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Description	

Assessment Aggregation in the Evidential Reasoning Approach to MADM under Uncertainty: Orthogonal Sum vs. Weighted Sum

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Abstract. In this paper, we revisit the evidential reasoning (ER) approach to multiple-attribute decision making (MADM) with uncertainty. The attribute aggregation problem in MADM under uncertainty is generally formulated as a problem of evidence combination. Then several new aggregation schemes are proposed and simultaneously their theoretical features are explored. A numerical example traditionally examined in published sources on the ER approach is used to illuminate the proposed techniques.

1 Introduction

So far, many attempts have been made to integrate techniques from artificial intelligence (AI) and operational research (OR) for handling uncertain information, e.g., [1, 4, 5, 8, 9, 11, 19]. During the last decade, an evidential reasoning (ER) approach has been proposed and developed for MADM under uncertainty in [20, 21, 23–25]. Essentially, this approach is based on an evaluation analysis model [26] and the evidence combination rule of the Dempster-Shafer (D-S) theory [14]. The ER approach has been applied to a range of MADM problems in engineering and management, including motorcycle assessment [21], general cargo ship design [13], system safety analysis and synthesis [17], retro-fit ferry design [22] among others.

Recently, due to a need of developing theoretically sound methods and tools for dealing with MADM problems under uncertainty, Yang and Xu [25] have proposed a system of four synthesis axioms within the ER assessment framework with which a rational aggregation process needs to satisfy. It has also been shown that the original ER algorithm only satisfies these axioms approximately. At the same time, guided by the aim exactly, the authors have proposed a new ER algorithm that satisfies all the synthesis axioms precisely.

It is worth emphasizing that the underlying basis of using Dempster's rule of combination is the independent assumption of information sources to be combined. However, in situations of multiple attribute assessment based on a multi-level structure of attributes, assumptions regarding the independence of attributes' uncertain evaluations may not be appropriate in general. In this paper, we reanalysis the previous ER approach in terms of D-S theory so that the attribute aggregation problem in MADM under uncertainty can be generally formulated as a problem of evidence combination. Then we propose a new aggregation scheme and simultaneously examine its theoretical features. For the purpose of the present paper, we take only qualitative attributes of an MADM problem with uncertainty into account, though quantitave attributes would be also included in a similar way as considered in [20, 21].

2 Background

2.1 Problem Description

This subsection describes an MADM problem with uncertainty through a tutorial example taken from [25].

Let us consider a problem of motorcycle evaluation [6]. To evaluate the quality of the *operation* of a motorcycle, the following set of distinct evaluation grades is defined

$$\mathcal{H} = \{poor\ (H_1), indifferent\ (H_2), average\ (H_3), good\ (H_4), excellent\ (H_5)\} \quad (1)$$

Because *operation* is a general technical concept and is not easy to evaluate directly, it needs to be decomposed into detailed concepts such as *handling*, *transmission*, and *brakes*. Again, if a detailed concept is still too general to assess directly, it may be further decomposed into more detailed concepts. For example, the concept of *brakes* is measured by *stopping power*, *braking stability*, and *feel at control*, which can probably be directly evaluated by an expert and therefore referred to as basic attributes (or basic factors).

Generally, a qualitative attribute y may be evaluated through a hierarchical structure of its subattributes. For instance, the hierarchy for evaluation of the *operation* of a motorcycle is depicted as in Fig. 1.

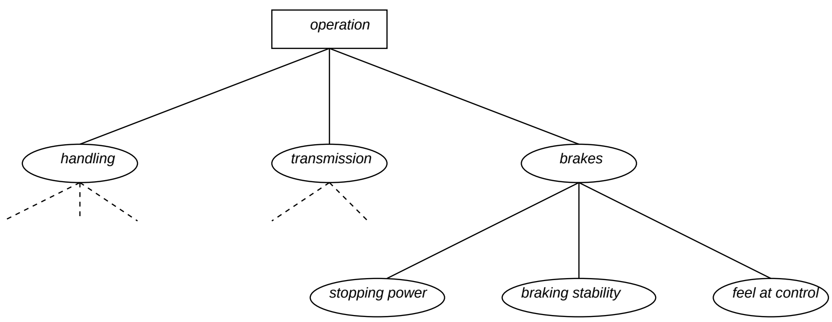


Fig. 1. Evaluation hierarchy for *operation*

In evaluation of qualitative attributes, judgments could be uncertain. For example, in the problem of evaluating different types of motorcycles, the following type of uncertain subjective judgments for the *brakes* of a motorcycle, say "Yamaha", was frequently used [6, 25]:

1. Its *stopping power* is *average* with a confidence degree of 0.3 and it is *good* with a confidence degree of 0.6.
2. Its *braking stability* is *good* with a confidence degree of 1.
3. Its *feel at control* is evaluated to be *good* with a confidence degree of 0.5 and to be *excellent* with a confidence degree of 0.5.

In the above statements, the confidence degrees represent the uncertainty in the evaluation. Note that the total confidence degree in each statement may be smaller than 1 as the case of the first statement. This may be due to incomplete of available information.

In a similar fashion, all basic attributes in question could be evaluated. Then the problem is to generate an overall assessment of the *operation* of a motorcycle by aggregating all uncertain judgments of its basic attributes in a rational way.

2.2 Evaluation Analysis Model

The evaluation analysis model was proposed in [26] to represent uncertain subjective judgments, such as statements specified in preceding subsection, in a hierarchical structure of attributes.

To begin with, let us suppose a simple hierarchical structure consisting of two levels with a general attribute, denoted by y, at the top level and a finite set E of its basic attributes at the bottom level. Let $E = \{e_1, \dots, e_i, \dots, e_L\}$ and assume the weights of basic attributes are given by $W = (w_1, \dots, w_i, \dots, w_L)$, where w_i is the relative weight of the ith basic attribute (e_i) with $0 \le w_i \le 1$.

Given the following set of evaluation grades

$$\mathcal{H} = \{H_1, \dots, H_n, \dots, H_N\}$$

designed as distinct standards for assessing an attribute, then an assessment for e_i of an alternative can be mathematically represented in terms of the following distribution [25]

$$S(e_i) = \{(H_n, \beta_{n,i}) \mid n = 1, \dots, N\}, \text{ for } i = 1, \dots, L \tag{2}$$

where $\beta_{n,i}$ denotes a degree of belief satisfying $\beta_{n,i} \ge 0$, and $\sum_{n=1}^{N} \beta_{n,i} \le 1$. An assessment $S(e_i)$ is called *complete* (respectively, *incomplete*) if $\sum_{n=1}^{N} \beta_{n,i} = 1$ (respectively, $\sum_{n=1}^{N} \beta_{n,i} < 1$).

