Unique Existence and Computability in Constructive Reverse Mathematics

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Abstract. We introduce, and show the equivalences among, relativized versions of Brouwer's fan theorem for detachable bars (FAN), weak König lemma with a uniqueness hypothesis (WKL!), and the longest path lemma with a uniqueness hypothesis (LPL!) in the spirit of constructive reverse mathematics. We prove that a computable version of minimum principle: if f is a real valued computable uniformly continuous function with at most one minimum on $\{0,1\}^{\mathbf{N}}$, then there exists a computable α in $\{0,1\}^{\mathbf{N}}$ such that $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$, is equivalent to some computably relativized version of FAN, WKL! and LPL!.

Keywords: unique existence, computability, Brouwer's fan theorem, weak König lemma, constructive mathematics, reverse mathematics

1 Introduction

The purpose of constructive reverse mathematics [8] is to classify various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations. Classifying mathematical theorems means finding logical principles and/or function existence axioms which are not only sufficient but also necessary to prove the theorems in a weak system. An informal approach [9] to constructive reverse mathematics, that is reverse mathematics in Bishop's constructive mathematics [3, 4, 6], seems to have started in Julian and Richman [11] proving that Brouwer's fan theorem for detachable bars is equivalent to a positivity property: every positively valued uniformly continuous function on $[0, 1]$ has a positive infimum. (See Veldman [21] and others [16] for a similar program of intuitionistic reverse mathematics.)

Ishihara [7] showed in the context of constructive reverse mathematics that the statement

MIN: *every real valued uniformly continuous function f on a compact metric space X attains its infimum, that is, there exists an x in X such that $f(x) = \inf f(X)$,* is equivalent to weak König's lemma

WKL: *every infinite binary tree has an infinite path.*

(The corresponding result in Friedman-Simpson program of (classical) reverse mathematics can be found in Simpson [19, IV.2].) Here a *binary tree* T is an inhabited subset of the set $\{0,1\}^*$ of finite binary sequences such that it is detachable in the sense that there exists a characteristic function χ_T of T (and hence $\forall a \in \{0,1\}^*(a \in T \vee a \notin T)$, which does not hold constructively in general), and downward closed with respect to the predecessor relation $\preceq$ defined, using the concatenation function $*$, by $a \preceq b := \exists c \in \{0,1\}^*(a * c = b)$. A binary tree T is *infinite* if for each n there is a in T with $|a| = n$, where $|a|$ denotes the length of a, and a binary sequence $\alpha \in \{0,1\}^{\mathbf{N}}$ is an *infinite path* of T if each finite initial segment $\overline{\alpha}n := (\alpha(0), \ldots, \alpha(n-1))$ of α is in T. (We will use a similar notation for a finite sequence $a = (a_0, \ldots, a_{m-1})$, that is, $\overline{a}n$ stands for $(a_0, \ldots, a_{n-1})$ if $n \leq m$, and a otherwise.)

A real valued function f on a metric space X *has at most one minimum* if

$$\forall x, y \in X[x \neq y \to \exists z \in X(f(z) < f(x) \vee f(z) < f(y)].$$

Note that if $m := \inf f(X)$ exists, the above condition is equivalent to the following condition in Berger et al. [1]:

$$\forall x, y \in X[x \neq y \to m < f(x) \vee m < f(y)],$$

and that the real valued function $f : y \mapsto d(x, y)$ on an inhabited subset S of a metric space (X, d) has at most one minimum if and only if x has at most one best approximation in S in the sense of Bridges [5] (see also [4, 7.2.11] and [1]). Berger et al. [1] showed that MIN with the uniqueness hypothesis

MIN!: *every real valued uniformly continuous function with at most one minimum on a compact metric space attains its infimum*

is equivalent to Brouwer's fan theorem for detachable bars

FAN: *every detachable bar is uniform.*

Here a *bar* B is a subset of $\{0,1\}^*$ such that for each α in $\{0,1\}^{\mathbf{N}}$ there exists n with $\overline{\alpha}n \in B$, and B is *uniform* if there exists n such that $\exists k \leq n(\overline{\alpha}k \in B)$ for all $\alpha \in \{0,1\}^{\mathbf{N}}$. As a corollary of the above results, we can see the implication from WKL to FAN, which was first proved indirectly in [7], and then directly in [10].

