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Description	

A New Approach to Belief Modeling*

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Abstract. It has been shown that, despite the differences in approach and interpretation, all belief function based models without the so-called *dynamic component* lead essentially to mathematically equivalent theories – at least in the finite case. In this paper, we first argue that at the logical level these models seem to share a common formal framework and various interpretations just come at the epistemic level. We then introduce a framework for belief modeling formally based on Dempster's structure with adopting Smets' view of the origin of beliefs. It is shown that the proposed model is more general than previous models, and may provide a suitable unified framework for belief modeling.

Keywords: Transferable belief model, Uncertainty, Dempster–Shafer theory, Propagable belief model.

1 Introduction

Dealing with uncertainty is a fundamental and unavoidable issue in AI researches. Undoubtedly, the Bayesian approach is the most widely-used approach to dealing with uncertainty. Although the Bayesian approach is strongly supported by relying on well-established techniques from probability theory as well as some philosophical justification, it has been widely criticized in the literature. So far numerous other approaches to dealing with uncertainty have been proposed, including Dempster-Shafer theory [2, 22], the transferable belief model [24, 28], the probability of modal propositions [21], various nonstandard and fuzzy logics [16, 10, 32], and the context model [7], among others. Of particular interest to us in this paper is based on the Dempster-Shafer-Smets model[3].

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[3] This name is used in [7] to reflect Smets' "non-probabilistic" view of using belief functions (including Dempster's rule of conditioning and Dempster's rule of combination) to model someone's belief.

From a mathematical point of view, a belief function can be treated as a mathematical object satisfying a certain set of axioms. Especially, the axioms for belief functions can be viewed as a weaker form of the Kolmogorov axioms that characterize probability functions. Under such a view, a number of authors have tried to characterize a belief function as a generalized probability function [5, 6] or in terms of probability functions [2, 3, 23]. On the other hand, belief functions have been also used to model someone's belief originated back to Shafer [22]. In belief modeling using belief functions, there are various views, even contrast, of the origin of beliefs. These have resulted in so many various interpretations of Dempster-Shafer theory and, at the same time, opened to criticism [29]. This paper does not aim at being a deal of debate regarding the existing approaches to belief modeling, also not presenting another interpretation of belief functions. Our main concern is on belief modeling itself. To this end, we adopt Smets' view of the origin of beliefs in the transferable belief model, inasmuch as it is not only based on a well-established axiomatic justification, but also supported by practical basis when someone intends to model subjective, personal beliefs. For example, in a medical diagnostic situation, it is easier and realizable for You, the doctor, to give basic belief masses on subsets of symptoms that may cause the unknown disease rather than to give (subjective) probabilities on single symptoms, even though such probabilities may exist[4]. On the other hand, we are highly motivated by the fact that the notion of a multivalued mapping may be a good mathematical tool for representing human beings' *cause-and-effect* view of reality. Thus our approach is based on Dempster's structure, but according to Kohlas and Monney's view of the multivalued mapping [13].

In the next section we will briefly present necessary notions from the Dempster-Shafer theory of evidence. Some belief function based models are recalled and analyzed in Section 3. We would like to emphasize that the model introduced in this paper should not be considered as a formally generalization of previous models, even though it may be. Thus not all interpretations of belief functions are analyzed here (see [29] for the details), but only models that we have been guided by our purpose are mentioned. A full description of the model for beliefs representation can be found in [22]. Other models can be found in, e.g. [21, 20, 15] for the modal logic based interpretation; [17] for the random set based interpretation; [7, 11] for the context model. In Section 4 we introduce the so-called propagable belief model, and conditioning as belief revision with certain evidence versus the one with uncertain evidence will be analyzed via the well-known *tree prisoners problem.* Finally, some concluding remarks and further work will be presented in Section 5.

2 Dempster-Shafer theory of evidence

We recall in this section necessary notions from the Dempster-Shafer theory of evidence (DS theory, for short). The theory aims at providing a mechanism for

[4] Note that this does not exclude the possibility of using correct probabilities whenever available.

representing and reasoning with uncertain, imprecise and incomplete information. It is based on Dempster's original work [2] on the modeling of uncertainty in terms of upper and lower probabilities induced by a multivalued mapping.

A multivalued mapping Γ from space Ω into space Θ associates to each element ω of Ω a subset $\Gamma(\omega)$ of Θ. The domain of Γ, denoted by $\mathsf{Dom}(\Gamma)$, is defined by

$$\mathsf{Dom}(\Gamma) = \{\omega \in \Omega | \Gamma(\omega) \neq \emptyset\}.$$

From a multivalued mapping Γ, a probability measure P on Ω can be propagated to Θ in such a way that for any subset T of Θ the lower and upper bounds of probabilities of T are defined as

$$P_*(T) = \frac{P(\Gamma_*(T))}{P(\mathsf{Dom}(\Gamma))} \tag{1}$$

$$P^*(T) = \frac{P(\Gamma^*(T))}{P(\mathsf{Dom}(\Gamma))} \tag{2}$$

where

$$\begin{aligned} \Gamma_*(T) &= \{\omega \in \Omega | \omega \in \mathsf{Dom}(\Gamma) \wedge \Gamma(\omega) \subseteq T\}, \\ \Gamma^*(T) &= \{\omega \in \Omega | \Gamma(\omega) \cap T \neq \emptyset\}. \end{aligned}$$

Clearly, P_*, P^* are well defined only when $P(\mathsf{Dom}(\Gamma)) \neq 0$.

