Unique normal form property of compatible term rewriting systems
– A new proof of Chew's theorem –

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Abstract

We present a new and complete proof of Chew's theorem, which states that a compatible term rewriting system has a unique normal form property, i.e., $a \leftrightarrow^* b$ implies $a \equiv b$ for any normal forms a, b.

1 Introduction

A term rewriting system (TRS) is a set of directed equations. As a computation/inference mechanism of an equational specification/logic, the natural question is whether its computation/inference terminates, and the next question is whether its result is unique.

The unique normal form property (UN), i.e., $a \leftrightarrow^* b$ implies $a \equiv b$ for any normal forms a, b, guarantees that the result is unique. If a TRS is terminating, UN can be tested by using the critical pair lemma [13]. However, UN is undecidable without termination [14,10].

A frequently used concept in proving UN is the *Church-Rosser property* (CR), which says that any two convertible terms t, s are joinable, i.e., $t \rightarrow^*; \leftarrow^* s$. CR obviously implies UN. When a TRS does not terminate, most of the known sufficient conditions for CR require that a TRS be:

- non-overlapping (i.e., it has no critical pairs) or its extensions, and
- left-linear.

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For instance, the next theorem is well-known and has been extended to loosen the non-overlapping restriction [30,13,31,28,29,25].

Theorem 1.1 [30] *A left-linear non-overlapping TRS is CR.*

However, the non-overlap assumption alone is not sufficient for concluding CR. In [13], the following two counter examples are presented:

$$R_1 = \left\{ \begin{array}{ll} d(x,x) & \to 0 \\ d(x,f(x)) & \to 1 \\ 2 & \to f(2) \end{array} \right\}, \qquad R_2 = \left\{ \begin{array}{ll} d(x,x) & \to 0 \\ f(x) & \to d(x,f(x)) \\ 1 & \to f(1) \end{array} \right\}.$$

In R_1 (from Huet), $d(2,2)$ possesses two distinct normal forms 0 and 1. Thus R_1 is not UN, hence R_1 is not CR. In R_2 (from Klop and Barendregt), 1 possesses two distinct reduction paths $1 \to^*_{R_2} 0$ and $1 \to^*_{R_2} f(0)$. Since the only possible reduction sequence from $f(0)$ is the infinite sequence

$$f(0) \to_{R_2} d(0,f(0)) \to_{R_2} d(0,d(0,f(0))) \to_{R_2} d(0,d(0,d(0,f(0)))) \to_{R_2} \cdots$$

0 and $f(0)$ cannot be joined, hence R_2 is not CR. Note that since 0 is a normal form and $f(0)$ has no normal forms they do not violate UN of R_2.

Chew's statement that a strongly non-overlapping TRS is UN [5] distinguishes between them. A TRS is called *strongly non-overlapping* if two linearizations of its rules have no critical pairs. For example, R_2 is strongly non-overlapping, so is UN, too; however, R_1 is not strongly non-overlapping because of an overlap between linearizations of the first and the second rules.

As an extension, Chew also stated that UN holds for a *compatible* TRS, where a TRS is compatible if, for each pair of rules, some linearization of a pair of rules is *almost non-overlapping*. However, there is a general feeling of doubt about the original proof in [5]. In fact, there is a "gap" in the proof of a key lemma (see Appendix for details).

There have been several attempts at providing a new proof of Chew's theorem, and partial answers have been obtained [37,22,32].

De Vrijer refined Chew's methodology in terms of *conditional linearization* [37]. The conditional linearization R^L of a TRS R is a semi-equational conditional TRS (CTRS) such that R^L and R are the same in convertibility and in the set of normal forms. Accordingly, UN of R is reduced to CR of R^L. Based on this observation, UN of combinatory logic with *parallel-conditional* is proved using a model-theoretic argument.

Toyama and Oyamaguchi proposed a variant of conditional linearization, called *left-right separated CTRS* (LRCTRS). They gave a sufficient condition for UN of *non-duplicating* TRSs [32].

Ogawa stated that UN holds for a larger class than compatible systems, called weakly compatible systems [22]; however, his proof is insufficient and the statement is still an open problem (see Note 7.9).

This paper presents a new complete proof of UN of compatible TRSs by showing CR of compatible LRCTRSs. We design a *peak elimination system* (PES) $\mathcal{P}_R$ for a reduction system R, which is a reduction system on *proofs* $t_1 \leftrightarrow^* t_n$ in R. If a proof is reduced to a normal form of $\mathcal{P}_R$, it must be valley-shaped $t_1 \rightarrow^*; \leftarrow^* t_n$, hence R is CR.

Then we introduce a binary relation on reduction steps called an *independence*. Intuitively, an independence of two reduction steps is a sufficient condition so that one reduction step does not go into a subproof (conditional part) of the other. We also show that the existence of an independence implies the termination of PES $\mathcal{P}_R$ of an (labeled) abstract reduction system (ARS) R under certain restrictions, whereas most conventional proofs of CR have direct weight constructions to show the diamond property [27,2].

Our construction of the independence is $\perp\!\!\!\perp_1$ for non-overlapping LRCTRSs and $\perp\!\!\!\perp_*$ for compatible LRCTRSs. Let α, β be reduction steps in a proof A. Intuitively, $\alpha \perp\!\!\!\perp_1 \beta$ means that α and β are separated by positions, that is, their positions are parallel to each other and no intermediate reduction steps *cover* either of them. An independence $\perp\!\!\!\perp_*$ is defined as $\perp\!\!\!\perp_1 \cup \perp\!\!\!\perp_2$, and $\alpha \perp\!\!\!\perp_2 \beta$ means that α and β are separated by a special term called *barrier*, that is, there is a barrier between α and β in a proof.

This paper is organized as follows. Section 2 provides basic definitions and states the main result. Section 3 introduces notions of LRCTRS and a conditional linearization. Section 4 demonstrates the peak elimination method of an (labeled) ARS, and we provide the criteria to show termination of a PES in terms of an independence. Section 5 presents our basic methodology of constructing an independence for an LRCTRS.

In Section 6, we construct an independence $\perp\!\!\!\perp_1$ and prove CR of a non-overlapping LRCTRS as an introductory application. In Section 7, we prove CR of a compatible LRCTRS by extending the independence $\perp\!\!\!\perp_1$ with $\perp\!\!\!\perp_2$ and by supplementing the argument of the overlapping case. As a result we derive UN of a compatible TRS.

Section 8 describes related work, and Section 9 concludes this paper. Appendix describes Chew's original proof and its gap.

2 Basic definitions and Main result

In this section, we introduce basic notions and then state our main result. The definitions and terminology of abstract reduction systems, terms, and term rewriting systems are taken from [16].

An *abstract reduction system* (ARS) $\langle D, \to \rangle$ is a tuple of a domain D and a binary relation $\to$ called a *reduction relation* on D. The domain D is often omitted. Each element of the reduction relation is called a *reduction step*, denoted by $d \to d'$. The symmetric closure, the reflexive transitive closure, and the reflexive transitive symmetric closure of $\to$ are written as $\leftrightarrow$, $\to^*$, and $\leftrightarrow^*$, respectively. If there is no d' such that $d \to d'$, then d is a *normal form* of $\to$. The set of all normal forms of $\to$ is denoted by $NF_{\to}$. If $d \to^* d'$ and $d' \in NF_{\to}$, then we say *d has a normal form d'*, and d' is called a normal form *of d*.

An ARS $\to$ has the *unique normal form property* (UN) if $d \leftrightarrow^* d'$ implies $d \equiv d'$ for each pair of normal forms d and d', where $\equiv$ is the identity on D. We say $\to$ has the *Church-Rosser property* (CR) if d_1 and d_2 have a common reduct d_3 (i.e., $d_1 \to^* d_3 \leftarrow^* d_2$) for any $d_1 \leftrightarrow^* d_2$.

An ARS $\to$ is *strongly normalizing* (or terminating, SN) if there is no infinite reduction sequence $d_1 \to d_2 \to \cdots$, and *weakly normalizing* (WN) if any $d \in D$ has a normal form.

The sequence $A : d_1 \leftrightarrow d_2 \leftrightarrow \cdots \leftrightarrow d_n$ is called a *proof* (of length $n-1$). A proof of length 0 is called an *empty proof*. We often write A as $d_1 \leftrightarrow^* d_n$. Each step in a proof is assumed to have a direction, that is, $\to$ or $\leftarrow$, even if it is not specified. A proof $d_i \leftrightarrow d_{i+1} \leftrightarrow \cdots \leftrightarrow d_j$ is a *subsequence* of a proof $d_1 \leftrightarrow d_2 \leftrightarrow \cdots \leftrightarrow d_n$ if $1 \leq i \leq j \leq n$. A proof of the form $d_1 \leftarrow d_2 \to d_3$ is called a *peak*.

Let F be a set of *function symbols*, and let V be a countably infinite set of *variables*. Each function symbol f is supposed to have its arity $ar(f)$. A function symbol c such that $ar(c) = 0$ is called a *constant symbol*. The set of all *terms* built from F and V is defined as follows:

(i) Constant symbols in F and variables in V are terms.
(ii) If $t_1, \ldots, t_n$ are terms, and f is a function symbol in F such that $ar(f) = n$, then $f(t_1, \ldots, t_n)$ is a term.

$V(t)$ denotes the set of variables occurring in a term t.

Let $\Box$ be a fresh special constant symbol. A *context* $C[\ ]$ is a term built from $F \cup \Box$ and V. When $C[\ ]$ is a context with n $\Box$s and $t_1, \cdots, t_n$ are terms,

$C[t_1, \cdots, t_n]$ denotes the term obtained by replacing the ith $\Box$ from the left in $C[\ ]$ with t_i for all $i = 1, \ldots, n$. For a term t, $C_t[\ ]$ is the context obtained by replacing all variables in t with $\Box$.

N denotes the set of all natural numbers and N^* denotes the set of all finite sequences of N. ϵ denotes the null sequence. For $p, p' \in \mathsf{N}^*$, $p; p'$ denotes their concatenation, and $|p|$ is the length of p. N^* encodes *positions* in a term. The set of all positions $Pos(t)$ in a term t, the subterm t/p occurring at p in t, and the head symbol $head(t) \in F \cup V$ are defined simultaneously as follows:

(i) If t is a constant or a variable, then

$$\begin{aligned} Pos(t) &= \{\epsilon\}, \\ head(t) &= t, \\ t/\epsilon &= t. \end{aligned}$$

(ii) If $t \equiv f(t_1, \ldots, t_n)$, then

$$\begin{aligned} Pos(t) &= \{\epsilon\} \cup \{iq \,|\, 1 \leq i \leq n \text{ and } q \in Pos(t_i)\}, \\ head(t) &= f, \\ t/\epsilon &\equiv t, \\ t/ip &\equiv t_i/p. \end{aligned}$$

For terms t, s and position $p \in Pos(t)$, $t[p \leftarrow s]$ is the term obtained from t by replacing the subterm at p with s.

A *substitution* is a set of the form $\{x_1 := t_1, \ldots, x_n := t_n\}$ with distinct variables x_i and terms t_i, and $t\{x_1 := t_1, \ldots, x_n := t_n\}$ denotes a term obtained by replacing all occurrences of x_i in t with t_i for all $i = 1, \ldots, n$.

For positions p_1 and p_2, $p_1 \leq p_2$ if p_1 is a prefix of p_2. We write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. When neither $p_1 \leq p_2$ nor $p_2 \leq p_1$, p_1 and p_2 are called *parallel*, denoted by $p_1 \parallel p_2$. $p_1 \wedge p_2$ denotes the longest common prefix of p_1 and p_2.

A *term rewriting system* (TRS) is a set R of *rewrite rules*. A rewrite rule is a pair of terms denoted by $l \rightarrow r$ satisfying two conditions: (1) l is not a variable and (2) $V(l) \supseteq V(r)$. We call l and r the left-hand side (LHS) and the right-hand side (RHS) of $l \rightarrow r$, respectively.

A TRS R defines the reduction relation $\rightarrow_R$ on the set of terms as

$$\rightarrow_R \; = \; \{C[l\theta] \rightarrow_R C[r\theta] \;|\; C[\ ] \text{ is a context, } \theta \text{ is a substitution, and } l \rightarrow r \in R\}.$$

A term $l\theta$ is called a *redex* of R if $l \to r \in R$. Suppose a redex $l_i\theta_i$ of a rule $l_i \to r_i$ occurs at position p_i in t $(i = 1, 2)$. If $p_1 \parallel p_2$, then $l_1\theta_1$ and $l_2\theta_2$ are called *parallel*. If $l_1/q \equiv x \in V$ and $p_2 \geq p_1; q$ for some q, then we say that $l_1\theta_1$ *nests* $l_2\theta_2$ and $l_2\theta_2$ occurs in a *substitution part* $x\theta_1$ of $l_1\theta_1$. Otherwise, $l_1\theta_1$ and $l_2\theta_2$ are *overlapping*.

When we think of a pair of rewrite rules $S_1 : l_1 \to r_1$ and $S_2 : l_2 \to r_2$, their variables are appropriately renamed so that S_1 and S_2 do not share variables.

For rewrite rules $S_1 : l_1 \to r_1, S_2 : l_2 \to r_2$, suppose there is a non-variable subterm l_2/p such that l_1 and l_2/p are unifiable with the most general unifier θ. Then, $\langle l_2[p \leftarrow r_1]\theta, r_2\theta\rangle$ is called a *critical pair* of S_1 and S_2 (obtained by an overlap of S_1 on S_2 at position p) unless S_1 and S_2 are the same rules modulo renaming of variables and $p = \epsilon$. A critical pair $\langle t_1, t_2\rangle$ is *trivial* if $t_1 \equiv t_2$. A critical pair of two rules in a TRS R is called a critical pair of R.

A TRS R is *non-overlapping* if it has no critical pairs. If each critical pair of a TRS R is an *overlay*, that is, an overlap at the head position ϵ, then R is called *overlay*.

A term t is *linear* if every variable occurs in t at most once. A rewrite rule (a TRS R, respectively) is *left-linear* if its LHS (the LHS of each rule in R, respectively) is linear.

Definition 2.1 A substitution ρ is called a *variable substitution* if $x\rho \in V$ for any $x \in V$. A variable substitution θ is called a *renaming of variables* if θ is injective, that is, $x\theta \not\equiv x'\theta$ for any distinct $x, x' \in V$.

Definition 2.2 A term $\bar{t}$ is a *linearization* of a term t if $\bar{t}$ is linear and there is a variable substitution ρ such that $\bar{t}\rho \equiv t$. For a rewrite rule $l \to r$, $\bar{l} \to \bar{r}$ is called a linearization of $l \to r$ if there is a variable substitution ρ such that

- $\bar{l}$ is a linearization of l satisfying $\bar{l}\rho \equiv l$, and
- $\bar{r}\rho \equiv r$.

Example 2.3 For a rewrite rule $f(x, x, y) \to g(y, x, x)$, all of the following are its linearizations: $f(x_1, x_2, y) \to g(y, x_1, x_1)$, $f(x_1, x_2, y) \to g(y, x_1, x_2)$, $f(x_1, x_2, y) \to g(y, x_2, x_1)$, and $f(x_1, x_2, y) \to g(y, x_2, x_2)$.

Definition 2.4 [5,36,37] A TRS R is *compatible*[1] if for any two rules $S_1 : l_1 \to r_1$ and $S_2 : l_2 \to r_2$ in R there exist linearizations $\bar{S}_i : \bar{l}_i \to \bar{r}_i$ of S_i $(i = 1, 2)$ such that $\bar{S}_1$ and $\bar{S}_2$ are *almost non-overlapping*, that is, any critical pair of $\bar{S}_1$ and $\bar{S}_2$ is a trivial overlay.

[1] De Vrijer's terminology [36] is used here. The corresponding notion in Chew's original paper is "strongly non-overlapping and compatible".

Example 2.5 Combinatory logic

$$\mathsf{CL} = \{Sxyz \to xz(yz), Kxy \to x, Ix \to x\}$$

can be regarded as a TRS by letting the symbols S, K, and I be constants and by introducing a new binary function symbol expressing the function application (e.g., *Ap* in [16]). A TRS $\mathsf{CL\text{-}pc}$, which is the union of CL and the *parallel-conditional* rules

$$\{CTxy \to x,\ CFxy \to y,\ Czxx \to x\}$$

is compatible since the linearization $CTxy \to x$ of the first rule and a linearization $Czx_1x_2 \to x_1$ of the third rule are almost non-overlapping, and $CFxy \to y$ and $Czx_1x_2 \to x_2$ are also almost non-overlapping.

The aim of this paper is to present a complete proof of Chew's theorem [5].

Main Theorem *A compatible TRS is UN.*

3 Left-right separated CTRS and conditional linearization

This section introduces conditional linearization using a slightly extended version of left-right separated conditional term rewriting systems [32]. The idea of conditional linearization originated with de Vrijer [36,37].

3.1 Left-right separated CTRS

Definition 3.1 A *left-right separated conditional term rewriting system* (LRCTRS) is a set of conditional rewrite rules with extra variables

$$l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$$

with $V(l) = \{x_1, \ldots, x_n\}$ and $V(r) \subseteq \{y_1, \ldots, y_n\}$ satisfying the following conditions[2].

(i) l is a linear non-variable term,
(ii) $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \emptyset$, and
(iii) $x_i \not\equiv x_j$ if $i \neq j$.[3]

[2] In the definition of LRCTRS in [32] there is an additional condition (iv) that restricts LRCTRSs to be "non-duplicating" (see Note in [32]).

[3] $y_i \equiv y_j$ may hold for $i \neq j$.

$l \to r$ is called the *unconditional part*, and $x_1 = y_1, \cdots, x_n = y_n$ is called the *condition part* of $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$. The unconditional part of R is the set of all unconditional parts of its rewrite rules.

For convenience,

(i) a condition part is often abbreviated to $Q, Q', \ldots$, and
(ii) variables in the left-hand side of a rule are indexed by N in a left-to-right manner, e.g., $f(x_1, g(x_2, x_3))$, $g(h(x'_1, x'_2), x'_3)$, etc.