A binary tree T *has at most one path* if

$$\forall \alpha, \beta \in \{0,1\}^{\mathbf{N}}[\alpha \neq \beta \to \exists n(\overline{\alpha}n \notin T \vee \overline{\beta}n \notin T)].$$

Berger and Ishihara [2] showed indirectly that the equivalence between FAN and WKL with the uniqueness hypothesis

WKL!: *every infinite binary tree with at most one path has an infinite path.*

A nice direct proof of the equivalence between FAN and WKL! can be found in Schwichtenberg [18]. Schuster [17] dealt with unique solutions in the context of constructive reverse mathematics.

In this paper, we introduce relativized versions of FAN, WKL! and the longest path lemma with the uniqueness hypothesis

LPL!: every binary tree with at most one path has a longest path,

where a *longest path* of a binary tree T is an infinite binary sequence such that $\forall a \in \{0,1\}^*(a \in T \rightarrow \overline{\alpha}|a| \in T)$, and prove the equivalences among them in the spirit of constructive reverse mathematics. (Note that the equivalence between WKL and the longest path lemma

LPL: every binary tree has a longest path

was proved in [10].) Then we show that the following computable version of MIN! for the Cantor space $\{0,1\}^{\mathbf{N}}$:

> *if f is a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$, then there exists a computable α in $\{0,1\}^{\mathbf{N}}$ such that $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$*

is equivalent to some (classically true) computably relativized versions of FAN, WKL!, and LPL!. Note that finding a zero of a real valued function g on $\{0,1\}^{\mathbf{N}}$ with $\inf |g(\{0,1\}^{\mathbf{N}})| = 0$ is reducible to MIN, and hence this result is related to the results in 6.3, especially Corollary 6.3.5, of [22].

2 Relativized versions of FAN, WKL! and LPL!

Let $\mathcal{C}$ be a subset of $\mathbf{N}^{\mathbf{N}}$. Then $\alpha \in \mathbf{N}^{\mathbf{N}}$ is *computable in* $\mathcal{C}$ if there exist an index e and $\beta \in \mathcal{C}$ such that

$$\forall n \exists z [T(e, \beta, n, z) \wedge U(z) = \alpha(n)],$$

where T is Kleene's T-predicate and U is the result-extracting function in [20, 3.7.6], and α is *computable* if it is computable in $\{\lambda x.0\}$. We say that $\mathcal{C}$ is *computably closed* if every $\alpha \in \mathbf{N}^{\mathbf{N}}$ computable in $\mathcal{C}$ is in $\mathcal{C}$. Let *Rec* be the smallest computably closed inhabited subset of $\{0,1\}^{\mathbf{N}}$, that is, consisting of all computable $\alpha \in \{0,1\}^{\mathbf{N}}$.

A subset $\mathcal{B}$ of $\{0,1\}^{\mathbf{N}}$ is *full* if $\{0,1\}^* \subseteq \{\overline{\alpha}n \mid \alpha \in \mathcal{B} \wedge n \in \mathbf{N}\}$. Note that every computably closed inhabited subset of $\{0,1\}^{\mathbf{N}}$ is full.

Let $\mathcal{B}$ be a full subset of $\{0,1\}^{\mathbf{N}}$. Then a detachable subset B of $\{0,1\}^*$ is a *bar in* $\mathcal{B}$ if for each α in $\mathcal{B}$ there exists n with $\overline{\alpha}n \in B$. Similarly, we may say a uniform bar B in $\mathcal{B}$ if there exists n such that $\exists k \leq n(\overline{\alpha}k \in B)$ for all $\alpha \in \mathcal{B}$. But, if a bar B is uniform in $\mathcal{B}$, then, since $\mathcal{B}$ is full, we have

$$(*) \quad \exists n \forall a \in \{0,1\}^*[|a| = n \rightarrow \exists k \leq n(\overline{a}k \in B)].$$

Conversely, if $(*)$ holds, then B is uniform in any full subset of $\{0,1\}^{\mathbf{N}}$. Therefore, the notion of the uniformity on bars is independent of an underlying full set, and so we adopt $(*)$ as the definition of the uniformity.

