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Well-quasi-orders and Regular ω -languages

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Abstract

In "On regularity of context-free languages, Theoretical Computer Science Vol.27, pp.311-332, 1983", Ehrenfeucht et al. showed that a set L of finite words is regular if and only if L is \leq -closed under some monotone well-quasi-order (WQO) \leq over finite words. We extend this result to regular ω -languages. That is,

- (1) an ω -language L is regular if and only if L is \leq -closed under a *periodic* extension \leq of some monotone WQO over finite words, and
- (2) an ω -language L is regular if and only if L is \leq -closed under a WQO \leq over ω -words that is a *continuous* extension of some monotone WQO over finite words.

Key words: ω -language, well-quasi-order, regularity.

1 Preliminaries

Throughout the paper, we will use A for a finite alphabet, A^* for a set of all (possibly empty) finite words on A, and A^{ω} for a set of all ω -words on A. A concatenation of two words u, v is denoted by u, v, an element-wise concatenation of two sets U, V of words by $U.V, \underbrace{V.V.\cdots V}_{i}$ by V^i , and $V.V.V.\cdots$ by V^{ω} . The length of a finite word u is denoted by |u|. As a convention, we will use ϵ for the empty word, u, v, w, \cdots for finite words, α, β, \cdots for ω -words, a_1, a_2, \cdots for elements in A, i, j, k, l, \cdots for indices, and U, V, \cdots (capital letters) for sets. We sometimes use x, y, \cdots for elements of a set.

A regular ω -language is a set of ω -words that are accepted by a (nondeterministic) *Büchi* automaton $\mathcal{A} = \{Q, q_0, \Delta, F\}$, where Q is a finite set of states, q_0 an initial state, $\Delta \subseteq Q \times A \times Q$ a transition relation, and F a set of final states. $\alpha = a_1 a_2 a_3 \cdots \in A^{\omega}$ is accepted by \mathcal{A} if its corresponding run $q_0 \xrightarrow[a_1]{} q_1 \xrightarrow[a_2]{} q_2 \xrightarrow[a_3]{} \cdots$ runs through some state of F infinitely often. A set of ω -words accepted by \mathcal{A} is denoted by $L(\mathcal{A})$. For states q, q' and $w \in A^*$, we write $q \xrightarrow[w]{} q'$ if there is a run of \mathcal{A} on w, and we write $q \xrightarrow[w]{} q'$ if there is a run of \mathcal{A} on w from q to q' such that the run runs through some state of F.

A congruence \sim is an equivalent relation over A^* preserved by concatenations. A congruence \sim is finite if there are only finitely many \sim -classes. Details are given elsewhere [3].

Definition 1.1 Let $L \subseteq A^{\omega}$ and let \sim be a congruence over A^* . We say that \sim saturates L if for each \sim -class $U, V, U.V^{\omega} \cap L \neq \phi$ implies $U.V^{\omega} \subseteq L$.

Lemma 1.2 For a *Büchi* automaton \mathcal{A} and $u, v \in A^*$, we define $u \sim_{\mathcal{A}} v$ if $(q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \land (q \xrightarrow{F} q' \Leftrightarrow q \xrightarrow{F} q')$ for each $q, q' \in Q$. Then $\sim_{\mathcal{A}}$ is a finite congruence that saturates $L(\mathcal{A})$.

Theorem 1.3 $L \subseteq A^{\omega}$ is regular if and only if some finite congruence saturates L.

Lemma 1.4 Let \sim be a finite congruence over A^* .

- (1) Let $\alpha = u_1 u_2 \cdots \in A^{\omega}$ and let $u(i, j) = u_i u_{i+1} \cdots u_{j-1}$ where $u_i \in A^*$. There exist a \sim -class V and $i_1 < i_2 < \cdots$ such that $u(i_j, i_k) \in V$ for each j, k with j < k.
- (2) Let U, V be ~-classes. There exist ~-classes U', V' such that $U.V^{\omega} \subseteq U'.V'^{\omega}, U'.V' \subseteq U'$, and $V'.V' \subseteq V'$.