Definition 3.2 Let $\hat{R}$ be an LRCTRS. The *reduction relation* $\overset{\nabla}{\to}_{\hat{R}_i}$ *in* $\hat{R}$ *at level* i is inductively defined as follows.

$$\begin{aligned}
\overset{\nabla}{\to}_{\hat{R}_0} &= \emptyset, \\
\overset{\nabla}{\to}_{\hat{R}_{i+1}} &= \left\{ C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{R}_{i+1}} C[\hat{r}\theta] \;\middle|\; \begin{array}{l} C[\;] \text{ is a context,} \\ \hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n \in \hat{R}, \text{ and} \\ x_j\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}_i} y_j\theta \text{ for } i = 1, \ldots, n. \end{array} \right\}.
\end{aligned}$$

Then, $\overset{\nabla}{\to}_{\hat{R}} = \cup_i \overset{\nabla}{\to}_{\hat{R}_i}$.

The proof $x_j\theta \overset{\nabla}{\leftrightarrow}{}^*_{R_i} y_j\theta$ is called the *jth subproof* of $C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{R}_{i+1}} C[\hat{r}\theta]$ for $j = 1, \ldots, n$. For the left-oriented reduction step $C[\hat{r}\theta] \overset{\nabla}{\leftarrow}_{\hat{R}_{i+1}} C[\hat{l}\theta]$, $y_j\theta \overset{\nabla}{\leftrightarrow}{}^*_{R_i} x_j\theta$ is called the $(n-j+1)$*-th subproof* for $j = 1, \ldots, n$.

Subproofs of reduction steps at level 1 are caused by taking the reflexive closure of the empty relation. Thus, $\overset{\nabla}{\to}_{\hat{R}_1}$ corresponds to a reduction relation of a possibly non-left-linear TRS. In the sequel, most definitions are only defined for right-directed reduction steps, but they are easily extended to left-directed reduction steps.

The reduction relation $\overset{\nabla}{\to}_{\hat{R}}$ is often treated as more than a relation; we assume that a reduction step $\alpha : C[\hat{l}\theta] \overset{\nabla}{\to}_{\hat{R}} C[\hat{r}\theta]$ due to $\hat{S} : \hat{l} \to \hat{r} \Leftarrow Q$, is implicitly associated with the following information: the rule $\hat{S}$, the position $pos(\alpha)$ of the redex $\hat{l}\theta$ in $C[\hat{l}\theta]$, and the subproofs. A reduction step in $\hat{R}$ is often denoted as $t \overset{\nabla}{\to}_{\hat{S}} t'$ if it is due to the rewrite rule $\hat{S} \in \hat{R}$.

We simultaneously define the set $Addr(A) \subseteq \mathsf{N}^*$ of *addresses* of the reduction steps in a proof $A : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$, and the function $red_A(\sigma)$ mapping an address $\sigma \in Addr(A)$ to the reduction step at σ in A:

$$\begin{aligned}
Addr(A) &= \{1, \ldots, n-1\} \cup \{ij\sigma \mid 1 \le i < n \text{ and } \sigma \in Addr(A_{ij})\}, \\
red_A(i) &= t_i \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1} \text{ for } i = 1, \ldots, n-1, \\
red_A(ij\sigma) &= red_{A_{ij}}(\sigma),
\end{aligned}$$

where A_{ij} is the jth subproof of $red_A(i)$.

Thus we say a reduction step α is *in* A when $\alpha = red_A(\sigma)$ for some $\sigma \in Addr(A)$. The reduction step $red_A(i)$ is called the ith *top reduction step* of A for $i = 1, \ldots, n-1$. A reduction step $red_A(\sigma)$ with $|\sigma| = 3$ is called a *subtop* reduction step of A. Top reduction steps and subtop reduction steps are called *visible*. For a proof A in $\hat{R}$, we define $size(A)$ as the number of elements in $Addr(A)$.

Here we introduce some notational conventions.

- We call t_i is the ith term *in* $A : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$, and write $t_i \in A$ for $i = 1, \ldots, n$.
- The set of all reduction steps in A is denoted by A itself when no confusion will arise, so we write, e.g., $\alpha \in A$. The set of all top (subtop, visible, respectively) reduction steps of A is denoted by $top(A)$ ($subtop(A)$, $visible(A)$, respectively).
- The set of all reduction steps in α, that is, α and all reduction steps in its subproofs, is denoted by $sub(\alpha)$. The set of all reduction steps in all proper subproofs of α is denoted by $sub^-(\alpha)$, i.e., $sub^-(\alpha) = sub(\alpha) \setminus \{\alpha\}$.
- When we refer to the position of a reduction step, we adopt 'automatic type-cast' and omit $pos(\,)$ when no confusion will arise. For instance, we write $\alpha < p$ instead of $pos(\alpha) < p$ for a reduction step α and a position p. Moreover, we write $A < p$ ($A \leq p$, $A \not< p$, $A \not\leq p$, respectively) if $\gamma < p$ ($\gamma \leq p$, $\gamma \not< p$, $\gamma \not\leq p$, respectively) for any reduction step $\gamma \in top(A)$.

3.2 Conditional linearization

In this section, we introduce (left-right separated) conditional linearization of a TRS. We show that UN of a TRS is reduced to CR of its conditional linearization.

Definition 3.3 For a rewrite rule $S : l \to r$, its *conditional linearization* $\hat{S} : \hat{l} \to \hat{r} \Leftarrow Q$ is a left-right separated conditional rewrite rule constructed as follows.

(i) $\hat{l}$ is a linearization of l such that $V(\hat{l}) \cap V(l) = \emptyset$ and $\hat{l}\rho \equiv l$ for some variable substitution ρ,
(ii) $\hat{r} \equiv r$, and
(iii) $x = x\rho$ is added to the condition part Q for all $x \in V(\hat{l})$.

Note that conditional linearizations of S are unique modulo renaming of variables in $\hat{l}$. The set of conditional linearizations of all rules in R is called *the conditional linearization of* R.

Example 3.4 The LRCTRS $\hat{R}$ below is the conditional linearization of R.

$$R = \left\{ \begin{array}{l} d(x,x) \to 0 \\ f(y) \to d(y, f(y)) \\ 1 \to f(1) \end{array} \right\}. \ \hat{R} = \left\{ \begin{array}{lll} d(x_1, x_2) \to 0 & \Leftarrow & x_1 = x, x_2 = x \\ f(y_1) \to d(y, f(y)) & \Leftarrow & y_1 = y \\ 1 \to f(1) & & \end{array} \right\}.$$

The following theorem is a slight extension of Theorem 18 in [32]. In fact, both theorems (and their proofs) are easy reformulations of Theorem 3.8 in [37].

Theorem 3.5 *Let $\hat{R}$ be the conditional linearization of a TRS R. If $\hat{R}$ is UN, then R is UN.*

Proof. We denote the sets of normal forms of R and $\hat{R}$ by NF_R and $NF_{\hat{R}}$, respectively. As with Theorem 3.8 in [37], the proof is done using the following claim.

Claim R is UN if all of these conditions hold:

(i) $\overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} \supseteq \leftrightarrow^*_R$.
(ii) $\hat{R}$ is UN.
(iii) $NF_{\hat{R}} \supseteq NF_R$.

Proof of Claim. Suppose there are distinct normal forms t and t' such that $t \leftrightarrow^*_R t'$. Then $t \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t'$ from (i). From (iii), $t, t' \in NF_{\hat{R}}$. Thus, $t \equiv t'$ from (ii). This leads to a contradiction. $\square$

Now we show that $\hat{R}$ and R satisfy the above properties. Since the reduction relation $\overset{\nabla}{\to}_{\hat{R}_1}$ of $\hat{R}$ at level 1 coincides with $\to_R$, (i) apparently holds. (ii) holds from the assumption of the theorem. We assume for (iii) that a term exists in NF_R but not in $NF_{\hat{R}}$, and derive a contradiction. Let t be such a term that is minimal wrt the number of function symbols in it. Then $t \equiv \hat{l}\theta$ for some substitution θ and a rule $\hat{S} : \hat{l} \to \hat{r} \Leftarrow Q \in \hat{R}$ that is the conditional linearization of $S : l \to r \in R$. Thus, there exist positions p_1 and p_2 such that $l/p_1 \equiv l/p_2 \equiv x$ and $t/p_1 \not\equiv t/p_2$ for some non-linear variable x in l; otherwise t would be a redex of R. $t/p_1, t/p_2 \in NF_R$ and $t/p_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t/p_2$. Since $\hat{R}$ is NF, either t/p_1 or t/p_2 is not in $NF_{\hat{R}}$. This contradicts the minimality of t. $\square$

We say an LRCTRS is non-overlapping (overlay, respectively) if its unconditional part is non-overlapping (overlay, respectively).

Definition 3.6 An LRCTRS $\hat{R}$ is *compatible* if for every two rules $\hat{S}_1 : \hat{l}_1 \to \hat{r}_1 \Leftarrow x_{11} = y_{11}, \ldots, x_{1n_1} = y_{1n_1}$ and $\hat{S}_2 : \hat{l}_2 \to \hat{r}_2 \Leftarrow x_{21} = y_{21}, \ldots, x_{2n_2} = y_{2n_2}$

in $\hat{R}$ there exist terms $\bar{r}_1, \bar{r}_2$ such that

(i) $\bar{r}_i\{x_{ij} := y_{ij} \mid j = 1, \ldots, n_i\} \equiv \hat{r}_i$ (i = 1, 2), and
(ii) the rewrite rules $\hat{l}_1 \to \bar{r}_1$ and $\hat{l}_2 \to \bar{r}_2$ are almost non-overlapping.

With Definition 2.4 and the above definition, it is easy to prove the following lemma.

Lemma 3.7 *A TRS R is compatible iff its conditional linearization $\hat{R}$ is compatible.*

In the above definition, $\bar{r}$ and $\bar{r}'$ are not generally unique. For convenience, we assume that the choice function χ maps a pair $\langle S_1, S_2\rangle$ of rules in a compatible LRCTRS to a pair $\langle \bar{r}_1, \bar{r}_2\rangle$ that satisfies the conditions of the above definition. We call $\chi(\langle S_1, S_2\rangle)$ the *standard pair of compatibility* of $\langle S_1, S_2\rangle$.

In Section 7, we prove CR of compatible LRCTRSs. Since CR implies UN, Main Theorem will be derived from Theorem 3.5.

3.3 Operations on proofs of an LRCTRS

In this section, we introduce operations on proofs that we freely use in later sections.

Definition 3.8 For proofs $A_1, \ldots, A_n$, a context $C[\ ]$ with n $\Box$'s, and a position p, we define operations, namely, concatenation $A_1; A_2$, embedding $C[\![A_1, \ldots, A_n]\!]$, and restriction A_1/p as the following. The operator ; is assumed to have higher priority than /. We associate a mapping called *parent* with each of them.

For proofs $A_1 : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, t_n$ and $A_2 : t_n \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, t_{n+m}$, the *concatenation* $A_1; A_2$ of A_1 and A_2 is the proof $t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, t_n \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, t_{n+m}$. The parent of $red_{A_1;A_2}(i\sigma)$ is $red_{A_1}(i\sigma)$ if $i < n$, otherwise $red_{A_2}(\{i+1-n\}\sigma)$.

For proofs $A_i : t_i \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, t'_i$ $(i = 1, \ldots, n)$, the *embedding* $C[\![A_1, \ldots, A_n]\!]$ of the proofs $A_1, \ldots, A_n$ *into* $C[\ ]$ is the concatenation $CA_1; \cdots; CA_n$, where

$$CA_i = C[t'_1, \ldots, t'_{i-1}, t_i, t_{i+1}, \ldots, t_n] \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}}\, C[t'_1, \ldots, t'_{i-1}, t'_i, t_{i+1}, \ldots, t_n],$$

and the kth top reduction step of CA_i and that of A_i are the same except for the context. We call CA_i the *A_i-segment* of the embedding. Suppose that $red_{C[\![A_1,\ldots,A_n]\!]}(m_i)$ is the first top reduction step of the A_i-segment. If $red_{C[\![A_1,\ldots,A_n]\!]}(\{m_i+k\}\sigma)$ is also in the A_i-segment, then its parent is $red_{A_i}(k\sigma)$ $(i = 1, \ldots, n)$.

Let $A_1 : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$ be a proof and let p be a position such that $A_1 \not< p$. Note that $p \in Pos(t_i)$ for $i = 1, \ldots, n$ since $A_1 \not< p$. The *restriction* A_1/p *of* A_1 *to* p is $t_1/p \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n/p$. Here, $red_{A_1}(i)$ is the parent of $t_i/p \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_{i+1}/p$ and they are the same except for the context if $pos(red_{A_1}(i)) \geq p$. Otherwise, $t_i/p \equiv t_{i+1}/p$, i.e., the 'step' is empty. If $red_{A_1}(i)$ is the parent of $red_{A_1/p}(j)$, then $red_{A_1}(i\sigma)$ is the parent of $red_{A_1/p}(j\sigma)$ $(i = 1, \ldots, n-1)$.

If a proof A occurs more than once in concatenations or embeddings, we assume that each occurrence is an isomorphic copy of A and we distinguish between them. Thus, the parent mapping is injective. For simplicity, we identify a reduction step with its parent in the sequel.

Now, we introduce an operation called *flattening* that flattens the top reduction steps, which will be extensively used in Section 6 and Section 7. This operation decreases the maximal level of a proof by 1.

Definition 3.9 For a rule $\hat{S} : \hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \ldots, x_n = y_n$ and a right-oriented reduction step $\alpha : C[C_{\hat{l}}[x_1\theta, \ldots, x_n\theta]] \overset{\nabla}{\to}_{\hat{S}} C[C_{\hat{r}}[y_{j_1}\theta, \ldots, y_{j_m}\theta]]$ with subproofs $AS_i : x_i\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} y_i\theta$ $(i = 1, \ldots, n)$, the *flattening* of α, denoted by $\alpha^\flat$, is an embedding of subproofs into the substitution parts of the LHS followed by the reduction step below:

$$C[\![C_{\hat{l}}[\![AS_1, \ldots, AS_n]\!]]\!]; C[C_{\hat{l}}[y_1\theta, \ldots, y_n\theta]] \to_{\hat{R}} C[C_{\hat{r}}[y_{j_1}\theta, \ldots, y_{j_m}\theta]],$$

where $\hat{S}$ is applied in the last step whose subproofs are trivial. In a similar way, we define the flattening of a left-oriented reduction step, that is, a reduction step with trivial subproofs followed by an embedding of subproofs. The flattening of a proof A, denoted by $A^\flat$, is obtained by replacing all top reduction steps of A with their flattenings.

Each right-oriented (left-oriented, respectively) reduction step $\alpha \in top(A)$ is identified with the last (first, respectively) reduction step of $\alpha^\flat$ in $A^\flat$. For any reduction step in any subproof of $\alpha \in top(A)$, the identification is defined according to the reduction step of the embedding. Accordingly, the position of $\beta \in subtop(A)$ is defined as the position of β in $A^\flat$.

We define $\flat_A$ to map each term $t_i \in A : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$ to an occurrence of the same term t_i in $A^\flat$. The first and last terms t_1, t_n are mapped by $\flat_A$ to the first and last terms in $A^\flat$, and the jth term t_j is mapped to the term between the flattenings of the $j-1$th and jth reduction steps for $j = 2, \ldots, n-1$.

Example 3.10 Let $\hat{R}$ be the same as the one in Example 3.4. For a reduction step $\alpha : d(d(1, f(1)), 1) \overset{\nabla}{\to}_{\hat{R}} 0$ with subproofs $d(1, f(1)) \leftarrow_{\hat{R}} f(1)$ and $1 \to_{\hat{R}} f(1)$, its flattening is $\alpha^\flat : d(d(1, f(1)), 1) \leftarrow_{\hat{R}} d(f(1), 1) \to_{\hat{R}} d(f(1), f(1)) \to_{\hat{R}} 0$, as shown in Fig. 1. The dash-arrows indicate identification of reduction

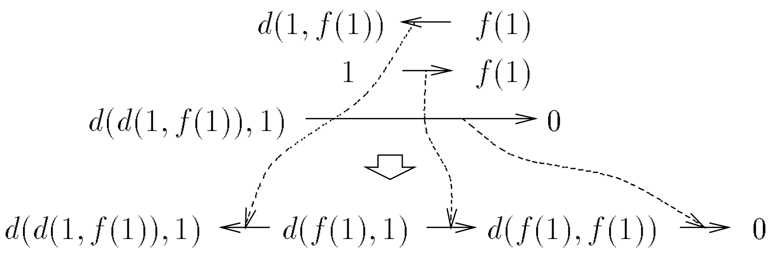

Fig. 1. Flattening

steps.

The next lemma follows from the definition of flattening and the fact that the left-hand side of each rewrite rule is not a variable.

Lemma 3.11 *Suppose* $\alpha \in top(A)$ *and* $\beta \in sub(\alpha) \cap subtop(A)$. *Then,* $\alpha < \beta$.

4 Abstract peak elimination

In this section we introduce a variant of ARS called LARS, which is an abstraction of conditional rewriting. We demonstrate a peak elimination method and present a sufficient condition for CR of a LARS.

4.1 Labeled ARS and Peak elimination system

Definition 4.1 A *(set-)Labeled ARS* (LARS) is a triple $R^\nabla = \langle D, \mathcal{H}, \rightarrow \rangle$ of a domain D, a set $\mathcal{H}$ of *tags*, and a relation $\rightarrow: D \times 2^{\mathcal{H}} \times D$ called a *labeled reduction relation*. Each element of a labeled reduction relation is called a *labeled reduction step*, denoted by $d \xrightarrow{H} d'$ ($d, d' \in D$, $H \subseteq \mathcal{H}$), and H is called the *label* of $d \xrightarrow{H} d'$. We suppose that each label contains a special tag $root(H)$ called the *root* of H.

Note that a label is non-empty since it has at least its root.

Example 4.2 *An LRCTRS* $\hat{R}$ *induces an LARS* $\hat{R}^\nabla$ *as the following: for every reduction step* $\alpha : t \xrightarrow{\nabla}_{\hat{R}} t'$, $\hat{R}^\nabla$ *has* $\alpha' : t \xrightarrow{H} t'$, *where* $H = sub(\alpha)$ *and* $root(H) = \alpha$. *That is, a reduction step in* $\hat{R}$ *corresponds to a tag of* $\hat{R}^\nabla$.

Once we forget the label, an LARS is a conventional ARS, so the notions related to proofs of LARSs are defined in the same way as ARSs. For example, if $d_i \xrightarrow{H_i} d_{i+1}$ or $d_i \xleftarrow{H_i} d_{i+1}$ for $i = 1, \ldots, n-1$, then the sequence $A : d_1 \overset{H_1}{\leftrightarrow} d_2 \overset{H_2}{\leftrightarrow} \cdots \overset{H_{n-1}}{\leftrightarrow} d_n$ is called a *proof* in R^∇. A subsequence $d_{i-1} \xleftarrow{H_{i-1}} d_i \xrightarrow{H_i} d_{i+1}$ is called a *peak*.