Remark 1. The equations (1) and (2) can be represented in the terms of conditional probabilities as follows

$$P_*(T) = P(\Gamma_*(T)|\mathsf{Dom}(\Gamma)), \quad P^*(T) = P(\Gamma^*(T)|\mathsf{Dom}(\Gamma)) \tag{3}$$

This presentation suggests us the idea of a new interpretation of conditional beliefs presented in Section 4.

Furthermore, Dempster also observed that, in the case that Θ is finite, these lower and upper probabilities are completely determined by the quantities

$$P(\{\omega \in \Omega | \Gamma(\omega) = T\}), \text{ for } T \in 2^{\Theta}.$$

As such Dempster implicitly gave the prototype of a mass function also called *basic probability assignment*. Shafer's contribution has been to explicitly define the basic probability assignment and to use it to represent evidence directly. Simultaneously, Shafer has reinterpreted Dempster's lower and upper probabilities as degrees of belief and plausibility respectively, and abandoned the idea that they arise as lower and upper bounds over classes of Bayesian probabilities [22].

Formally, the definitions of these measures are given as follows:

1. A function $bel : 2^{\Theta} \to [0, 1]$ is called a *belief measure* over Θ if
   B1. $bel(\emptyset) = 0, bel(\Theta) = 1$
   B2. For any finite family $\{A_i\}_{i=1}^n$ in 2^{Θ},

$$bel(\bigcup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} bel(\bigcap_{i \in I} A_i)$$

2. A function $pl : 2^{\Theta} \to [0,1]$ is called a *plausibility measure* if
   P1. $pl(\emptyset) = 0, pl(\Theta) = 1$
   P2. For any finite family $\{A_i\}_{i=1}^n$ in 2^{Θ},
$$pl(\bigcap_{i=1}^{n} A_i) \leq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} pl(\bigcup_{i \in I} A_i)$$

It should be noted that belief and plausibility measures form a dual pair, namely
$$pl(A) = 1 - bel(\overline{A}), \text{ for any } A \in 2^{\Theta}$$

In the case of a finite universe S, a function $m : 2^{\Theta} \to [0,1]$ is called a *basic probability assignment* if $m(\emptyset) = 0$ and
$$\sum_{A \in 2^{\Theta}} m(A) = 1$$
A subset $A \in 2^{\Theta}$ with $m(A) > 0$ is called a *focal element* of m. The difference between $m(A)$ and $bel(A)$ is that while $m(A)$ is our belief committed to the subset A excluding any of its proper subsets, $bel(A)$ is our degree of belief in A as well as all of its subsets. Consequently, $pl(A)$ represents the degree to which the evidence fails to refute A. Furthermore, the belief and plausibility measures are in an one-to-one correspondence with basic probability assignments. Namely, given a basic probability assignment m, the corresponding belief measure bel and its dual plausibility measure pl are determined by
$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$$
$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$$
Conversely, given a belief measure bel, the corresponding basic probability assignment m is determined via Möbius inversion as follows
$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} bel(B)$$

In the next section we will briefly present several various interpretations of the DS model, namely Kohlas and Monney's hint model [14], Fagin and Halpern's model [5] and Smets' transferable belief model [28]. In this paper we ourselves confine the consideration to only the finite structures.

3 Belief function based models

Since Shafer introduced the model in the seminal work "*A Mathematical Theory of Evidence*" [22], many interpretations of it have been proposed. According to Smets [29], any model for belief has at least two components: one *static* that describes our state of belief, and the other *dynamic* that explains how to update our belief given new pieces of information. It has been clear that by restricting to the static component, various models for belief, despite the differences in approach and interpretation, lead essentially to mathematically equivalent forms.

3.1 The hint model

The hint model proposed by Kohlas [12] and developed further by Kohlas and Monney [13, 14] begins with Dempster's original structure $(\Omega, P, \Gamma, \Theta)$ where Ω and Θ are two sets, P is a probability measure on Ω and Γ is a multivalued mapping from Ω into Θ.

The authors assume a certain question, whose answer is unknown. The set Θ called the *frame of discernment* is the set of possible answers to the question. One and only element of Θ is the correct answer but unknown. Ω is interpreted as the set of *possible interpretations* allowed from the light of the available information. Exactly one of the elements $\omega \in \Omega$ must be the correct interpretation, but it is unknown which one. Furthermore, the assumption that not all possible interpretations are equally likely induces the known probability measure P on Ω. In the simplest case, one can assume that if ω is the correct interpretation, then the correct answer θ must be within some nonempty subset $\Gamma(\omega)$ of Θ, the *focal set* of the interpretation. Alternately, for any possible interpretation ω, the family $\mathcal{S}(\omega)$ of the subsets of Θ (considered as propositions) **implied** by ω can be considered. $\mathcal{S}(\omega)$ called a *filter* is simply the family of supersets of the focal set $\Gamma(\omega)$ and has the following properties:

(1) $H \in \mathcal{S}(\omega)$ and $H \subseteq H'$ imply $H' \in \mathcal{S}(\omega)$.
(2) $H_1, H_2 \in \mathcal{S}(\omega)$ imply $H_1 \cap H_2 \in \mathcal{S}(\omega)$.
(3) Θ belongs to $\mathcal{S}(\omega)$, $\emptyset$ does not belong to $\mathcal{S}(\omega)$.