For example, the three assessments 1.–3. given in preceding subsection can be represented in the form of distributions defined by (2) as

$$\begin{aligned} S(\textit{stopping power}) &= \{(H_3, 0.3), (H_4, 0.6)\} \\ S(\textit{braking stability}) &= \{(H_4, 1)\} \\ S(\textit{feel at control}) &= \{(H_4, 0.5), (H_5, 0.5)\} \end{aligned}$$

where only grades with nonzero degrees of belief are listed in the distributions.

Let us denote β_n the degree of belief to which the general attribute y is assessed to the evaluation grade of H_n. The problem now is to how to generate β_n, for $n = 1, \ldots, N$, by combinating the assessments for all associated basic attributes e_i $(i = 1, \ldots, L)$ as given in (2). However, before continuing the discussion, it is necessary to briefly review the basis of D-S theory of evidence in the next subsection.

2.3 Dempster-Shafer Theory of Evidence

In D-S theory, a problem domain is represented by a finite set Θ of mutually exclusive and exhaustive hypotheses, called *frame of discernment* [14]. Formally, a basic probability assignment (BPA, for short) is a function $m : 2^\Theta \to [0,1]$ verifying

$$m(\emptyset) = 0, \text{ and } \sum_{A \in 2^\Theta} m(A) = 1$$

The quantity $m(A)$ can be interpreted as a measure of the belief that is committed exactly to A, given the available evidence. A subset $A \in 2^\Theta$ with $m(A) > 0$ is called a *focal element* of m.

Two useful operations that play a central role in the manipulation of belief functions are *discounting* and *Dempster's rule of combination* [14]. The discounting operation is used when a source of information provides a BPA m, but one knows that this source has probability α of reliable. Then m is discounted by a factor of $(1-\alpha)$, resulting in a new BPA m^α defined by

$$m^\alpha(A) = \alpha m(A), \quad \text{for any } A \subset \Theta \tag{3}$$

$$m^\alpha(\Theta) = (1-\alpha) + \alpha m(\Theta) \tag{4}$$

Consider now two pieces of evidence on the same frame Θ represented by two BPAs m_1 and m_2. Dempster's rule of combination is then used to generate a new BPA, denoted by $(m_1 \oplus m_2)$ (also called the orthogonal sum of m_1 and m_2), defined as follows

$$(m_1 \oplus m_2)(\emptyset) = 0, (m_1 \oplus m_2)(A) = \frac{1}{K} \sum_{B,C \subseteq \Theta : B \cap C = A} m_1(B) m_2(C) \tag{5}$$

where

$$K = 1 - \sum_{B,C \subseteq \Theta : B \cap C = \emptyset} m_1(B) m_2(C) \tag{6}$$

Note that the orthogonal sum combination is only applicable to such two BPAs that verify the condition $K > 0$.

As we will see in the following sections, these two operation essentially play an important role in the ER approach to MADM under uncertainty developed in, e.g., [20, 21, 25]. Although the discounting operation has not been mentioned explicitly in these published sources.

3 The Evidential Reasoning Approach

Let us return to the two-level hierarchical structure with a general attribute y at the top level and a finite set $E = \{e_1, \ldots, e_i, \ldots, e_L\}$ of its basic attributes at the bottom level. Denote β_n the degree of belief to which the general attribute y is assessed to the evaluation grade of H_n, for $n = 1, \ldots, N$.

3.1 The Original ER Algorithm

The original ER algorithm proposed in [20] has been used for the purpose of obtaining β_n $(n = 1, \ldots, N)$ by aggregating the assessments of basic attributes given in (2). The summary of the algorithm in this subsection is taken from [25].

Given the assessment $S(e_i)$ of a basic attribute e_i $(i = 1, \ldots, L)$, let $m_{n,i}$ be a basic probability mass representing the belief degree to which the basic attribute e_i supports the hypothesis that the attritute y is assessed to the evaluation grade H_n. Let $m_{\mathcal{H},i}$ be the remaining probability mass unassigned to any individual grade after all the N grades have been considered for assessing the general attribute y as far as e_i is concerned. These quantities are defined as follows

$$m_{n,i} = w_i \beta_{n,i}, \text{ for } n = 1, \ldots, N \tag{7}$$

$$m_{\mathcal{H},i} = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - w_i \sum_{n=1}^{N} \beta_{n,i} \tag{8}$$

Let $E_{I(i)} = \{e_1, \ldots, e_i\}$ be the subset of first i basic attributes. Let $m_{n,I(i)}$ be a probability mass defined as the belief degree to which all the basic attributes in $E_{I(i)}$ supports the hypothesis that y is assessed to H_n. Let $m_{\mathcal{H},I(i)}$ be the remaining probability mass unassigned to individual grades after all the basic attributes in $E_{I(i)}$ have been assessed. The quantities $m_{n,I(i)}$ and $m_{\mathcal{H},I(i)}$ can be generated by combining the basic probability masses $m_{n,j}$ and $m_{\mathcal{H},j}$ for all $n = 1, \ldots, N$, and $j = 1, \ldots, i$.

With these notations, the key step in the original ER algorithm is to inductively calculate $m_{n,I(i+1)}$ and $m_{\mathcal{H},I(i+1)}$ as follows

$$m_{n,I(i+1)} = K_{I(i+1)}(m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{\mathcal{H},i+1} + m_{\mathcal{H},I(i)} m_{n,i+1}) \tag{9}$$

$$m_{\mathcal{H},I(i+1)} = K_{I(i+1)}(m_{\mathcal{H},I(i)} m_{\mathcal{H},i+1}) \tag{10}$$

for $n = 1, \ldots, N$, $i = 1, \ldots, L-1$, and $K_{I(i+1)}$ is a normalizing factor defined by

$$K_{I(i+1)} = \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)} m_{j,i+1} \right]^{-1} \tag{11}$$

Then we obtain

$$\begin{aligned} \beta_n &= m_{n,I(L)}, \text{ for } n = 1, \ldots, N \\ \beta_{\mathcal{H}} &= m_{\mathcal{H},I(L)} = 1 - \sum_{n=1}^{N} \beta_n \end{aligned} \tag{12}$$

3.2 Synthesis Axioms and the Modified ER Algorithm

Inclined to developing theoretically sound methods and tools for dealing with MADM problems under uncertainty, Yang and Xu [25] have recently proposed a system of four synthesis axioms with which a rational aggregation process needs to satisfy. These axioms are symbolically stated as below.

