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Description	

# Improvement of the robustness on geographical networks by adding shortcuts

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## Abstract

In a topological structure affected by geographical constraints on linking, the connectivity is weakened by constructing local stubs with small cycles, a something of randomness to bridge them is crucial for the robust network design. In this paper, we numerically investigate the effects of adding shortcuts on the robustness in geographical scale-free network models under a similar degree distribution to the original one. We show that a small fraction of shortcuts is highly contribute to improve the tolerance of connectivity especially for the intentional attacks on hubs. The improvement is equivalent to the effect by fully rewirings without geographical constraints on linking. Even in the realistic Internet topologies, these effects are virtually examined.

*Key words:* Complex network; Geographical constraint; Overhead bridge; Robust connectivity; Efficient routing

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## 1 Introduction

In many social, technological, and biological networks, there exist several common topological characteristics which can be emerged by simple generation mechanisms [1,2], and a small fraction of networks crucially influences the communication properties for the routing and the robustness of connectivity. One of the sensational facts is the **small-world** (SW) phenomenon: each pair of nodes are connected through relatively small number of hops to a huge network size defined by the total number of nodes. This favorable phenomenon has been explained by the SW model with a small fraction of random rewirings on a one-dimensional lattice [3]. Another fact is the **scale-free** (SF) structure that follows a power-law degree distribution  $P(k) \sim k^{-\gamma}$ ,  $2 < \gamma < 3$ , consists

of many nodes with low degrees and a few hubs with high degrees. The heterogeneous networks are drastically broken into many isolated clusters, when only a small fraction of high degree nodes are removed as the intentional attacks [4,5]. However, the SF structure is robust against random failures [4,6], and well balanced in the meaning of both economical and efficient communication by small number of hops in a connected network as few links as possible [7].

On the other hands, real complex networks, such as power-grid, airline flight-connection, and the Internet, are embedded in a metric space, and long-range links are restricted [8,9] for economical reasons. Based on the connection probability according to a penalty of distance  $r$  between two nodes and on random triangulation, the generation mechanisms of geographical SF networks have been proposed in **lattice-embedded scale-free** (LESF) [10] and **random Apollonian** (RA) [11,12] network models. Unfortunately, the vulnerability of connectivity has been numerically found in both networks [13,14,15]. Moreover, it has been theoretically predicted [13] in a generating function approach to more general networks with any degree distribution that a geographical constraint on linking decreases the tolerance to random failures, since the percolation threshold is increased by the majority of small-order cycles that locally connected with a few hops. As the smallest-order, triangular cycles tend to be particularly constructed by a geographical constraint.

In contrast, it has been suggested that higher-order cycles connected with many hops improve the robustness in the theoretical analysis on a one-dimensional SW model modified by adding shortcuts between two nodes out of the connected neighbors [16,17]. Similarly, the expected delivery time of any decentralized search algorithm without global information is decreased on a two-dimensional lattice whose each node has a shortcut with the connection probability proportional to the power of distance  $r^{-\alpha}$ ,  $\alpha > 0$  [18]. These results support the usefulness of shortcuts for maintaining both the robust connectivity and the communication efficiency, however the network structures are almost regular and far from the realistic SF. Recently, it has been numerically shown [15] that the robustness is improved by fully random rewirings under a same degree distribution [19] in typical geographical network models: **Delaunay triangulation** (DT) [20,21], RA, and **Delaunay-like scale-free** (DLSF) networks [15]. Instead of rewirings, we expect the shortcut effect on the improvement of robustness in such geographical SF networks. Adding shortcuts is practically more natural rather than rewirings, because the already constructed links are not wastefully discarded. Thus, we investigate how large connected component still remains at a rate of removed nodes as the random failures and the targeted attacks on hubs in the geographical SF networks with shortcuts, comparing the original ones without shortcuts. We show that a small quantity of geographical randomness highly contributes to maintain both the communication efficiency and the robustness under almost invariant degree distributions to the original ones. It is not trivial that the improvement

of robustness is equivalent to the effect by fully random rewirings.