Furthermore, one can also look at the family $\mathcal{P}(\omega)$ of the propositions which are **possible** under ω. That is, a subset H of Θ is considered as possible if H does have a nonempty intersection with the focal set $\Gamma(\omega)$. The family $\mathcal{P}(\omega)$ has the following properties:

(1') $H \in \mathcal{P}(\omega)$ and $H \subseteq H'$ imply $H' \in \mathcal{P}(\omega)$.
(2') $H_1, H_2 \in \mathcal{P}(\omega)$ imply $H_1 \cup H_2 \in \mathcal{P}(\omega)$.
(3') Θ belongs to $\mathcal{P}(\omega)$, $\emptyset$ does not belong to $\mathcal{P}(\omega)$.

Under such an analysis, the quadruple $\mathcal{H} = (\Omega, P, \Gamma, \Theta)$ is called a **hint**.

Now if a proposition $H \subseteq \Theta$ is fixed as a hypothesis about the correct answer, then this hypothesis should be judged in the light of a hint $\mathcal{H}$. That is, one can look at the subsets of interpretations under which H is implied, $u(H)$, or possible, $v(H)$

$$\begin{aligned} u(H) &= \{\omega \in \Omega | H \in \mathcal{S}(\omega)\} \\ v(H) &= \{\omega \in \Omega | H \in \mathcal{P}(\omega)\} \end{aligned} \tag{4}$$

Then the *degree of credibility* (or *support*), denoted by $sp(H)$, and the *degree of plausibility*, denoted by $pl(H)$ are defined as follows

$$\begin{aligned} sp(H) &= P(u(H)) \\ pl(H) &= P(v(H)) \end{aligned} \tag{5}$$

As such the hint model is based on Dempster's original approach and in this model degrees of supports (or equivalently, beliefs) are deduced from a filter-valued mapping and a probability measure on the space of possible interpretations.

Remark 2. **(i)** In the hint model one may implicitly assume a propositional language $\mathcal{L}_\Theta$ that is derived from the question of concern and is semantically interpreted by the Boolean algebra of 2^Θ. The filter-valued mapping induced from Γ plays an important role in forming credibility in the light of a hint. Thus, at the logical level a hint may be seen as a quadruple $(\Omega, \Gamma, \Theta, \mathcal{L}_\Theta)$.

(ii) In our opinion, the assumption "not all possible interpretations are equally likely" is not always available in general, once it is available it should be considered as the supplemental information and then the probability measure P on Ω is added to the hint to quantify degrees of credibility in the light of the hint. Furthermore, although a probability function is assumed on Ω, the hint model does not explicitly assume there is a probability function on Θ as upper and lower probabilities model does. Thus the hint model may be considered as a logical based interpretation associated with supplemental probabilistic information of the DS model.

3.2 Fagin and Halpern's model

In [5] Fagin and Halpern introduced a new probabilistic approach to dealing with uncertainty by using the standard mathematical notions of *inner measure* and *outer measure* induced by the probability measure [8]. Interestingly, inner measures induced by probability measures turn out to correspond in a precise sense to DS belief functions. The model is interpreted as follows.

Let $\Phi = \{p_1, \ldots, p_n\}$ be a finite set of primitive propositions thought of as corresponding to basic events concerning with the situation we want to reason about. The set $\mathcal{L}(\Phi)$ of *propositional formulas* is the closure of Φ under the Boolean operations $\wedge$ and $\neg$. For convenience we assume also that there is a special formula *true*, and we abbreviate $\neg$*true* by $false$. To get mutually exclusive events, we can consider all the formulas of the form $p'_1 \wedge \ldots p'_n$ called *atoms*[5], where p'_i is either p_i or $\neg p_i$. Let At denote the set of atoms over Φ.

In Nilsson's probabilistic logic [16], a probability distribution is assumed on At. Then the *probabilistic truth value* of a formula φ can be computed by using the finite additivity property of the probability measure and the equivalent representation of the formula φ as a disjunction of atoms. This formally forms a probability space of the form $(At, 2^{At}, P)$[6] called a *Nilsson structure*. Given a Nilsson structure $N = (At, 2^{At}, P)$ and a formula φ, let $W_N(\varphi)$ denote the probabilistic truth value (or shortly, *weight*) of φ in N, which is defined to be $P(At(\varphi))$, where $At(\varphi)$ is the set of atoms whose disjunction is equivalent to φ.

Fagin and Halpern have proposed a more general approach by taking a *probability structure* as a quadruple $(S, \mathcal{X}, P, \pi)$, where $(S, \mathcal{X}, P)$ is a probability space, π associates with each $s \in S$ a truth assignment $\pi(s) : \Phi \to \{\mathbf{true}, \mathbf{false}\}$. The equation $\pi(s)(p) = \mathbf{true}$ means that p is *true at s*. The set S is thought of as consisting of the possible states of the world. We can associate with each state s

[5] The terminology by Fagin and Halpern, also called *interpretations* in the logic literature.

[6] The notation P is used here instead of μ as in [5] to denote a probability measure.

in S a unique atom describing the truth values of the primitive propositions in s. Further, there may be several states associated with the same atom. Using the usual rules of propositional logic we can easily extend $\pi(s)$ to a truth assignment on all formulas.