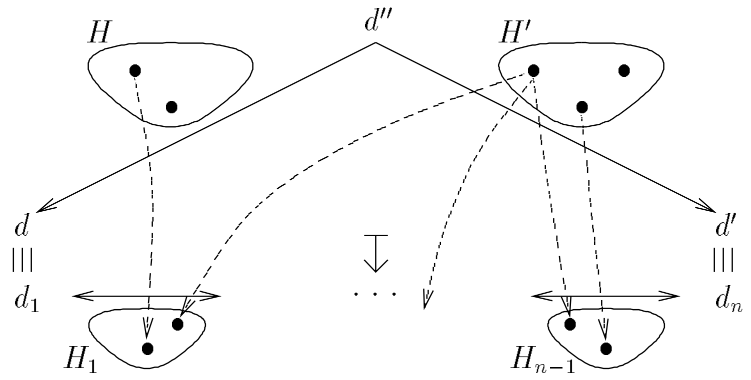

Fig. 2. Peak rewrite rule

For any proof $A : d_1 \overset{H_1}{\leftrightarrow} d_2 \overset{H_2}{\leftrightarrow} \cdots \overset{H_{n-1}}{\leftrightarrow} d_n$, we assume for convenience that H_i and H_j are disjoint for any $i \neq j$, and define the *label* of A as the disjoint union $H_1 \cup \cdots \cup H_{n-1}$, denoted by $label(A)$.

Definition 4.3 A *peak elimination rule of* an LARS R^∇ is a triple $\langle A, J, A' \rangle$ that satisfies the following properties.

(i) A is a peak, say $d \overset{H}{\leftarrow} d'' \overset{H'}{\rightarrow} d'$.
(ii) A' is a proof $d_1 \overset{H_1}{\leftrightarrow} \cdots \overset{H_{n-1}}{\leftrightarrow} d_n$ such that $d_1 \equiv d$ and $d_n \equiv d'$.
(iii) J is a mapping, called *ancestor mapping* from $label(A')$ to $label(A)$. If $J(h') = h$, then h is the *ancestor* of h', and h' is a *descendant* of h.

A peak elimination rule is denoted as $A \overset{J}{\mapsto} A'$ (J is often omitted), and A and A' are called the left-hand side (LHS) and the right-hand side (RHS) of the rule, respectively. We also call A' a *replacement sequence* of A. Fig. 2 illustrates a peak elimination rule and each dash-arrow goes from an ancestor to its descendants.

Let $B : d_1 \overset{H_1}{\leftrightarrow} \cdots \overset{H_{n-1}}{\leftrightarrow} d_n$ be a proof with a peak $A : d_{i-1} \overset{H_{i-1}}{\leftarrow} d_i \overset{H_i}{\rightarrow} d_{i+1}$ and let $P : A \overset{J}{\mapsto} A'$ be a peak elimination rule. Suppose that B' is obtained from B by replacing the peak A with A'. Then $B \overset{J'}{\mapsto} B'$ is called a *peak elimination step*, where the ancestor mapping J' coincides with J on the label of the replacement sequence and J' is the identity mapping on $H_1 \cup \cdots \cup H_{i-2} \cup H_{i+1} \cup \cdots \cup H_{n-1}$.

Definition 4.4 Suppose that $\mathcal{P}_{R^\nabla}$ is a set of peak elimination rules of R^∇, and for any peak A there is $P \in \mathcal{P}_{R^\nabla}$ such that A is the LHS of P. Then $\mathcal{P}_{R^\nabla}$ is called a *peak elimination system* (PES) of R^∇. A peak elimination step of $\mathcal{P}_{R^\nabla}$ is a peak elimination step of a rule in $\mathcal{P}_{R^\nabla}$.

Definition 4.5 If $A^i \mapsto A^{i+1}$ is a peak elimination step in a PES $\mathcal{P}_{R^\nabla}$ for $i = 1, 2, \ldots$, then the sequence $A^1 \mapsto A^2 \mapsto \cdots$ is called a *peak elimination sequence*. The ancestor mapping of a finite peak elimination sequence $A^j \overset{J^j}{\mapsto}$

$\cdots \overset{J^{k-1}}{\mapsto} A^k$ $(k \geq j)$ is $J^j \circ \cdots \circ J^{k-1}$. If $h' \in label(A^k)$ and $J^j \circ \cdots \circ J^{k-1}(h') = h$, then h is the *ancestor* of h', and h' is a *descendant* of h.

As a special case, a tag h is the ancestor and the descendant of h itself with respect to an empty peak elimination sequence.

A PES $\mathcal{P}_{R^\nabla}$ can be regarded as an ARS with the set of all proofs in R^∇ as the domain. Since a PES can replace *any* peak by definition, a proof A is a normal form of a $\mathcal{P}_{R^\nabla}$ iff A is 'valley-shaped', that is, of the form $d_1 \overset{H_1}{\to} \cdots \overset{H_{i-1}}{\to} d_i \overset{H_i}{\leftarrow} \cdots \overset{H_{n-1}}{\leftarrow} d_n$. Thus, the next lemma follows.

Lemma 4.6 *If a PES of an LARS R^∇ is WN, then R^∇ is CR.*

4.2 Termination of PES by independence

We present a sufficient condition for SN of a PES (hence CR of an LARS) in terms of a certain binary relation on tags, called *independence*.

In Section 6 and Section 7, we introduce PESs of non-overlapping and compatible LRCTRSs, whose rules are classified into three categories: parallel, nest, and critical. The former two cases are *injective*, i.e., the ancestor mapping is injective, and *simple*, i.e., one step divergence closes with at most one step valley. The difficulties are in the last case; some reduction steps may be multiplied by a critical peak elimination. The independence is a sufficient condition so that one reduction step does not go into a subproof (i.e., conditional part) of the other. Thus, it guarantees that the multiplication of reduction steps does not create *unexpected* peaks.

In this section, we present this idea in an axiomatic way and show how to construct the well-founded weight on proofs that decreases at each step of the PES if the independence exists. Therefore, SN of a PES (hence CR of an LARS) is reduced to the construction of the independence, which is the main topic in Section 6 and Section 7. We denote $\{u \in S \mid u \notin S'\}$ by $S \setminus S'$.

Definition 4.7 Suppose a binary relation $\perp\!\!\!\perp$ is defined on $label(A)$ for any proof A in an LARS R^∇. Then $\perp\!\!\!\perp$ is called an *independence* for a PES $\mathcal{P}_{R^\nabla}$ of R^∇ if the following properties hold for any proof $A : d_1 \overset{H_1}{\leftrightarrow} \cdots \overset{H_{n-1}}{\leftrightarrow} d_n$ in R^∇ and tags $h, g \in label(A)$.

- (i) (dominance) $root(H_i) \not\!\perp\!\!\!\perp h$ if $h \in H_i$.
- (ii) (adherence) If $h \perp\!\!\!\perp root(H_j)$ and $g \in H_j$, then $h \perp\!\!\!\perp g$.
- (iii) (non-incest) Suppose that $A \mapsto A'$ and that $h', h'' \in label(A')$ are distinct descendants of h. Then $h' \perp\!\!\!\perp h''$.

(iv) (preservation) Suppose that $A \mapsto A'$ and that h' and g' in $label(A')$ are descendants of h and g, respectively. Then $h \perp\!\!\!\perp g$ implies $h' \perp\!\!\!\perp g'$.

Definition 4.8 Let $P : A \overset{J}{\mapsto} A'$ be a peak elimination rule with $A = d \overset{H}{\leftarrow} d'' \overset{H'}{\rightarrow} d'$. We say P is *injective* if J is injective. We say P is *root-erasing* if neither $root(H)$ nor $root(H')$ have descendants in the $label(A')$. A peak elimination step due to an injective (root-erasing, respectively) peak elimination rule is called injective (root-erasing, respectively).

In the rest of this section, $A^1 \overset{J^1}{\mapsto} A^2 \overset{J^2}{\mapsto} \cdots$ is a peak elimination sequence in a PES $\mathcal{P}_{R^\triangledown}$ with an independence $\perp\!\!\!\perp$.

Definition 4.9 The *origin* of a tag $h \in label(A^j)$ $(j = 1, 2, \ldots)$ is the ancestor in $label(A^1)$ of h, denoted by $orig(h)$.

We use [and] to represent multisets and $\uplus$ for the multiset union. For example, $[a, a, b] \uplus [a, b, c] = [a, a, a, b, b, c]$.

Definition 4.10 We define the *weight* for the proofs A^i for $i = 1, 2, \ldots$ and their tags.

(i) The weight $w(h)$ of a tag h is defined as
  (a) $w(h) = label(A^1)$ for $h \in label(A^1)$.
  (b) Let $h \in label(A^{k+1})$ $(k = 1, 2, \ldots)$. Assume that $A^k \overset{J^k}{\mapsto} A^{k+1}$ eliminates a peak $d \overset{H}{\leftarrow} d'' \overset{H'}{\rightarrow} d'$. If $A^k \overset{J^k}{\mapsto} A^{k+1}$ is root-erasing, then

$$w(h) = \begin{cases} w(J^k(h)) \backslash \{orig(root(H))\} & \text{if } J^k(h) \in H, \\ w(J^k(h)) \backslash \{orig(root(H'))\} & \text{if } J^k(h) \in H', \\ w(J^k(h)) & \text{otherwise.} \end{cases}$$

  Otherwise, $w(h) = w(J^k(h))$.

(ii) Then $w(A^k) = \biguplus_{h \in label(A^k)} [w(h)]$ for $k = 1, 2, \ldots$

Let $\subset_{mul}$ be the *multiset extension* [9] of the proper subset relation $\subset$ on finite sets, and let $\subseteq_{mul}$ be the reflexive closure of $\subset_{mul}$. Note that $\subset_{mul}$ is well-founded.

Lemma 4.11 *If $A^k \overset{J^k}{\mapsto} A^{k+1}$ is injective, then $w(A^k) \supseteq_{mul} w(A^{k+1})$ for $k =$ $1, 2, \ldots$*

Proof. Since $w(A^k) \supseteq w(A^{k+1})$, the result follows. □

Lemma 4.12 *Let $h^k \in H$. Suppose that a root-erasing peak elimination rule is applied to a peak $d \overset{H}{\leftarrow} d'' \overset{H'}{\rightarrow} d'$ in $A^k \overset{J^k}{\mapsto} A^{k+1}$. If $h^k \in H$ (H', respectively)*

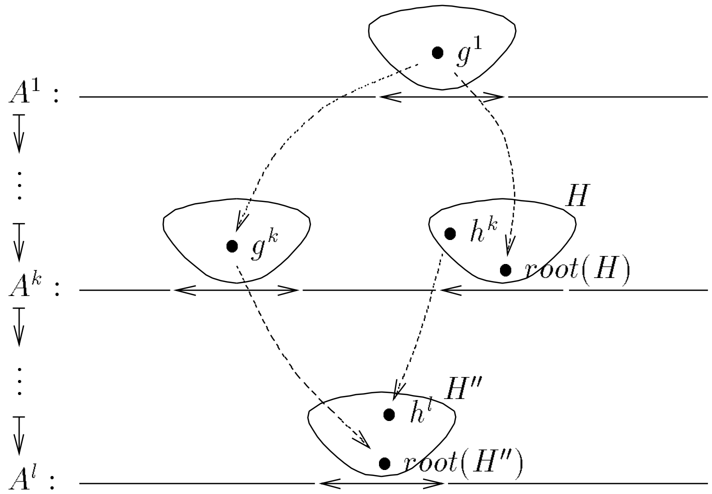

Fig. 3. Proof of Lemma 4.12

has a descendant h^l in a label H'' of a reduction step in A^l for some $l > k$, then $orig(root(H)) \neq orig(root(H''))$ ($orig(root(H')) \neq orig(root(H''))$, respectively).

Proof. We assume that $orig(root(H)) = orig(root(H'')) = g^1$ and derive a contradiction. Let $g^k \in label(A^k)$ be the ancestor of $root(H'')$.

Since the peak elimination rule of $A^k \mapsto A^{k+1}$ is root-erasing, $root(H)$ has no descendant after A^{k+1}, so $g^k \neq root(H)$. Thus, g^k and $root(H)$ are distinct descendants of g^1, and $g^k \perp\!\!\!\perp root(H)$ due to the property of non-incest and preservation. Since $h^k \in H$, $g^k \perp\!\!\!\perp h^k$ because of adherence. Hence, $root(H'') \perp\!\!\!\perp h^l$ according to preservation. However, $root(H'') \not\perp\!\!\!\perp h^l$ due to the property of dominance. This leads to a contradiction. The case $h^k \in H'$ is similarly proved. □

Lemma 4.13 *If $A^k \overset{J^k}{\mapsto} A^{k+1}$ is root-erasing, then $w(A^k) \supset_{mul} w(A^{k+1})$ for $k = 1, 2, \ldots$*

Proof. Let $d \overset{H}{\leftarrow} d'' \overset{H'}{\rightarrow} d'$ be the eliminated peak in $A^k \overset{J^k}{\mapsto} A^{k+1}$. Note that $H \cup H'$ is non-empty.

If the replacement sequence in A^{k+1} is an empty proof, then the result is obvious. Let h be any tag of the replacement sequence. Suppose $J^k(h) \in H$. Then $w(h) = w(J^k(h)) \backslash \{orig(root(H))\}$, and from Lemma 4.12 $orig(root(H)) \in w(J^k(h))$. Thus, $w(J^k(h)) \supset w(h)$. When $J^k(h) \in H'$, this is similarly proved. Therefore, $w(A^k) \supset_{mul} w(A^{k+1})$. □

For an injective and non-root-erasing peak elimination, we prepare simpleness as defined below.

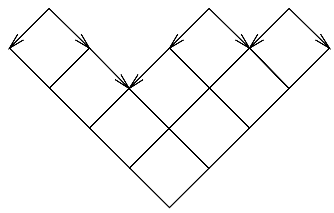

Fig. 4. Mass

Definition 4.14 A proof of the form $d \to^{\equiv} d' \leftarrow^{\equiv} d''$ is called *simple*, where $\to^{\equiv}$ is the reflexive closure of $\to$. A peak elimination rule is simple if its RHS is simple. A peak elimination step resulting from a simple peak elimination rule is also called simple.

Definition 4.15 A labeled reduction step α is left-oriented (right-oriented, respectively) if α is $d \overset{H}{\leftarrow} d'$ ($d \overset{H}{\rightarrow} d'$, respectively) for some d, d', and H. For a proof A, the $height(A) \geq 0$ and $mass(A) \geq 0$ is simultaneously defined as the following.

(i) If A is an empty proof, then $height(A) = mass(A) = 0$.
(ii) If $A = A'; \alpha$ for some proof A and reduction step α, then
  - $height(A) = height(A') + 1$ and $mass(A) = mass(A')$ if α is left-oriented, and
  - $height(A) = height(A')$ and $mass(A) = mass(A') + height(A')$ if α is right-oriented.

In other words, the mass is the number of tiles as shown in Fig. 4.

Lemma 4.16 *If $A^k \mapsto A^{k+1}$ is simple, then $mass(A) > mass(A')$.*

Theorem 4.17 *Suppose every peak elimination rule in a PES $\mathcal{P}_{R^\triangledown}$ is either root-erasing or simple and injective. If $\mathcal{P}_{R^\triangledown}$ has an independence $\perp\!\!\!\perp$, then $\mathcal{P}_{R^\triangledown}$ is SN.*

Proof. Let $\sqsubset$ be the lexicographic extension of $\subset_{mul}$ and $<$. Then $\langle w(A^k), mass(A^k)\rangle \sqsupset \langle w(A^{k+1}), mass(A^{k+1})\rangle$ for $k = 1, 2, \ldots$ from Lemma 4.11, Lemma 4.13, and Lemma 4.16. Therefore, the result follows. $\square$

5 Basic construction of independence for LRCTRSs

Before constructing the independences of non-overlapping and compatible LRCTRSs, we prepare the methodology for their simplification. That is, we first define the binary relation $\perp\!\!\!\perp$ on visible reduction steps, and next extend $\perp\!\!\!\perp$ using the subproof closure $\perp\!\!\!\perp^s$, which we introduce in this section. Finally, we check whether $\perp\!\!\!\perp^s$ satisfies the conditions in Definition 4.7, where most of the conditions are reduced to those in terms of $\perp\!\!\!\perp$. For LRCTRSs, Definition 4.7 is reformulated in the manner in Example 4.2.[4]

Definition 5.1 An independence $\perp\!\!\!\perp$ is a binary symmetric relation on reduction steps satisfying the following conditions.

(i) (dominance) $\alpha \not\perp\!\!\!\perp \beta$ if $\alpha \in sub(\beta)$.
(ii) (adherence) If $\alpha \perp\!\!\!\perp \beta$ and $\gamma \in sub(\beta)$, then $\alpha \perp\!\!\!\perp \gamma$.
(iii) (non-incest) Suppose that $A \mapsto A'$ and that $\alpha, \beta \in A'$ are distinct descendants of γ. Then $\alpha \perp\!\!\!\perp \beta$.
(iv) (preservation) Suppose that $A \mapsto A'$ and that α' and β' in A' are descendants of α and β in A, respectively. Then $\alpha \perp\!\!\!\perp \beta$ implies $\alpha \perp\!\!\!\perp \beta$.

The constructions of the independence in Section 6 and Section 7 are context-sensitive, i.e., $\alpha \perp\!\!\!\perp \beta$ (or $\alpha \perp\!\!\!\perp^s \beta$) cannot be decided without mentioning the proof A, which contains α and β. If $\alpha \perp\!\!\!\perp \beta$ is invariant under concatenation, restriction, and embedding, we say it is structural. If $\alpha \perp\!\!\!\perp \beta$ is also invariant under the subproof, we say it is local.

Section 5.1 shows that dominance is inherited from $\perp\!\!\!\perp$ to $\perp\!\!\!\perp^s$ (Lemma 5.3), and this section gives a sufficient condition for $\perp\!\!\!\perp^s$ as local in terms of $\perp\!\!\!\perp$ (Lemma 5.6). The adherence of $\perp\!\!\!\perp^s$ immediately follows from the construction of the subproof closure. Section 5.2 gives a sufficient condition for the preservation of $\perp\!\!\!\perp^s$ in terms of $\perp\!\!\!\perp$ (Lemma 5.10). Proving preservation of independences is the main technical difficulty in Section 6 and Section 7.