The organization of this paper is as follows. In Sec. 2, we briefly introduce the geographical networks based on planar triangulation and embedding in a lattice under a given power-law degree distribution. In Sec. 3, we numerically investigate the effects of shortcuts on the optimal paths in two measures of distance/hop and the robustness in the geographical networks. In particular, we show that a degree cutoff enhances the improvement of the error and attack tolerance. Moreover, we virtually examine the effects for realistic data of the Internet topologies. Finally, in Sec. 4 we summarize these results.

## 2 Geographical SF networks

### 2.1 Planar network models

Planar networks without crossing of links are suitable for efficient geographical routings, since we can easily find the shortest path from a set of edges of the faces that intersect the straight line between the source and terminal. In computer science, online routing algorithms [22] that guarantee delivery of messages using only local information about positions of the source, terminal, and the adjacent nodes to a current node are well-known. As a connection to SF networks, we consider Delaunay triangulation (DT) and random Apollonian (RA) network models based on planar triangulation of a polygonal region. DT is the optimal planar triangulation in some geometric criteria [20] with respect to the maximin angle and the minimax circumcircle of triangles on a two-dimensional space. In addition, the ratio of the shortest path length is bounded by a constant factor to the direct Euclidean distance between any source and terminal [23], while RA network belongs to both SF and planar networks [12,11], however long-range links inevitably appear near the edges of an initial polygon. To reduce the long-range links, Delaunay-like scale-free (DLSF) network has been proposed [15].

On the preliminaries, just like overhead highways, we add shortcuts between randomly chosen two nodes excluding self-loops and multi-links after constructing the above networks. For adding shortcuts, the routing algorithm can be extended as mentioned in Appendix 1. Note that the added shortcuts contribute to create some higher-order cycles which consists of a long path and the overhead bridge in the majority of triangular cycles. The original degree distributions without shortcuts follow a power-law with the exponent nearly 3 in RA, log-normal in DT, and power-law with an exponential cutoff in DLSF networks [15]. Note that the lognormal distribution has an unimodal shape as similar to one in Erdős-Renyi random networks. Thus, RA and DLSF networks

are vulnerable because of double constraints of planarity and geographical distances on the linkings, but DT networks are not so. We have confirmed that the degree distributions have only small deviation from the original ones at shortcut rate up to the amount of 30% of the total links.

## 2.2 Lattice-embedded SF networks

Let us consider a  $d$ -dimensional lattice of size  $R$  with the periodic boundary conditions. The LESF network model [10] combines the configuration model [24] for any degree distribution with a geographical constraint on a lattice. Although the homogeneous positioning of nodes differs from a realistic spacial distribution such as in the Internet routers according to the population density [8], it has been studied as a fundamental spacial model. Note that the spatial distribution of nodes is restricted on the regular lattice, some links are crossed, therefore LESF networks are not planar.

In the following simulation, we assign a degree taken from the distribution  $P(k) \sim k^{-3}$  to each node on a two-dimensional lattice of the network size  $N = 32 \times 32$ , where  $\gamma = 3$ ,  $d = 2$ , and  $R = 32$ . The networks have the average numbers  $M \approx 1831$  of the total links at  $A = 1$  and  $M \approx 2673$  at  $A = 3$  for comparison. The case of  $A \rightarrow \infty$  is equivalent to the Warren-Sander-Sokolov model [25] whose degree distribution follows a pure power-law, however a cut-off is rather natural in real networks with something of constraints on linkings [26]. As similar to the previously mentioned planar networks, there are little deviation from the original power-law distributions with strong and weak cut-offs at  $A = 1$  and  $A = 3$ , respectively. The detailed configuration procedures for RA, DT, DLSF, and LESF networks are summarized in Appendix 2.