Given a probability structure $M = (S, \mathcal{X}, P, \pi)$, we now associate with each formula φ the set $\varphi^M = \{s \in S | \pi(s)(\varphi) = \mathbf{true}\}$ with assuming that $true^M = S$. If p^M is measurable for every primitive propositions $p \in \Phi$ then so is φ^M for every formula φ. In that case we say that M is a *measurable probability structure*. In general, we can not talk about the probabilistic truth value of a formula φ if φ^M is not measurable. In such a case, Fagin and Halpern proposed to use its inner measure and outer measure as these are defined for all subsets. Intuitively, the inner and outer measure provide lower and supper bounds on the probabilistic truth value of φ. Particularly, if φ^M is not measurable, we define $W_M(\varphi)$ to be the inner measure of φ in M as follows

$$W_M(\varphi) \stackrel{def}{=} P_*(\varphi^M) = \sup\{P(X) | X \subseteq \varphi^M, X \in \mathcal{X}\}$$

A proof given in [5] following from a more general result in [23] shows that P_* is indeed a belief measure. On the basis of the ideas above, the authors also developed a new notion of *conditional belief* which plays the same role for DS belief functions as conditional probability does for probability functions [6, 9].

It is of interest that Fagin and Halpern's model can be viewed as a special case of Dempster's structure at least in the finite case as follows.

If S is a finite set, it is easy to see that $\mathcal{X}$ has a *basis*, i.e. a family $\mathcal{B}$ of nonempty and disjoint subsets of S such that every member of $\mathcal{X}$ is a union of members of $\mathcal{B}$. Furthermore, the basis $\mathcal{B}$ forms a partition of S, say $\mathcal{B} = \{B_1, \ldots, B_k\}$. We can now associate with each B_i a so-called *situation* t_i, which may be thought of as a realization of the possible states in B_i. Let T denote the set $\{t_1, \ldots, t_k\}$. In addition we define a probability distribution P_T on T as $P_T(t_i) = P(B_i)$, and a multivalued mapping Γ from T into S by $\Gamma(t_i) = B_i$. Then it is easy to see that the Dempster structure (T, P_T, Γ, S) induces a belief function that coincides with Fagin and Halpern's proposal via the inner measure above.

3.3 Transferable belief model

The transferable belief model (TBM, for short) introduced in [24, 28] provides a model for the representation of quantified belief. This model is based on the assumption that beliefs manifest themselves at two mental levels: the *credal* level where beliefs are entertained and the *pignistic* level where beliefs are used to make decisions (from *credo*, I believe and *pignus*, a bet both in Latin). Especially, the TBM justifies the use of belief functions to model subjective, personal beliefs even in the cases where every probability concept is absent at the credal level. Once probabilities are defined everywhere the TBM is reduced to the Bayesian model [29]. The TBM is briefly described as follows.

Let $\mathcal{L}$ be a finite propositional language, and $W = \{w_1, w_2, \ldots, w_n\}$ be the set of possible worlds that correspond to the interpretations of $\mathcal{L}$. The set W is called the *frame of discernment*. Each proposition in $\mathcal{L}$ identifies a subset of W, and two propositions are logically equivalent iff they identify the same subset. Given a partition Π of W, we build the *Boolean algebra* $\mathcal{R}$ of subsets of W generated from Π. The elements of Π are called the *atoms* of $\mathcal{R}$, and the pair $(W, \mathcal{R})$ is called a *propositional space*.

Now assume that You is an ideal rational agent, and all beliefs entertained by You at time t about which world is the actual world ϖ are defined relative to a given *evidential corpus* ($\mathrm{EC}_t^{\mathrm{Y}}$). By the **Basic Assumption**, the TBM assume a *basic belief assignment* $m : \mathcal{R} \to [0,1]$ with

$$\sum_{A \in \mathcal{R}} m(A) = 1, \quad m(\emptyset) = 0.$$

For $A \in \mathcal{R}$, $m(A)$ is a part of Your belief that supports A, i.e. that the actual world ϖ is in A, and that, due to the lack of information, does not support any strict subproposition of A. The difference with probability models here is that masses can be given to any proposition of $\mathcal{R}$ instead of only to atoms of $\mathcal{R}$. In the TBM, once some further evidence becomes available to You and implies that B is true, the mass $m(A)$ initially allocated to A is transferred to $A \cap B$. This transfer of belief in the TBM satisfies the so-called *Dempster rule of conditioning* and results in $m_B : \mathcal{R} \to [0,1]$ with

$$m_B(A) = \begin{cases} c \sum\limits_{X \subseteq \overline{B}} m(A \cup X) & \text{for} \quad A \subseteq B, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$c = \frac{1}{1 - \sum\limits_{X \subseteq \overline{B}} m(X)}$$

Given a propositional space $(W, \mathcal{R})$ and a basic belief assignment m, the belief function $bel : \mathcal{R} \to [0,1]$ is defined as usual by

$$bel(A) = \sum_{\mathcal{R} \ni X \subseteq A} m(X).$$

The triple $(W, \mathcal{R}, bel)$ is then called a *credibility space*.

At this juncture we can see that, given the evidence available on a situation You want to reason about, the TBM claims the existence of a belief function that describes Your credal state on the frame of discernment. Suppose now a *decision* must be made based on this credal state. As is well known [1] that decisions will be coherent if the underlying uncertainties can be described by a probability distribution defined on 2^W. Based on the *Generalized Insufficient Reason Principle* [28], the *pignistic probability distribution* derived from bel at the pignistic state via the so-called *pignistic transformation* is defined as follows

$$BetP(x) = \sum_{x \subseteq A \in \mathcal{R}} \frac{m(A)}{|A|} = \sum_{A \in \mathcal{R}} m(A) \frac{|x \cap A|}{|A|} \tag{6}$$

where x is an atom in $\mathcal{R}$ and $|A|$ is the number of atoms of $\mathcal{R}$ in A.