Axiom 1. (*Independency*) If $\beta_{n,i} = 0$ for all $i = 1, \ldots, L$, then $\beta_n = 0$.
Axiom 2. (*Consensus*) If $\beta_{k,i} = 1$ and $\beta_{n,i} = 0$, for all $i = 1, \ldots, L$, and $n = 1, \ldots, N$, $n \neq k$, then $\beta_k = 1$, $\beta_n = 0$, for $n = 1, \ldots, N$, $n \neq k$.
Axiom 3. (*Completeness*) Assume $\mathcal{H}^+ \subset \mathcal{H}$ and denote $I^+ = \{n | h_n \in \mathcal{H}^+\}$. If $\sum_{n \in I^+} \beta_{n,i}(>0) = 1$, for all $i = 1, \ldots, L$, then $\sum_{n \in I^+} \beta_n(>0) = 1$ as well.
Axiom 4. (*Incompleteness*) If there exists $i \in \{1, \ldots, L\}$ such that $\sum_{n=1}^{N} \beta_{n,i} < 1$, then $\sum_{n=1}^{N} \beta_n < 1$.

It is easily seen from (9–12) that the original ER algorithm naturally follows the independency axiom. However, it has been shown in [25] that the original ER algorithm only satisfies the consensus axiom approximately, and does not satisfy the completeness axiom.

In [25], Yang and Xu proposed a new ER algorithm that satisfies all the synthesis axioms. Its main features are summarized as follows

1) *Weight normalization.* In the new ER algorithm, the weights w_i ($i = 1, \ldots, L$) of basic attributes are normalized such that: $0 \leq w_i \leq 1$ and $\sum_{i=1}^{L} w_i = 1$.
2) *Aggregation process.* First, the probability mass $m_{\mathcal{H},i}$ given in (8) is decomposed into two parts: $m_{\mathcal{H},i} = \tilde{m}_{\mathcal{H},i} + \overline{m}_{\mathcal{H},i}$, where

$$\overline{m}_{\mathcal{H},i} = 1 - w_i, \text{ and } \tilde{m}_{\mathcal{H},i} = w_i \left(1 - \sum_{n=1}^{N} \beta_{n,i} \right) \quad (13)$$

Then, with the notations as in preceding section, the process of aggregating the first i assessments with the $(i+1)$th assessment is recursively carried out as follows

$$m_{n,I(i+1)} = K_{I(i+1)}[m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{\mathcal{H},i+1} + m_{\mathcal{H},I(i)} m_{n,i+1}] \quad (14)$$
$$m_{\mathcal{H},I(i)} = \tilde{m}_{\mathcal{H},I(i)} + \overline{m}_{\mathcal{H},I(i)}, \ n = 1, \ldots, N$$
$$\tilde{m}_{\mathcal{H},I(i+1)} = K_{I(i+1)}[\tilde{m}_{\mathcal{H},I(i)} \tilde{m}_{\mathcal{H},i+1} + \overline{m}_{\mathcal{H},I(i)} \tilde{m}_{\mathcal{H},i+1} + \tilde{m}_{\mathcal{H},I(i)} \overline{m}_{\mathcal{H},i+1}] \quad (15)$$
$$\overline{m}_{\mathcal{H},I(i+1)} = K_{I(i+1)}[\overline{m}_{\mathcal{H},I(i)} + \overline{m}_{\mathcal{H},i+1}] \quad (16)$$

where $K_{I(i+1)}$ is defined as same as in (11).
For assigning the assessment $S(y)$ for the general attribute y, after all L assessments of basic attributes have been aggregated, the algorithm finally

defines

$$\beta_n = \frac{m_{n,I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}}, \text{ for } n = 1, \dots, N \tag{17}$$

$$\beta_{\mathcal{H}} = \frac{\tilde{m}_{\mathcal{H},I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}} \tag{18}$$

and then

$$S(y) = \{(H_n, \beta_n), n = 1, \dots, N\} \tag{19}$$

The following theorems are due to Yang and Xu [25] that are taken for granted to develop the new ER algorithm above.

Theorem 1. *The degrees of belief defined by (17) and (18) satisfy the following*

$$0 \leq \beta_n, \beta_{\mathcal{H}} \leq 1, n = 1, \dots, N$$
$$\sum_{n=1}^{N} \beta_n + \beta_{\mathcal{H}} = 1$$

Theorem 2. *The aggregated assessment for y defined by (19) exactly satisfies all four synthesis axioms.*

Although proofs of these theorems given in [25] are somehow complicated, however, by analysing the ER approach in terms of D-S theory in the next section, we show that these theorems follow quite simply.

4 A Reanalysis of the ER Approach

Let us remind ourselves the available information given to an assessment problem in the two-level hierarchical structure:

- the assessments $S(e_i)$ for basic attributes e_i $(i = 1, \dots, L)$, and
- the weights w_i of the basic attributes e_i $(i = 1, \dots, L)$.

Given the assessment $S(e_i)$ of a basic attribute e_i $(i = 1, \dots, L)$, we now defines a corresponding BPA, denoted by m_i, which quantifies the belief about the performance of e_i as follows

$$m_i(H_n) \triangleq \beta_{n,i}, \text{ for } n = 1, \dots, N \tag{20}$$

$$m_i(\mathcal{H}) \triangleq 1 - \sum_{n=1}^{N} m_i(H_n) = 1 - \sum_{n=1}^{N} \beta_{n,i} \tag{21}$$

The quantity $m_i(H_n)$ represents the belief degree that supports for the hypothesis that e_i is assessed to the evaluation grade H_n. While $m_i(\mathcal{H})$ is the remaining probability mass unassigned to any individual grade after all evaluation grades have been considered for assessing e_i. If $S(e_i)$ is a complete assessment, m_i is a probability distribution. Otherwise, m_i quantifies the ignorance resulted in $m_i(\mathcal{H}) > 0$.

