathbf{LK}$ by first deleting contraction rules and then adding both initial sequents for 1 and 0, rules for logical constants and rules for $\cdot$, which are given as follows (see [18]);

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \cdot \beta, \Gamma \Rightarrow \Delta} (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Sigma \Rightarrow \Theta, \beta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \alpha \cdot \beta} (\Rightarrow \cdot)$$

In other words, $\mathbf{CFL_{ew}}$ is obtained from $\mathbf{FL_{ew}}$ by extending each of its rule to one with sequents of the form $\Gamma \Rightarrow \Delta$. An important feature of $\mathbf{CFL_{ew}}$ is that the sequent $\neg\neg\alpha \Rightarrow \alpha$ is provable in it. Note that the sequent $\alpha \Rightarrow \neg\neg\alpha$ is provable already in $\mathbf{FL_{ew}}$. Algebraic structures for $\mathbf{CFL_{ew}}$ are *involutive* integral commutative residuated lattices, i.e. integral commutative residuated lattices that satisfy $\neg\neg x \leq x$. Recall that $\neg x$ is defined as an abbreviation of $x \to 0$.

An obvious modification is necessary in the definition of $\preceq$ when we introduce *Gentzen structures for* $\mathbf{CFL_{ew}}$. That is, $\preceq$ must be defined as a binary relation on $Q^*$. Thus, the conditions corresponding to rules of $\mathbf{CFL_{ew}}$ should be expressed by using this $\preceq$. For instance, the condition corresponding to the rule $(\cdot \Rightarrow)$, mentioned above, becomes

$$abx \preceq y \text{ implies } (a \cdot b)x \preceq y$$

for $x, y \in Q^*$. Since the set $(Q^*)^2$ is regarded as a direct product of a commutative monoid $Q^*$, it forms also a commutative monoid with the unit $(\varepsilon, \varepsilon)$. Hence, $\mathbf{C'}_{(\mathbf{Q}^*)^2}$ is a complete commutative residuated lattice for any closure operator $C'$ on $\wp((Q^*)^2)$ (see Section 5). When we define the closed base, we need to take $[x; y]$ for $x, y \in Q^*$, instead of taking $[x; a]$. Define $[x; y]$ by

$$[x; y] = \{(u, w) \in (Q^*)^2 : ux \preceq yw\},$$

and then define a closure operator $C$ on $\wp((Q^*)^2)$ by using them. Though the proof goes basically in the same way as the case for $\mathbf{FL_{ew}}$, we give some explanations on points where further consideration might be necessary.

The unit element $I$ of $\mathbf{C}_{(\mathbf{Q}^*)^2}$ is defined obviously by $I = C(\{(\varepsilon, \varepsilon)\}) = \bigcap\{[x; y] : x \preceq y\}$. For its zero element $O$, we take

$$O = \bigcap\{[x; y] : (x, y) \in I\} = \bigcap\{[x; y] : \text{for any } u, v, u \preceq v \text{ implies } ux \preceq vy\}.$$

Let $\sim X = X \Rightarrow O$. Then, we can show that $\sim X = \bigcap\{[x;y] : (x,y) \in X\}$, and hence $\sim I = O$. Note that $\sim X$ is always $C$-closed since it is represented as an intersection of members in the closed base.

We show now that $\mathbf{C}_{(\mathbf{Q}^*)^2}$ is involutive, i.e. $\sim\sim X = X$ for any $C$-closed subset $X$. To prove this, it is enough to show that $\sim\sim X = C(X)$ for any subset $X$ of $(Q^*)^2$. It is easy to see that $X \subseteq \sim\sim X$. Hence, $C(X) \subseteq \sim\sim X$ since $\sim\sim X$ is $C$-closed. Conversely, suppose that $(u,v) \in \sim\sim X$ and that $X \subseteq [w;z]$. The latter implies that for any $(x,y) \in X$, $xw \preceq zy$ holds, and hence $(w,z) \in [x;y]$. Thus, $(w,z) \in \sim X$. But, since $(u,v) \in \sim\sim X$, $(u,v) \in [w;z]$. This shows that $\sim\sim X \subseteq C(X)$. As a corollary, we have that $\sim\sim X = X$ for any $C$-closed $X$. As we can show that it is also integral, $\mathbf{C}_{(\mathbf{Q}^*)^2}$ turns out to be an involutive, integral commutative residuated lattice.

Our definition of $O$ given here works for other classical substructural logics like $\mathbf{CFL_e}$ for linear logic (without exponentials). But, in the case of $\mathbf{CFL_{ew}}$, we can show that $O = C(\emptyset)$ since $u \preceq v$ implies $ux \preceq vy$ holds in any Gentzen structure for $\mathbf{CFL_{ew}}$.

We show next how to modify Theorem 5.3, Lemma 5.4 and their proofs. We define a map $k : Q \to C(\wp((Q^*))^2)$ by $k(a) = [\varepsilon; a] = \{(u,v) : u \preceq av\}$. Then we show Theorem 5.3 for classical substructural logics in the following way.