Throughout this section, we assume $\perp\!\!\!\perp$ is a symmetric binary relation defined in the reduction steps in a proof A for any proof A in an LRCTRS.

5.1 Subproof closure

Definition 5.2 The *subproof closure* $\perp\!\!\!\perp^s$ of $\perp\!\!\!\perp$ is inductively defined as the following.

(base) If $\alpha \perp\!\!\!\perp \beta$, then $\alpha \perp\!\!\!\perp^s \beta$.
(adhere) Suppose that $\alpha \perp\!\!\!\perp^s \beta$.
  (a) If $\beta' \in sub(\beta)$, then $\alpha \perp\!\!\!\perp^s \beta'$.
  (b) If $\alpha' \in sub(\alpha)$, then $\alpha' \perp\!\!\!\perp^s \beta$.
(subproof) Suppose that $\alpha, \beta \in sub(\gamma)$ for some $\gamma \in A$.
  (a) If α and β are in distinct subproofs of γ, then $\alpha \perp\!\!\!\perp^s \beta$ in A.
  (b) If $\alpha, \beta \in B$ for a subproof B of γ, then $\alpha \perp\!\!\!\perp^s \beta$ in A if $\alpha \perp\!\!\!\perp^s \beta$ in B.

[4] The symmetric assumption is added that will be used in Lemma 5.3.

Note that $\perp\!\!\!\perp^s$ is also symmetric and the subproof closure operation commutes with the union, i.e., $\perp\!\!\!\perp^s \cup \perp\!\!\!\perp'^s = (\perp\!\!\!\perp \cup \perp\!\!\!\perp')^s$.

The next lemma guarantees that dominance in Definition 5.1 is preserved by the subproof closure.

Lemma 5.3 *If $\perp\!\!\!\perp$ satisfies dominance, then the subproof closure $\perp\!\!\!\perp^s$ of $\perp\!\!\!\perp$ also satisfies dominance.*

Proof. We assume that the result does not hold. Suppose A is a minimal proof wrt $size(A)$ in which $\alpha \perp\!\!\!\perp^s \beta$ for some $\beta \in sub(\alpha)$. Also suppose that $\alpha = red_A(\sigma)$ and $\beta = red_A(\sigma')$ are minimal wrt $|\sigma| + |\sigma'|$. According to the assumption of the lemma, $\alpha \perp\!\!\!\perp^s \beta$ is inductively generated using (adhere) or (subproof). Let us consider the last step of its generation.

Neither (subproof)-(a) from $\beta \in sub(\alpha)$ nor (subproof)-(b) from the minimality of $size(A)$ performs the last step. Suppose that (adhere)-(a) performs the last step. The case of (adhere)-(b) is treated symmetrically. Then $\alpha \perp\!\!\!\perp^s \beta$ is derived from $\alpha \perp\!\!\!\perp^s \beta'$ such that $\beta \in sub(\beta')$ for some β'. From $\beta \in sub(\alpha)$, either $\beta' = \alpha$, $\beta' \in sub(\alpha)$, or $\alpha \in sub(\beta')$. However, in any case a contradiction is derived from the minimality of $|\sigma| + |\sigma'|$ using the symmetry of $\perp\!\!\!\perp^s$. □

We identify a parent of a reduction step α (wrt concatenation, embedding, and restriction) with α. If a binary relation is local, then the relation is well-defined under such an identification.

Definition 5.4 If $\perp\!\!\!\perp$ satisfies

(i) $\alpha \perp\!\!\!\perp \beta$ in A iff $\alpha \perp\!\!\!\perp \beta$ in $A'; A; A''$ for any reduction steps $\alpha, \beta \in A$, and proofs A, A', A'', and
(ii) $\alpha \perp\!\!\!\perp \beta$ in A iff $\alpha \perp\!\!\!\perp \beta$ in A/p for any reduction steps $\alpha, \beta \in A/p$, proof A, and position p satisfies $A \not\leq p$,

then $\perp\!\!\!\perp$ is called *structural*. If $\perp\!\!\!\perp$ subsequently satisfies

(iii) $\alpha \perp\!\!\!\perp \beta$ in A iff $\alpha \perp\!\!\!\perp \beta$ in B for any reduction steps α, β in any subproof B of any reduction step in any proof A,

then $\perp\!\!\!\perp$ is called *local*.

The following technical lemma is prepared for proving Lemma 5.6.

Lemma 5.5 *Let $\perp\!\!\!\perp^s$ be the subproof closure of $\perp\!\!\!\perp$. Assume that*

(i) $\alpha \not\perp\!\!\!\perp^s \beta$ if $\alpha \in sub(\beta)$ for any reduction steps α, β, and
(ii) $\alpha \perp\!\!\!\perp \beta$ in A implies $\alpha \perp\!\!\!\perp \beta$ in B for any reduction steps α, β in any subproof B of any reduction step in any proof A.

Then $\alpha \perp\!\!\!\perp^s \beta$ in A implies $\alpha \perp\!\!\!\perp^s \beta$ in B for any subproof B of a reduction step in a proof A.

Proof. We assume that the result does not hold. Suppose A is a minimal proof wrt $size(A)$ such that $\alpha \perp\!\!\!\perp^s \beta$ in A and $\alpha \not\perp\!\!\!\perp^s \beta$ in B. From (ii), $\alpha \perp\!\!\!\perp^s \beta$ in A is not derived by (base). Thus, $\alpha \perp\!\!\!\perp^s \beta$ in A is inductively generated using (adhere) or (subproof) of the subproof closure. Also suppose that $\alpha = red_A(\sigma)$ and $\beta = red_A(\sigma')$ are minimal wrt $|\sigma| + |\sigma'|$.

Suppose that (subproof)-(a) performs the last step. Then, α and β are in distinct subproofs of some reduction step γ in A. Since B is a subproof of a reduction step in A, $\gamma \in B$. Thus, $\alpha \perp\!\!\!\perp^s \beta$ in B is derived by (subproof)-(a), which leads to a contradiction. The last step is not (subproof)-(b) from the minimality of $size(A)$. Thus, we suppose that (adhere)-(a) performs the last step. The case of (adhere)-(b) is treated symmetrically. Then $\alpha \perp\!\!\!\perp^s \beta'$ in A and $\beta \in sub(\beta')$ for some β'. If $\beta' \notin B$, B must be in some subproof of β' since $\beta \in B$. Thus $\alpha \in B$ implies $\alpha \in sub(\beta')$, which violates $\alpha \perp\!\!\!\perp^s \beta'$ by (i), and $\beta' \in B$. As a result, $\alpha \perp\!\!\!\perp^s \beta'$ in B from the minimality of $|\sigma| + |\sigma'|$. Then, $\alpha \perp\!\!\!\perp^s \beta$ in B, which leads to a contradiction. □

The next lemma presents a sufficient condition for that a subproof closure is local.

Lemma 5.6 *Let $\perp\!\!\!\perp^s$ be the subproof closure of $\perp\!\!\!\perp$. Assume that*

(i) $\perp\!\!\!\perp$ satisfies dominance,
(ii) $\perp\!\!\!\perp$ is structural, and
(iii) $\alpha \perp\!\!\!\perp \beta$ in A implies $\alpha \perp\!\!\!\perp \beta$ in B for any reduction step in any subproof B of any reduction step in any proof A.

Then $\perp\!\!\!\perp^s$ is local.

Proof. From (i), $\perp\!\!\!\perp^s$ satisfies dominance by Lemma 5.3. Let B be a subproof of a reduction step in A. From (ii), it is easy to see that $\perp\!\!\!\perp^s$ is structural. Through (subproof)-(b), $\alpha \perp\!\!\!\perp^s \beta$ in B implies $\alpha \perp\!\!\!\perp^s \beta$ in A. From (iii), $\alpha \perp\!\!\!\perp^s \beta$ in A implies $\alpha \perp\!\!\!\perp^s \beta$ in B by Lemma 5.5. □

5.2 Simplifying case analysis of preservation

Here we show how to simplify the exhaustive case analysis in the proof of preservation. When proving the preservation of $\alpha \perp\!\!\!\perp^s \beta$ such that α and β are not in the same subproof of a top reduction step, we restrict ourselves to $\alpha \perp\!\!\!\perp \beta$ for $\alpha, \beta \in visible(A)$.

Definition 5.7 If $\perp\!\!\!\perp$ is a relation on visible reduction steps, that is, $\perp\!\!\!\perp \subseteq \bigcup_A visible(A) \times visible(A)$, then $\perp\!\!\!\perp$ is called a *visible relation.*

Definition 5.8 For a reduction step α in A, $\pi_A(\alpha) = \alpha$ if $\alpha \in visible(A)$; otherwise, $\pi_A(\alpha)$ is the subtop reduction step such that $\alpha \in sub(\pi_A(\alpha))$.

Lemma 5.9 *Let $\perp\!\!\!\perp^s$ be the subproof closure of a visible relation $\perp\!\!\!\perp$. For any $\alpha, \beta, \gamma \in visible(A)$, assume that*

(i) $\alpha \perp\!\!\!\perp \beta$ and $\beta' \in sub(\beta)$ imply $\alpha \perp\!\!\!\perp \beta'$, and
(ii) $\alpha \perp\!\!\!\perp \beta$ for α and β in distinct subproofs of a top reduction step in A.

Then for any $\alpha, \beta \in A$ not in the same subproof of a top reduction step in A, $\alpha \perp\!\!\!\perp^s \beta$ iff $\pi_A(\alpha) \perp\!\!\!\perp \pi_A(\beta)$.

Proof. If-part is trivial. We assume that the only-if part does not hold. Suppose A is a minimal proof wrt $size(A)$ in which $\alpha \perp\!\!\!\perp^s \beta$ and $\pi_A(\alpha) \not\perp\!\!\!\perp \pi_A(\beta)$ for some α, β not in the same subproof of a top reduction step in A. Also suppose that $\alpha = red_A(\sigma)$ and $\beta = red_A(\sigma')$ are minimal wrt $|\sigma| + |\sigma'|$. Since $\perp\!\!\!\perp$ is a visible relation and $\pi_A(\alpha) \not\perp\!\!\!\perp \pi_A(\beta)$, $\alpha \perp\!\!\!\perp^s \beta$ cannot be derived by (base). Then $\alpha \perp\!\!\!\perp^s \beta$ is inductively generated using (adhere) or (subproof) of the subproof closure. Consider the last step of its generation.

Neither (subproof)-(a) from (ii) nor (subproof)-(b) performs the last step since α and β are not in the same subproof. Suppose that the last step is (adhere)-(a). The case of (adhere)-(b) is treated symmetrically. Then $\alpha \perp\!\!\!\perp^s \beta'$ and $\beta \in sub(\beta')$ for some β'. From the minimality of $|\sigma| + |\sigma'|$, $\pi_A(\alpha) \perp\!\!\!\perp \pi_A(\beta')$. Since $\beta \in sub(\beta')$, either $\pi_A(\beta) = \pi_A(\beta')$ or $\pi_A(\beta) \in subtop(\beta')$. In either case, $\pi_A(\alpha) \perp\!\!\!\perp \pi_A(\beta)$ holds from (i), which leads to a contradiction. □

Lemma 5.10 *Let $\perp\!\!\!\perp^s$ be the subproof closure of a visible relation $\perp\!\!\!\perp$. Let $A \mapsto A'$ be a peak elimination rule with proofs A, A' in an LRCTRS. Assume that*

(i) $\alpha \perp\!\!\!\perp \beta$ and $\beta' \in sub(\beta)$ imply $\alpha \perp\!\!\!\perp \beta'$ for $\alpha, \beta, \beta' \in visible(A)$,
(ii) $\alpha \perp\!\!\!\perp \beta$ for visible reduction steps α and β in distinct subproofs of a top reduction step in A,
(iii) for any $\gamma \in A \setminus visible(A)$, if γ has a descendant γ' in A' then $\pi_A(\gamma)$ has a descendant γ'' in A' with $\gamma' \in sub(\gamma'')$, and
(iv) for any $\alpha, \beta \in visible(A)$ not in the same subproof of a top reduction step in A, if α and β have descendants α' and β', respectively, then $\alpha \perp\!\!\!\perp \beta$ implies $\alpha' \perp\!\!\!\perp^s \beta'$.

Then, $A \mapsto A'$ preserves $\alpha_1 \perp\!\!\!\perp^s \alpha_2$ for any $\alpha_1, \alpha_2 \in A$ not in the same subproof of a top reduction step in A.

Proof. Suppose reductions $\alpha_1, \alpha_2 \in A$ are not in the same subproof of a top reduction step and $\alpha_1 \perp\!\!\!\perp^s \alpha_2$. Also suppose that α_i has a descendant α'_i for $i = 1, 2$. By Lemma 5.9, $\pi_A(\alpha_1) \perp\!\!\!\perp \pi_A(\alpha_2)$ from (i) and (ii). If α_i is a visible reduction step then let $\alpha''_i = \alpha'_i$; otherwise, let α''_i be a descendant of $\pi_A(\alpha_i)$ such that $\alpha'_i \in sub(\alpha''_i)$ (from (iii)) for $i = 1, 2$. Then, $\alpha''_1 \perp\!\!\!\perp^s \alpha''_2$ from (iv), and we obtain $\alpha'_1 \perp\!\!\!\perp^s \alpha'_2$ by (adhere) of the subproof closure. □

6 The Church-Rosser property of non-overlapping LRCTRS

In this section, we define a PES of a non-overlapping LRCTRS and construct its independence $\perp\!\!\!\perp_1$. Therefore, by the result of Section 4, we complete SN of the PES and hence CR of a non-overlapping LRCTRS. For CR of non-overlapping semi-equational CTRSs without extra variables, see [1]. Throughout this section, $\hat{R}$ denotes a non-overlapping LRCTRS.

6.1 Peak elimination of non-overlapping LRCTRS

Lemma 6.1 *Let AS_i be the ith subproof of a reduction step with a rule $\hat{S}$: $\hat{l} \to \hat{r} \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ for $i = 1, \ldots, n$. Suppose that $\bar{r}\{x_i := y_i \mid i = 1, \ldots, n\} \equiv \hat{r}$ and $\bar{r} \equiv C_{\bar{r}}[x_{j_1}, \ldots x_{j_m}]$. Then $C_{\hat{r}}[\![AS_{j_1}, \ldots, AS_{j_m}]\!]$ is a proof of the form $\bar{r}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{S}} \hat{r}\theta$.*

Proof. Let $\hat{r} \equiv C_{\hat{r}}[y_{k_1}, \ldots y_{k_m}]$. Note that $C_{\bar{r}}[\ ] \equiv C_{\hat{r}}[\ ]$. Then, $y_{j_i} \equiv x_{j_i}\{x_i := y_i \mid i = 1, \ldots, n\} \equiv y_{k_i}$ for $i = 1, \ldots, m$, and the result follows. □

Definition 6.2 We define a PES $\mathcal{P}_{\hat{R}}$ of $\hat{R}$ as the following. Let $\hat{S}_i : \hat{l}_i \to \hat{r}_i \Leftarrow x_{i1} = y_{i1}, \cdots, x_{in_i} = y_{in_i}$ be any rewrite rule in $\hat{R}$ ($i = 1, 2$), and let $A : t_1 \overset{\nabla}{\leftarrow}_{\hat{S}_1} t_2 \overset{\nabla}{\rightarrow}_{\hat{S}_2} t_3$ be any peak, where $t_1 \equiv C[\hat{r}_1\theta]$, $t_2 \equiv C[\hat{l}_1\theta] \equiv C'[\hat{l}_2\theta]$, and $t_3 \equiv C'[\hat{r}_2\theta]$. We denote $t_1 \overset{\nabla}{\leftarrow}_{\hat{S}_1} t_2$ and $t_2 \overset{\nabla}{\rightarrow}_{\hat{S}_2} t_3$ as γ_1 and γ_2, respectively. Then, $\mathcal{P}_{\hat{R}}$ has the peak elimination rule $A \overset{J}{\mapsto} A'$ as follows. Peak elimination rules are classified into three categories; $P_{||}$, $P_{<}$, and $P_{\sharp}$, according to the relative positions of the redexes of γ_1 and γ_2.

($P_{||}$) **[parallel]** If the redexes of γ_1 and γ_2 are parallel, then $A' = t_1 \overset{\nabla}{\rightarrow}_{\hat{S}_2} t'_2 \overset{\nabla}{\leftarrow}_{\hat{S}_1} t_3$ is obtained by exchanging the order of γ_1 and γ_2. For any reduction step $\alpha \in A'$,

$$J(\alpha) = \begin{cases} red_A(1\sigma) & \text{if } \alpha = red_A(2\sigma), \\ red_A(2\sigma) & \text{if } \alpha = red_A(1\sigma). \end{cases}$$

$(P_<)$ **[nest]** Suppose that γ_2's redex nests γ_1's and that the redex of γ_1 occurs below the substitution part $x_{2k}\theta$. Also suppose that $C'[\,]/p \equiv \Box$ and $\hat{l}_2/q \equiv x_{2k}$. Then, $A' = t_1 \overset{\nabla}{\rightarrow}_{\hat{S}_2} t_3$, which is the same as γ_2 except for the kth subproof modified into $t_1/p; q \overset{\nabla}{\leftarrow}_{\hat{S}_1} x_{2k}\theta \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} y_{2k}\theta$. For any reduction step $\alpha \in A'$,

$$J(\alpha) = \begin{cases} red_A(1\sigma) & \text{if } \alpha = red_{A'}(1k1\sigma), \\ red_A(2kj\sigma) & \text{if } \alpha = red_{A'}(1k\{j+1\}\sigma) \text{ for some } j \geq 2, \\ red_A(2\sigma) & \text{otherwise, assume } \alpha = red_{A'}(1\sigma). \end{cases}$$

The case when γ_1's redex nests γ_2's is dealt with symmetrically.

$(P_\sharp)$ **[critical]** Suppose that the redexes of γ_1 and γ_2 are overlapping. This kind of peak is called *critical*. Since $\hat{R}$ is non-overlapping, $C[\,] \equiv C'[\,]$, and S_1 and S_2 are the same modulo renaming of variables. Since different subproofs yield different reducts in LRCTRSs, t_1 and t_3 can still be different.