## 3 Shortcut effects

### 3.1 Shortest distance and minimum hops

For the shortcut rates from 0% to 30%, we investigate four combinations of distance/hops and two kinds of the optimal paths with respect to the shortest **distance** and the minimum number (or called **length**) of hops: the average distance  $\langle D \rangle$  on the shortest paths, the distance  $\langle D' \rangle$  on the paths of the minimum hops, the average number of hops  $\langle L \rangle$  on these paths, and the number of hops  $\langle L' \rangle$  on the shortest paths between any two nodes in the geographical networks. The prime denotes the cross relation to the case of no prime in the combinations of the measures and the two kinds of paths. The distance is

defined by a sum of link lengths on the path, and the average means a statistical ensemble over the optimal paths in the above two criteria. Note that the shortest path and the path of the minimum hops may be distinct, these are related to the link cost or delay and the load for transfer of a message. It is better to shorten both the distance and the number of hops, however the constraints are generally conflicted.

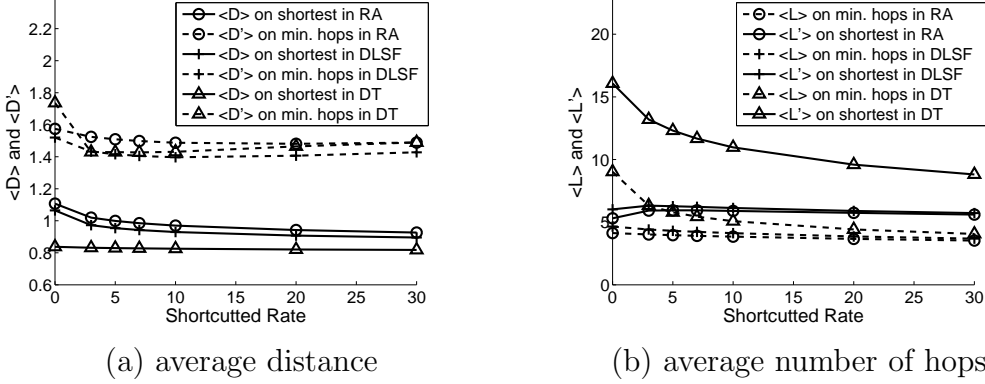


Fig. 1. The average distance and the number of hops on two kinds of the optimal paths in DT (triangle), RA (circle), DLSF (plus) networks. Solid lines guide the decreasing or increasing of  $\langle D \rangle$  and  $\langle L' \rangle$  on the shortest paths, dashed lines guide that of  $\langle D' \rangle$  and  $\langle L \rangle$  on the paths of the minimum hops.

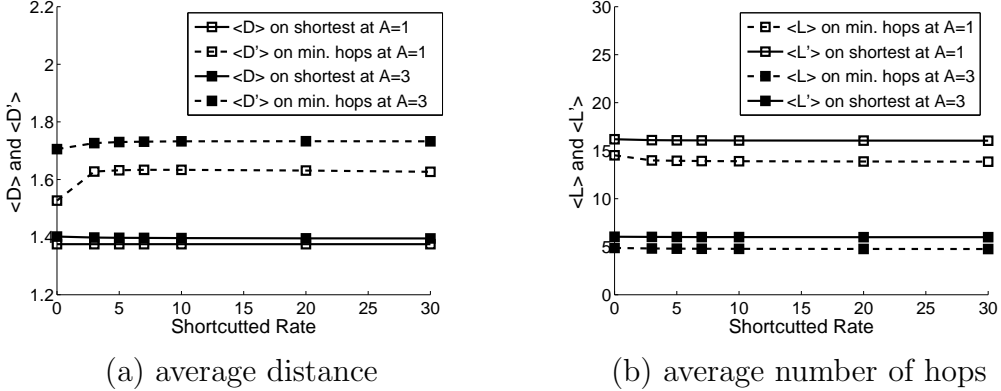


Fig. 2. The average distance and the number of hops on two kinds of the optimal paths in LESF networks at  $A = 1$  (open rectangle) and  $A = 3$  (filled rectangle). Solid lines guide the decreasing or increasing of  $\langle D \rangle$  and  $\langle L' \rangle$  on the shortest paths, dashed lines guide that of  $\langle D' \rangle$  and  $\langle L \rangle$  on the paths of the minimum hops.