THEOREM 6.3. *Suppose that $a, b \in Q$ and that $U$ and $V$ are arbitrary $C$-closed subsets of $(Q^*)^2$ such that $(a,\varepsilon) \in U \subseteq k(a)$ and $(b,\varepsilon) \in V \subseteq k(b)$. Then for each $\star \in \{\wedge, \vee, \cdot, \to\}$, $(a \star b, \varepsilon) \in U \star_C V \subseteq k(a \star b)$, where $\star_C$ denotes $\cap, \cup_C, *_C$ and $\Rightarrow$, respectively.*

We will give here a proof of the case when $\star$ is $\to$. To show that $(a \to b, \varepsilon) \in U \Rightarrow V$, it suffices to prove that $(a \to b, \varepsilon) * k(a) \subseteq V$, since $U \subseteq k(a)$. Take any $(u,v) \in k(a)$ and any $[z;w]$ such that $V \subseteq [z;w]$. Then, both $u \preceq va$ and $bz \preceq w$ hold, since $(b,\varepsilon) \in [z;w]$. Then, using the condition corresponding to $(\to\Rightarrow)$ of $\mathbf{CFL_{ew}}$, we have $(a \to b)uz \preceq vw$ and thus $((a \to b)u, v) \in [z;w]$. Since $V$ is $C$-closed, this implies that $((a \to b)u, v) \in V$. Therefore, $(a \to b, \varepsilon) * k(a) \subseteq V$. Next, suppose that $(u,v) \in U \Rightarrow V$. Then $(u,v) * U \subseteq V \subseteq k(b)$. Since $(a,\varepsilon) \in U$, we have $(au, v) \in k(b)$, and therefore $au \preceq bv$. It follows that $u \preceq (a \to b)v$ and thus $(u,v) \in k(a \to b)$. This proves $U \Rightarrow V \subseteq k(a \to b)$.

Lemma 5.4 is modified as follows. We note here that the valuation $g$ satisfies $g(0) = O$ and $g(1) = I$.

LEMMA 6.4. *For any term $r$, $(f(r), \varepsilon) \in g(r) \subseteq k(f(r))$.*

We will give a proof of the above lemma only when $r$ is 0. Suppose that for given $x$ and $y$, $u \preceq v$ implies $ux \preceq vy$ for any $u, v$. Since $0 \preceq \varepsilon$ holds, we have $0x \preceq y$, i.e. $(0, \varepsilon) \in [x; y]$. Thus, $(f(0), \varepsilon) \in O = g(0)$. To show that $g(0) = O \subseteq [\varepsilon; 0] = k(f(0))$, it is enough to prove that $u \preceq v$ implies $u\varepsilon \preceq v0$, by the definition of $O$. But, this is shown to hold, by using $(\Rightarrow 0)$ rule for $\mathbf{CFL_{ew}}$.

We have now come to the final step of our proof of cut elimination for $\mathbf{CFL_{ew}}$. A sequent of the form $s_1, \ldots, s_m \Rightarrow t_1, \ldots, t_n$ is said to be *valid* in an involutive, integral commutative residuated lattice $\mathbf{P}$ if for any valuation $h$, $(h(s_1) \cdot \ldots \cdot h(s_m)) \leq (h(t_1) + \ldots + h(t_n))$ holds in $\mathbf{P}$. Here, $a + b$ is defined by $a + b = \neg(\neg a \cdot \neg b)$ for any $a, b \in P$.

We assume that a sequent $s_1, \ldots, s_m \Rightarrow t_1, \ldots, t_n$ is not valid in a Gentzen structure $\mathbf{Q}$ for $\mathbf{CFL_{ew}}$ under a valuation $f$, which means that $\langle f(s_1), \ldots, f(s_m)\rangle \preceq \langle f(t_1), \ldots, f(t_n)\rangle$ doesn't hold. Moreover, we suppose to the contrary that it valid in $\mathbf{C}_{(\mathbf{Q}^*)^2}$. Then, in particular

$$g(s_1) *_C \cdots *_C g(s_m) \subseteq \sim(\sim g(t_1) *_C \cdots *_C \sim g(t_n)).$$

By Lemma 6.4, $(f(s_i), \varepsilon) \in g(s_i)$ holds for each $i$ and $g(t_j) \subseteq k(f(t_j))$ holds for each $j$. Therefore,

$$(\langle f(s_1), \ldots, f(s_m)\rangle, \varepsilon) \in \sim(\sim k(f(t_1)) *_C \cdots *_C \sim k(f(t_n)))$$

holds. That is, for any $x, y$,

(1) if $(x, y) \in \sim k(f(t_1)) *_C \cdots *_C \sim k(f(t_n))$ then $\langle f(s_1), \ldots, f(s_m)\rangle x \preceq y$.