Let $AS_{1j} : y_{1j}\theta \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} x_{1j}\theta$ be the jth subproof of γ_1 ($j = 1, \ldots, n_1$), and let $AS_{2j'} : x_{2j'}\theta \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} y_{2j'}\theta$ be the j'th subproof of γ_2 ($j' = 1, \ldots, n_2$). Let $\hat{r}_i \equiv C_{\hat{r}_i}[y_{i1}, \ldots, y_{ik}]$ ($i = 1, 2$). $C[\![C_{\hat{r}_1}[\![AS_{1j_1}, \ldots, AS_{1j_k}]\!]]\!]$ is of the form $t_1 \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} C[C_{\hat{r}_1}[x_{1j_1}\theta, \ldots, x_{j_k}\theta]]$, and $C[\![C_{\hat{r}_2}[\![AS_{2j_1}, \ldots, AS_{2j_k}]\!]]\!]$ is of the form $C[C_{\hat{r}_2}[x_{2j_1}\theta, \ldots, x_{2j_k}\theta]] \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_3$ by Lemma 6.1. Moreover, $C[C_{\hat{r}_1}[x_{1j_1}\theta, \ldots, x_{j_k}\theta]] \equiv C[C_{\hat{r}_2}[x_{2j_1}\theta, \ldots, x_{2j_k}\theta]]$. Thus, we define A' as

$$C[\![C_{\hat{r}_1}[\![AS_{1j_1}, \ldots, AS_{1j_k}]\!]]\!]; C[\![C_{\hat{r}_2}[\![AS_{2j_1}, \ldots, AS_{2j_k}]\!]]\!].$$

For a reduction step $\alpha \in A'$, if $\alpha = red_{AS_{ij}}(\sigma)$, then $J(\alpha) = red_A(ij\sigma)$.

Note 6.3 (i) γ_1 and γ_2 have no descendants when $P_\sharp$ is applied to a critical peak $\gamma_1; \gamma_2$.

(ii) α may have multiple descendants only when $P_\sharp$ is applied to a critical peak $\gamma_1; \gamma_2$ and $\alpha \in sub^-(\gamma_1) \cup sub^-(\gamma_2)$.

We often refer to the rules in $P_{||}$ ($P_<$, $P_\sharp$, respectively) as simply $P_{||}$ ($P_<$, $P_\sharp$, respectively). Observe that $P_{||}$ and $P_<$ are simple and injective while $P_\sharp$ is root-erasing.

Example 6.4 We illustrate the peak elimination steps shown above in examples. Let $\hat{R}$ be the same as the one in Example 3.4 and let the ith rule from the top be $\hat{S}_i$ ($i = 1, 2, 3$). Suppose that there are proofs $AS_1 : 1 \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} s$, $AS_2 : t \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} s$, and $AS_3 : u \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t$ for some terms s, t, and u. In the following, a line under a subterm (a line over a subterm, respectively) indicates the redex of the left-oriented (right-oriented, respectively) reduction step in the peak.

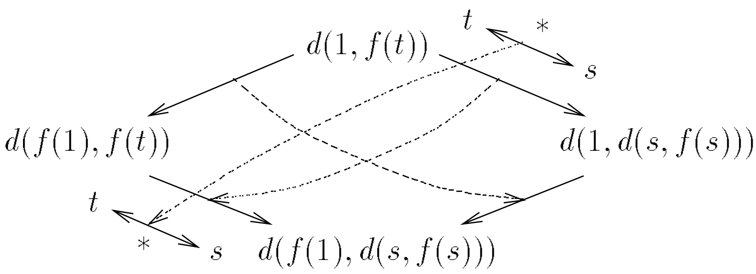

Fig. 5. Rule $P_{\|}$

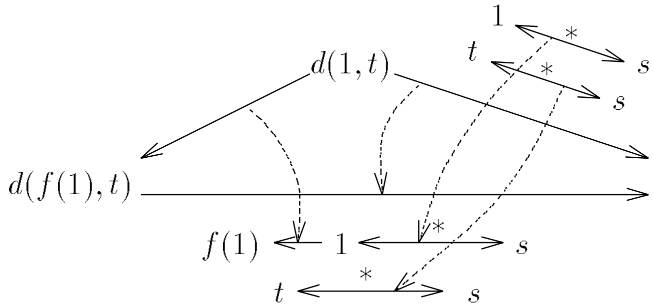

Fig. 6. Rule $P_<$

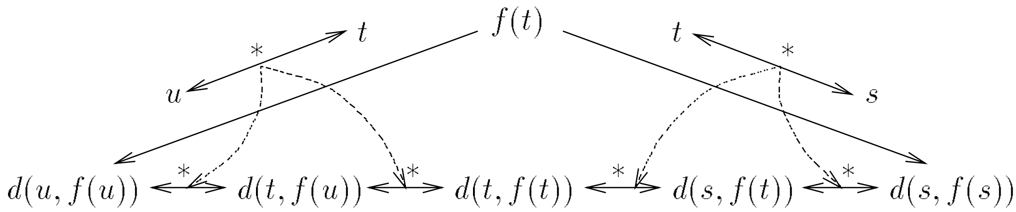

Fig. 7. Rule $P_\sharp$

($P_{\|}$) Let us consider a peak $d(f(1), f(t)) \overset{\nabla}{\leftarrow}_{\hat{S}_3} d(\underline{1}, \overline{f(t)}) \overset{\nabla}{\rightarrow}_{\hat{S}_2} d(1, d(s, f(s)))$, where the right-oriented reduction step has a subproof AS_2. Then, it is replaced with $d(f(1), f(t)) \overset{\nabla}{\rightarrow}_{\hat{S}_2} d(f(1), d(s, f(s))) \overset{\nabla}{\leftarrow}_{\hat{S}_3} d(1, d(s, f(s)))$ using $P_{\|}$ as shown in Fig. 5.

($P_<$) Consider a peak $d(f(1), t) \overset{\nabla}{\leftarrow}_{\hat{S}_3} \overline{d(\underline{1}, t)} \overset{\nabla}{\rightarrow}_{\hat{S}_1} 0$, where the right-oriented reduction step has the subproofs AS_1 and AS_2. Using $P_<$, the peak is replaced with $d(f(1), t) \overset{\nabla}{\rightarrow}_{\hat{S}_1} 0$ as shown in Fig. 6.

($P_\sharp$) With the different subproofs AS_2 and AS_3, $\hat{S}_2$ has a critical peak $d(u, f(u)) \overset{\nabla}{\leftarrow}_{\hat{S}_2} \underline{\overline{f(t)}} \overset{\nabla}{\rightarrow}_{\hat{S}_2} d(s, f(s))$. The peak is replaced with the concatenation of $d(u, f(u)) \overset{\nabla}{\leftrightarrow}_{\hat{R}} d(t, f(u)) \overset{\nabla}{\leftrightarrow}_{\hat{R}} d(t, f(t))$ and $d(t, f(t)) \overset{\nabla}{\leftrightarrow}_{\hat{R}} d(s, f(t)) \overset{\nabla}{\leftrightarrow}_{\hat{R}} d(s, f(s))$ using $P_\sharp$ as shown in Fig. 7.

The following lemma directly follows from the definition of peak elimination rules.

Lemma 6.5 *The PES $\mathcal{P}_{\hat{R}}$ of a non-overlapping LRCTRS $\hat{R}$ defined above works monotonically downwards wrt the position, that is, if a peak $\gamma_1; \gamma_2$ is* *replaced with A by $\mathcal{P}_{\hat{R}}$, then $\alpha \geq \gamma_1$ or $\alpha \geq \gamma_2$ for any reduction step $\alpha \in top(A)$.*

6.2 Independence $\perp\!\!\!\perp_1$ of non-overlapping LRCTRS

Here we introduce a relation $\perp\!\!\!\perp_1$, which is proved to be an independence for a PES $\mathcal{P}_{\hat{R}}$ of a non-overlapping LRCTRS $\hat{R}$. Intuitively, $\alpha \perp\!\!\!\perp_1 \beta$ means that α and β are separated by positions, that is, their positions are parallel to each other and no intermediate reduction steps *cover* either of them. This is first defined for the top reduction steps, then extended to the visible reduction steps using flattening, and finally extended to all reduction steps by the subproof closure.

Definition 6.6 Let $A : t_1 \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_n$ be a proof in $\hat{R}$, and let α_i be the ith top reduction step. Suppose $j \neq k$. An *open interval* $A(\alpha_j, \alpha_k)$ is the subsequence $t_{j+1} \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_k$ of A if $j < k$, and otherwise $t_{k+1} \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_j$. A *closed interval* $A[\alpha_j, \alpha_k]$ is $t_j \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_{k+1}$ if $j < k$, and otherwise $t_k \overset{\nabla}{\leftrightarrow}{}^{*}_{\hat{R}} t_{j+1}$.

In an open interval, instead of the reduction step α_i, we admit the term t_i. They are defined as the following: $A(\alpha_j, \alpha_k) = A(\alpha_j, t_k) = A(t_{j+1}, \alpha_k) = A(t_{j+1}, t_k)$ if $j < k$, and otherwise $A(\alpha_j, \alpha_k) = A(\alpha_j, t_{k+1}) = A(t_j, \alpha_k) = A(t_j, t_{k+1})$.

Definition 6.7 For any proof A in $\hat{R}$ and $\alpha, \beta \in top(A)$, $\alpha \perp\!\!\!\perp_1^{\mathrm{T}} \beta$ *in* A if $A[\alpha, \beta] \not\leq \alpha \wedge \beta$, that is, no reduction steps in $A[\alpha, \beta]$ occur above or equal to $\alpha \wedge \beta$.

$\alpha \not\perp\!\!\!\perp_1^{\mathrm{T}} \alpha$ in A for any $\alpha \in top(A)$ since $A[\alpha, \alpha] = \alpha$ and $pos(\alpha) = \alpha \wedge \alpha$. If the proof A is clear from the context, "in A" is often omitted. There are some direct consequences of the above definition: (1) $\perp\!\!\!\perp_1^{\mathrm{T}}$ is symmetric, and (2) $\alpha \perp\!\!\!\perp_1^{\mathrm{T}} \beta$ implies $\alpha \parallel \beta$. In addition, $\gamma \not\leq \alpha \wedge \beta$ if $\gamma \parallel \alpha$ or $\gamma \parallel \beta$.

Lemma 6.8 *Let $\alpha, \beta, \gamma \in top(A)$. Suppose that $\beta \in A(\alpha, \gamma)$ and $\beta \leq A[\alpha, \beta]$. Then $\alpha \perp\!\!\!\perp_1^{\mathrm{T}} \gamma$ iff $\beta \perp\!\!\!\perp_1^{\mathrm{T}} \gamma$.*

Proof. If $\alpha \perp\!\!\!\perp_1 \gamma$, $\beta \not\leq \alpha \wedge \gamma$. Since $\beta \leq A[\alpha, \beta]$, $\beta \leq \alpha$. Thus, $\beta \wedge \gamma = \alpha \wedge \gamma$, and $\beta \perp\!\!\!\perp_1 \gamma$.

If $\beta \perp\!\!\!\perp_1 \gamma$, then $\beta \parallel \gamma$. Since $\beta \leq A[\alpha, \beta]$, $\beta \leq \alpha$, then $\beta \wedge \gamma = \alpha \wedge \gamma$. Since $\beta \wedge \gamma < \beta \leq A[\alpha, \beta]$, then $\alpha \perp\!\!\!\perp_1 \gamma$. □

We extend $\perp\!\!\!\perp_1^{\mathrm{T}}$ to visible reduction steps using flattening. Note that $\flat_A(\alpha) \in top(A^\flat)$ iff $\alpha \in visible(A)$.

Definition 6.9 For $\alpha, \beta \in visible(A)$, $\alpha \perp\!\!\!\perp_1^{\mathrm{M}} \beta$ in A if $\flat_A(\alpha) \perp\!\!\!\perp_1^{\mathrm{T}} \flat_A(\beta)$ in $A^\flat$.

Lemma 6.10 $\perp\!\!\!\perp_1^{\mathrm{T}}$ *and* $\perp\!\!\!\perp_1^{\mathrm{M}}$ *are structural.*

Note 6.11 By definition, it is easy to see that the PES and $\perp\!\!\!\perp_1^{\mathrm{M}}$ satisfy the assumptions (i), (ii), (iii) in Lemma 5.10. In the sequel, we will apply Lemma 5.10 without mentioning these assumptions.

Definition 6.12 We define $\perp\!\!\!\perp_1$ as the subproof closure of $\perp\!\!\!\perp_1^{\mathrm{M}}$.

Example 6.13 For proofs $A_i : s_i \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s'_i$, let us consider an embedding $A : C[\![A_1, \ldots, A_n]\!]$. Let α be a reduction step in an A_i-segment of the embedding and β be one in an A_j-segment such that $i \neq j$. If both α and β are top reduction steps in A, then $\alpha \perp\!\!\!\perp_1 \beta$ since $A \not\leq \alpha \wedge \beta$. Otherwise, $\alpha \perp\!\!\!\perp_1 \beta$ is derived by (adhere) of the subproof closure. Therefore, $\alpha \perp\!\!\!\perp_1 \beta$ in A.

We show that a non-overlapping LRCTRS is CR by proving that $\perp\!\!\!\perp_1$ is an independence.

Theorem 6.14 *The relation* $\perp\!\!\!\perp_1$ *is an independence for the PES* $\mathcal{P}_{\hat{R}}$ *of a non-overlapping LRCTRS* $\hat{R}$.

To prove the above theorem, we show in the following that $\perp\!\!\!\perp_1$ satisfies the four properties in Definition 5.1. The hardest part is (iv) preservation, which will be proved in Section 6.3.

Lemma 6.15 $\perp\!\!\!\perp_1$ *satisfies dominance.*

Proof. Let $\alpha, \beta \in visible(A)$. If $\beta \in sub(\alpha)$ then $\flat_A(\alpha) \leq \flat_A(\beta)$ form Lemma 3.11. Thus, $\alpha \not\perp\!\!\!\perp_1^{\mathrm{M}} \beta$. Therefore, the dominance of $\perp\!\!\!\perp_1$ follows from Lemma 5.3. □

Lemma 6.16 $\perp\!\!\!\perp_1$ *is local.*

Proof. By Lemma 5.6 and Lemma 6.10. □

Lemma 6.17 $\perp\!\!\!\perp_1$ *satisfies adherence.*

Proof. Obvious since $\perp\!\!\!\perp_1$ satisfies (adhere) of the subproof closure. □

Lemma 6.18 $\perp\!\!\!\perp_1$ *satisfies non-incest.*

Proof. By using Note 6.3, it is sufficient to consider the descendants of α when $P_\sharp$ is applied to some critical peak $\gamma_1; \gamma_2$ and $\alpha \in sub^-(\gamma_1) \cup sub^-(\gamma_2)$. However, two distinct descendants of α, α_1, α_2 reside in different segments of the embedding in the replacement sequence for the peak. Therefore, $\alpha_1 \perp\!\!\!\perp_1 \alpha_2$. (See Example 6.13.) □

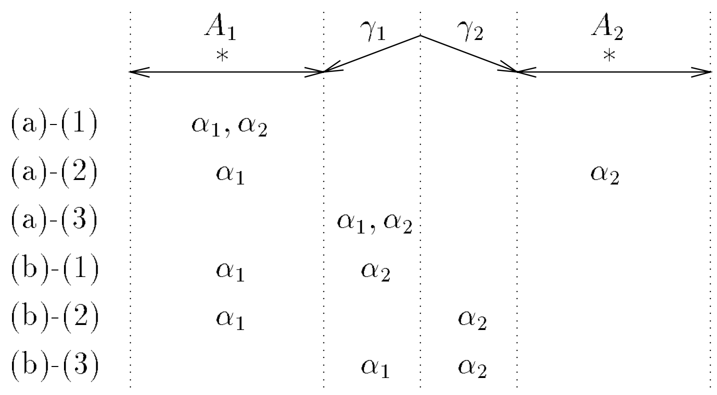

Fig. 8. Case analysis in Lemma 6.16

6.3 Preservation of independence $\perp\!\!\!\perp_1$

Lemma 6.19 $\perp\!\!\!\perp_1$ *satisfies preservation.*

Proof. Let α_1 and α_2 be reduction steps in a proof A. Suppose that $A \mapsto A'$, and that α'_i in A' is a descendant of α_i $(i = 1, 2)$.

Let $A = A_1; \gamma_1; \gamma_2; A_2$, where $\gamma_1; \gamma_2$ is the eliminated peak. Let $B_1; B_2$ be the replacement sequence. We perform a case analysis depending on where α_1 and α_2 occur in A. The following cases must be considered:

(a)-(1) $\alpha_1, \alpha_2 \in A_1$.
(a)-(2) $\alpha_i \in A_i$ $(i = 1, 2)$.
(a)-(3) $\alpha_1, \alpha_2 \in sub(\gamma_1)$.
(b)-(1) $\alpha_1 \in A_1$ and $\alpha_2 \in sub(\gamma_1)$.
(b)-(2) $\alpha_1 \in A_1$ and $\alpha_2 \in sub(\gamma_2)$.
(b)-(3) $\alpha_i \in sub(\gamma_i)$ $(i = 1, 2)$.

Fig. 8 shows an analysis of the above cases. The other cases are obtained by exchanging α_1 and α_2 and/or reversing A. Claim 6.20, Claim 6.21, Claim 6.22, and Claim 6.23 prove preservation.

Claim 6.20 *Preservation of* $\perp\!\!\!\perp_1$ *holds in cases (a)-(1) to (3).*

Proof. In case (a)-(1), preservation follows since $\perp\!\!\!\perp_1$ is local by Lemma 6.16. In case (a)-(2), preservation holds since $\mathcal{P}_{\hat{R}}$ works monotonically downwards because of Lemma 6.5. In case (a)-(3), if α_1 and α_2 are in the same subproof and α'_1 and α'_2 are in the same copy of that subproof, preservation again follows since $\perp\!\!\!\perp_1$ is local by Lemma 6.16. Otherwise, α'_1 and α'_2 are in different segments of the embedding, so $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$. $\square$

Claim 6.21 *Preservation of* $\perp\!\!\!\perp_1$ *holds in cases (b)-(1) to (3) when* $P_{\|}$ *is applied.*

Proof. When $P_{\|}$ is applied, B_i is the same as γ_{3-i} except for the context ($i = 1, 2$). In case (b)-(1), preservation is obvious since $B_1 \parallel B_2$. In case (b)-(2), preservation is obvious since the descendant of γ_1 simply disappears from $A'(\alpha'_1, \alpha'_2)$. In case (b)-(3), $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$ in A' since $B_1 \perp\!\!\!\perp_1^{\mathrm{T}} B_2$ in A'. □

Claim 6.22 *Preservation of $\perp\!\!\!\perp_1$ holds in cases (b)-(1) to (3) when $P_<$ is applied.*

Proof. Without loss of generality, we can assume that $\alpha_1, \alpha_2 \in \mathit{visible}(A)$ and $\alpha_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha_2$ by Lemma 5.10.