Remark 3. If we denote $\mathcal{F}(x)$ the *principal filter* generated by an atom x in the Boolean algebra $\mathcal{R}$ [19], the pignistic probability distribution $BetP$ derived from m is represented as

$$BetP(x) = \sum_{A \in \mathcal{F}(x)} \frac{m(A)}{|A|} \tag{7}$$

This may show a logical relation implicitly behind the TBM and the hint model even though the primitive concepts of these two models are different. While in the hint model, the primitive concept is the hint from which degrees of supports are deduced, the TBM assume the degrees of belief as a primitive concept from which the pignistic probability function is derived. At the same time, as mentioned in [28] (page 200), the important concept in a propositional space $(W, \mathcal{R})$ is the algebra $\mathcal{R}$ (so is the partition Π), not the set of worlds W. Formally, similar as mentioned above in Fagin and Halpern's model, we can view the TBM in the terms of Dempster's structure without, however, reference to any probability concepts.

For the axiomatic justifications and more details on the TBM as well as its applications, the reader could be referred to, e.g. [4, 25, 27, 29–31].

4 The propagable belief model

In this section we introduce a model called *propagable belief model* (PBM, for short) that aims at presenting a new approach to modeling subjective, personal beliefs in the spirit of the TBM. Essentially, our model is based on Dempster's structure, except the assumption of a underlying probability distribution is not assumed. Instead of this we adopt the *basic assumption* as in the TBM.

4.1 The model

The PBM concerns the same concepts as considered by previous models that are specified as follows.

Let $W = \{w_1, w_2, \ldots, w_n\}$ be the set of possible states of the world concerning a situation we want to reason about. We call W the frame of discernment and may think of elements of W as interpretations of a underlying propositional language, or possible answers to a given question, or the like. Practically, due to the complexity of the reasoning situation and/or lack of information, the information on W may be encoded into a nonempty *finite set of possible observations* $\mathcal{O}$. Each observation Ob in $\mathcal{O}$ can cover several possible states of the world, a subset $\Gamma(Ob)$ of W. In addition, we assume that all the available information allow us to allocate *belief masses* to subsets of the set of possible observations. For $O \in 2^{\mathcal{O}}$, $m_{\mathcal{O}}(O)$ is the belief degree that supports that the true state of the world ϖ is covered in the set of observations O. That is, due to lack of information, in some cases a belief mass is only assigned in a combined view of several observations but not any strict subset of these observations. For the discussion on the origin of the basic belief masses, we could be referred to [28].

Example 1. Assume You, the detective, are dealing with a case of murder. You may determine a basic evidential structure consisting of W as the set of suspects who had potential to be the killer, $\mathcal{O}$ as the set of observed evidence in which each observed evidence supposes several suspects to be the killer, and $m_{\mathcal{O}} : 2^{\mathcal{O}} \to [0,1]$ as the basic belief assignment, where $m_{\mathcal{O}}(O)$, for $O \in 2^{\mathcal{O}}$, quantifies Your belief degree supporting that observed evidence in O constitute the murder.

Example 2. In a medical diagnostic situation, You, the doctor, may determine a basic evidential structure for diagnosis consisting of W as the set of possible diseases which the present patient may get, $\mathcal{O}$ as the set of observed symptoms from the patient in which, according to Your experience, each symptom may occur in several diseases, and $m_{\mathcal{O}} : 2^{\mathcal{O}} \to [0,1]$ as the basic belief assignment, where $m_{\mathcal{O}}(O)$, for $O \in 2^{\mathcal{O}}$, quantifies Your belief degree supporting that symptoms in O causes the unknown disease.

Formally, we define a *basic evidential structure* as a quadruple $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$, where $\mathcal{O}$ is the finite set of possible observations, $m_{\mathcal{O}}$ is an initially basic belief assignment on $2^{\mathcal{O}}$, W is the frame of discernment, and Γ is a multivalued mapping from $\mathcal{O}$ into W that associates to each element Ob in $\mathcal{O}$ a subset $\Gamma(Ob)$ of W. For any $O \in 2^{\mathsf{Dom}(\Gamma)}$ we call the set $\Gamma(O)$ to be *observable* in W, and if A is observable we denote

$$\Gamma^{-1}(A) = \{Ob \in \mathsf{Dom}(\Gamma) | \cup \Gamma(Ob) = A\}.$$

An observation $Ob \in \mathcal{O}$ is said to be *irrelevant* (resp., *relevant*) if $\Gamma(Ob) = \emptyset$ (resp., $\Gamma(Ob) \neq \emptyset$). Naturally, we do not consider any irrelevant observations in the basic evidential structure, i.e. that we assume as an assumption that every observations in the basic evidential structure is relevant. However, irrelevant observations may occur once the conditioning information from a new piece of evidence becomes available. The set of observations $\mathcal{O}$ is said to be *complete* in the basic evidential structure if

$$\bigcup_{Ob \in \mathcal{O}} \Gamma(Ob) = W,$$

and *mutually exclusive* if $\Gamma(Ob) \cap \Gamma(Ob') = \emptyset$ for any $Ob, Ob' \in \mathcal{O}$ and $Ob \neq Ob'$. Intuitively, the set of observations $\mathcal{O}$ is incomplete when the available observations do not cover completely the situation, and the true state of the world may be in $W \setminus \Gamma(\mathcal{O})$. Consequently, a positive belief mass may be assigned to $\emptyset$, i.e. $m_{\mathcal{O}}(\emptyset) > 0$ that corresponds to the so-called *open world assumption*. Hereafter we accept the *closed world assumption*, namely $m_{\mathcal{O}}(\emptyset) = 0$.