Proof

- (1) Since \sim has only finitely many \sim -classes, this is a direct consequence of (infinite) Ramsey Theorem.
- (2) Note that for each ~-class $U_1, \dots, U_m, W, U_1, \dots, U_n \cap W \neq \emptyset$ implies $U_1, \dots, U_n \subseteq W$. Since ~ has only finitely many ~-classes, from (infinite) Ramsey Theorem there exist a ~-class V' and $i_1 < i_2 < \cdots$ such that $V^{i_k-i_j} \subseteq V'$ for each j, k with j < k and $V'.V' \subseteq V'$. Let U' be a ~-class that includes $U.V^{i_1}$. Then $U.V^{\omega} \subseteq U'.V'^{\omega}, U'.V' \subseteq U'$, and $V'.V' \subseteq V'$.

We denote a quasi-order (QO, i.e., reflexive transitive binary relation) over a set S by (S, \leq) . If S is clear from the context, we simply denote by \leq . As a convention, a QO over finite words is denoted it by \leq , and a QO over ω -words is denoted by \leq .

Definition 1.5 For a QO (S, \leq) and $L \subseteq S$, L is \leq -closed if for each $x \in L$ $x \leq y$ implies $y \in L$.

Definition 1.6 A QO (S, \leq) is a well-quasi-order (WQO) if for any infinite sequence x_1, x_2, \cdots in S, there exist i, j such that i < j and $x_i \leq x_j$.

A QO (A^*, \leq) is monotone if $u \leq v$ implies $w_1 u w_2 \leq w_1 v w_2$ for each $u, v, w_1, w_2 \in A^*$.

2 First theorem

Definition 2.1 A QO (A^{ω}, \preceq) is a periodic extension of (A^*, \leq) if the following conditions are satisfied:

- For each $u_i, v_i \in A^*$, $u_i \leq v_i$ for any *i* implies $u_1 u_2 u_3 \cdots \leq v_1 v_2 v_3 \cdots$.
- For each $\alpha \in A^{\omega}$, there exist $u, v \in A^*$ such that $\alpha \preceq u.v^{\omega}$ and $\alpha \succeq u.v^{\omega}$.

Theorem 2.2 Let $L \subseteq A^{\omega}$. *L* is regular if and only if *L* is \preceq -closed under a periodic extension (A^{ω}, \preceq) of a monotone WQO (A^*, \leq) .

For instance, the embedding over ω -words is the periodic extension of the embedding over finite words. Note that a periodic extension of a monotone WQO over A^* is a WQO over A^{ω} . We will prove Theorem 2.2 below.

Lemma 2.3 Let ~ be a finite congruence on A^* and let U, V be ~-classes. For $u, v \in A^*$, if $uv^{\omega} \in U.V^{\omega}$, $U.V \subseteq U$, and $V.V \subseteq V$, there exist $w_1 \in U$ and $w_2 \in V$ such that $w_1w_2^{\omega} = uv^{\omega}$.

Proof Let $uv^{\omega} = u'v'_1v'_2\cdots$ satisfying $u' \in U$ and $v'_i \in V$, and let $w(i, j) = v'_i \cdots v'_{j-1}$ for i < j. Let $k_j \equiv |w(1, j)| \pmod{|v|}$. Then there exist k_{j_1} and k_{j_2} such that $k_{j_1} < k_{j_2}$ and $k_{j_1} \equiv k_{j_2} \pmod{|v|}$. Since there are infinitely many such pairs, we can assume that $|u| \leq |u'w(1, j_1 - 1)|$. Let $w_1 = u'.w(1, j_1 - 1)$ and $w_2 = w(j_1, j_2 - 1)$. Since $U.V \subseteq U$ and $V.V \subseteq V$, $w_1 \in U$, $w_2 \in V$ and $uv^{\omega} = w_1w_2^{\omega}$.

Lemma 2.4 For a *Büchi* automaton \mathcal{A} and $\alpha \in A^{\omega}$, let $[\![\alpha]\!] = \{U.V^{\omega} \mid \alpha \in U.V^{\omega}\}$ where U, V are $\sim_{\mathcal{A}}$ -classes. We define $\alpha \preceq' \beta$ if $[\![\alpha]\!] \cap [\![\beta]\!] \neq \emptyset$. Then,

- (1) $L(\mathcal{A})$ is \preceq' -closed.
- (2) $u_i \sim_{\mathcal{A}} v_i$ for each *i* imply $u_1 u_2 \cdots \preceq' v_1 v_2 \cdots$.