Suppose γ_2 nests γ_1. Then B_1 is empty and B_2 is γ_2 with γ_1 prefixed to the kth subproof for some k. Let AS_i be the ith subproof of γ_2 for $i = 1, \ldots, n$. Then, the kth subproof of B_2 is $\gamma_1/p; AS_k$ for a position p. The descendant γ'_1 of γ_1 is a subtop reduction step in A' and $pos(\gamma_1) = pos(\gamma'_1)$.

The case $\alpha_2 = \gamma_1$ from Lemma 3.11 and Lemma 6.8 is sufficient for case (b)-(1). Then, $\alpha_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha_2$ implies $\alpha'_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha'_2$ since $\beta \parallel \flat_{B_2}(\alpha'_2)$ for any reduction step β in an AS_i-segment in $B_2^\flat$ ($i = 1, \ldots, k-1$).

In case (b)-(2), if α'_2 is in AS_i-segment such that $i \neq k$, then the preservation is obvious since $\gamma'_1 \parallel \alpha'_2$. If α'_2 is in the AS_k-segment, the preservation follows from $pos(\gamma_1) = pos(\gamma'_1)$.

Let us consider (b)-(3). We borrow notations from Definition 6.2, so $\gamma_2 = C_2[\hat{l}_2\theta] \xrightarrow{\nabla}_{\hat{S}_2} C_2[\hat{r}_2\theta]$. If $\alpha_2 = \gamma_2$, then $\alpha_1 \not\perp\!\!\!\perp_1 \alpha_2$. Suppose $\alpha_2 \in AS_j$. If $j \neq k$, then $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$ in A'. If $j = k$, then

$$\begin{aligned}
& \alpha_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha_2 \text{ in } A, \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } A^\flat && \text{(by definition)}, \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } \gamma_1^\flat; C_2[\![C_{\hat{l}_2}[\![AS_1, \ldots, AS_{n_1}]\!]]\!], \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } \gamma_1^\flat/p; AS_k && \text{(since } \perp\!\!\!\perp_1^{\mathrm{T}} \text{ is structural)}, \\
\Leftrightarrow\ & \alpha'_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha'_2 \text{ in } \gamma_1/p; AS_k, \\
\Rightarrow\ & \alpha'_1 \perp\!\!\!\perp_1 \alpha'_2 \text{ in } A' && \text{(by definition)}.
\end{aligned}$$

In the step from the third line to the fourth line, we used the fact that $\gamma_1^\flat/p; AS_k = \gamma_1^\flat; C_2[\![C_{\hat{l}_2}[\![AS_1, \ldots, AS_{n_1}]\!]]\!]/p$.

Next suppose that γ_1 nests γ_2. The case $\alpha_2 = \gamma_1$ from Lemma 6.8 is sufficient for cases (b)-(1) and (2), so preservation is obvious. In case (b)-(3), the result is proved in the same way as when γ_2 nests γ_1. □

Claim 6.23 *Preservation of $\perp\!\!\!\perp_1$ holds in cases (b)-(1) to (3) when $P_\sharp$ is applied.*

Proof. When $P_\sharp$ is applied, B_i consists of multiple copies of subproofs of γ_i $(i = 1, 2)$. We borrow notations in $(P_\sharp)$ of Definition 6.2, so $B_1 = C[\![C_{\hat{r}_1}[\![AS_{1j_1}, \ldots, AS_{1j_k}]\!]]\!]$ and $B_2 = C[\![C_{\hat{r}_2}[\![AS_{2j_1}, \ldots, AS_{2j_k}]\!]]\!]$. Without loss of generality, we can assume that $\alpha_1, \alpha_2 \in \mathit{visible}(A)$, and $\alpha_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha_2$ by Lemma 5.10.

The case of $\alpha_2 = \gamma_1$ from Lemma 3.11 and Lemma 6.8 is sufficient for cases (b)-(1) and (2). Thus, preservation follows since $\mathcal{P}_{\hat{R}}$ works monotonically downwards according to Lemma 6.5.

In case (b)-(3), suppose that α_1' is in an AS_{1j_n}-segment and α_2' is in an AS_{2j_m}-segment. If $n \neq m$, then $\alpha_1' \perp\!\!\!\perp_1 \alpha_2'$. Otherwise,

$$
\begin{aligned}
& \alpha_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha_2 \text{ in } A, \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } A^\flat && \text{(by definition)}, \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } C[\![C_{\hat{l}_1}[\![AS_{11}, \ldots, AS_{1n_1}]\!]]\!]; C[\![C_{\hat{l}_2}[\![AS_{21}, \ldots, AS_{2n_2}]\!]]\!], \\
\Leftrightarrow\ & \alpha_1 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2 \text{ in } AS_{1j_n}; AS_{2j_n} && \text{(since } \perp\!\!\!\perp_1^{\mathrm{T}} \text{ is structural)}, \\
\Leftrightarrow\ & \alpha_1' \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2' \text{ in } B_1; B_2 && \text{(since } \perp\!\!\!\perp_1^{\mathrm{T}} \text{ is structural)}, \\
\Leftrightarrow\ & \alpha_1' \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_2' \text{ in } A',
\end{aligned}
$$

We used the fact that $AS_{1j_n}; AS_{2j_n}$ is a restriction of $C[\![C_{\hat{l}_1}[\![AS_{11}, \ldots, AS_{1n_1}]\!]]\!]$; $C[\![C_{\hat{l}_2}[\![AS_{21}, \ldots, AS_{2n_2}]\!]]\!]$ in the step from the third line to the fourth line, and the fact that it is also a restriction of $B_1; B_2$ in the step from the fourth line to the fifth line. □

Proof of Theorem 6.14 By Lemma 6.15, Lemma 6.17, Lemma 6.18, and Lemma 6.19. □

Theorem 6.24 *A non-overlapping LRCTRS is CR.*

Proof. By Lemma 4.6, Theorem 4.17, and Theorem 6.14. □

7 The Church-Rosser property of compatible LRCTRS

We now extend the results in the previous section to compatible LRCTRSs by supplementing the argument of the overlapping case. As a consequence, we derive CR of a compatible LRCTRS, and hence UN of a compatible TRS. In this section, $\hat{R}$ denotes a compatible LRCTRS.

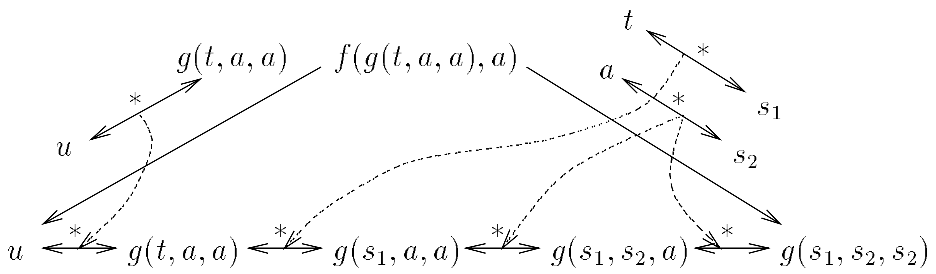

Fig. 9. Rule $P_\sharp$

7.1 Peak elimination of compatible LRCTRS

Definition 7.1 The definition of a PES $\mathcal{P}_{\hat{R}}$ of a compatible LRCTRS $\hat{R}$ is obtained by replacing the description of $P_\sharp$ in Definition 6.2 with the following.

($P_\sharp$) **[critical]** Suppose redexes of γ_1 and γ_2 are overlapping. This kind of peak is called *critical*. Since $\hat{R}$ is compatible, $C[\,] \equiv C'[\,]$. Let $AS_{1j} : y_{1j}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} x_{1j}\theta$ be the jth subproof of γ_1 ($j = 1, \ldots, n_1$), and let $AS_{2j'} : x_{2j'}\theta \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} y_{2j'}\theta$ be the j'th subproof of γ_2 ($j' = 1, \ldots, n_2$). Let $\langle \bar{r}_1, \bar{r}_2 \rangle$ be the standard pair of compatibility of $\langle S_1, S_2 \rangle$. Let $\bar{r}_1 \equiv C_{\hat{r}_1}[x_{1j_1}, \ldots, x_{1j_k}]$ and let $\bar{r}_2 \equiv C_{\hat{r}_2}[x_{2j'_1}, \ldots, x_{2j'_{k'}}]$. Note that $\bar{r}_1\theta \equiv \bar{r}_2\theta$ as a result of compatibility. Then

$$A' = C[\![C_{\hat{r}_1}[\![AS_{1j_1}, \ldots, AS_{1j_k}]\!]]\!]; C[\![C_{\hat{r}_2}[\![AS_{2j'_1}, \ldots, AS_{2j'_{k'}}]\!]]\!].$$

For the reduction step $\alpha \in A'$, if $\alpha = red_{AS_{ij}}(\sigma)$, then $J(\alpha) = red_A(ij\sigma)$.

Observe that the $P_\sharp$ is root-erasing. Lemma 6.5 also holds for a compatible case.

Lemma 7.2 *The PES $\mathcal{P}_{\hat{R}}$ of a compatible LRCTRS $\hat{R}$ defined above works monotonically downwards wrt the position.*

Example 7.3 Let $\hat{R}$ be the following compatible LRCTRS:

$$\hat{R} = \left\{ \begin{array}{lll} \hat{S}_1 : f(x_1, a) \to y_1 & \Leftarrow x_1 = y_1, \\ \hat{S}_2 : f(g(x'_1, a, a), x'_2) \to g(y'_1, y'_2, y'_2) & \Leftarrow x'_1 = y'_1, x'_2 = y'_2 \end{array} \right\}.$$

The standard pair of compatibility of $\langle \hat{S}_1, \hat{S}_2 \rangle$ is $\langle x_1, g(x'_1, x'_2, x'_2) \rangle$. Suppose that $u \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(t, a, a)$, $a \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_2$, and $t \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_1$. Then, there is a critical peak $u \overset{\nabla}{\leftarrow}_{\hat{S}_1} f(g(t, a, a), a) \overset{\nabla}{\rightarrow}_{\hat{S}_2} g(s_1, s_2, s_2)$. Using $P_\sharp$, the peak is replaced with $u \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(t, a, a) \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(s_1, a, a) \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(s_1, s_2, a) \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(s_1, s_2, s_2)$ as shown in Fig. 9.

Unfortunately, the independence $\perp\!\!\!\perp_1$ defined in the previous sections does not work for compatible systems. In the above example, let us consider the reduction steps α_1 in $u \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(t,a,a)$ corresponding to $x_1 = y_1$, and α_2 in $a \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_2$ corresponding to $x'_2 = y'_2$. Then $\alpha_1 \perp\!\!\!\perp_1 \alpha_2$ in the peak. However, if α_1 touches the head position (i.e., $pos(\alpha_1) = \epsilon$), then $\alpha'_1 \not\perp\!\!\!\perp_1 \alpha'_2$ for any descendants α'_i of α_i $(i = 1, 2)$.

In the next section, we introduce a modified version of an independence $\perp\!\!\!\perp_*$. Before introducing $\perp\!\!\!\perp_*$, we will explain its key idea in Example 7.12 by using Example 7.3 again.

Definition 7.4 A term t is a *head normal form* of $\hat{R}$ if s is not a redex of $\hat{R}$ for all s such that $t \overset{\nabla}{\rightarrow}{}^*_{\hat{R}} s$.[5]

Lemma 7.5 *Let $\hat{l}$ be the left-hand side of any rule in a compatible LRCTRS $\hat{R}$. For any non-variable proper subterm t of $\hat{l}$ and any substitution θ, $t\theta$ is a head normal form of $\hat{R}$.*

Proof. This is obtained by a straightforward induction on the length of the reduction sequence $t\theta \overset{\nabla}{\rightarrow}{}^*_{\hat{R}} s$ since $\hat{R}$ is an overlay. $\square$

In fact, such a property holds for any overlay semi-equational CTRS.

Definition 7.6 For unifiable terms t_1 and t_2, the set of *minimal variable positions* MV_{t_1,t_2} is the set of all minimal elements wrt $\leq$ in $\{p \mid t_1/p \in V$ or $t_2/p \in V\}$.

Note that $MV_{t_1,t_2} = MV_{t_2,t_1}$.

Lemma 7.7 *Let $\hat{S}_i : \hat{l}_i \to \hat{r}_i \Leftarrow Q_i$ be the rewrite rules in a compatible LRCTRS $\hat{R}$ for $i = 1, 2$. Suppose $\hat{l}_1$ and $\hat{l}_2$ are unifiable and $\langle \bar{r}_1, \bar{r}_2 \rangle$ is the standard pair of compatibility of $\langle \hat{S}_1, \hat{S}_2 \rangle$. Assume that $\tilde{q} \in MV_{\bar{r}_1,\bar{r}_2}$ and $\bar{r}_1/\tilde{q} \equiv \hat{l}_1/\tilde{p} \in V$. Let θ be any unifier of $\hat{l}_1$ and $\hat{l}_2$.*

(i) Suppose that $\bar{r}_2/\tilde{q}; q \equiv x \in V$. If $x \in V(\hat{l}_2/\tilde{p})$, then $\hat{l}_2/\tilde{p}; q \equiv x$. Otherwise, $x\theta$ is a ground normal form of $\hat{R}$.
(ii) Suppose that $\bar{r}_2/\tilde{q}; q \notin V$. Then $(\bar{r}_2/\tilde{q}; q)\theta$ is a head normal form of $\hat{R}$.

Proof. Let μ be the most general unifier of $\hat{l}_1$ and $\hat{l}_2$ constructed as

$$\{x := \hat{l}_2/p \mid p \in MV_{\hat{l}_1,\hat{l}_2} \text{ and } x \equiv \hat{l}_1/p\}$$
$$\cup \{x := \hat{l}_1/p \mid p \in MV_{\hat{l}_1,\hat{l}_2}, x \equiv \hat{l}_2/p, \text{ and } \hat{l}_1/p \notin V\}.$$

[5] The notion of a head normal form is the same as that of *a root-stable form* of a TRS in [21].

$\theta = \mu\theta'$ for some substitution θ'. Then $(\bar{r}_2/\tilde{q})\mu \equiv (\bar{r}_1/\tilde{q})\mu$ from the compatibility of $\hat{R}$, and $(\hat{l}_1/\tilde{p})\mu \equiv \hat{l}_2/\tilde{p}$ from $\hat{l}_1/\tilde{p} \in V$. Since $\hat{l}_1/\tilde{p} \equiv \bar{r}_1/\tilde{q}$, $(\bar{r}_2/\tilde{q})\mu \equiv (\hat{l}_1/\tilde{p})\mu \equiv \hat{l}_2/\tilde{p}$.

(i) Suppose that $\bar{r}_2/\tilde{q}; q \equiv x \in V$ and $x \in V(\hat{l}_2/\tilde{p})$. Since $\hat{l}_1/\tilde{p} \in V$, $x\mu \equiv x$ from the construction of μ. Thus $\hat{l}_2/\tilde{p}; q \equiv x$ follow from $(\bar{r}_2/\tilde{q})\mu \equiv \hat{l}_2/\tilde{p}$.
Next assume that $x \notin V(\hat{l}_2/\tilde{p})$. Since $(\bar{r}_2/\tilde{q})\mu \equiv \hat{l}_2/\tilde{p}$, $x \notin V(\hat{l}_2/\tilde{p})$ implies that $x := \hat{l}_2/\tilde{p}; q$ is in μ. From $x \in V(\bar{r}_2/\tilde{q})$, $x\mu$ is a proper subterm of $\hat{l}_1$ according to the definition of μ. Since $\hat{l}_1$ and $\hat{l}_2$ share no variables and $x\mu \equiv \hat{l}_2/\tilde{p}; q$, $x\mu$ cannot contain variables. Therefore, $x\theta \equiv x\mu$ and $x\theta$ is a ground normal form of $\hat{R}$ by Lemma 7.5.

(ii) Suppose that $\bar{r}_2/\tilde{q}; q \notin V$. Since $\hat{l}_1/\tilde{p} \in V$, $\tilde{p} \neq \epsilon$ from the definition of an LRCTRS. Thus $(\bar{r}_2/\tilde{q}; q)\mu \equiv \hat{l}_2/\tilde{p}; q$ is a proper non-variable subterm of $\hat{l}_2$. Therefore, $(\bar{r}_2/\tilde{q}; q)\theta$ is a head normal form of $\hat{R}$ by Lemma 7.5. □

Definition 7.8 We borrow notations from Definition 7.1 and Lemma 7.7. Let $\tilde{q} \in MV_{\bar{r}_1,\bar{r}_2}$. Suppose that $\bar{r}_1/\tilde{q} \equiv \hat{l}_1/\tilde{p} \in V$. If $x_{2m} \in V(\bar{r}_2/\tilde{q}) \cap V(\hat{l}_2/\tilde{p})$, then the AS_{2m}-segment in A' is called a *preserved segment.* If $x_{2m} \in V(\bar{r}_2/\tilde{q}) \backslash V(\hat{l}_2/\tilde{p})$, then the AS_{2m}-segment in A' is called a *skewed segment.* When $\bar{r}_2/\tilde{q} \equiv \hat{l}_2/\tilde{p} \in V$, preserved segments and skewed segments are defined symmetrically.

Note 7.9 It is easy to see that any segment in a replacement sequence is a preserved segment with a non-overlapping LRCTRS. The proof of UN of a weakly compatible TRS in [22] is insufficient since the methodology neglects skewed segments.

The following example explains the key idea of the modified independence $\perp\!\!\!\perp_* = \perp\!\!\!\perp_1 \cup \perp\!\!\!\perp_2$, where $\perp\!\!\!\perp_2$ is a new relation defined in the next section.