Given a basic evidential structure $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$, as our main concern is on W, the initially basic belief assignment $m_{\mathcal{O}}$ should be propagated to W in a natural way similar to the case of Dempster's approach. For any $A \in 2^W$, the set $\Gamma_*(A) = \{Ob \in \mathcal{O} | \Gamma(Ob) \subseteq A\}$[7] consists of all observations that, according to available evidence, support (imply) the proposition "ϖ is in A", and the set $\Gamma^*(A) = \{Ob \in \mathcal{O} | \Gamma(Ob) \cap A \neq \emptyset\}$ consists of all observations in which the proposition is possible. It is clearly that any nonempty subsets of $\Gamma_*(A)$ also support the proposition, whilst any subsets of $\mathcal{O}$ having a nonempty intersection with $\Gamma^*(A)$ cause the proposition possible. Thus we can define the *degree of support* and the *degree of plausibility* for A, denoted by $Sp(A)$ and $Pl(A)$, respectively, as follows

$$Sp(A) = bel_{\mathcal{O}}(\Gamma_*(A)) \tag{8}$$

$$Pl(A) = pl_{\mathcal{O}}(\Gamma^*(A)) \tag{9}$$

where $bel_{\mathcal{O}}$ and $pl_{\mathcal{O}}$ respectively are the belief function and the plausibility function defined on $2^{\mathcal{O}}$ from $m_{\mathcal{O}}$.

Remark 4.
- When the belief is probabilistic ([28], page 222), a basic evidential structure becomes a Dempster's structure. More especially, the ideal situation where $\mathcal{O}$ is finest, i.e. each observation covers exactly one possible state of the world, induces a probability model.
- If the set of observations $\mathcal{O}$ in the structure $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ is complete and mutually exclusive, the model is reduced to the TBM as shown below.
- If the set of observations $\mathcal{O}$ is complete and mutually exclusive and $bel_{\mathcal{O}}$ is a probability function, then observable subsets in W become measurable events and the PBM without the dynamic component is equivalent to Fagin and Halpern's inner and outer measures model.

Interestingly enough, we have the following theorem.

Theorem 1. *Let $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ be a basic evidential structure. Then we have $Sp : 2^W \to [0, 1]$ with $Sp(A) = bel_{\mathcal{O}}(\Gamma_*(A))$ is a belief function.*

Proof. Clearly Sp satisfies B1, i.e. $Sp(\emptyset) = 0$ and $Sp(W) = 1$, so it suffices to show that it satisfies B2. Given subsets $A_1, A_2, \ldots, A_n \in 2^W$, we now show that

$$Sp(\bigcup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Sp(\bigcap_{i \in I} A_i).$$

Indeed, by definition we have

$$\Gamma_*(\bigcup_{i=1}^{n} A_i) \supseteq \bigcup_{i=1}^{n} \Gamma_*(A_i)$$

[7] Note that, by assumption, $\mathsf{Dom}(\Gamma) = \mathcal{O}$.

and, for any $\emptyset \neq I \subseteq \{1, \ldots, n\}$,

$$\Gamma_*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \Gamma_*(A_i).$$

These follow that

$$\begin{aligned} Sp(\bigcup_{i=1}^{n} A_i) = bel_{\mathcal{O}}(\Gamma_*(\bigcup_{i=1}^{n} A_i)) &\geq bel_{\mathcal{O}}(\bigcup_{i=1}^{n} \Gamma_*(A_i)) \\ &\geq \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|+1} bel_{\mathcal{O}}(\bigcap_{i\in I} \Gamma_*(A_i)) \\ &= \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|+1} bel_{\mathcal{O}}(\Gamma_*(\bigcap_{i\in I} A_i)) \\ &= \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|+1} Sp(\bigcap_{i\in I} A_i) \end{aligned}$$

This proves the theorem.

Note that Sp and Pl also form a dual pair, i.e. $Sp(A) = 1 - Pl(\overline{A})$ for any $A \in 2^W$. Thus Pl is a plausibility function.

The duality shows an one-to-one correspondence between these two functions. The plausibility function is just another way of presenting the same information as the support function doing and so could be forgotten.

4.2 Conditioning as belief revision

Now assume that some further evidence becomes available and implies that $B \in 2^W$ is surely true. In the PBM, an observation Ob in $\mathcal{O}$ such that $\Gamma(Ob) \cap B = \emptyset$ becomes irrelevant in the light of new evidence. More particularly, the conditioning on B means that the mapping $\Gamma : \mathcal{O} \to 2^W$ has been transformed into the mapping $\Gamma_B : \mathcal{O} \to 2^W$ with $\Gamma_B(Ob) = \Gamma(Ob) \cap B$[8]. As B is surely true in the light of new evidence, Your evidential corpus ($\mathrm{EC}_t^{\mathrm{Y}}$) must be revised according to B. Thus the new evidence should be propagated back to $2^{\mathcal{O}}$ and results in, following Smets' proposal, the mass $m_{\mathcal{O}}(O)$ initially allocated to O is then transferred to $O \cap \Gamma^*(B)$. Clearly, $\Gamma^*(B) = \mathsf{Dom}(\Gamma_B)$. The initially basic belief assignment $m_{\mathcal{O}}$ is transformed into $m_{\mathcal{O}}(\cdot|\Gamma^*(B)) : 2^{\mathcal{O}} \to [0, 1]$ with

$$m_{\mathcal{O}}(O|\Gamma^*(B)) = \begin{cases} c \sum\limits_{X\subseteq\overline{\Gamma^*(B)}} m_{\mathcal{O}}(O \cup X) & \text{for } \quad O \subseteq \Gamma^*(B), \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where

$$c = \frac{1}{1 - \sum\limits_{X\subseteq\overline{\Gamma^*(B)}} m_{\mathcal{O}}(X)}.$$