In the original networks without shortcuts, we note the tendencies [15]: RA networks have a path connected by a few hops but the path length tend to be long including some long-range links, while DT networks have a zig-zag path connected by many hops but each link is short, in addition, DLSF networks have the intermediately balanced properties. Figure 1 shows numerical results in adding shortcuts to the planar networks. We find that, from the solid and dashed lines in Figs. 1(a)(b), the average distance  $\langle D \rangle$  and the number of hops  $\langle L \rangle$  become shorter as increasing the shortcut rate. In particular, the

shortcuts are effective for the distance in RA and DLSF (solid lines marked with circles and pluses) networks, and also for the number of hops in DTs (dashed line marked with triangles). On another measures of  $\langle D' \rangle$  and  $\langle L' \rangle$ , the dashed lines for DTs (marked with triangles) and RA (with circles) networks in Fig. 1(a) and the solid lines for RA (with circles) and DLSF (with pluses) networks in Fig. 1(b) approach to each other. Thus, the shortcuts even around 10% decrease both  $\langle D \rangle$  and  $\langle L \rangle$ , and maintain small  $\langle D' \rangle$  and  $\langle L' \rangle$ . On the other hand, as shown in Fig. 2, the average distance and the number of hops are almost constant in LESF networks. Probably, the links emanated from hubs already act as shortcuts on the lattice. These results are obtained from ensembles over 100 realizations for each network model.

### 3.2 Robustness of connectivity

The fault tolerance and attack vulnerability are known as the typical properties of SF networks [4], which are further affected by geographical constraints. We investigate the tolerance of connectivity in the giant component (GC) of the geographical networks with shortcuts comparing with that of the original ones without shortcuts. The size  $S$  of GC and the average size  $\langle s \rangle$  of isolated clusters are obtained from ensembles over 100 realizations for each network model. Figure 3 shows the typical results that a small fraction of shortcuts suppresses the breaking of the GC against random failures. It seems to be enough in less than 10%. In other DLSF, LESF ( $A = 3$ ) networks, the effects are similar. Thus, the added shortcuts strengthen the tolerance in comparison with each original network.

Figures 4 and 5 show the effect of shortcuts on the robustness against the targeted attacks on hubs. In particular, around the shortcuts rate 10%, the extremely vulnerable RA and DLSF networks are improved up to the similar level to DTs. We compare the critical values of fraction  $f_c$  of removed nodes at the peak of the average size  $\langle s \rangle$ , as the GC is broken off. As shown in Figure 7(b), the critical values  $f_c$  in RA and DLSF networks reach to 0.3 at the level of the original DTs without shortcuts. It is consistent with the effect in evolving networks with local preferential attachment [27] that the tolerance becomes higher as increasing the cutoff under the same average degree  $\langle k \rangle$  and size  $N$ . We emphasize that, by adding shortcuts around 10% under almost invariant degree distributions, the robustness against the intentional attacks can be considerably improved up to the similar level to the fully rewired networks by ignoring the geographical constraints [15].

The effects on LESF networks by adding shortcuts are also obtained in Fig. 6. The case of  $A = 3$  is more robust because of less geographical constraint with a larger number  $M$  of the total links. Figures 7(c)(d) shows the improvement