On the other hand, if $(u, v) \in k(f(t_j)) = [\varepsilon; f(t_j)]$ then $u \preceq v f(t_j)$, and hence $(\varepsilon, f(t_j)) \in [u; v]$. This implies that $(\varepsilon, f(t_j)) \in \sim k(f(t_j))$. Hence,

(2) $(\varepsilon, \langle f(t_1), \ldots, f(t_n)\rangle) \in \sim k(f(t_1)) *_C \cdots *_C \sim k(f(t_n))$.

From (1) and (2) it follows that $\langle f(s_1), \ldots, f(s_m)\rangle \preceq \langle f(t_1), \ldots, f(t_n)\rangle$. But this is a contradiction.

THEOREM 6.5. *Cut elimination holds for the sequent system* $\mathbf{CFL_{ew}}$.

When we discuss cut elimination of sequent systems for classical modal logics, we need to introduce $\Diamond X$ for a subset $X$ of $(Q^*)^2$ by

$$\Diamond X = \bigcap\{[\Box x; \Diamond y] : X \subseteq [x; y]\},$$

where $\Box x$ ($\Diamond x$) is the element $\{\Box a_1, \ldots, \Box a_m\}$ ($\{\Diamond a_1, \ldots, \Diamond a_m\}$, respectively) of $Q^*$ when $x$ is $\{a_1, \ldots, a_m\}$. By using this, we can get an algebraic proof of cut elimination of sequent systems of some for basic modal logics, including **K**, **KT** and **S4**.

It is easy to see that we can apply our method to one-sided sequent systems and tableau systems. The idea of introducing one-sided sequent systems is based on the fact that in *classical* systems, the provability of a sequent $\Gamma \Rightarrow \Delta$ is equivalent to that of $\Rightarrow \neg\Gamma, \Delta$, where $\neg\Gamma$ denotes $\neg\alpha_1, \ldots, \neg\alpha_m$ when $\Gamma$ is $\alpha_1, \ldots, \alpha_m$. Furthermore, we write simply $\neg\Gamma, \Delta$ instead of $\Rightarrow \neg\Gamma, \Delta$. By this translation, the initial sequent $\alpha \Rightarrow \alpha$ becomes $\neg\alpha, \alpha$. In such a formal system, it is convenient to take the negation $\neg$ as a primitive symbol, for which we take the rule $(\neg)$ shown below. Also, the rule $(\Rightarrow \wedge)$ and the cut rule, for instance, will be expressed as follows.

$$\frac{\alpha, \Gamma}{\neg\neg\alpha, \Gamma}\ (\neg) \qquad \frac{\alpha, \Gamma \quad \beta, \Gamma}{\alpha \wedge \beta, \Gamma}\ (\wedge) \qquad \frac{\neg\alpha, \Gamma \quad \alpha, \Sigma}{\Gamma, \Sigma}\ (cut)$$

Gentzen structures for these systems can be defined in the same way as before. In these cases, we may take a subset $\mathsf{Pr}$ of $Q^*$ instead of using a relation $\preceq$, and write $x \in \mathsf{Pr}$ whenever $\varepsilon \preceq x$. Then, we can show cut elimination for these one-sided sequent systems, as before.

Tableau systems can be defined as *duals* of one-sided sequent systems (without the cut rule, by definition). In this case, a sequent $\Gamma \Rightarrow \Delta$ is represented as $\Gamma, \neg\Delta$ in tableau systems. In other words, instead of searching for a proof of a sequent $\Rightarrow \alpha$, we try to show that $\neg\alpha$ is *refutable* in a tableau system. By the standard convention, rules in tableau systems are written upside down. For example, $(\neg)$ and $(\wedge)$ in tableau systems are expressed as follows:

$$\frac{\neg\neg\alpha, \Gamma}{\alpha, \Gamma}\ (\neg) \qquad \frac{\neg(\alpha \wedge \beta), \Gamma}{\neg\alpha, \Gamma \mid \neg\beta, \Gamma}\ (\wedge)$$

Gentzen structures for tableau systems can be defined in the same way as those for one-sided sequent system. This time, we take a subset $\mathsf{Ref}$ of $Q^*$ and define $x \in \mathsf{Ref}$ when $x \preceq \varepsilon$. Thus, conditions on $\mathsf{Ref}$ corresponding to initial sequents and the above two rules become as follows.

- $\langle \neg a, a \rangle \in \mathsf{Ref}$,
- $ax \in \mathsf{Ref}$ implies $(\neg\neg a)x \in \mathsf{Ref}$,

- $(\neg a)x \in \mathsf{Ref}$ and $(\neg b)x \in \mathsf{Ref}$ imply $\neg(a \wedge b)x \in \mathsf{Ref}$.

Using quasi-completions, we can show the completeness of these tableau systems. Now let us define $\mathsf{Con}$ to be the complement of $\mathsf{Ref}$ with respect to $Q^*$. Then, the above conditions can be obviously rewritten as follows.