Proof From Lemma 1.2, $\sim_{\mathcal{A}}$ saturates L and $U.V^{\omega} \subseteq L$ for each $U.V^{\omega} \in [\![\alpha]\!]$. Thus L is \preceq' -closed. From Lemma 1.4 (i), there exist a $\sim_{\mathcal{A}}$ -class V and $i_1 < i_2 < \cdots$ such that $u(i_j, i_k) \in V$ for each j < k. Let U be a $\sim_{\mathcal{A}}$ -class such that $u(1, i_1) \in U$. (We borrow the notation from Lemma 1.4 (i).) Since $\sim_{\mathcal{A}}$ is a congruence, $v(1, i_1) \in U$ and $v(i_j, i_k) \in V$ for each j < k. Thus $u_1 u_2 \cdots \in U.V^{\omega}$ implies $v_1 v_2 \cdots \in U.V^{\omega}$, and $\alpha \preceq \beta$.

Definition 2.5 [1] For $u, v \in A^*$, we define $u \approx_L v$ if $w(w_1 u w_2)^{\omega} \in L \Leftrightarrow w(w_1 v w_2)^{\omega} \in L$ and $w_1 u w_2 w^{\omega} \in L \Leftrightarrow w_1 v w_2 w^{\omega} \in L$ for each $w, w_1, w_2 \in A^*$.

Proof of Theorem 2.2

Only-if part: Assume L is regular. Let \mathcal{A} be a *Büchi* automaton such that $L = L(\mathcal{A})$. Since $\sim_{\mathcal{A}}$ is a finite congruence, $(A^*, \sim_{\mathcal{A}})$ is a monotone WQO. Define \preceq as the transitive closure of \preceq' (defined in Lemma 2.4), then (A^{ω}, \preceq) is a periodic extension of $(A^*, \sim_{\mathcal{A}})$ and $L(\mathcal{A})$ is \preceq -closed.

If part: Assume that L is \preceq -closed where \preceq is a periodic extension of a monotone WQO \leq . First, we show that \approx_L is a finite congruence. Assume that $\{u_i\}$ is an infinite set in A^* such that $u_i \not\approx_L u_j$ for $i \neq j$. Since (A^*, \leq) is a WQO, there exists an infinite ascending subsequence $\{u_{k_i}\}$.

Let $F(u) = \{(v, v_1, v_2, w_1, w_2, w) \in A^* \times A^* \times A^* \times A^* \times A^* \times A^* | v(v_1uv_2)^\omega \in L \land w_1uw_2w^\omega \in L\}$. Since \preceq is a periodic extension of \leq and L is \preceq -closed, each F(u) is $\leq \times \leq \times \leq \times \leq \times \leq \times \leq \times \leq$ -closed and hence $F(u_{k_i}) \subseteq F(u_{k_j})$ for i < j. Since $u_{k_i} \not\approx_L u_{k_j}$ for $i \neq j$, $F(u_{k_i}) \neq F(u_{k_j})$, thus $F(u_{k_i}) \subset F(u_{k_j})$. Then there exists an infinite sequence in which each pair of different elements is incomparable. Since $\leq \times \leq \times A^* \times A^* \times A^* \times A^*$, this is a contradiction.

Second, we show that \approx_L saturates L. Assume that some \approx_L -classes U, V satisfy $U.V^{\omega} \cap L \neq \phi$ and $U.V^{\omega} \not\subseteq L$. From Lemma 1.4 (ii), we can assume that $U.V \subseteq U$ and $V.V \subseteq V$.

Let $\alpha \in U.V^{\omega} \cap L$ and $\beta \in U.V^{\omega} \setminus L$. Since (A^{ω}, \preceq) is a periodic extension, from Lemma 2.3 there exist $u, u' \in U$ and $v, v' \in V$ such that $\alpha = uv^{\omega}$ and $\beta = u'v'^{\omega}$. By definition of \approx_L , $uv^{\omega} \in L$ and $u'v'^{\omega} \notin L$ are contradictory.