Let $\mathcal{B}$ and $\mathcal{D}$ be subset of $\{0,1\}^{\mathbf{N}}$ such that $\mathcal{B}$ is full. Then we have the following relativized version of Brouwer's fan theorem for detachable bars.

$\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$: Every $\mathcal{D}$-bar in $\mathcal{B}$ is uniform,

where $\mathcal{D}$-bar is a detachable bar whose characteristic function is in $\mathcal{D}$ (here we assume a coding of finite binary sequences into natural numbers). Similarly, we will say $\mathcal{D}$-tree for a binary tree whose characteristic function is in $\mathcal{D}$. Note that Kleene [12, 13] showed that $\mathrm{FAN}_{Rec}(Rec)$ is refutable (see also [20, 4.7.6]).

A binary tree T *has at most one path in* $\mathcal{B}$ if

$$\forall\alpha,\beta\in\mathcal{B}[\alpha\neq\beta\rightarrow\exists n(\overline{\alpha}n\notin T\vee\overline{\beta}n\notin T)].$$

Similar to FAN, we have the following relativized version of WKL! and LPL!.

$\mathrm{WKL!}_{\mathcal{D}}(\mathcal{B})$: *Every infinite $\mathcal{D}$-tree with at most one path in $\mathcal{B}$ has an infinite path in $\mathcal{B}$.*

$\mathrm{LPL!}_{\mathcal{D}}(\mathcal{B})$: *Every $\mathcal{D}$-tree with at most one path in $\mathcal{B}$ has a longest path in $\mathcal{B}$.*

Before discussing relations among $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$, $\mathrm{WKL!}_{\mathcal{D}}(\mathcal{B})$ and $\mathrm{LPL!}_{\mathcal{D}}(\mathcal{B})$, we introduce a notion of uniformly having at most one path, which is similar to the notion of having uniformly at most one minimum introduced in [17] for non-negative functions; see also [14] and [15, 4.1]. A binary tree T *has uniformly at most one path* if

$$(**)\quad \forall k\exists n\geq k\forall a,b\in\{0,1\}^*[|a|=|b|=n\wedge\overline{a}k\neq\overline{b}k\rightarrow a\notin T\vee b\notin T].$$

Again, we may define this notion relative to a full subset $\mathcal{B}$ of $\{0,1\}^{\mathbf{N}}$: a binary tree T has uniformly at most one path in $\mathcal{B}$ if

$$\forall k\exists n\geq k\forall\alpha,\beta\in\mathcal{B}[\overline{\alpha}k\neq\overline{\beta}k\rightarrow\overline{\alpha}n\notin T\vee\overline{\beta}n\notin T].$$

But we can see that if a binary tree T has uniformly at most one path in some full set, then $(**)$ holds, and if $(**)$ holds, then T has uniformly at most one path in any full set. Hence this notion is independent of an underlying full set.

We show the following proposition in a similar way to [18].

Proposition 1. *Let $\mathcal{B}$ and $\mathcal{D}$ be computably closed inhabited subsets of* $\{0,1\}^{\mathbf{N}}$, *and assume* $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$. *Then every $\mathcal{D}$-tree with at most one path in $\mathcal{B}$ has uniformly at most one path.*

Proof. Let T be a $\mathcal{D}$-tree with at most one path in $\mathcal{B}$. Then

$$\forall\alpha,\beta\in\mathcal{B}[\exists k(\overline{\alpha}k\neq\overline{\beta}k)\rightarrow\exists n(\overline{\alpha}n\notin T\vee\overline{\beta}n\notin T)],$$

and hence

$$\forall k\forall\alpha,\beta\in\mathcal{B}[\overline{\alpha}k\neq\overline{\beta}k\rightarrow\exists n(\overline{\alpha}n\notin T\vee\overline{\beta}n\notin T)].$$

Therefore, for given k, we have

$$\forall \alpha, \beta \in \mathcal{B} \exists n[\overline{\alpha}k \neq \overline{\beta}k \to \overline{\alpha}n \notin T \vee \overline{\beta}n \notin T],$$

and so, since T is downward closed, we have

$$\forall \alpha, \beta \in \mathcal{B} \exists n \geq k[\overline{\overline{\alpha}n}k \neq \overline{\overline{\beta}n}k \to \overline{\alpha}n \notin T \vee \overline{\beta}n \notin T].$$