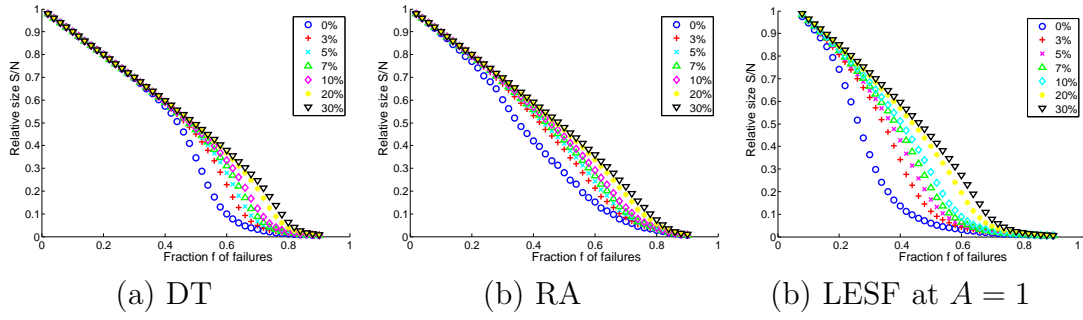


Fig. 3. (Color online) Typical results of the relative size  $S/N$  of the GC against random failures in (a) DT, (b) RA, and (c) LESF ( $A = 1$ ) networks at the shortcuts rates in legend.

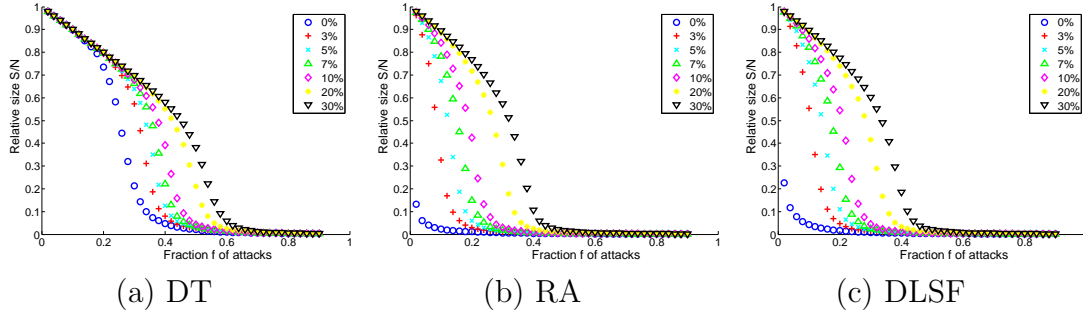


Fig. 4. (Color online) Relative size  $S/N$  of the GC against intentional attacks in (a) DT, (b) RA, and (c) DLSF networks at the shortcuts rates in legend.

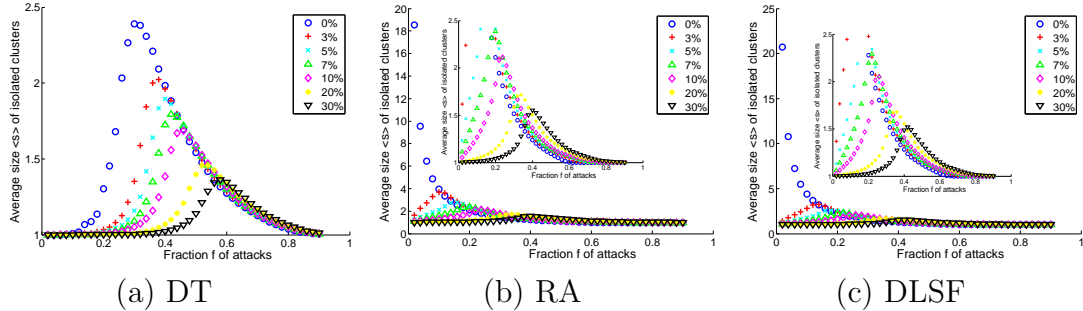


Fig. 5. (Color online) Average size  $\langle s \rangle$  of isolated clusters except the GC against intentional attacks in (a) DT, (b) RA, and (c) DLSF networks at the shortcuts rates in legend. Inset shows the peaks enlarged by other scale of the vertical axis.

of the critical values  $f_c$ ; the increase is remarkable in less than the shortcuts rate 10% as similar to Figs. 7(a)(b). These results are also obtained in the LESF networks without the periodic boundary conditions.