3 Second theorem

Definition 3.1 For a monotone QO (A^*, \leq) , a QO (A^{ω}, \preceq) is a *continuous* extension if the following conditions are satisfied.

(1) For each $u, v \in A^*$ and $\alpha, \beta \in A^{\omega}, u \leq v$ and $\alpha \preceq \beta$ imply $u\alpha \preceq v\beta$.

(2) Let $u_j, v_j \in A^*$ for each j and let $\alpha_i = v_1 \cdots v_{i-1} u_i \cdots$ for each i and $\alpha_{\infty} = v_1 v_2 \cdots$. For $\beta \in A^{\omega}$, if $u_i \leq v_i$ and $\alpha_i \leq \beta$ for each i, then $\alpha_{\infty} \leq \beta$, and if $u_i \geq v_i$ and $\alpha_i \geq \beta$ for each i, then $\alpha_{\infty} \geq \beta$.

Theorem 3.2 Let $L \subseteq A^{\omega}$. *L* is regular if and only if *L* is \preceq -closed under a WQO (A^{ω}, \preceq) that is a continuous extension of a monotone WQO (A^*, \leq) .

For the embedding \leq over finite words, let (A^*, \leq°) be defined as $u \leq^\circ v$ if and only if $u \leq v$ and elt(u) = elt(v), where $elt(u) = \{a_i \mid u = a_1a_2\cdots a_j\}$. Since the embedding \leq over finite words is a WQO from Higman's lemma, \leq° is also a WQO. Then the embedding over A^{ω} is a continuous extension of \leq° . Note that the embedding over A^{ω} is a continuous extension of the embedding \leq over finite words. Actually, any continuous extension of the embedding \leq over finite words is a trivial WQO (i.e., $A^{\omega} \times A^{\omega}$). For instance, given $\alpha, \beta \in A^{\omega}$. Let $\alpha(1, i)$ be the prefix of α of the length i and $\alpha_i = \alpha(1, i)$. β for each i. Since $\alpha(1, i) \geq \epsilon$, $\alpha_i \succeq \beta$ for each i. Thus, by definition of continuity, $\alpha_{\infty} = \alpha \succeq \beta$.

Definition 3.3 Let $u, v \in A^*$ and let $L \subseteq A^{\omega}$. We write

- $u \simeq_L^1 v$ if and only if $\forall w \in A^*, \forall \alpha \in A^{\omega}. wu\alpha \in L \Leftrightarrow wv\alpha \in L$,
- $u \simeq_L^2 v$ if and only if $\forall w \in A^*$. $wu^{\omega} \in L \Leftrightarrow wv^{\omega} \in L$, and
- $u \simeq_L v$ if and only if $u \simeq_L^1 v$ and $u \simeq_L^2 v$.

Proof of Theorem 3.2

Only-if part: Assume L is regular. Let \mathcal{A} be a Büchi automaton such that $L = L(\mathcal{A})$. Since $\sim_{\mathcal{A}}$ is a finite congruence, $(A^*, \sim_{\mathcal{A}})$ is a monotone WQO. Define \preceq as the transitive closure of \preceq' (defined in Lemma 2.4), then $L(\mathcal{A})$ is \preceq -closed. Since \preceq' is symmetric, (A^{ω}, \preceq) is a continuous extension of $(A^*, \sim_{\mathcal{A}})$ from Lemma 2.4 (ii). For the index n of $\sim_{\mathcal{A}}$, the number of \preceq -classes is bound by 2^{n^2} . Thus \preceq is a WQO.

If part: First, we show that \simeq_L is a finite congruence. Assume that $\{u_i\}$ is an infinite set in A^* such that $u_i \not\simeq_L u_j$ for $i \neq j$. Since (A^*, \leq) is a WQO, there exists an infinite ascending subsequence $\{u_{k_i}\}$.