For $\alpha \in \{0,1\}^{\mathbf{N}}$, let $E_\alpha(n) := \alpha(2n)$ and $O_\alpha(n) := \alpha(2n+1)$. Then, since $\mathcal{B}$ is computably closed, if $\alpha \in \mathcal{B}$, then $E_\alpha, O_\alpha \in \mathcal{B}$, and hence

$$\forall \alpha \in \mathcal{B} \exists n \geq k[\overline{\overline{E_\alpha}n}k \neq \overline{\overline{O_\alpha}n}k \to \overline{E_\alpha}n \notin T \vee \overline{O_\alpha}n \notin T].$$

For $a = (a_0, a_1, \dots, a_{n-1}) \in \{0,1\}^*$, let $E(a) := (a_0, \dots, a_{2m-2})$ and $O(a) := (a_1, \dots, a_{2m-1})$, where $m = \lfloor n/2 \rfloor$. Note that $\overline{E_\alpha}n = E(\overline{\alpha}(2n))$ and $\overline{O_\alpha}n = O(\overline{\alpha}(2n))$. Then we have

$$\forall \alpha \in \mathcal{B} \exists n \geq k[\overline{E(\overline{\alpha}(2n))}k \neq \overline{O(\overline{\alpha}(2n))}k \to E(\overline{\alpha}(2n)) \notin T \vee O(\overline{\alpha}(2n)) \notin T].$$

Let

$$a \in B := 2k \leq |a| \wedge [\overline{E(a)}k \neq \overline{O(a)}k \to E(a) \notin T \vee O(a) \notin T].$$

Then, since $\mathcal{D}$ is computably closed, the characteristic function of B is in $\mathcal{D}$, and $\forall \alpha \in \mathcal{B} \exists n(\overline{\alpha}(2n) \in B)$, that is B is a bar in $\mathcal{B}$. By $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$, there exists m such that

$$\forall a \in \{0,1\}^*[|a| = m \to \exists j \leq m(\overline{a}j \in B)].$$

Choose n such that $m \leq 2n$. Then, taking $0^m := (0, \dots, 0)$ with $|0^m| = m$, there exists $j \leq m$ such that $\overline{0^m}j \in B$, and hence $2k \leq |\overline{0^m}j| = j \leq m \leq 2n$. Thus $k \leq n$. Let $a = (a_0, \dots, a_{n-1})$ and $b = (b_0, \dots, b_{n-1})$ be in $\{0,1\}^*$ with $\overline{a}k \neq \overline{b}k$. Then, setting $c := (a_0, b_0, \dots, a_{n-1}, b_{n-1})$, we have $m \leq 2n = |c|$, and there exists j with $j \leq m$ such that $\overline{\overline{c}m}j = \overline{c}j \in B$. Since $2k \leq j \leq m \leq 2n$, we have $\overline{E(\overline{c}j)}k = \overline{a}k \neq \overline{b}k = \overline{O(\overline{c}j)}k$, and hence either $E(\overline{c}j) \notin T$ or $O(\overline{c}j) \notin T$. In the former case, since $E(\overline{c}j) \preceq a$, we have $a \notin T$. Similarly, in the latter case, we have $b \notin T$. Thus T has uniformly at most one path.

The following proposition shows that a binary tree with uniformly at most one path has a longest path.

Proposition 2. *Let $\mathcal{D}$ be a computably closed inhabited subset of $\{0,1\}^{\mathbf{N}}$. Then every $\mathcal{D}$-tree with uniformly at most one path has a longest path in $\mathcal{D}$.*

Proof. Let T be a $\mathcal{D}$-tree with uniformly at most one path, and define a relation $\mathrm{big}(c, n)$ by

$$\mathrm{big}(c, n) := \forall d \in \{0,1\}^*[|d| = n \to c * d \notin T].$$

Note that the characteristic function of big is computable in $\mathcal{D}$. Then for given $c \in \{0,1\}^*$, since T has uniformly at most one path, there exists n such that

$$\forall a, b \in \{0,1\}^*[|a| = |b| = n \to c * (0) * a \notin T \vee c * (1) * b \notin T],$$