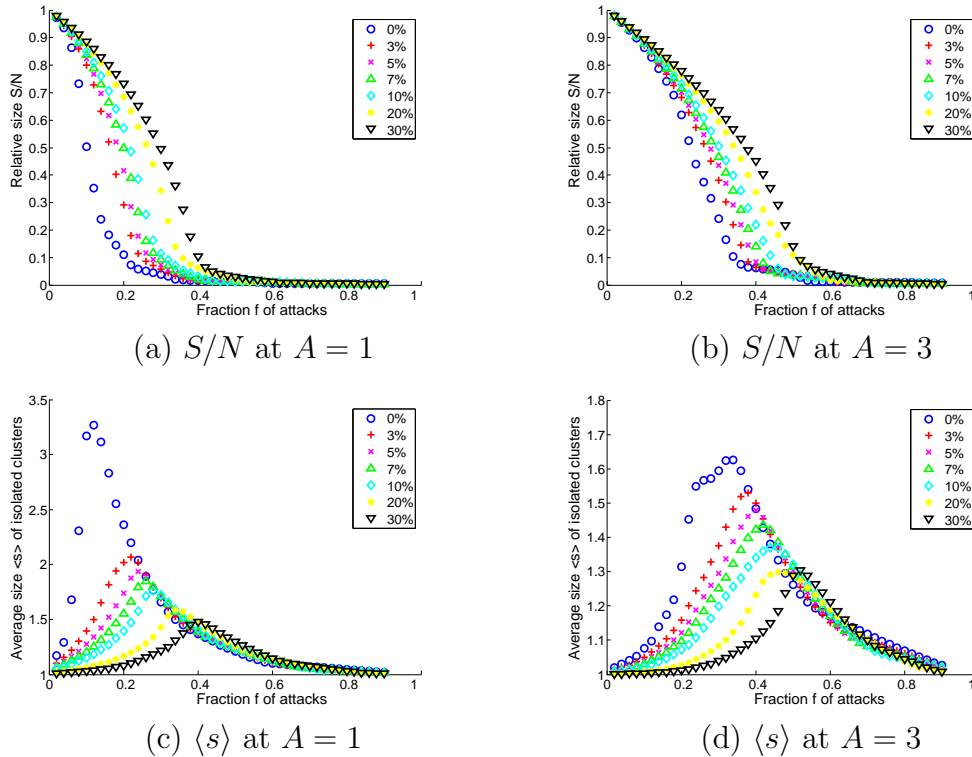


Fig. 6. (Color online) Relative size  $S/N$  of the GC against intentional attacks in LESF networks at (a)  $A = 1$  and (b)  $A = 3$ . Average size  $\langle s \rangle$  of isolated clusters except the GC against intentional attacks in LESF networks at (c)  $A = 1$  and (d)  $A = 3$ .

### 3.3 Simulation for AS networks

Historically, in the 1960s, the Internet was motivated to design a self-organized computer network with the survival capability for communication that is highly resilient to local failures [28]. Today, it evolves to one of the worldwide large scale systems, whose topology belongs to a SF network with a power-law degree distribution [29]. The SF nature of the Internet exhibits both error tolerance and attack vulnerability [4,5,6,28]. Moreover, the geographical constraints on the topological linkings [8,9] implicitly affect the robustness, indeed, the numerical study [4] has been shown more serious result in the Internet than that in a relational SF network called Barabási-Albert model without geographical constraints. For the realistic case, we examine an improvement of the robustness against the attacks in particular, when some shortcuts are virtually added to the Internet. We use the topology data [30] at the level of autonomous system (AS) derived from RouteViews BGP table snapshots by CAIDA (Cooperative Association for Internet Data Analysis).