- $\langle \neg a, a \rangle \notin \mathsf{Con}$,
- $(\neg\neg a)x \in \mathsf{Con}$ implies $ax \in \mathsf{Con}$,
- $\neg(a \wedge b)x \in \mathsf{Con}$ implies $(\neg a)x \in \mathsf{Con}$ or $(\neg b)x \in \mathsf{Con}$.

When $Q$ is the set of formulas, such a set $\mathsf{Con}$ that satisfies these conditions is called a *consistency property* in Fitting [4]. In the paper, it is shown that any member $S$ of a consistency property is satisfiable, by constructing a Kripke model in which $S$ is true. In this way, the completeness of these tableau systems with respect to *Kripke semantics* is obtained. It would be interesting to see whether there exists a relation between Fitting's construction of Kripke frames from consistency properties and our construction of algebras given here, in particular, in the case of modal logics. This topic will be discussed elsewhere in the future.

## 7. Finite model property

In this section, we will give a proof of the finite model property of the logic $\mathbf{FL_{ew}}$. By the finite model property of $\mathbf{FL_{ew}}$, we mean that if a sequent $\Gamma \Rightarrow \delta$ is not provable in $\mathbf{FL_{ew}}$, then there exists a *finite* integral commutative residuated lattice in which this sequent does not hold.

Our proof of the finite model property of $\mathbf{FL_{ew}}$ given below is of algebraic character, and it is given by modifying our algebraic proof of the cut elimination theorem. We owe the idea of the present proof to ones by Lafont [12] and Okada-Terui [17], though the presentation is different from them.

Suppose that $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \to, 0, 1 \rangle$ is a Gentzen structure for $\mathbf{FL_{ew}}$ such that the closed base $\mathcal{B}$ determined by $\preceq$ is finite. Then, the set of all $C$-closed subsets is also finite, where $C$ is the closure operator determined by $\mathcal{B}$, since each $C$-closed subset is obtained as an intersection of some of members of $\mathcal{B}$. Thus we have the following lemma, by observing how $\mathbf{C_{Q^*}}$ is constructed in the previous section.

Lemma 7.1. *Let* $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \to, 0, 1 \rangle$ *be a Gentzen structure for* $\mathbf{FL_{ew}}$ *such that the closed base given by* $\preceq$ *is finite. Then the quasi-completion* $\mathbf{C_{Q^*}}$ *of* $\mathbf{Q}$ *is also finite.*

Now let $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ be a Gentzen structure for $\mathbf{FL_{ew}}$, and let $(x, a)$ be a fixed member of the set $Q^* \times (Q \cup \{\varepsilon\})$. We define a subset $\mathcal{P}_{(x,a)}$ of $Q^* \times (Q \cup \{\varepsilon\})$ as follows. Each member of $\mathcal{P}_{(x,a)}$ is called a *predecessor* of $(x, a)$.

1. $(x, a) \in \mathcal{P}_{(x,a)}$.
2. Suppose that $(w, b) \in \mathcal{P}_{(x,a)}$. If "$u \preceq c$ implies $w \preceq b$" is an instance of one of conditions for $\preceq$ in a Gentzen structure for $\mathbf{FL_{ew}}$ for some $u \in Q^*$ and $c \in Q \cup \{\varepsilon\}$, then $(u, c)$ is a member of $\mathcal{P}_{(x,a)}$. Similarly, if "$u \preceq c$ and $v \preceq d$ imply $w \preceq b$" is an instance of one of conditions for $\preceq$ for some $u, v \in Q^*$ and $c, d \in Q \cup \{\varepsilon\}$, then both $(u, c)$ and $(v, d)$ are members of $\mathcal{P}_{(x,a)}$.
3. Every member of $\mathcal{P}_{(x,a)}$ is obtained in this way.

An intuitive proof-theoretic meaning of the set $\mathcal{P}_{(x,a)}$ is the set of all "sequents" which may appear in a cut-free proof of the "sequent" $x \preceq a$. For a finite subset $S$ of $Q^* \times (Q \cup \{\varepsilon\})$, let $\mathcal{P}_S$ be the union of $\mathcal{P}_{(x,a)}$ such that $(x, a) \in S$. We say that the set $S$ is *finitely based*, when $\mathcal{P}_S$ is finite. The following lemma shows that any finitely based subset of $Q^* \times (Q \cup \{\varepsilon\})$ can be *embedded* into a Gentzen structure for $\mathbf{FL_{ew}}$ with the same underlying set $Q$ such that the closed base determined by it is finite.