As such with L basic attributes e_i, we obtain L corresponding BPAs m_i as quantified beliefs of the assessments for basic attributes. The problem now is how to generate an assessment for y, i.e. $S(y)$, represented by a BPA m from m_i and w_i $(i = 1, \ldots, L)$. Formally, we aim at obtaining the BPA m that combines all m_i's with taking weights w_i's into account in the form of the following

$$m = \bigoplus_{i=1}^{L} (w_i \otimes m_i) \tag{22}$$

where $\otimes$ is a product-type operation and $\oplus$ is a sum-type operation in general.

Under such a reformulation, we may have different schemes for obtaining the BPA m represented the generated assessment $S(y)$.

4.1 The Discounting-and-Orthogonal Sum Scheme

Let us first consider $\otimes$ as the discounting operation and $\oplus$ as the orthogonal sum in D-S theory. Then, for each $i = 1, \ldots, L$, we have $(w_i \otimes m_i)$ is a BPA (refer to (3–4)) defined by

$$(w_i \otimes m_i)(H_n) \stackrel{\triangle}{=} m_i^{w_i}(H_n) = w_i m_i(H_n) = w_i \beta_{n,i}, \text{ for } i = 1, \ldots, L \tag{23}$$

$$\begin{aligned}(w_i \otimes m_i)(\mathcal{H}) &\stackrel{\triangle}{=} m_i^{w_i}(\mathcal{H}) = (1 - w_i) + w_i m_i(\mathcal{H}) \\ &= (1 - w_i) + w_i(1 - \sum_{n=1}^{N} \beta_{n,i}) = 1 + w_i \sum_{n=1}^{N} \beta_{n,i}\end{aligned} \tag{24}$$

With this formulation, we consider each m_i as the belief quatified from the information source $S(e_i)$ and the weight w_i as the "probability" of $S(e_i)$ supporting the assessment of y.

Now Dempster's rule of combination allows us to combine BPAs $m_i^{w_i}$ $(i = 1, \ldots, L)$ under the independent assumption of information sources for generating the BPA m for the assessment of y. Namely,

$$m = \bigoplus_{i=1}^{L} m_i^{w_i} \tag{25}$$

where, with an abuse of the notation, $\oplus$ stands for the orthogonal sum.

It would be worth noting that two BPAs $m_i^{w_i}$ and $m_j^{w_j}$ are combinable, i.e. $(m_i^{w_i} \oplus m_j^{w_j})$ does exist, if and only if

$$\sum_{t=1}^{N} \sum_{\substack{n=1 \\ n \neq t}}^{N} m_i^{w_i}(H_n) m_j^{w_j}(H_t) < 1$$

For example, assume that we have two basic attributes e_1 and e_2 with

$$S(e_1) = \{(H_1, 0), (H_2, 0), (H_3, 0), (H_4, 1), (H_5, 0)\}$$
$$S(e_2) = \{(H_1, 0), (H_2, 0), (H_3, 1), (H_4, 0), (H_5, 0)\}$$

and both are equally important, or $w_1 = w_2$. If the weights w_1 and w_2 are normalized so that $w_1 = w_2 = 1$, then $(m_1^{w_1} \oplus m_2^{w_2})$ does not exist.

Note further that, by definition, focal elements of each $m_i^{w_i}$ are either singleton sets or the whole set $\mathcal{H}$. It is easy to see that m also verifies this property if applicable. Interestingly, the commutative and associative properties of Dempster's rule of combination with respect to a combinable collection of BPAs $m_i^{w_i}$ $(i = 1, \ldots, L)$ and the mentioned property essentially form the basis for the ER algorithms developed in [20, 25]. More particularly, with the same notations as in preceding section, we have

$$m(H_n) = m_{n,I(L)}, \text{ for } n = 1, \ldots, N \tag{26}$$
$$m(\mathcal{H}) = m_{\mathcal{H},I(L)} \tag{27}$$

Further, by a simple induction, we easily see that the following holds

Lemma 1. *With the quantity $\overline{m}_{\mathcal{H},I(L)}$ inductively defined by (16), we have*

$$\overline{m}_{\mathcal{H},I(L)} = K_{I(L)} \prod_{i=1}^{L} (1 - w_i) \tag{28}$$

where $K_{I(L)}$ is inductively defined by (11).

Except the weight normalization, the key difference between the original ER algorithm and the modified ER algorithm is nothing but the way of assignment of β_n $(n = 1, \ldots, N)$ and $\beta_{\mathcal{H}}$ after obtained m. That is, in the original ER algorithm, the BPA m is directly used to define the assessment for y by assigning

$$\beta_n = m(H_n) = m_{n,I(L)}, \text{ for } n = 1, \ldots, N \tag{29}$$
$$\beta_{\mathcal{H}} = m(\mathcal{H}) = m_{\mathcal{H},I(L)} \tag{30}$$

While in the modified ER algorithm, after obtained the BPA m, instead of using m to define the assessment for y as in the original ER algorithm, it defines a BPA m' derived from m as follows

$$m'(H_n) = \frac{m(H_n)}{1 - \overline{m}_{\mathcal{H},I(L)}}, \text{ for } n = 1, \ldots, N \tag{31}$$
$$m'(\mathcal{H}) = \frac{(m(\mathcal{H}) - \overline{m}_{\mathcal{H},I(L)})}{1 - \overline{m}_{\mathcal{H},I(L)}} = \frac{\tilde{m}_{\mathcal{H},I(L)}}{1 - \overline{m}_{\mathcal{H},I(L)}} \tag{32}$$

Then the assessment for y is defined by assigning

$$\beta_n = m'(H_n), \text{ for } n = 1, \ldots, N \tag{33}$$
$$\beta_{\mathcal{H}} = m'(\mathcal{H}) \tag{34}$$

By (31)–(32), Theorem 1 straightforwardly follows as m is a BPA.

Lemma 2. *If all assessments $S(e_i)$ $(i = 1, \dots, L)$ are complete, we have*

$$m(\mathcal{H}) = \overline{m}_{\mathcal{H},I(L)} = K_{I(L)} \prod_{i=1}^{L} (1 - w_i) \tag{35}$$

i.e., $\tilde{m}_{\mathcal{H},I(L)} = 0$; and, consequently, $S(y)$ defined by (33) is also complete.