Example 7.10 Let us consider Example 7.3 again. Suppose that the reduction steps α_1, α_2, and α_3 are in subproofs $u \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} g(t,a,a)$, $a \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_2$, and $t \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} s_1$ of the peak, respectively. Let α'_1, α'_2, and α'_3 be descendants (by $P_\#$) of α_1, α_2, and α_3, respectively. The intuition behind the modified independence is as follows.

- The reduction steps α'_1 and α'_2 are in preserved segments. Since relative positions of preserved segments are preserved by using (i) of Lemma 7.7, we can prove preservation of the independence of α_1 and α_2 by using an argument similar to that of the non-overlapping case. This case is handled with $\perp\!\!\!\perp_1$.
- The reduction step α'_3 is in a skewed segment in the replacement sequence. Then the position of α'_3 may overlap with the position of α'_1. However, the intermediate term $g(t,a,a)$ plays the role of *barrier* between α'_1 and α'_3 so that they do not interfere with each other. Note that $g(t,a,a)$ is a head normal form and a is a ground normal form by (ii) of Lemma 7.7. This case

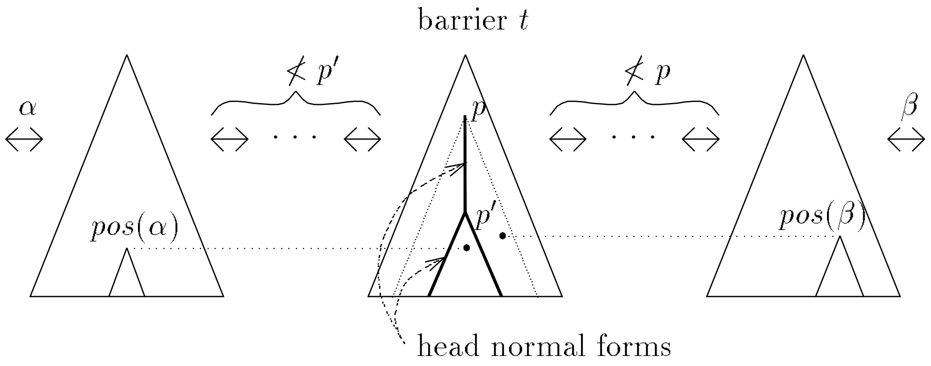

Fig. 10. Relation $\perp\!\!\!\perp_2^{\mathrm{T}}$ when $\alpha \geq p'$ and $A(\alpha, t) \not< p'$ hold

will be handled with $\perp\!\!\!\perp_2$.

In the following sections, we will formally discuss the above idea.

7.2 Independence $\perp\!\!\!\perp_*$ of compatible LRCTRS

In this section we introduce a binary relation $\perp\!\!\!\perp_*$, which is the union of $\perp\!\!\!\perp_1$ and $\perp\!\!\!\perp_2$. Then $\perp\!\!\!\perp_*$ is proved to be an independence for a PES $\mathcal{P}_{\hat{R}}$ of a compatible LRCTRS $\hat{R}$. Intuitively, $\alpha \perp\!\!\!\perp_2 \beta$ means that α and β are separated by a special term called a *barrier*.

Definition 7.11 For any proof A in $\hat{R}$ and $\alpha, \beta \in top(A)$, $\alpha \not\perp\!\!\!\perp_2^{\mathrm{T}} \beta$ in A if a term $t \in A(\alpha, \beta)$ and a position $p \in Pos(t)$ exist such that

(i) $p \leq \alpha \wedge \beta$,
(ii) $A(\alpha, \beta) \not< p$, and
(iii) there exists $p' \geq p$ satisfying $\alpha \geq p'$ and $A(\alpha, t) \not< p'$ such that t/q is a head normal form for each $q \in Pos(t/p)$ with $q \not\parallel p'$.

Then t is called a *barrier between* α and β. We also say t/p is the body and t/p' is the rock of the barrier t in order to make the positions p and p' explicit. Fig. 10 illustrates $\alpha \not\perp\!\!\!\perp_2^{\mathrm{T}} \beta$. We write $\alpha \perp\!\!\!\perp_2^{\mathrm{T}} \beta$ if either $\alpha \not\perp\!\!\!\perp_2^{\mathrm{T}} \beta$ or $\beta \not\perp\!\!\!\perp_2^{\mathrm{T}} \alpha$.

Example 7.12 Let $\hat{R}$ be the same as the one in Example 7.3. Consider the following proof in $\hat{R}$:

$$
\begin{array}{ll}
 & \underline{f(g(f(a,a),a,a),a)} \\
\stackrel{\alpha_1}{\rightarrow}_{\hat{S}'} & g(\underline{f(a,a)},a,a) \\
\stackrel{\alpha_2}{\rightarrow}_{\hat{S}} & g(a,a,a) \\
\stackrel{\alpha_3}{\leftarrow}_{\hat{S}} & g(a,\underline{f(a,a)},a) \\
\stackrel{\alpha_4}{\leftarrow}_{\hat{S}} & g(a,f(a,a),\underline{f(a,a)}),
\end{array}
$$

where the underlined parts are the contracted redexes. Then, $\alpha_3 \not\perp\!\!\!\perp_2^{\mathrm{T}} \alpha_1$ and $\alpha_4 \not\perp\!\!\!\perp_2^{\mathrm{T}} \alpha_1$ with a barrier $g(f(a,a),a,a)$ or $g(a,a,a)$. Note that $g(f(a,a),a,a) \equiv g(x,a,a)\{x := f(a,a)\}$ and $g(a,a,a) \equiv g(x,a,a)\{x := a\}$, so $g(f(a,a),a,a)$ and $g(a,a,a)$ are head normal forms and a is a ground normal form of $\hat{R}$. We also have $\alpha_2 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_3$, $\alpha_2 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_4$, and $\alpha_3 \perp\!\!\!\perp_1^{\mathrm{T}} \alpha_4$, but $\alpha_1 \not\perp\!\!\!\perp_i^{\mathrm{T}} \alpha_2$ for $i = 1, 2$.

A similar property to Lemma 6.8 holds for $\perp\!\!\!\perp_2^{\mathrm{T}}$.

Lemma 7.13 *Let $\alpha, \beta, \gamma \in top(A)$. Suppose that $\beta \in A(\alpha, \gamma)$ and $\beta \leq A[\alpha, \beta]$. Then $\alpha \perp\!\!\!\perp_2^{\mathrm{T}} \gamma$ with a barrier $t \in A(\beta, \gamma)$ iff $\beta \perp\!\!\!\perp_2^{\mathrm{T}} \gamma$.*

Proof. If-part follows from $\beta \leq A[\alpha, \beta]$. Let us consider the only-if-part. Suppose that t is a barrier between α and γ. Let t/p be the body and let t/p' be the rock. If $\alpha \not\perp\!\!\!\perp_2^{\mathrm{T}} \gamma$, then $\alpha \geq p'$ and $A(\alpha, t) \not\leq p'$. Since $\beta \leq A[\alpha, \beta]$ and $\beta \in A(\alpha, t)$, $\beta \leq p'$ and t is a barrier for $\beta \not\perp\!\!\!\perp_2^{\mathrm{T}} \gamma$. If $\gamma \not\perp\!\!\!\perp_2^{\mathrm{T}} \alpha$, then $\alpha \geq p$. Since $\beta \leq A[\alpha, \beta]$ and $\beta \in A(\alpha, t)$, $\beta \geq p$ and t is a barrier for $\gamma \not\perp\!\!\!\perp_2^{\mathrm{T}} \beta$. □

We extend $\perp\!\!\!\perp_2^{\mathrm{T}}$ to visible reduction steps in a similar way to Definition 6.9.

Definition 7.14 For $\alpha, \beta \in visible(A)$, $\alpha \perp\!\!\!\perp_2^{\mathrm{M}} \beta$ if $\alpha \perp\!\!\!\perp_2^{\mathrm{T}} \beta$ in $A^\flat$ with a barrier $\flat_A(t)$ for some $t \in A$. We also call t a barrier between α and β.

Lemma 7.15 $\perp\!\!\!\perp_2^{\mathrm{T}}$ *and* $\perp\!\!\!\perp_2^{\mathrm{M}}$ *are structural.*

Definition 7.16 Let $\perp\!\!\!\perp_*^{\mathrm{M}} = \perp\!\!\!\perp_1^{\mathrm{M}} \cup \perp\!\!\!\perp_2^{\mathrm{M}}$. We define $\perp\!\!\!\perp_2$ and $\perp\!\!\!\perp_*$ as the subproof closure of $\perp\!\!\!\perp_2^{\mathrm{M}}$ and $\perp\!\!\!\perp_*^{\mathrm{M}}$, respectively.

Note 7.17 *By definition, it is easy to see that $\perp\!\!\!\perp_2^{\mathrm{M}}$ and $\perp\!\!\!\perp_*^{\mathrm{M}}$ satisfy the assumptions (i), (ii), and (iii) in Lemma 5.10. In the sequel, we will apply Lemma 5.10 without mentioning these assumptions. Note also that $\perp\!\!\!\perp_* = \perp\!\!\!\perp_1 \cup \perp\!\!\!\perp_2$.*

Theorem 7.18 *The relation $\perp\!\!\!\perp_*$ is an independence for the PES $\mathcal{P}_{\hat{R}}$ of a compatible LRCTRS $\hat{R}$.*

In order to prove the above theorem, we show in the following that $\perp\!\!\!\perp_*$ satisfies the four properties in Definition 5.1. The hardest part is the (iv) preservation, which will be proved in Section 7.3.

Lemma 7.19 $\perp\!\!\!\perp_*$ *satisfies dominance.*

Proof. If $\beta \in sub(\alpha)$, there is no term $t \in A$ such that $\flat_A(t) \in A^\flat(\beta, \alpha)$. Thus, $\alpha \not\perp\!\!\!\perp_2^{\mathrm{M}} \beta$, and the result follows from Lemma 5.3 and Lemma 6.15. □

Lemma 7.20 $\perp\!\!\!\perp_*$ *is local.*

Proof. By Lemma 5.6, Lemma 6.16, and Lemma 7.15. □

Lemma 7.21 $\perp\!\!\!\perp_*$ *satisfies adherence.*

Proof. Obvious since $\perp\!\!\!\perp_*$ satisfies (adhere) of the subproof closure. $\square$

Lemma 7.22 $\perp\!\!\!\perp_*$ *satisfies non-incest.*

Proof. From the same argument in Lemma 6.18, $\alpha_1 \perp\!\!\!\perp_1 \alpha_2$ for any two distinct descendants α_1, α_2 of any reduction step α in a critical peak. Therefore, the result follows. $\square$

7.3 Preservation of independence $\perp\!\!\!\perp_*$

The first lemma is used to push barriers out of the eliminated peaks.

Lemma 7.23 *(Push-out Lemma) Let* $A : t_1 \overset{\nabla}{\leftrightarrow}_{\hat{R}} \cdots \overset{\nabla}{\leftrightarrow}_{\hat{R}} t_n$ *be a proof in* $\hat{R}$*, and let* $\alpha, \beta \in visible(A)$*. Suppose* $\alpha \perp\!\!\!\perp_2^{\mathrm{M}} \beta$ *and* t_i *is a barrier between* α *and* β*. Suppose also that the ith reduction step of* A *is right-oriented (i.e.,* $t_i \to_{\hat{R}} t_{i+1}$*) and that* $\flat_A(t_{i+1}) \in A^\flat(\alpha, \beta)$*. Then* t_{i+1} *is also a barrier between* α *and* β*. Such a property also holds for left-oriented reduction steps.*

Proof. Let $\gamma = t_i \to_{\hat{R}} t_{i+1}$, and let t_i/p be the body of the barrier t_i. Then, $\gamma \not\leq p$. Therefore, t_{i+1}/q is a head normal form for any q such that $p \leq q \leq \alpha \wedge \beta$, and the result follows. $\square$

Lemma 7.24 $\perp\!\!\!\perp_*$ *satisfies preservation.*

Proof. Let α_1 and α_2 be reduction steps in a proof $A : t_1 \overset{\nabla}{\leftrightarrow}{}^*_{\hat{R}} t_n$. Suppose that $A \mapsto A'$ and that α'_i in A' is a descendant of α_i $(i = 1, 2)$.

Let $A = A_1; \gamma_1; \gamma_2; A_2$, where $\gamma_1; \gamma_2$ is the eliminated peak with the reduction steps $\gamma_1 : t_{k-1} \overset{\nabla}{\leftarrow}_{\hat{R}} t_k$ and $\gamma_2 : t_k \overset{\nabla}{\to}_{\hat{R}} t_{k+1}$. Let $B_1; B_2$ be the replacement sequence. We perform the same case analysis as Lemma 6.19 (see Fig. 8). Claim 7.25, Claim 7.26, Claim 7.27, and Claim 7.28 establish the preservation. $\square$

Claim 7.25 *Preservation of* $\perp\!\!\!\perp_*$ *holds in cases (a)-(1) to (3).*

Proof. In case (a)-(1), preservation follows since $\perp\!\!\!\perp_*$ is local by Lemma 7.20. In case (a)-(2), we can assume $\alpha_1, \alpha_2 \in visible(A)$ and $\alpha_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha_2$ because of Lemma 5.10. Thus, if t_k is a barrier between α_1 and α_2, then t_{k-1} is also a barrier by the Push-out Lemma. Thus, preservation follows since $\mathcal{P}_{\hat{R}}$ works monotonically downwards according to Lemma 7.2. In case (a)-(3), if α_1 and α_2 are in the same subproof and α'_1 and α'_2 are in the same copy of that subproof, preservation follows since $\perp\!\!\!\perp_*$ is local by Lemma 7.20. Otherwise,

α'_1 and α'_2 are in different segments of embedding, so $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$. $\Box$

Claim 7.26 *Preservation of $\perp\!\!\!\perp_*$ holds in cases (b)-(1) to (3) when $P_{\|}$ is applied.*

Proof. Due to Claim 6.21 and Lemma 5.10, we can prove $\alpha'_1 \perp\!\!\!\perp_* \alpha'_2$ under the assumptions $\alpha_1, \alpha_2 \in visible(A)$ and $\alpha_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha_2$. In case (b)-(1), $\alpha'_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha'_2$ is obvious since $B_1 \parallel B_2$. In case (b)-(2), if t_k is a barrier between α_1 and α_2, then t_{k-1} is also a barrier according to the Push-out Lemma, so $\alpha'_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha'_2$ holds. In case (b)-(3), $\alpha'_1 \perp\!\!\!\perp_1^{\mathrm{M}} \alpha'_2$ in A', so $\alpha'_1 \perp\!\!\!\perp_* \alpha'_2$ follows. $\Box$

Claim 7.27 *Preservation of $\perp\!\!\!\perp_*$ holds in cases (b)-(1) to (3) when $P_<$ is applied.*

Proof. Suppose γ_2 nests γ_1. Then, B_1 is empty, and B_2 is γ_2 with γ_1 prefixed to the kth subproof for some k. Let AS_i be the ith subproof of γ_2 for $i = 1, \ldots, n$. Then, the kth subproof of B_2 is $\gamma_1/p; AS_k$ for some position p. Without loss of generality, we can assume $\alpha_1, \alpha_2 \in visible(A)$ and $\alpha_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha_2$ by using Claim 6.22 and Lemma 5.10.

The case $\alpha_2 = \gamma_1$ from Lemma 3.11 and Lemma 7.13 is sufficient for case (b)-(1). Then $\alpha'_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha'_2$ since any reduction step β in an AS_i-segment in $B_2^\flat$ $(i = 1, \ldots, k-1)$ satisfies $\beta \parallel \alpha'_2$.

In case (b)-(2), we can assume that a barrier between α_1 and α_2 is in A_1 by using the Push-out Lemma. Thus, $\alpha'_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha'_2$.

Let us consider (b)-(3). We borrow notations in Definition 6.2, so $\gamma_2 = C_2[\hat{l}_2\theta] \xrightarrow{\nabla}_{\hat{S}_2} C_2[\hat{r}_2\theta]$. If $\alpha_2 = \gamma_2$, then $\alpha_1 \not\perp\!\!\!\perp_* \alpha_2$. Suppose that $\alpha_2 \in AS_j$. If $j \neq k$, then $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$ in A'. If $j = k$, t_k is the barrier between α_1 and α_2. Then,

$$
\begin{array}{lll}
 & \alpha_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha_2 \text{ in } A \text{ with } t_k \text{ as the barrier,} & \\
\Leftrightarrow & \alpha_1 \perp\!\!\!\perp_2^{\mathrm{T}} \alpha_2 \text{ in } A \text{ with } \flat_A(t_k) \text{ as the barrier} & \text{(by definition),} \\
\Leftrightarrow & \alpha_1 \perp\!\!\!\perp_2^{\mathrm{T}} \alpha_2 \text{ in } \gamma_1^\flat; C_2[\![C_{\hat{l}_2}[\![AS_1, \ldots, AS_{n_1}]\!]]\!] \text{ with } \flat_A(t_k) \text{ as the barrier,} & \\
\Leftrightarrow & \alpha_1 \perp\!\!\!\perp_2^{\mathrm{T}} \alpha_2 \text{ in } \gamma_1^\flat/p; AS_k \text{ with } t_k/p \text{ as the barrier} & \text{(since } \perp\!\!\!\perp_2^{\mathrm{T}} \text{ is structural),} \\
\Leftrightarrow & \alpha'_1 \perp\!\!\!\perp_2^{\mathrm{M}} \alpha'_2 \text{ in } \gamma_1/p; AS_k \text{ with } t_k/p \text{ as the barrier,} & \\
\Rightarrow & \alpha'_1 \perp\!\!\!\perp_* \alpha'_2 \text{ in } A' & \text{(by definition).}
\end{array}
$$

In the step from the third line to the fourth line, we used the fact that $\gamma_1^\flat/p; AS_k = \gamma_1^\flat; C_2[\![C_{\hat{l}_2}[\![AS_1, \ldots, AS_{n_1}]\!]]\!]/p$.