Let $F(u) \subseteq A^* \times A^\omega \times A^*$ be a set such that $(w, \alpha, v) \in F(u) \Leftrightarrow wu\alpha \in L \wedge vu^\omega \in L$. Then, each F(u) is $\leq \times \leq \times \leq$ -closed and hence $F(u_{k_i}) \subseteq F(u_{k_j})$ for i < j. Since $u_{k_i} \not\simeq L u_{k_j}$ for $i \neq j$, $F(u_{k_i}) \neq F(u_{k_j})$, thus $F(u_{k_i}) \subset F(u_{k_j})$. Then there exists an infinite sequence in which each pair of different elements is incomparable. Since $\leq \times \leq \times \leq$ is a WQO over $A^* \times A^\omega \times A^*$, this is a contradiction.

Second, we show that \simeq_L saturates L. Assume that some \simeq_L -classes U, V

satisfy $U.V^{\omega} \cap L \neq \phi$ and $U.V^{\omega} \not\subseteq L$. From Lemma 1.4 (2), we can assume that $V.V \subseteq V$.

Let $\alpha = uv_1v_2\cdots$ be a minimal element (wrt \preceq) in $U.V^{\omega} \cap L$, and let $\beta = u'v'_1v'_2\cdots \in U.V^{\omega} \setminus L$ such that $u, u' \in U$ and $v_i, v'_i \in V$. Let $\{\bar{v}_l\}$ be sets of minimal elements of V wrt \leq . Since (V, \leq) is a WQO, $\{\bar{v}_l\}$ are finite.

Let $\alpha'(j, j+k) = v_j \cdots v_{j+k}$. Since \bar{v}_l are finitely many, from (infinite) Ramsey Theorem there exist l and an ascending sequence $0 < j_1 < j_2 < \cdots$ such that $\alpha'(j_m, j_{m+1} - 1) \ge \bar{v}_l$ for any m > 0.

Let $\alpha_m = u \ \alpha'(1, j_1 - 1) \ \bar{v}_l^{m-1} \ \alpha'(j_m, j_{m+1} - 1) \cdots$. Obviously, $\alpha_m \preceq \alpha$ and $\alpha_m \in U.V^{\omega} \cap L$. Since α is minimal in $U.V^{\omega} \cap L$, $\alpha_m \succeq \alpha$. By definition of the continuous extension, $\alpha_{\infty} = u \ \alpha'(1, j_1 - 1) \ \bar{v}_l^{\omega} \succeq \alpha$. Thus since L is \preceq -closed, $\alpha_{\infty} \in U.V^{\omega} \cap L$.

Let $\beta'(j, j+k) = v'_j \cdots v'_{j+k}$. Since \bar{v}_l are finitely many, from (infinite) Ramsey Theorem there exist l' and an ascending sequence $0 < j'_1 < j'_2 < \cdots$ such that $\beta'(j'_m, j'_{m+1} - 1) \geq \bar{v}_{l'}$ for any m > 0. Let $\beta_{\infty} = u' \beta'(1, j_1 - 1) \bar{v}_{l'}^{\omega}$. By definition of the continuous extension, $\beta_{\infty} \preceq \beta$. Since L is \preceq -closed, $\beta \notin L$ implies $\beta_{\infty} \notin L$. Thus $\bar{\beta} \in U.V^{\omega} \setminus L$.

Since $u \simeq_L^1 u'$ and $\bar{v}_j \simeq_L^2 \bar{v}_{j'}$ for each j, repeated applications of \simeq_L^1 and an application of \simeq_L^2 imply that $\alpha_{\infty} \in L \iff \beta_{\infty} \in L$. This contradicts $\alpha_{\infty} \in L$ and $\beta_{\infty} \notin L$.

Example 3.4 Either the periodic or continuous assumption cannot be dropped. Let $\beta = abaabaaabaaaab\cdots$ and let $L(\beta)$ be the set of ω -words that have a common suffix with β . For $\alpha \in A^{\omega}$, let $p_{\beta}(\alpha) = 1$ if $\alpha \in L(\beta)$ and let $p_{\beta}(\alpha) = 0$ if $\alpha \notin L(\beta)$. Define $\alpha \preceq \alpha' \Leftrightarrow p_{\beta}(\alpha) \le p_{\beta}(\alpha')$. Then \preceq is a WQO over ω -words and $L(\beta)$ is \preceq -closed, but $L(\beta)$ is not regular.

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