As if $w_i = 0$ then the BPA $m_i^{w_i}$ immediately becomes the *vacuous* BPA, and, consequently, plays no role in the aggregation. Thus, without any loss of generality, we assume that $0 < w_i < 1$ for all $i = 1, \dots, L$. Under this assumption, we are easily to see that if the assumption of the completeness axiom holds, then

$$\mathcal{F}_{m_i^{w_i}} = \{\{h_n\} | n \in I^+\} \cup \{\mathcal{H}\}, \text{ for } i = 1, \dots, L \tag{36}$$

where $\mathcal{F}_{m_i^{w_i}}$ denotes the family of focal elements of $m_i^{w_i}$. Hence, by a simple induction, we also have

$$\mathcal{F}_m = \{\{h_n\} | n \in I^+\} \cup \{\mathcal{H}\} \tag{37}$$

Note that the assumption of the consensus axiom is the same as that of the completeness axiom with $|I^+| = 1$.

Therefore, the consensus and completeness axioms immediately follow from Lemma 2 along with (31)–(34) and (37).

It is also easily seen that

$$m(\mathcal{H}) = K_{I(L)} \prod_{i=1}^{L} m_i^{w_i}(\mathcal{H}) = K_{I(L)} \prod_{i=1}^{L} [w_i m_i(\mathcal{H}) + (1 - w_i)] \tag{38}$$

and in addition, if there is an incomplete assessment $S(e_j)$ then $w_j m_j(\mathcal{H}) > 0$, resulting in

$$w_j m_j(\mathcal{H}) \prod_{\substack{i=1 \\ i \neq j}}^{L} (1 - w_i) > 0$$

This directly implies $m'(\mathcal{H}) > 0$. Consequently, the incompleteness axiom follows as (33)–(34).

4.2 The Discounting-and-Averaging Scheme

In this subsection, instead of applying the the orthogonal sum operation after discounting m_i's, we apply the averaging operation over L BPAs $m_i^{w_i}$ $(i = 1, \dots, L)$ to obtain a BPA $\overline{m}$ defined by

$$\overline{m}(H) = \frac{1}{L} \sum_{i=1}^{L} m_i^{w_i}(H), \text{ for any } H \subseteq \mathcal{H} \tag{39}$$

Therefore, we have

$$\overline{m}(H) = \begin{cases} \frac{1}{L}\sum_{i=1}^{L} w_i\beta_{n,i}, & \text{if } H = \{H_n\} \\ \frac{1}{L}\sum_{i=1}^{L}\left(1 - w_i \sum_{n=1}^{N}\beta_{n,i}\right), & \text{if } H = \mathcal{H} \\ 0, & \text{otherwise} \end{cases} \tag{40}$$

After obtaining the aggregated BPA $\overline{m}$, the problem now is to use $\overline{m}$ for generating the aggregated assessment for the general attribute y. Naturally, we can assign

$$\beta_n = \overline{m}(H_n) = \frac{1}{L}\sum_{i=1}^{L} w_i\beta_{n,i}, \text{ for } n = 1, \ldots, N \tag{41}$$

$$\beta_{\mathcal{H}} = \overline{m}(\mathcal{H}) = \frac{1}{L}\sum_{i=1}^{L}\left(1 - w_i \sum_{n=1}^{N}\beta_{n,i}\right) \tag{42}$$

Then the assessment for y is defined by

$$S(y) = \{(H_n, \beta_n) | n = 1, \ldots, N\} \tag{43}$$

Regarding the synthesis axioms, we easily see that the first axiom holds for the assessment (43). For the next two axioms, we have the following

Theorem 3. *The assessment (43) defined via (41)–(42) satisfies the consensus axiom and/or the completeness axiom if and only if $w_i = 1$ for all $i = 1, \ldots, L$.*

The assessment for y according to this aggregation scheme also satisfies the incompleteness axiom trivially due to the nature of discounting-and-averaging.

Unfortunately, the requirement of $w_i = 1$ for all i to satisfy the consensus axiom and the completeness axiom would not be appropriate in general. This is due to the allocation of the average of discount rates

$$\overline{\alpha} \triangleq \left(1 - \frac{\sum_{i=1}^{L} w_i}{L}\right)$$

to $\mathcal{H}$ as a part of unassigned probability mass. This dilemma can be resolved in a similar way as in the modified algorithms above. Interestingly, this modification leads to the weighted sum scheme as shown in the following.

4.3 Weighted Sum as the Modified Discounting-and-Averaging Scheme

By applying the discounting-and-averaging scheme, we obtain the BPA $\overline{m}$ as defined by (40). Now, guided by the synthesis axioms, instead of making direct use of $\overline{m}$ in defining the generated assessment $S(y)$ (i.e., allocating the average discount rate $\overline{\alpha}$ to $\beta_{\mathcal{H}}$ as a part of unassigned probability mass) as above, we define a new BPA denoted by $\overline{m}'$ derived from $\overline{m}$ by making use of $(1-\overline{\alpha})$ as a normalization factor. More particularly, we define

$$\overline{m}'(H_n) = \frac{\overline{m}(H_n)}{1-\overline{\alpha}}, \text{ for } n = 1, \ldots, N \tag{44}$$

$$\overline{m}'(\mathcal{H}) = \frac{\overline{m}(\mathcal{H}) - \overline{\alpha}}{1-\overline{\alpha}} \tag{45}$$

Then by (40) and a simple transformation, we easily obtain

$$\overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i \beta_{n,i}, \text{ for } n = 1, \ldots, N \tag{46}$$

$$\overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) \tag{47}$$

where

$$\overline{w}_i = \frac{w_i}{\sum\limits_{i=1}^{L} w_i}, \text{ for } i = 1, \ldots, L$$

Let us turn back to the general scheme of combination given in (22). Under the view of this general scheme, the above BPA $\overline{m}'$ is nothing but an instance of it by simply considering $\otimes$ as the multiplication and $\oplus$ as the weighted sum. Namely, we have

$$\overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i m_i(H_n), \text{ for } n = 1, \ldots, N \tag{48}$$

$$\overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i m_i(\mathcal{H}) \tag{49}$$

where relative weights $\overline{w}_i$ are normalized as above so that $\sum_i \overline{w}_i = 1$. It is of interest to note that the possibility of using such an operation has previously been mentioned in, for example, [18]. Especially, the weighted sum operation of two BPAs has been used for the integration of distributed databases for purposes of data mining [10].