LEMMA 7.2. *Suppose that* $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ *is a Gentzen structure for* $\mathbf{FL_{ew}}$ *and that* $S$ *is a finitely based subset of* $Q^* \times (Q \cup \{\varepsilon\})$. *Then, there exists a subset* $\preceq^\star$ *of* $Q^* \times (Q \cup \{\varepsilon\})$ *which satisfies the following conditions.*

1. *if* $(w, b) \in S$ *then* $w \preceq^\star b$ *iff* $w \preceq b$,
2. *the structure* $\mathbf{Q}^\star = \langle Q, \preceq^\star, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ *forms a Gentzen structure for* $\mathbf{FL_{ew}}$,
3. *the closed base determined by* $\preceq^\star$ *of* $\mathbf{Q}^\star$ *is finite.*

**Proof.** Note that the set $\mathcal{P}_S$ is finite by our assumption. We define a subset $\preceq^\star$ of $Q^* \times (Q \cup \{\varepsilon\})$ as follows. For $w \in Q^*$ and $b \in Q \cup \{\varepsilon\}$, if $(w, b) \in \mathcal{P}_S$ then $w \preceq^\star b$ iff $w \preceq b$, and otherwise $w \preceq^\star b$ holds always. Clearly, this relation $\preceq^\star$ satisfies the first condition of our Lemma 7.2.

To show that $\mathbf{Q}^\star$ is a Gentzen structure, it is enough to check that $\preceq^\star$ satisfies all the conditions in the definition of Gentzen structures for $\mathbf{FL_{ew}}$. Let us assume that one of conditions (for $\preceq^\star$), say ($\sharp$), is of the form "$u \preceq^\star c$ and $v \preceq^\star d$ implies $w \preceq^\star b$". To show that this holds for $\preceq^\star$, we suppose that $w \preceq^\star b$ does not hold. By the definition of $\preceq^\star$, this happens only when $(w, b) \in \mathcal{P}_S$ but $w \preceq b$ does not hold. In this case, both $(u, c)$ and $(v, d)$ must belong to $\mathcal{P}_S$. Since "$u \preceq c$ and $v \preceq d$ implies $w \preceq b$" is the condition ($\sharp$) for $\preceq$ which must be true, at least one of $u \preceq c$ and $v \preceq d$ does not hold. Therefore, at least one of $u \preceq^\star c$ and $v \preceq^\star d$ does not hold either. This means that the condition ($\sharp$) holds for $\preceq^\star$. In this way, we can show that all the conditions holds for $\preceq^\star$. Thus $\mathbf{Q}^\star$ is a Gentzen structure for $\mathbf{FL_{ew}}$.

The closed base determined by $\preceq^\star$ consists of all sets of the following form, where $x \in Q^*$ and $a \in Q \cup \{\varepsilon\}$:

$$[x; a]^\star = \{w \in Q^* : xw \preceq^\star y\}.$$

We show that $[x; a]^\star = Q^*$ holds for all but finitely many pairs $(x, a)$. To prove this, it is enough to show that if $(x, a) \notin \mathcal{P}_S$ then $[x; a]^\star = Q^*$. In fact, if $(x, a) \notin \mathcal{P}_S$ then $x \preceq^\star a$ by the definition of $\preceq^\star$, and hence $xw \preceq^\star a$ holds for any $w$ by using a condition (which corresponds to the weakening rule) for Gentzen structures. Thus, $[x; a]^\star = Q^*$, and it follows that the closed base is finite.

Now we are ready to prove the following.

Theorem 7.3. *The logic $\mathbf{FL_{ew}}$ has the finite model property.*

**Proof.** Suppose that a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is not provable in $\mathbf{FL_{ew}}$. Obviously, this is not provable in $\mathbf{FL_{ew}}$ without using cut. Then, by the proof of Lemma 4.1, $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq \beta$ doesn't hold in the *free* Gentzen structure $\mathbf{Q}^+$ for $\mathbf{FL_{ew}}$ whose universe $Q^+$ is the set of all terms, under the valuation $i$ which is the identity mapping on the set of all term variables. (When $m = 0$, $\langle \alpha_1, \ldots, \alpha_m \rangle$ denotes the empty sequence $\varepsilon$. Also, the term $\beta$ denotes $\varepsilon$ when the right hand side of the sequent is empty.) We show that the singleton set $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is finitely based. To see this, define the "length" of any element of $(Q^+)^* \times (Q^+ \cup \{\varepsilon\})$ as follows. Let $\ell(\delta)$ denote the length of a given term $\delta$. For an element $(\langle \gamma_1, \ldots, \gamma_m \rangle, \delta)$ of $(Q^+)^* \times (Q^+ \cup \{\varepsilon\})$, its length is defined to be the sum $\ell(\gamma_1) + \cdots + \ell(\gamma_m) + \ell(\delta)$. Then we can show that if $(\langle \gamma_1, \ldots, \gamma_m \rangle, \gamma)$ is a predecessor of $(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)$, then the length is smaller than or equal to the length of $(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)$ and moreover, any of $\gamma_1, \ldots, \gamma_m$ and $\delta$ is a subterm of any one of $\alpha_1, \ldots, \alpha_m, \beta$. Thus, the number of predecessors of $(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)$ must be finite.