Figure 8(a) shows a power-law degree distribution in the AS networks with a few huge hubs. We also find small deviation  $P(k)$  for the shortcuts added into

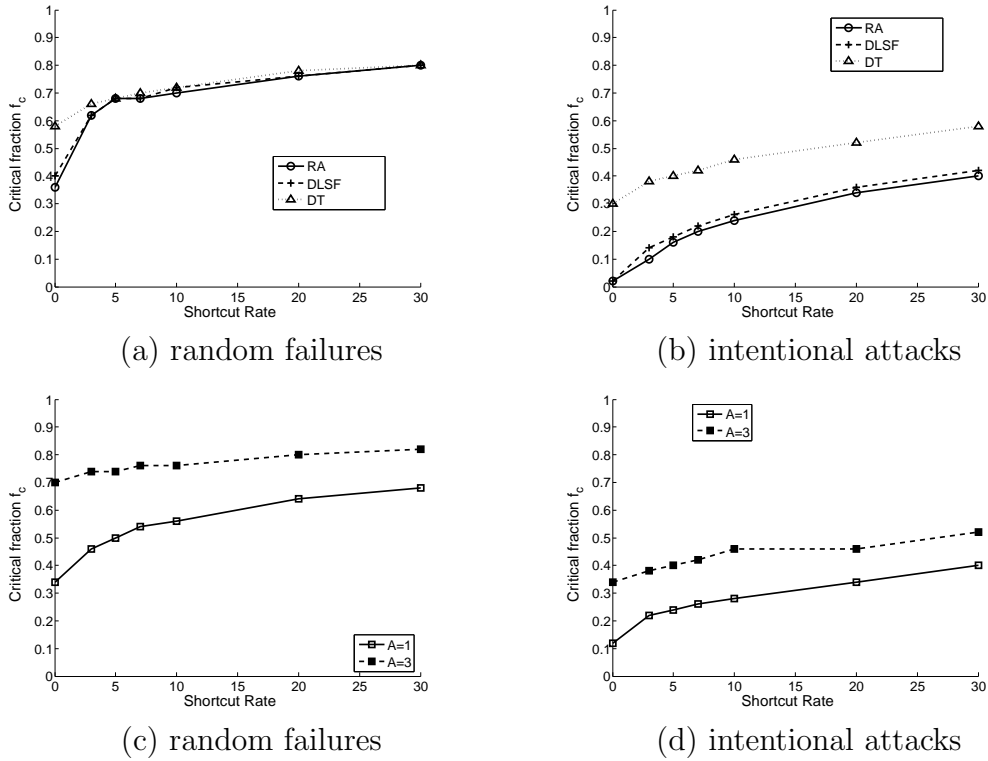


Fig. 7. The critical value  $f_c$  of removed nodes vs. shortcut rate in (a)(b) RA, DLSF, DT, and (c)(d) LESF networks. The piece-wise linear lines guide the increasing.

these data, however the linearity in log-log plot is almost invariant. Figures 8(b)(c) show the effect of shortcuts on the tolerance of connectivity against the targeted attacks; the GC survives even in a double amount of attacks at the breaking of the original networks without shortcuts, and the peak of  $\langle s \rangle$  is slightly shifted to right. The breaking around the attack rate 3% is consistent with the previous simulations [4,5,28]. Since a smaller average degree  $\langle k \rangle$  is improper for maintaining the connectivity in spite of a small average clustering coefficient  $\langle C \rangle$  as shown in Table 1, these results may be related to a structural vulnerability including tree-like stubs.

## 4 Conclusion

To improve the weakened connectivity by cycles in a theoretical prediction [13,14], we investigate effects of shortcuts on the robustness in geographical SF networks. Something of randomness [16,17,18,15] is expected to relax the geographical constraints that tend to make cycles locally. Since many real complex systems belong to SF networks [1,2] and are embedded in a metric space [8,9], in addition, planar networks are suitable for efficient routings [22], we consider a family of planar SF network models called RA [12,11], DT