Now we quite naturally define the assessment for y by assigning

$$\beta_n = \overline{m}'(H_n) = \sum_{i=1}^{L} \overline{w}_i m_i(H_n), \text{ for } n = 1, \ldots, N \tag{50}$$

$$\beta_{\mathcal{H}} = \overline{m}'(\mathcal{H}) = \sum_{i=1}^{L} \overline{w}_i m_i(\mathcal{H}) \tag{51}$$

Appealingly simple as it is, we can see quite straightforwardly that the following holds.

Proposition 1. *The degrees of belief generated using (50)–(51) satisfy the following*

$$0 \leq \beta_n, \beta_{\mathcal{H}} \leq 1, \text{ for } n = 1, \ldots, N$$
$$\sum_{n=1}^{N} \beta_n + \beta_{\mathcal{H}} = 1$$

Furthermore, we have the following theorem.

Theorem 4. *The aggregated assessment for y defined as in (50)–(51) exactly satisfies all four synthesis axioms.*

4.4 Expected Utility in the ER Approaches

In the tradition of decision making under uncertainty [12], the notion of expected utility has been mainly used to rank alternatives in a particular problem. That is one can represent the preference relation $\succeq$ on a set of alternatives X with a single-valued function $u(x)$ on X, called *expected utility*, such that for any $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Maximization of $u(x)$ over X provides the solution to the problem of selecting x.

In the ER approach, we assume a utility function $u' : \mathcal{H} \to [0, 1]$ satisfying

$$u'(H_{n+1}) > u'(H_n) \text{ if } H_{n+1} \text{ is preferred to } H_n.$$

This utility function u' may be determined using the probability assignment method [8] or using other methods as in [20, 25].

If all assessments for basic attributes are complete, Lemma 2 shows that the assessment for y is also complete, i.e. $\beta_{\mathcal{H}} = 0$. Then the expected utility of an alternative on the attribute y is defined by

$$u(y) = \sum_{n=1}^{N} \beta_n u'(H_n) \quad (52)$$

An alternative a is strictly preferred to another alternative b if and only if $u(y(a)) > u(y(b))$.

Due to incompleteness, in general, in basic assessments, the assessment for y may result in incomplete. In such a case, in [25] the authors defined three measures, called minimum, maximum and average expected utilities, and proposed a ranking scheme based on these measures (see, e.g., [25] for more details).

In this paper, based on the *Generalized Insufficient Reason Principle*, we define a probability function P_m on $\mathcal{H}$ derived from m for the purpose of making decisions via the *pignistic transformation* [15]. Namely,

$$P_m(H_n) = m(H_n) + \frac{1}{N} m(\mathcal{H}) \text{ for } n = 1, \ldots, N \quad (53)$$

That is, as in the two-level language of the so-called *transferable belief model* [15], the aggregated BPA m itself represented the belief is entertained based on the available evidence at the *credal level*, and when a decision must be made, the belief at the credal level induces the probability function P_m defined by (53) for decision making. Particularly, the approximately assessment for y for the purpose of decision making is then defined as

$$\beta'_n = P_m(H_n) = \beta_n + \frac{1}{N}\beta_{\mathcal{H}}, \text{ for } n = 1, \ldots, N \tag{54}$$

Therefore, the expected utility of an alternative on the attribute y is straightforwardly defined by

$$u(y) = \sum_{n=1}^{N} \beta'_n u'(H_n) = \sum_{n=1}^{N} (\beta_n + \frac{1}{N}\beta_{\mathcal{H}}) u'(H_n) \tag{55}$$

In fact, while the amount of belief $\beta_{\mathcal{H}}$ (due to ignorance) is allocated either to the least preferred grade H_1 or to the most preferred grade H_N to define the expected utility interval in Yang's approach [25], it is uniformly allocated to every evaluation grade H_n, guided by the Generalized Insufficient Reason Principle [15], to define an approximately assessment for y and, hence, a single-valued expected utility function.

5 An Example: Motorcycle Assessment Problem

The problem is to evaluate the performance of four types of motorcycles, namely *Kawasaki*, *Yamaha*, *Honda*, and *BMW*.

The overall performance of each motorcycle is evaluated based on three major attributes which are *quality of engine*, *operation*, *general finish*. The process of attribute decomposition for the evaluation problem of motorcycles results in a hierarchy graphically depicted in Fig. 2, where the relative weights of attributes at a single level associated with the same upper level attribute are defined by w_i, w_{ij}, and w_{ijk}, respectively.

Using the five-grade evaluation scale as given in (1), the assessment problem of motorcycles is given in Table 1, where P, I, A, G, and E are the abbreviations of *poor*, *indifferent*, *average*, *good*, and *excellent*, respectively, and a number in bracket denoted the degree of belief to which an attribute is assessed to a grade. For example, $E(0.8)$ means "*excellent* to a degree of 0.8".

Further, all relevant attributes are assumed to be of equal relative important [25]. That is

$$\begin{aligned}
w_1 = w_2 = w_3 &= 0.3333 \\
w_{11} = w_{12} = w_{13} = w_{14} = w_{15} &= 0.2 \\
w_{21} = w_{22} = w_{23} &= 0.3333 \\
w_{211} = w_{212} = w_{213} = w_{214} &= 0.25 \\
w_{221} = w_{222} &= 0.5 \\
w_{231} = w_{232} = w_{233} &= 0.3333 \\
w_{31} = w_{32} = w_{33} = w_{34} = w_{35} &= 0.2
\end{aligned}$$

Table 1: Generalized Decision Matrix for Motorcycle Assessment [25]