The rule of conditioning is expressed in terms of the belief function $bel_{\mathcal{O}}$ as follows

$$bel_{\mathcal{O}}(O|\Gamma^*(B)) = \frac{bel_{\mathcal{O}}(O \cup \overline{\Gamma^*(B)}) - bel_{\mathcal{O}}(\overline{\Gamma^*(B)})}{1 - bel_{\mathcal{O}}(\overline{\Gamma^*(B)})}$$

[8] This goes back to Dempster [2].

On the other hand, the conditioning on B with respect to Sp yields, according to Dempster's rule of conditioning, the following

$$Sp(A|B) = \frac{Sp(A \cup \overline{B}) - Sp(\overline{B})}{1 - Sp(\overline{B})} \tag{11}$$

The following theorem shows that the propagation of conditioning is consistent with the transfer of beliefs. Hence the name PBM.

Theorem 2. *Let $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ be a basic evidential structure. Then the rule of conditioning as belief revision above is consistent with the transfer of beliefs.*

Proof. Given $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ and a new piece of evidence that implies B is surely true. Then by the rule of conditioning as belief revision we have

$$\begin{aligned} Sp_B(A) &= bel_{\mathcal{O}}(\Gamma_{B*}(A)|\Gamma^*(B)) \\ &= \tfrac{bel_{\mathcal{O}}(\Gamma_{B*}(A) \cup \overline{\Gamma^*(B)}) - bel_{\mathcal{O}}(\overline{\Gamma^*(B)})}{1 - bel_{\mathcal{O}}(\overline{\Gamma^*(B)})} \end{aligned} \tag{12}$$

On the other hand, it follows by definition that

$$\begin{aligned} Sp(\overline{B}) &= bel_{\mathcal{O}}(\Gamma_*(\overline{B})) \\ &= bel_{\mathcal{O}}(\overline{\Gamma^*(B)}) \end{aligned} \tag{13}$$

Furthermore, it is easy to check the following holds

$$\Gamma_*(A \cup \overline{B}) = \Gamma_{B*}(A) \cup \overline{\Gamma^*(B)}$$

That immediately implies

$$Sp(A \cup \overline{B}) = bel_{\mathcal{O}}(\Gamma_{B*}(A) \cup \overline{\Gamma^*(B)}) \tag{14}$$

The equations (13) and (14) imply that $Sp_B(A) = Sp(A|B)$ (review (11) and (12)). In terms of basic belief assignments, we obtain the following schema

$$\begin{array}{lcl} m_{\mathcal{O}} \xrightarrow{\text{propagation}} m(Sp) & \xrightarrow{\text{transfer}} & m_B(Sp_B) \\ \downarrow \text{transfer} & & \parallel \\ m_{\mathcal{O}}(\cdot|\Gamma^*(B)) & \xrightarrow{\text{propagation}} & m(\cdot|B)(Sp(\cdot|B)) \end{array}$$

This concludes the proof.

Remark 5. The PBM is reduced to the TBM when once the set of observations $\mathcal{O}$ in a basic evidential structure $(\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ is complete and mutually exclusive. Indeed, since $\mathcal{O}$ is complete and mutually exclusive, i.e. $\{\Gamma(Ob)|Ob \in \mathcal{O}\}$ forms a partition of W, hence the set of observable subsets in W forms a Boolean algebras that is isomorphic to $2^{\mathcal{O}}$. Thus, it is legitimate to define $m : 2^W \longrightarrow [0,1]$ as follows

$$m(A) = \begin{cases} m_{\mathcal{O}}(\Gamma^{-1}(A)) & \text{if } A \text{ is observable,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if A is observable then $\Gamma^{-1}(A) = \Gamma^*(A) = \Gamma_*(A)$. Consequently, $(W, \mathcal{R}, Sp)$ is a credibility space in the sense of Smets and Kennes [28], where $\mathcal{R}$ is the Boolean algebra of the observable subsets of W generated by $\Gamma(\mathcal{O})$.

Remark 6. In the case where focal elements of $m_{\mathcal{O}}$ are exactly singletons, i.e. that $bel_{\mathcal{O}}$ is a probability function, a basic evidential structure becomes a Dempster's structure. Then Dempster considered that the conditioning on $B \subseteq W$ means the transformation of Γ into Γ_B, and also postulated that the knowledge of the conditioning event B does not modify $bel_{\mathcal{O}}$. This opened to criticism [29]. Surprisingly, whilst Dempster defined the lower probability of an event A in W as the conditional probability of $\Gamma_*(A)$ given the set of only relevant observations $\mathsf{Dom}(\Gamma)$ in $\mathcal{O}$ (review (3)), he did not take the idea into account once the conditioning information becomes available.