and hence $\neg\mathrm{big}(c * (0), n) \to \mathrm{big}(c * (1), n)$. Therefore for each $c \in \{0,1\}^*$ there exists n such that $\mathrm{big}(c * (0), n) \vee \mathrm{big}(c * (1), n)$, and so the function σ defined by

$$\sigma(c) := \min_n [\mathrm{big}(c * (0), n) \vee \mathrm{big}(c * (1), n)]$$

is computable in $\mathcal{D}$. Define functions δ and τ by

$$\delta(c) := \begin{cases} 0 \text{ if } \mathrm{big}(c * (1), \sigma(c)), \\ 1 \text{ if } \neg\mathrm{big}(c * (1), \sigma(c)), \end{cases}$$

and

$$\tau(0) := (),$$
$$\tau(n+1) := \tau(n) * \delta(\tau(n)).$$

Then, clearly, δ and τ are computable in $\mathcal{D}$. Let $\alpha(n) := \delta(\tau(n))$. Then $\alpha \in \{0,1\}^{\mathbf{N}}$ is computable in $\mathcal{D}$, and, since $\mathcal{D}$ is computably closed, α is in $\mathcal{D}$. Note that, by induction, $\overline{\alpha}n = \tau(n)$.

We prove that α is a longest path of T. Suppose that $\overline{\alpha}n = \tau(n) \notin T$. Then we show that $\forall d \in \{0,1\}^*(|d| = n \to d \notin T)$, or $\mathrm{big}(\tau(0), n)$. To this end, we prove by induction that

$$\forall k \leq n[\mathrm{big}(\tau(n-k), k)].$$

If $k = 0$, then, trivially, $\mathrm{big}(\tau(n), 0)$. Assume that $\mathrm{big}(\tau(n-k), k)$. Then $\mathrm{big}(\tau(n-k-1) * \alpha(n-k-1), k)$, and hence $\sigma(\tau(n-k-1)) \leq k$. Either $\alpha(n-k-1) = 0$ or $\alpha(n-k-1) = 1$. In the former case, $\mathrm{big}(\tau(n-k-1) * (0), k)$ and, since $\mathrm{big}(\tau(n-k-1) * (1), \sigma(\tau(n-k-1)))$, we have $\mathrm{big}(\tau(n-k-1) * (1), k)$. Hence $\mathrm{big}(\tau(n-k-1), k+1)$. In the latter case, $\mathrm{big}(\tau(n-k-1) * (1), k)$, and, since $\neg\mathrm{big}(\tau(n-k-1) * (1), \sigma(\tau(n-k-1)))$, we have $\mathrm{big}(\tau(n-k-1) * (0), \sigma(\tau(n-k-1)))$. Therefore $\mathrm{big}(\tau(n-k-1) * (0), k)$, and so $\mathrm{big}(\tau(n-k-1), k+1)$.

We omit proof of the following proposition which is an easy adaptation of the proof in [2] or [18].

Proposition 3. *Let $\mathcal{B}$ and $\mathcal{D}$ be computably closed inhabited subsets of $\{0,1\}^{\mathbf{N}}$. Then* $\mathrm{WKL!}_{\mathcal{D}}(\mathcal{B})$ *implies* $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$.

The aforementioned propositions culminate in the following theorem.

Theorem 1. *Let $\mathcal{B}$ and $\mathcal{D}$ be computably closed inhabited subsets of $\{0,1\}^{\mathbf{N}}$ with $\mathcal{D} \subseteq \mathcal{B}$. Then the following statements are equivalent.*

1. $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$.
2. *Every $\mathcal{D}$-tree with at most one path in $\mathcal{B}$ has a longest path in $\mathcal{D}$.*
3. *Every infinite $\mathcal{D}$-tree with at most one path in $\mathcal{B}$ has an infinite path in $\mathcal{D}$.*
4. $\mathrm{LPL!}_{\mathcal{D}}(\mathcal{B})$.
5. $\mathrm{WKL!}_{\mathcal{D}}(\mathcal{B})$.

Proof. (1) $\to$ (2): Assume $\mathrm{FAN}_{\mathcal{D}}(\mathcal{B})$, and let T be a $\mathcal{D}$-tree with at most one path in $\mathcal{B}$. Then, by Proposition 1, T has uniformly at most one path, and hence T has a longest path in $\mathcal{D}$, by Proposition 2. (2) $\to$ (3) and (4) $\to$ (5): Trivial. (2) $\to$ (4) and (3) $\to$ (5): By $\mathcal{D} \subseteq \mathcal{B}$. (5) $\to$ (1): By Proposition 3.