General attributes			**Basic attributes**	**types of motor cycle (alternatives)**			
				Kawasaki (a_1)	**Yamaha (a_2)**	**Honda (a_3)**	**BMW (a_4)**
Overall performance	engine		responsiveness	E (0.8)	G (0.3) E (0.6)	G (1.0)	I(1.0)
			fuel economy	A(1.0)	I(1.0)	I(0.5) A(0.5)	E(1.0)
			quietness	I(0.5) A(0.5)	A(1.0)	G(0.5) E(0.3)	E(1.0)
			vibration	G (1.0)	I(1.0)	G(0.5) E(0.5)	P(1.0)
			starting	G (1.0)	A(0.6) G(0.3)	G (1.0)	A(1.0)
	operation	handling	steering	E(0.9)	G(1.0)	A(1.0)	A(0.6)
			bumpy bends	A(0.5) G(0.5)	G (1.0)	G(0.8) E(0.1)	P(0.5) I(0.5)
			maneuverability	A(1.0)	E(0.9)	I(1.0)	P(1.0)
			top speed stability	E(1.0)	G(1.0)	G(1.0)	G(0.6) E(0.4)
		transmission	clutch operation	A(0.8)	G(1.0)	E(0.85)	I(0.2) A(0.8)
			gearbox operation	A(0.5) G(0.5)	I(0.5) A(0.5)	E(1.0)	P(1.0)
		brakes	stopping power	G(1.0)	A(0.3) G(0.6)	G(0.6)	E(1.0)
			braking stability	G(0.5) E(0.5)	G(1.0)	A(0.5) G(0.5)	E(1.0)
			feel at control	P(1.0)	G(0.5) E(0.5)	G(1.0)	G(0.5) E(0.5)
	general		quality of finish	P(0.5) I(0.5)	G(1.0)	E(1.0)	G(0.5) E(0.5)
			seat comfort	G(1.0)	G(0.5) E(0.5)	G(0.6)	E(1.0)
			headlight	G(1.0)	A(1.0)	E(1.0)	G(0.5) E(0.5)
			mirrors	A(0.5) G(0.5)	G(0.5) E(0.5)	E(1.0)	G(1.0)
			horn	A(1.0)	G(1.0)	G(0.5) E(0.5)	E(1.0)

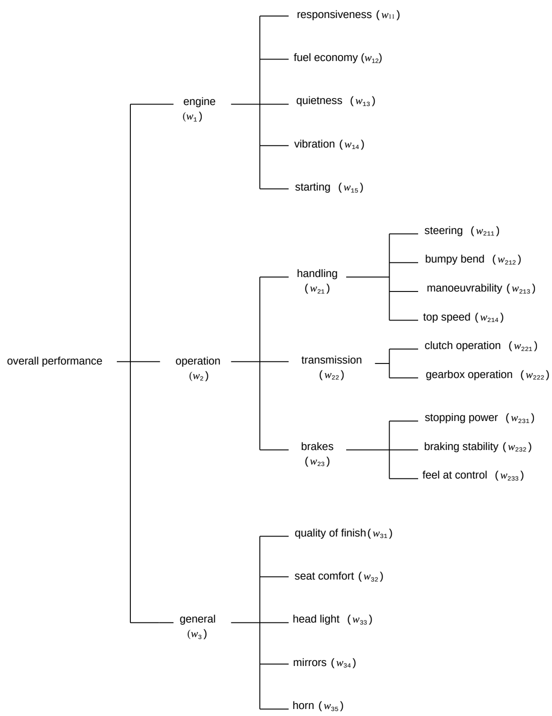


Fig. 2. Evaluation hierarchy for motorcycle performance assessment [25]

In the sequent, for the purpose of comparison, we generate two different results of aggregation via the modified ER approach (refer to (33)–(34)), and the new approach taken in this paper (refer to (50)–(51)).

By applying the modified ER approach, the distributed assessments for overall performance of four types of motorcycles are given in Table 2. These four distrubutions are graphically shown as in Fig. 3 (a).

At the same time, by applying the weighted sum aggregation scheme, we obtain the distributed assessments for overall performance of four types of motorcycles as shown in Table 3 (graphically depicted in Fig. 3 (b)).

As we can easily see, it is not much difference between the result obtained by the modified ER algorithm and that obtained by our method, especially the behavior of correspondingly assessment distributions as Fig. 3 has shown.

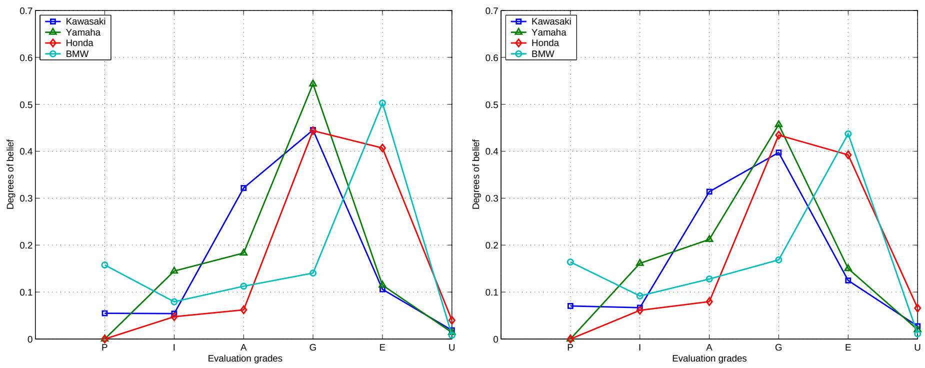
Now, as mentioned above, for the purpose of making decisions we apply the *pignistic transformation* (refer to (53)) to obtain the approximately assessment for overall performance of motorcycles given in Table 4 below.

Assume the same utility function $u' : \mathcal{H} \rightarrow [0, 1]$ as in [25] defined by

$$u'(P) = 0, u'(I) = 0.35, u'(A) = 0.55, u'(G) = 0.85, u'(E) = 1$$

	Poor(*P*)	*Indifference*(*I*)	*Average*(*A*)	*Good*(*G*)	*Excellent*(*E*)	*Unknown*(*U*)
Kawasaki	0.0547	0.0541	0.3216	0.4452	0.1058	0.0186
Yamaha	0.0	0.1447	0.1832	0.5435	0.1148	0.0138
Honda	0.0	0.0474	0.0621	0.4437	0.4068	0.0399
BMW	0.1576	0.0792	0.1124	0.1404	0.5026	0.0078

Table 2. Aggregated assessments for four types of motocycles obtained by using the modified ER approach [25]



(a) The Modified ER Method (b) The Weighted Sum Method

Fig. 3. Overall Evaluation of Motorcycles

Using (55), we easily obtain the expected utility of motorcycles given as

$$u(Kawasaki) = 0.6733, u(Yamaha) = 0.7223$$
$$u(Honda) = 0.8628, \quad u(BMW) = 0.6887$$

Consequently, the ranking of the four types of motorcycles is given by

$$Honda \succ Yamaha \succ BMW \succ Kawasaki$$

which exactly coincides with that obtained by the expected utility interval and the ranking scheme by Yang and Xu [25].