4.3 Refinements and the three prisoners problem

Let us consider two basic evidential structures $BE_1 = (\mathcal{O}_1, m_{\mathcal{O}_1}, W, \Gamma_1)$ and $BE = (\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$ on the same frame of discernment W. We call BE_1 is a *refinement* of BE if there is a surjection $f : \mathcal{O}_1 \rightarrow \mathcal{O}$ such that $\Gamma = \Gamma_1 \circ f^{-1}$ and $bel_{\mathcal{O}}(O) = bel_{\mathcal{O}_1}(f^{-1}(O))$, for any $O \in 2^{\mathcal{O}}$. An illustrated example is depicted as below.

We would like to close this section by analyzing the well-known *three prisoners problem*, that is one of the most quoted examples concerning the applicability of Dempster's rule of conditioning, e.g. [18, 7]. The problem is stated as follows[9].

> Let a, b and c be three prisoners. Two of the prisoners are chosen by the warden to be executed but a does not know which. He therefore says to the jailer: "Since either b or c is certainly going to be executed, you will give me no information about my own chances if you give me the name of one man, either b or c, who is going to be executed." Accepting this argument, the jailer truthfully replies: "b will be executed." Thereupon a feels happier because before the jailer replied, his own chance of execution was two-thirds, but afterwards there are only two people, himself and c, who could be the one not executed, and so his chance of execution is one-half.

Is the prisoner a justified in believing that his chance of escaping has improved?

Before analyzing the problem in terms of a basic evidential structure. We note that, as discussed in [6], in order for a to believe that his own chance of execution was two-thirds before the jailer replied, he seems to be implicitly assuming that the one to get pardoned is chosen at random from among a, b and c. This assumption means that each prisoner would be randomly selected with probability $\frac{1}{3}$ to be pardoned. Further, following [6] we model a possible state by a pair (x, y), where $x, y \in \{a, b, c\}$, that represents a state where x is pardoned and the jailer replies that y will be executed to a's question. Since the jailer answers truthfully and will never tell a directly that a will be executed, we have the set of possible states is $W = \{(a, b), (a, c), (b, c), (c, b)\}$.

[9] This description of the story is taken from [6] and our discussion is based on that of Fagin and Halpern [6] and Smets [26]

We now construct a basic evidential structure for a before getting the answer from the jailer as $BE = (\mathcal{O}, m_{\mathcal{O}}, W, \Gamma)$, where $\mathcal{O} = \{Ob_a, Ob_b, Ob_c\}$ with Ob_x corresponds to "x is pardoned", $m_{\mathcal{O}}(Ob_x) = \frac{1}{3}$, for every $x \in \{a, b, c\}$, and $\Gamma(Ob_a) = \{(a, b), (a, c)\}$, $\Gamma(Ob_b) = \{(b, c)\}$, $\Gamma(Ob_c) = \{(c, b)\}$. Let us denote *says-b* the event $\{(a, b), (c, b)\}$ corresponding to the jailer's answer. Then two situations could be arisen when the jailer gave the answer to a's question [26].

Context 1. a has learnt that the jailer's answer is surely true (e.g., the jailer saw the result of the selection from the judge), i.e. that *says-b* is surely true. Then a should revise his belief by conditioning on *says-b* from BE, that results in

$$Sp_{says\text{-}b}(\{(a, b), (a, c)\}) = \frac{1}{2}.$$

Hence he feels happier realistically.

Context 2. a has learnt that the jailer chooses at random between saying b and c if a is pardoned. This is because the jailer would like to satisfy the prisoner a while making sure that the answer does not change a's belief about his chance of saving. Then a has been just updated a piece of uncertain information that "the probability that jailer chooses at saying b will be executed is $\frac{1}{2}$". This uncertain information helps a just refining his basic evidential structure, say $BE' = (\mathcal{O}', m_{\mathcal{O}'}, W, \Gamma')$, that is a refinement of BE, where $\mathcal{O}' = \{Ob_{ab}, Ob_{ac}, Ob_b, Ob_c\}$ with Ob_{ax} corresponds to "a is pardoned and the jailer says x", $m_{\mathcal{O}'}(Ob_{ax}) = \frac{1}{6}$, $m_{\mathcal{O}'}(Ob_x) = \frac{1}{3}$ for $x \in \{b, c\}$, and $\Gamma'(Ob_{ab}) = \{(a, b)\}$, $\Gamma'(Ob_{ac}) = \{(a, c)\}$, $\Gamma'(Ob_b) = \{(b, c)\}$, $\Gamma'(Ob_c) = \{(c, b)\}$. This yields a probability model, and then one gets

$$Sp_{says\text{-}b}(\{(a, b), (a, c)\}) = \frac{Sp(\{(a, b)\})}{Sp(\{(a, b), (c, b)\})} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

5 Conclusions

In this paper we have proposed a new approach to belief modeling based on Smets' view of the origin of beliefs and the notion of a multivalued mapping. Interestingly enough, the model also induces a belief function that quantifies our degrees of support in subsets of the frame of discernment given a basic evidential structure. Furthermore, it has been shown that the propagation of conditioning in the model is consistent with the transfer of beliefs. As we have mentioned in Remarks 4–5, the approach proposed in this paper has provided a generalization of a number of existing models. This may allow us to understand their commonalities and differences, and to facilitate the formal comparison of these models. We do hope that this will also support a better understanding of existing models of beliefs and serve as a bridge of the gap between well-known approaches. More details on the model as well as the problem on the combination of evidence in the model will be presented in a forthcoming paper.

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