Then by Lemma 7.2, $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is embedded into a Gentzen structure $(\mathbf{Q}^+)^\star$ with a relation $\preceq^\star$ for $\mathbf{FL_{ew}}$ such that the closed base determined by $\preceq^\star$ is finite. Moreover, $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^\star \beta$ doesn't hold in $(\mathbf{Q}^+)^\star$.

Now, using Lemma 7.1, the quasi-completion $\mathbf{R}$ of $(\mathbf{Q}^{+})^{\star}$ is finite. Since $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^{\star} \beta$ doesn't hold in $(\mathbf{Q}^{+})^{\star}$, $(\alpha_1 \cdot \ldots \cdot \alpha_m) \leq \beta$ doesn't hold either in $\mathbf{R}$, which is a finite, integral commutative residuated lattice, as shown just above Theorem 5.6. This completes the proof of the finite model property.

A key of our proof of the finite model property given here is the fact that the set $\mathcal{P}_{(x,a)}$ of all predecessors of any given $(x, a)$ is finite. In a syntactic term, this means that the *proof search tree* of any sequent in the cut-free sequent system $\mathbf{FL_{ew}}$ is always finite. Here, by a proof search tree of a given sequent $\Gamma \Rightarrow \delta$ we mean a proof search procedure represented in a tree-like form which searches for a (cut-free) proof of $\Gamma \Rightarrow \delta$ and can always find it as long as it is provable. Thus the finiteness of the proof search procedure means that after finitely many steps of the proof search we can always see whether a given sequent is provable or not. We have constructed a finite algebra in which a given unprovable sequent does not hold, by using the finite proof search tree of the sequent. Thus, our proof of the finite model property also works for other logics with cut-free sequent systems, as long as the proof search tree of any sequent in them is always finite. Theorem 8.1 in the next section is a good example that confirms the above argument, since it says that even the predicate logic $\mathbf{QFL_{ew}}$ has the finite model property.

Usually, the finite model property is proved in order to derive the decidability. On the other hand, as mentioned above our method uses the existence of a decision procedure to prove the finite model property. This may sound strange, but it is not unusual in the study of substructural logics, where decidability results for most of the basic substructural logics are obtained as simple consequences of cut elimination and therefore can be proved much earlier than the finite model property (see [19, 15, 12, 17]). We note here that recently a promising way of showing the finite model property of some of substructural logics has been developed in the paper Blok-van Alten [3], where the *finite embeddability property* of the class of algebras for a given substructural logic is used.

Our algebraic proof of cut elimination and its application to the finite model property seems to work well for various sequent systems. But this doesn't mean that we can prove most of the consequences of cut elimination algebraically. For instance, though induction on the length of formulas is a basic tool in syntactic arguments, it sometimes happens that we cannot find any substitute for it in algebraic arguments. For, in algebra mathematical objects are not always distinguished from their representations. Thus sometimes it becomes necessary to introduce algebraic substitutes for syntactic objects, like free Gentzen structures, to fill this gap. (To see this, for example, consider the role of the free Gentzen structure in the proof of Theorem 7.3.)

## 8. Finite model property of predicate logic $\mathbf{QFL_{ew}}$

This section is devoted to an outline of the proof of the following theorem. One may skip the proof, as it assumes certain familiarity with proof-theoretic arguments. What we need to mention here is that the proof relies on the finiteness proof of the proof search procedure in $\mathbf{QFL_{ew}}$ shown in Komori [10], from which he derived the decidability of $\mathbf{QFL_{ew}}$ (without function symbols).

THEOREM 8.1. *The predicate logic* $\mathbf{QFL_{ew}}$ *(without function symbols) has the finite model property. More precisely, if a sequent* $\Gamma \Rightarrow \beta$ *is not provable in* $\mathbf{QFL_{ew}}$ *then there exists an algebraic frame* $\langle \mathbf{P}, D\rangle$ *for* $\mathbf{QFL_{ew}}$ *with a finite integral commutative residuated lattice* $\mathbf{P}$ *and a finite individual domain* $D$ *in which* $\Gamma \Rightarrow \beta$ *is not valid.*

A related result is shown for the predicate extension of $\mathbf{CFL_{ew}}$ by Grishin [8], who proved that there exists an algebraic frame $\langle \mathbf{P}, D\rangle$ with a *simple* (but not necessarily finite) algebra $\mathbf{P}$ and a finite individual domain $D$ in which a given unprovable sequent is not valid.

For a given sequent $\Gamma \Rightarrow \beta$, we consider whether it is provable in $\mathbf{QFL_{ew}}$ or not. If it contains free variables, we introduce *new* constant symbols and replace all of the free variables by these distinct constant symbols. This doesn't affect the provability in $\mathbf{QFL_{ew}}$. Therefore, from the outset we can assume that $\Gamma \Rightarrow \beta$ has no free variables. Let $E = \{e_1, \ldots, e_k\}$ be the set of all constant symbols appearing in $\Gamma \Rightarrow \beta$.