3 A computable version of MIN!

We assume that a real number x is given by a Cauchy sequence $(p_n)_n$ of rationals with a fixed modulus, that is, $\forall m, n(|p_m - p_n| < 2^{-m} + 2^{-n})$. For a real number $x := (p_n)_n$, we write $(x)_n$ for p_n. See [3, 4, 6, 20] for more on constructive theory of the real numbers.

A uniformly continuous function f from the Cantor space $\{0,1\}^{\mathbf{N}}$ to $\mathbf{R}$ is *computable* if there exists an index e and a computable $M \in \mathbf{N}^{\mathbf{N}}$ such that

$$\forall \alpha \in \{0,1\}^{\mathbf{N}} \forall n \exists z [T(e, \alpha, n, z) \wedge U(z) = (f(\alpha))_n]$$

and

$$\forall k \forall \alpha, \beta \in \{0,1\}^{\mathbf{N}} [\overline{\alpha} M(k) = \overline{\beta} M(k) \to |f(\alpha) - f(\beta)| < 2^{-k}].$$

We show that the following computable version of MIN! for the Cantor space $\{0,1\}^{\mathbf{N}}$:

> *if f is a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$, then there exists a computable α in $\{0,1\}^{\mathbf{N}}$ such that $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$*

is equivalent to the classically true relativized versions $\mathrm{WKL!}_{Rec}(\{0,1\}^{\mathbf{N}})$, $\mathrm{FAN}_{Rec}(\{0,1\}^{\mathbf{N}})$, and $\mathrm{LPL!}_{Rec}(\{0,1\}^{\mathbf{N}})$. We start with showing the following propositions.

Proposition 4. *Let T be an infinite Rec-tree with at most one path in $\{0,1\}^{\mathbf{N}}$. Then there exists a real valued computable uniformly continuous function f on the Cantor space $\{0,1\}^{\mathbf{N}}$ such that if $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$, then α is an infinite path of T.*

Proof. Define a real valued function f on the Cantor space $\{0,1\}^{\mathbf{N}}$ by

$$(f(\alpha))_n := \begin{cases} 2^{-n} & \text{if } \overline{\alpha} n \in T, \\ (f(\alpha))_{n-1} & \text{if } \overline{\alpha} n \notin T. \end{cases}$$

Then f is a computable uniformly continuous function with $\inf f(\{0,1\}^{\mathbf{N}}) = 0$. Let α, β be in $\{0,1\}^{\mathbf{N}}$ with $\alpha \neq \beta$. Then, since T has at most one path in $\{0,1\}^{\mathbf{N}}$, there exists n such that $\overline{\alpha} n \notin T$ or $\overline{\beta} n \notin T$, and hence either $0 < f(\alpha)$ or $0 < f(\beta)$. Thus f has at most one minimum. If $f(\alpha) = 0$, then $(f(\alpha))_n \leq 2^{-n}$ for all n, and hence $\overline{\alpha} n \in T$ for all n.

Proposition 5. *Let f be a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$. Then there exists an infinite Rec-tree T with at most one path in $\{0,1\}^{\mathbf{N}}$ such that if α is an infinite path of T, then $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$.*

Proof. We may assume without loss of generality that $\inf f(\{0,1\}^{\mathbf{N}}) = 0$. Let $M \in \mathbf{N}^{\mathbf{N}}$ be a computable function such that

$$\forall k \forall \alpha, \beta \in \{0,1\}^{\mathbf{N}}[\overline{\alpha}M(k) = \overline{\beta}M(k) \rightarrow |f(\alpha) - f(\beta)| < 2^{-k}].$$

We may assume further that M is strictly increasing and $0 < M(0)$. For $a \in \{0,1\}^*$, we write $f(a)$ for $f(a * (\lambda x.0))$, where the concatenation is extended to concatenation of a finite sequence with an infinite sequence. Define a subset T of $\{0,1\}^*$ by

$$a \in T := \forall k[M(k) \leq |a| \rightarrow (f(\overline{a}M(k)))_k < 2^{-|a|} + 2^{-k+1}].$$