Next suppose γ_1 nests γ_2. In cases (b)-(1) and (2), we can assume that a barrier between α_1 and α_2 is in A_1 by using Lemma 3.11 and the Push-out

Lemma. Thus, it is enough to consider the case $\alpha_2 = \gamma_1$ from Lemma 7.13, and then $\alpha'_1 \perp\!\!\!\perp^{\mathrm{M}}_2 \alpha'_2$. In case (b)-(3), the result is proved in the same way as in the case when γ_2 nests γ_1. □

Claim 7.28 *Preservation of $\perp\!\!\!\perp_*$ holds in cases (b)-(1) to (3) when $P_\sharp$ is applied.*

Proof. We borrow notations in Definition 7.1 and Lemma 7.7, so $B_1 = C[\![C_{\hat{r}_1}[\![AS_{1j_1}, \ldots, AS_{1j_k}]\!]]\!]$ and $B_2 = C[\![C_{\hat{r}_2}[\![AS_{2j'_1}, \ldots, AS_{2j'_{k'}}]\!]]\!]$. Without loss of generality, we can assume that $\alpha_1, \alpha_2 \in \mathit{subtop}(A)$ and $\alpha_1 \perp\!\!\!\perp^{\mathrm{M}}_* \alpha_2$ by using Lemma 5.10.

The case of $\alpha_2 = \gamma_1$ from Lemma 3.11, Lemma 6.8, and Lemma 7.13 is sufficient for cases (b)-(1) and (2). By the Push-out Lemma, we can assume a barrier is in A_1. Since $\mathcal{P}_{\hat{R}}$ works monotonically downwards by Lemma 7.2, $\alpha'_1 \perp\!\!\!\perp^{\mathrm{M}}_* \alpha'_2$.

Let us consider (b)-(3). Let $p_0 = pos(\gamma_1)(= pos(\gamma_2))$. Let $\langle \bar{r}_1, \bar{r}_2 \rangle$ be the standard pair of compatibility of $\langle S_1, S_2 \rangle$. Every variable in $\hat{r}_i$ occurs below some position in $MV_{\bar{r}_1,\bar{r}_2}$ $(i = 1, 2)$. Since subproofs of the peak are embedded in subproof parts, there is a unique $q_i \in MV_{\bar{r}_1,\bar{r}_2}$ such that $\alpha'_i \geq p_0; q_i$ for $i = 1, 2$. If $q_1 \neq q_2$, then $\alpha'_1 \perp\!\!\!\perp_1 \alpha'_2$. Otherwise, let $q_1 = q_2 = \tilde{q}$ and let us consider a restriction $B_1; B_2/p_0; \tilde{q}$.

From the definition of $MV_{\bar{r}_1,\bar{r}_2}$, either $\bar{r}_1/\tilde{q} \in V$ or $\bar{r}_2/\tilde{q} \in V$. Suppose that $\bar{r}_1/\tilde{q} \equiv x_{1j}$. The other case is treated symmetrically.

Let $\bar{r}_2/\tilde{q} \equiv C_{\hat{r}_2/\tilde{q}}[x_{2m_1}, \ldots, x_{2m_g}]$. Then

$$B_1; B_2/p_0; \tilde{q} = AS_{1j}; C_{\hat{r}_2/\tilde{q}}[\![AS_{2m_1}, \ldots, AS_{2m_g}]\!].$$

Let $\tilde{p}$ be the position such that $\hat{l}_1/\tilde{p} \equiv x_{1j}$. Recall that

$$B^l = C[\![C_{\hat{l}_1}[\![AS_{11}, \ldots, AS_{1n_1}]\!]]\!]; C[\![C_{\hat{l}_1}[\![AS_{21}, \ldots, AS_{2n_2}]\!]]\!]$$

is a subsequence of the flattening of the peak, hence

$$B^l/p_0; \tilde{p} = AS_{1j}; C_{\hat{l}_2/\tilde{p}}[\![AS_{2l_1}, \ldots, AS_{2l_h}]\!]$$

for some $l_1, \ldots, l_h$. Then $\alpha_1 \perp\!\!\!\perp^{\mathrm{T}}_* \alpha_2$ in $B^l/p_0; \tilde{p}$.

Suppose that α'_2 is in the AS_{2m_j} segment, hence $\bar{r}_2/\tilde{q}; q \equiv x_{2m_j}$.

Suppose that $x_{2m_j} \in V(\hat{l}_2/\tilde{p})$. With (i) of Lemma 7.7, $\hat{l}_2/\tilde{p}; q \equiv x_{2m_j}$, that is, the position of AS_{2m_j} in $B^l/p_0; \tilde{p}$ is preserved in $B_1; B_2/p_0; \tilde{q}$. Thus $\alpha_1 \perp\!\!\!\perp^{\mathrm{T}}_* \alpha_2$ in $B^l/p_0; \tilde{p}$ iff $\alpha'_1 \perp\!\!\!\perp^{\mathrm{T}}_* \alpha'_2$ in $B_1; B_2/p_0; \tilde{q}$. Therefore, preservation holds since $\perp\!\!\!\perp^{\mathrm{T}}_*$ is structural by Lemma 6.10 and Lemma 7.15.

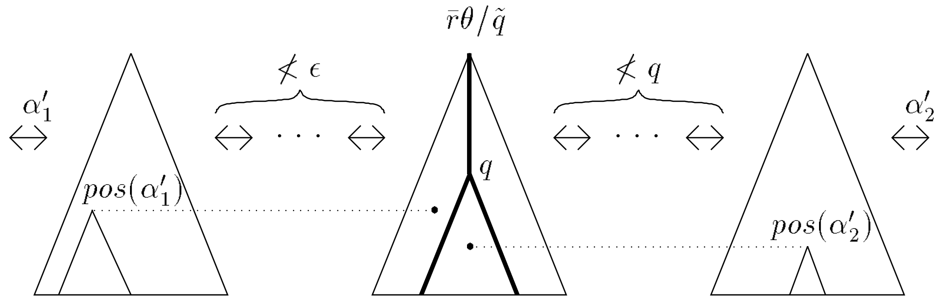

Fig. 11. Skewed segment in $B_1; B_2/p_0; \tilde{q}$

Suppose that $x_{2m_j} \notin V(\hat{l}_2/\tilde{p})$. Then, the AS_{2m_j}-segment is a skewed segment. By Lemma 6.1, the term between AS_{1j} and $C_{\hat{r}_2/\tilde{q}}[\![AS_{2m_1}, \ldots, AS_{2m_g}]\!]$ is $C[\bar{r}_2\theta]/p_0; \tilde{q} \equiv \bar{r}_2\theta/\tilde{q}$. For each $q' < q$, $\bar{r}_2\theta/\tilde{q}; q'$ is a head normal form by (ii) of Lemma 7.7. Moreover, $\bar{r}_2\theta/\tilde{q}; q$ is a ground normal form by (i) of Lemma 7.7. We also have $\alpha'_2 \geq q$, $\alpha'_1 \geq \epsilon$, $B_2/p_0; \tilde{q} \not\leq q$, and $B_1/p_0; \tilde{q} \not\leq \epsilon$ in $B_1; B_2/p_0; \tilde{q}$. Thus, $\bar{r}_2\theta/\tilde{q}$ is a barrier for $\alpha'_2 \not\perp\!\!\!\perp_2^{\mathrm{T}} \alpha'_1$ in $B_1; B_2/p_0; \tilde{q}$ as shown in Fig. 11. Therefore, preservation holds since $\perp\!\!\!\perp_2^{\mathrm{T}}$ is structural as a result of Lemma 7.15. □

Proof of Theorem 7.18. By Lemma 7.19, Lemma 7.21, Lemma 7.22, and Lemma 7.24. □

Theorem 7.29 *A compatible LRCTRS is CR.*

Proof. By Lemma 4.6, Theorem 4.17, and Theorem 7.18. □

Proof of Main Theorem. By Theorem 3.5 and Theorem 7.29. □

8 Related work

8.1 λ-calculus with nonlinear rules

Many of UN results are derived from studies on extensions of the untyped λ-calculus. Klop proved in his pioneering work [15] (see also [3,7]) that CR fails for the extension of λ_β with any of the following rules:

$\mathsf{D_h}: \{D_h zz \to z\}$,

$\mathsf{D_s}: \{D_s zz \to E\}$,

$\mathsf{D_k}: \{D_k zz \to Ez\}$,

$\mathsf{PC}: \{CTxy \to x,\ CFxy \to y,\ Czxx \to x\}$,

$\mathsf{SP}: \{D_0(Dxy) \to x,\ D_1(Dxy) \to y, D(D_0x)(D_1x) \to x\}$.

He also proved that UN holds for $\lambda_\beta + \mathsf{D_h}$, $\lambda_\beta + \mathsf{D_s}$, and $\lambda_\beta + \mathsf{D_k}$. Later, de Vrijer and Klop proved that UN holds for $\lambda_\beta + \mathsf{SP}$ [17,35]. Although UN of $\lambda_\beta + \mathsf{PC}$ has not been explicitly referred to previously, it is proved by an easy reformulation of the argument in Section 4 of [37].

The general statement that a strongly non-overlapping higher-order rewriting system is UN [18] derives UN of $\lambda_\beta + \mathsf{D_h}$, $\lambda_\beta + \mathsf{D_s}$, and $\lambda_\beta + \mathsf{D_k}$.

8.2 Non-ω-overlapping TRS

Two terms are *infinitely unifiable* if they are unifiable with an infinite unifier, and it is decidable using the unification algorithm without the occur check [6,19]. For instance, $d(x, x)$ and $d(y, f(y))$ are infinitely unifiable with an infinite unifier $\{x := f(f(f(\cdots))), y := f(f(f(\cdots)))\}$, whereas $d(x, x)$ and $d(g(y), f(y))$ are not because of the clash between $g(y)$ and $f(y)$. A TRS R is non-ω-overlapping when l_1 and l_2/p are not infinitely unifiable for any rules $S_1 : l_1 \to r_1, S_2 : l_2 \to r_2 \in R$ and any non-variable subterm l_2/p of l_2, unless S_1 and S_2 are identical and $p = \epsilon$. For instance, R_2 in Introduction is non-ω-overlapping. One of the authors posed the problem in [23] of whether a non-ω-overlapping TRS is UN, which is Problem 79 in [8]. In [20] a partial answer is obtained: a non-ω-overlapping and depth-preserving TRS is UN. [6]

Verma stated in [34] that every non-overlapping uniquely consistent non-duplicating TRS in which every non-overlap is an I-non-overlap is UN, and this implies another partial answer, i.e., UN of a non-ω-overlapping non-duplicating TRS. However, the proof of the key observation, Theorem 6, in [34] is inconclusive. In [33], there is a similar statement, Claim 8, but its proof is omitted.

[6] They also show CR of a non-ω-overlapping TRS with a stronger restriction [11,12].

8.3 The Church-Rosser property of right-linear TRS

Instead of the left-linearity, the right-linearity is also expected to recover CR. Toyama, Oyamaguchi, and Ohta have shown a partial answer [32,24].

Theorem 8.1 *A simple-right-linear and strongly non-overlapping TRS is CR.*

Here, a TRS is called *simple-right-linear* if each rule is right-linear and the non-linear variables in the left-hand side do not appear in the right-hand side. The full statement that *a right-linear and strongly-nonoverlapping TRS is CR* is still open. Note that there is a counter example when the right-linearity is relaxed to the the non-duplicating condition.

9 Conclusion

We have presented a new proof of Chew's theorem, which we believe is the first complete proof of the theorem.

Our next step will be UN of a weakly compatible TRS. However, this will require a further extension of the current framework of independence, since weak compatibility lacks the head normal constraints on barriers, so that it becomes difficult to show *dominance*.

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[Appendix] Scenario of Chew's original proof and the gap

Here we present an outline of Chew's original proof [5], and show a "gap" in it pointed out by van Oostrom [26]. In this Appendix, we follow the original notation in [5].

Let G be a TRS and let G' be the set of all linearizations of all rules in G. For example, if $g(h(x,x)) \to h(x,x) \in G$, then $g(h(x_1,x_2)) \to h(x_1,x_1)$, $g(h(x_1,x_2)) \to h(x_1,x_2)$, $g(h(x_1,x_2)) \to h(x_2,x_1)$, and $g(h(x_1,x_2)) \to h(x_2,x_2)$ are in G'. To avoid difficulties caused by non-left-linearity, Chew introduced *closure* and *marker*.

The *closure* $\to_{\bar{G}}$ *of* $\to_G$ *with respect to* G' is defined as the following conditional TRS obtained from G':

$$\left.\begin{array}{l} g(h(x_1,x_2)) \to h(x_1,x_1) \\ g(h(x_1,x_2)) \to h(x_1,x_2) \\ g(h(x_1,x_2)) \to h(x_2,x_1) \\ g(h(x_1,x_2)) \to h(x_2,x_2) \end{array}\right. \quad \text{if there is a redex } M \text{ of } G \text{ s.t. } M \to^*_{\bar{G}} g(h(x_1,x_2)).$$

where $M \overset{nr\,*}{\to}_{\bar{G}} g(h(x_1,x_2))\theta$ is a reduction sequence from M to $g(h(x_1,x_2))\theta$ such that no reduction step occurs at the head position ϵ.[7]

Two fresh symbols α and β of the variable arity called *markers* (corresponding to the right direction and the left direction, respectively, as will become clear) are introduced to represent "all the possible choices of variables in the

[7] Strictly speaking, $\bar{G}$ should be defined by an inductive generation since the rule defining $\to_{\bar{G}}$ contains $\to_{\bar{G}}$ itself.

linearization" in one rewrite rule. For example, $g(h(x,x)) \to h(x,x)$ is transformed into the following rule using α:

$$g(h(x_1,x_2)) \to \alpha(h(x_1,x_1), h(x_1,x_2), h(x_2,x_1), h(x_2,x_2)).$$

The reduction system obtained by such a transformation from G is denoted by αG. The system βG is defined similarly using the symbol β instead of α. The following additional reduction rules are also introduced to simulate $\to_G$: copying reduction rules $\to_{\alpha+}$, $\to_{\beta+}$, selecting reduction rules $\to_{\alpha-}$, $\to_{\beta-}$, and distributing reduction rules $\to_{\alpha d}$, $\to_{\beta d}$. For instance,

$$\begin{array}{rcl} h(t_1,t_2) & \to_{\alpha+} & \alpha(h(t_1,t_2), h(t_1,t_2)), \\ \alpha(h(t_1,t_2), h(t_3,t_4)) & \to_{\alpha-} & h(t_1,t_2) \text{ or } h(t_3,t_4), \\ g(\alpha(h(t_1,t_2), h(t_3,t_4))) & \to_{\alpha d} & g(h(\alpha(t_1,t_3), \alpha(t_2,t_4)))). \end{array}$$

A reduction relation $\to_{\alpha G^c}$ ($\to_{\beta G^c}$, respectively) is the closure of G with respect to αG (βG, respectively) using $\overset{nr}{\to}{}^*_R$ in the condition part, where $\to_R \; = \; \to_{\alpha G^c} \cup \to_{\beta G^c} \cup \to_{\alpha+} \cup \to_{\beta+} \cup \to_{\alpha-} \cup \to_{\beta-} \cup \to_{\alpha d} \cup \to_{\beta d}$. Let $\to_S \; = \; \to_{\alpha G^c} \cup \to_{\alpha d} \cup \to_{\beta+} \cup \to_{\beta-}$, and $\to_T \; = \; \to_{\beta G^c} \cup \to_{\beta d} \cup \to_{\alpha+} \cup \to_{\alpha-}$.

An outline of Chew's original proof is as follows. At first, similar to what de Vrijer observed, UN of $\to_G$ is reduced to CR of $\to_{\bar{G}}$. Then $\to_S$ and $\to_T$ are shown to be commutative. Finally, CR of $\to_{\bar{G}}$ is proved by the following steps: given a proof $t \leftrightarrow^*_{\bar{G}} t'$,

(i) transform $t \leftrightarrow^*_{\bar{G}} t'$ into $t \leftrightarrow^*_G t'$ (since $\to_{\bar{G}}$ and $\to_G$ are the same in convertibility),
(ii) replace each $\to_G$ with $\to_{\alpha G^c} \cdot \leftarrow_{\alpha+}$ ($\in \to_S \cdot \leftarrow_T$) and replace each $\leftarrow_G$ with $\to_{\beta+} \cdot \leftarrow_{\beta G^c}$ ($\in \to_S \cdot \leftarrow_T$),
(iii) $t \to^*_T \cdot \leftarrow^*_S t'$ through commutativity of $\to_S$ and $\to_T$,
(iv) $t \to^*_{\bar{G}} \cdot \leftarrow^*_{\bar{G}} t'$ by "stripping" α's and β's.

The key Lemma 6.1 in [5] is necessary in the final step, that is, (iii) to (iv). This lemma states that if A is a redex of $\to_{\alpha G^c}$ (by definition, this means that a redex B of G exists such that $B \overset{nr}{\to}{}^*_R A$), then any $\to_{\alpha-} \cup \to_{\beta-}$-normal form $\bar{A}$ of A is a redex of $\to_{\bar{G}}$. Chew proves the lemma by induction on the length of $B \overset{nr}{\to}{}^*_R A$. However, there is a gap that seems difficult to remedy.

The induction does not work for $\to_{\alpha d}$ [26]. Let us consider the following example:

$$B \to^*_R \underset{(\equiv B')}{g(\alpha(h(t_1,t_2), h(t_3,t_4)))} \to_{\alpha d} \underset{(\equiv A)}{g(h(\alpha(t_1,t_3), \alpha(t_2,t_4)))},$$

where t_i is an arbitrary term containing neither α nor β for $i = 1, \ldots, 4$. By removing the markers by $\rightarrow_{\alpha -}$ and $\rightarrow_{\beta -}$, we obtain $C_{B'} = \{g(h(t_1, t_2)), g(h(t_3, t_4))\}$ from B', and $C_A = \{g(h(t_1, t_2)), g(h(t_1, t_4)), g(h(t_3, t_2)), g(h(t_3, t_4))\}$ from A. In the induction step, it must be shown that $s_{B'} \in C_{B'}$ exists such that $s_{B'} \rightarrow^*_{\bar{G}} s_A$ for each $s_A \in C_A$. However, this is impossible due to a "cross product", that is, $s_A \equiv g(h(t_1, t_4))$ or $g(h(t_3, t_2))$.

Chew's thesis [4] also contains a similar gap. In the proof of Lemma 4.28 in the thesis, the property corresponding to Lemma 6.1 in [5] is implicitly assumed. The definition of the closure is different from that of [5], that is, the condition part of the definition of the closure accepts not only $M \rightarrow^*_{\bar{G}} g(h(x_1, x_2))$ but also $M \leftarrow^*_{\bar{G}} g(h(x_1, x_2))$. However, induction also fails at the step of $\rightarrow_{\alpha d}$ with forward reduction, and $\rightarrow_{\alpha -}$ with backward reduction if we try to prove the corresponding property.