Let $z_1, \ldots, z_m$ be all bound variables in $\Gamma \Rightarrow \beta$. We can assume without loss of generality that any two distinct occurrences of quantifiers in $\Gamma \Rightarrow \beta$ are followed by distinct bound variables. Let us take a set $V = \{v_1, \ldots, v_m\}$ of variables which are distinct from $z_1, \ldots, z_m$. Define $V$-subformulas of a formula $\delta$ in $\Gamma \Rightarrow \beta$ in the same way as the usual definition of subformulas, except for the following. Suppose that $\sharp z\psi$ is a $V$-subformula of $\delta$ where $\sharp$ is either $\forall$ or $\exists$. Then only formulas of the form $\psi[u/z]$ are $V$-subformulas of $\delta$ if $u \in E \cup V$. Roughly speaking, a $V$-subformula of $\delta$ is a subformula of $\delta$ containing only free variables in $V$ and constant symbols in $E$. Let $\Theta$ be the set of all $V$-subformulas of a formula in $\Gamma \Rightarrow \beta$. Clearly, $\Theta$ is finite.

Next we define *predecessors* of $\Gamma \Rightarrow \beta$. Predecessors are defined essentially in the same way as those in the previous section, except that we take sequents themselves for them, instead of taking elements of $Q^* \times (Q \cup \{\varepsilon\})$. We add moreover the following.

1. If $\Sigma \Rightarrow \forall z_i \theta$ is a predecessor of $\Gamma \Rightarrow \beta$ then $\Sigma \Rightarrow \theta[v_i/z_i]$ is also a predecessor of it.
2. If $\exists z_i \theta, \Sigma \Rightarrow \delta$ is a predecessor of $\Gamma \Rightarrow \beta$ then $\theta[v_i/z_i], \Sigma \Rightarrow \delta$ is also a predecessor of it.
3. If $\forall z_j \theta, \Sigma \Rightarrow \delta$ is a predecessor of $\Gamma \Rightarrow \beta$ then any sequent of the form $\theta[u/z_j], \Sigma \Rightarrow \delta$ is also a predecessor of it, where $u$ is $v_j$ or any member of $E$ or any variable $v \in V$ appearing in $\forall z_j \theta, \Sigma \Rightarrow \delta$.
4. If $\Sigma \Rightarrow \exists z_j \theta$ is a predecessor of $\Gamma \Rightarrow \beta$ then $\Sigma \Rightarrow \theta[u/z_j]$ is also a predecessor of it, where $u$ is $v_j$ or any member of $E$ or any variable $v \in V$ appearing in $\Sigma \Rightarrow \exists z_j \theta$.

Then it is easily seen that every predecessor of $\Gamma \Rightarrow \beta$ consists of formulas in $\Theta$, and that any cut-free proof of $\Gamma \Rightarrow \beta$, if it exists, can be transformed into one consisting only of predecessors (in the above sense) of $\Gamma \Rightarrow \beta$. In this way, we can show the following result due to Komori [10].

PROPOSITION 8.2. *A sequent $\Gamma \Rightarrow \beta$ is provable in $\mathbf{QFL_{ew}}$ if and only if it has a cut-free proof which consists only of predecessors of $\Gamma \Rightarrow \beta$.*

Since the system $\mathbf{QFL_{ew}}$ doesn't have contraction rule and no function symbols are contained in our language, the length of (each of) the upper sequent(s), i.e. the total number of symbols in it, is strictly smaller than the length of the lower sequent in each rule of $\mathbf{QFL_{ew}}$, except the cut rule. Combining this with the fact that the set $\Theta$ is finite, we have that the total number of predecessors of a given $\Gamma \Rightarrow \beta$ is finite. Therefore by exhaustive check of all possible proofs, we can decide whether a given sequent $\Gamma \Rightarrow \beta$ is provable in $\mathbf{QFL_{ew}}$ or not. Thus, the following result due to Komori can be shown.

PROPOSITION 8.3. *The predicate logic $\mathbf{QFL_{ew}}$ is decidable.*

To prove the finite model property, we suppose that a sequent $\Gamma \Rightarrow \beta$ is not provable in $\mathbf{QFL_{ew}}$. Depending on this sequent $\Gamma \Rightarrow \beta$, we define a Gentzen structure for $\mathbf{QFL_{ew}}$ $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ as follows. (Here, we still assume that $\Gamma \Rightarrow \beta$ satisfies conditions on variables and constants mentioned above.) First, we introduce additional new constants $d_1, \ldots, d_m$, and let $D = \{e_1, \ldots, e_k, d_1, \ldots, d_m\}$. We take $D$ as the individual domain.