Then T is a *Rec*-tree. For given n, choose $\alpha \in \{0,1\}^{\mathbf{N}}$ such that $f(\alpha) < 2^{-n}$, and set $a := \overline{\alpha}n$. Then, for each k with $M(k) \leq |a| = n$, we have

$$\begin{aligned}(f(\overline{a}M(k)))_k &\leq f(\overline{a}M(k)) + 2^{-k} = f(\overline{\alpha}M(k)) + 2^{-k} \\ &< f(\alpha) + 2^{-k+1} < 2^{-|a|} + 2^{-k+1},\end{aligned}$$

and hence $a \in T$. Therefore T is infinite. Let α, β be in $\{0,1\}^{\mathbf{N}}$ with $\alpha \neq \beta$. Since f has at most one minimum, there exists n such that either $5 \cdot 2^{-n} < f(\alpha)$ or $5 \cdot 2^{-n} < f(\beta)$. In the former case, since $n \leq M(n)$, we have

$$\begin{aligned}(f(\overline{\alpha}M(n)))_n &\geq f(\overline{\alpha}M(n)) - 2^{-n} > f(\alpha) - 2^{-n+1} \\ &> 2^{-n} + 2^{-n+1} \geq 2^{-M(n)} + 2^{-n+1},\end{aligned}$$

and hence $\overline{\alpha}M(n) \notin T$. Similarly, in the latter case, we have $\overline{\beta}M(n) \notin T$. Therefore T has at most one path. If α is an infinite path of T, then, for each n, we have

$$\begin{aligned}f(\alpha) &< f(\overline{\alpha}M(n)) + 2^{-n} \leq (f(\overline{\alpha}M(n)))_n + 2^{-n+1} \\ &< 2^{-M(n)} + 2^{-n+2} \leq 5 \cdot 2^{-n},\end{aligned}$$

and hence $f(\alpha) = 0$.

The above propositions and Theorem 1 culminate in the following theorem.

Theorem 2. *The following statements are equivalent.*

1. *If f is a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$, then there exists a computable α in $\{0,1\}^{\mathbf{N}}$ such that $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$.*
2. *If f is a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$, then there exists α in $\{0,1\}^{\mathbf{N}}$ such that $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$.*

3. $\mathrm{WKL!}_{Rec}(\{0,1\}^{\mathbf{N}})$.
4. $\mathrm{LPL!}_{Rec}(\{0,1\}^{\mathbf{N}})$.
5. $\mathrm{FAN}_{Rec}(\{0,1\}^{\mathbf{N}})$.

Proof. Note that (1) $\rightarrow$ (2) is trivial. Then, by Theorem 1, it is enough to show that (2) $\rightarrow$ (3) and (3) $\rightarrow$ (1).

(2) $\rightarrow$ (3): By Proposition 4. (3) $\rightarrow$ (1): Let f be a real valued computable uniformly continuous function with at most one minimum on the Cantor space $\{0,1\}^{\mathbf{N}}$. Then, by Proposition 5, there exists an infinite *Rec*-tree T with at most one path in $\{0,1\}^{\mathbf{N}}$ such that if α is an infinite path of T, then $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$. By Theorem 1, there exists an infinite path α of T in *Rec*, that is computable, and hence $f(\alpha) = \inf f(\{0,1\}^{\mathbf{N}})$.

4 A concluding remark

Let $\mathcal{B}$ be a subset in between *Rec* and $\{0,1\}^{\mathbf{N}}$, say the set of characteristic functions of computably enumerable sets. Then, since $\mathrm{FAN}_{Rec}(Rec)$ is refutable as mentioned before, it is natural to ask whether $\mathrm{FAN}_{Rec}(\mathcal{B})$ is still refutable or is derivable from $\mathrm{FAN}_{Rec}(\{0,1\}^{\mathbf{N}})$. In the latter case, since it is trivial that $\mathrm{FAN}_{Rec}(\mathcal{B})$ implies $\mathrm{FAN}_{Rec}(\{0,1\}^{\mathbf{N}})$, we could see the equivalence between $\mathrm{FAN}_{Rec}(\mathcal{B})$ and $\mathrm{FAN}_{Rec}(\{0,1\}^{\mathbf{N}})$.

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