6 Concluding Remarks

In this paper, we have reanalysed the ER approach to MADM under uncertainty. Theoretically, the analysis provides a general formulation for the attribute aggregation problem in MADM under uncertainty. With this new formulation, the previous aggregation scheme becomes, as a consequence, a particular instance of it, along with a simple understanding of the technical proofs. Furthermore, as

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)	*Unknown*(U)
Kawasaki	0.0703	0.0667	0.3139	0.3972	0.1247	0.0272
Yamaha	0.0	0.1611	0.2122	0.4567	0.1501	0.0198
Honda	0.0	0.0611	0.0796	0.4344	0.3922	0.0659
BMW	0.1639	0.0917	0.1278	0.1685	0.437	0.0111

Table 3. Aggregated assessments for four types of motocycles obtained by using the weighted sum aggregation scheme

	Poor(P)	*Indifference*(I)	*Average*(A)	*Good*(G)	*Excellent*(E)
Kawasaki	0.07574	0.07214	0.31934	0.40264	0.13014
Yamaha	0.00396	0.16506	0.21616	0.46066	0.15406
Honda	0.01318	0.07428	0.09278	0.44758	0.40538
BMW	0.16612	0.09392	0.13	0.17072	0.43922

Table 4. Approximately assessments for four types of motocycles obtained by using the pignistic transformation

another result of the new formulation, a new aggregation scheme based on the weighted sum operation has been also proposed. This scheme of aggregation allows us to handle incomplete uncertain information in a simple and proper manner when the assumption regarding the independence of attributes' uncertain evaluations is not appropriate. For the purpose of decision making, an approximate method of uncertain assessments based on the so-called pignistic transformation [15] has been applied to define the expected utility function, instead of using the expected utility interval proposed previously. A tutorial example has been examined to illustrate the discussed techniques.

References

1. G. Balestra, A. Tsoukias, Multicriteria analysis represented by artificial intelligence techniques, *European Journal of Operational Research* **41** (1990) 419–430.
2. V. Belton, Comparison of the analytic hierarchy process and a simple multi-attribute value function, *European Journal of Operational Research* **26** (1986) 7–21.
3. V. Belton, T. J. Stewart, *Multiple Criteria Decision Analysis: An Integrated Approach* (Kluwer, Norwell, MA, 2002).
4. B. G. Buchanan, E. H. Shortliffe, *Rule-Based Expert Systems* (Addison-Wesley, Reading, MA, 1984).
5. S. J. Chen, C. L. Hwang, *Fuzzy Multiple Attribute Decision Making-Methods and Applications*. Berlin: Springer, 1992.
6. T. Isitt, The Sports Tourers, *Motor Cycle International* **64** (1990) 19–27.

7. C. L. Hwang and K. Yoon, *Multiple Attribute Decision Making Methods and Applications* (Springer-Verlag, New York, 1981).
8. R. L. Keeney, H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs* (John Wiley & Sons, New York, 1976).
9. L. Levrat, A. Voisin, S. Bombardier, J. Bremont, Subjective evaluation of car seat comfort with fuzzy set techniques, *International Journal of Intelligent Systems* **12** (1997) 891–913.
10. S. McClean, B. Scotney, Using evidence theory for the integration of distributed databases, *International Journal of Intelligent Systems* **12** (1997) 763–776.
11. M. Roubens, Fuzzy sets and decision analysis, *Fuzzy Sets and Systems* **90** (1997) 199–206.
12. L. J. Savage, *The Foundations of Statistics* (John Wiley and Sons, New York, 1954).
13. P. Sen, J. B. Yang, Multiple criteria decision making in design selection and synthesis, *Journal of Engineering Design* **6** (1995) 207–230.
14. G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, Princeton, 1976)
15. P. Smets, R. Kennes, The transferable belief model, *Artificial Intelligence* **66** (1994) 191–234.
16. T. J. Stewart, A critical survey on the status of multiple criteria decision making theory and practice, *OMEGA International Journal of Management Science* **20** (5–6) (1992) 569–586.
17. J. Wang, J. B. Yang, P. Sen, Multi-person and multi-attribute design evaluations using evidential reasoning based on subjective safety and cost analysis, *Reliability Engineering and System Safety* **52** (1996) 113–127.
18. R. R. Yager, Non-monotonic reasoning with belief structures, in R. R. Yager, M. Fedrizzi, J. Kacprzyk (Eds.), *Advances in the Dempster-Shafer Theory of Evidence* (Wiley, New York, 1994) pp. 534–554.
19. R. R. Yager, Decision-making under various types of uncertainties, *Journal of Intelligent & Fuzzy Systems* **3** (1995) 317–323.
20. J. B. Yang, M. G. Singh, An evidential reasoning approach for multiple attribute decision making with uncertainty, *IEEE Transactions on Systems, Man, and Cybernetics* **24** (1994) 1–18.
21. J. B. Yang, P. Sen, A general multi-level evaluation process for hyprid multiple attribute decision making with uncertainty, *IEEE Transactions on Systems, Man, and Cybernetics* **24** (1994) 1458–1473.
22. J. B. Yang, P. Sen, Multiple attribute design evaluation of large engineering products using the evidential reasoning approach, *Journal of Engineering Design* **8** (1997) 211–230.
23. J. B. Yang, Rule and utility based evidential reasoning approach for multiple attribute decision analysis under uncertainty, *European Journal of Operational Research* **131** (1) (2001) 31–61.
24. J. B. Yang, D. L. Xu, Nonlinear information aggregation via evidential reasoning in multiattribute decision analysis under uncertainty, *IEEE Transactions on Systems, Man, and Cybernetics* **32** (2002) 376–393.
25. J. B. Yang, D. L. Xu, On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty, *IEEE Transactions on Systems, Man, and Cybernetics* **32** (2002) 289–304.
26. Z. J. Zhang, J. B. Yang, D. L. Xu, A hierarchical analysis model for multiobjective decision making, in *Analysis, Design and Evaluation of Man-Machine Systems*, selected papers from the 4th IFAC/IFIP/ IFORS/IEA Conference, Xian, P.R. China, September 1989. Pergamon, Oxford, UK, 1990, pp. 13–18.