Let $Q$ be the set of all closed formulas of the language $\mathcal{L}_D$ with constant symbols in $D$. Operations $\wedge, \vee, \cdot, \rightarrow$ and logical constants $0, 1$ are defined in the obvious way. Also, we suppose that both $\bigwedge\{\alpha_d : d \in D\}$ and $\bigvee\{\alpha_d : d \in D\}$ are defined only when $\alpha_d$ is of the form $\alpha[d/z]$ for each $d$, and when defined they are equal to $\forall z\alpha$ and $\exists z\alpha$, respectively.

It remains to define the relation $\preceq$. Let $\mathcal{S}$ be the set of all sequents, each of which consists only of closed formulas in $\mathcal{L}_D$. We define a subset $\mathcal{S}_0$ of $\mathcal{S}$ as follows. A sequent $\Sigma \Rightarrow \delta$ in $\mathcal{S}$ belongs to $\mathcal{S}_0$ if and only if it is a substitution instance of a predecessor $\Sigma^\dagger \Rightarrow \delta^\dagger$ of $\Gamma \Rightarrow \beta$. Note that $\mathcal{S}_0$ is finite. Now we define a subset $\preceq$ of $Q^* \times (Q \cup \{\varepsilon\})$ as follows. For each $\Sigma \in Q^*$ and each $\delta \in Q \cup \{\varepsilon\}$,

1. if $\Sigma \Rightarrow \delta$ belongs to $\mathcal{S}_0$, $\Sigma \preceq \delta$ holds if and only if $\Sigma \Rightarrow \delta$ is provable in $\mathbf{QFL_{ew}}$,
2. otherwise, $\Sigma \preceq \delta$ holds always.

It remains to show that this relation $\preceq$ satisfies conditions of Gentzen structures for $\mathbf{QFL_{ew}}$, described in Section 6. Here, we show only that the second condition holds. To show this, it is enough to consider the case where $\Sigma \preceq \alpha[d/z_i]$ for any $d \in D$ and $\Sigma \Rightarrow \forall z_i\alpha$ belongs to $\mathcal{S}_0$. For, otherwise $\Sigma \preceq \forall z_i\alpha$ holds trivially by the definition of $\preceq$. Now, by the definition of $\mathcal{S}_0$, there exists a predecessor $\Sigma^\dagger \Rightarrow (\forall z_i\alpha)^\dagger$ of $\Gamma \Rightarrow \beta$, and both $\Sigma$ and $\forall z_i\alpha$ are obtained from $\Sigma^\dagger$ and $(\forall z_i\alpha)^\dagger$, respectively, by a substitution $\sigma$ which replaces each free variable in them by a member of $D$. Note that $\alpha$ is obtained from $\alpha^\dagger$ also by this substitution $\sigma$, and also that the variable $v_i$ does not appear in $\Sigma^\dagger \Rightarrow (\forall z_i\alpha)^\dagger$. Then, we can show that $\Sigma \Rightarrow \alpha[d/z_i]$ is obtained from $\Sigma^\dagger \Rightarrow \alpha^\dagger[v_i/z_i]$ by the substitution $\sigma$ and by substituting $d$ for $v_i$ in addition. We note that $\Sigma^\dagger \Rightarrow \alpha^\dagger[v_i/z_i]$ is a predecessor of $\Gamma \Rightarrow \beta$. Thus, $\Sigma \Rightarrow \alpha[d/z_i]$ belongs to $\mathcal{S}_0$ for each $d$. By our assumption with the definition of $\preceq$, $\Sigma \Rightarrow \alpha[d/z_i]$ is provable for each $d$. Let us suppose that the substitution $\sigma$ replaces $n(\leq m)$ variables in $V$ by members of $E$. Then $n$ must be smaller than $m$ since $v_i$ is not replaced by $\sigma$. On the other hand, $D$ contains $m$ constants other than $e_1, \ldots, e_k$. Thus, at least one member $d_i \in D$ is not in the range of $\sigma$. Since $\Sigma \Rightarrow \alpha[d_i/z_i]$ is provable, there exists a proof $P$ of $\Sigma \Rightarrow \alpha[d_i/z_i]$. By replacing all occurrences of $d_i$ in $P$ by $v_i$, we have a proof of $\Sigma \Rightarrow \alpha[v_i/z_i]$. Hence, $\Sigma \Rightarrow \forall z_i\alpha$ is also provable in $\mathbf{QFL_{ew}}$. Thus, $\Sigma \preceq \forall z_i\alpha$ holds. Similarly, we can show that other three conditions hold.

In this Gentzen structure $\mathbf{Q}$, $\Sigma \preceq \delta$ holds except for finitely many pairs $(\Sigma, \delta)$, and hence the closed base determined by $\preceq$ is finite. Therefore, by

Lemma 7.1, the quasi-completion $\mathbf{C}_{\mathbf{Q}^*}$ of $\mathbf{Q}$, in which $\Gamma \Rightarrow \beta$ is not valid, is also finite. Thus, we have proved our theorem.

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