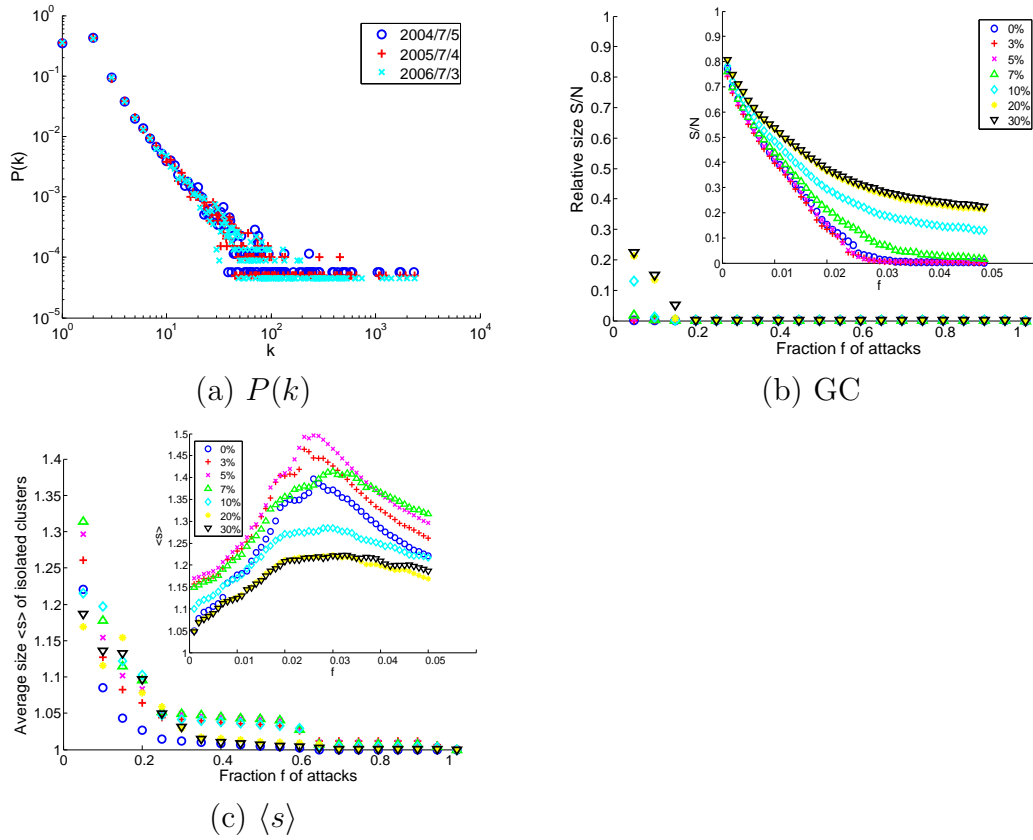


Fig. 8. (Color online) Results for the AS networks. (a) Degree distribution  $P(k)$ . Each mark corresponds the year/month/day. (b) Relative size  $S/N$  of the GC and (c) the average size  $\langle s \rangle$  of isolated clusters except the GC against intentional attacks on the network in 2006. Insets shows the enlarged parts for the small fraction  $f$ . These are similar in other two years.

[20,21], and DLSF [15], a non-planar basic geographical model called LESF [10], and a real data of the Internet at the AS level [30] as an example for the virtual examination. Our numerical results show that the robustness is improved by shortcuts around 10% rate maintaining the small distance  $\langle D \rangle$  and number of hops  $\langle L \rangle$  on the optimal paths (with respect to the shortest and the minimum number of hops, respectively) in each network, under similar degree distributions to the original ones. In particular, the improvement is remarkable in the intentional attacks on hubs. However, some cases exhibit weak effects which depend on the values of other topological characteristics such as  $\langle k \rangle$  and  $\langle C \rangle$ . We will further study for comprehending the properties. On the other hand, these results give an insight for practical constructing of a geographical network, since the robustness can be effectively increased by adding a small fraction of shortcuts.

Network	$N$	$M$	$\langle k \rangle$	$\langle C \rangle$	$\langle L \rangle$	$P(k)$
DT	1000	2993	5.986	0.441	9.02	lognormal
RA	1000	2993	5.986	0.767	4.13	power-law
DLSF	1000	2993	5.986	0.726	4.65	with cutoff
LESF						
( $A = 1$ )	1024	1831	3.576	0.342	14.5	with cutoff
( $A = 3$ )	1024	2673	5.221	0.104	4.87	with cutoff
AS04	17509	35829	4.0926	0.234	3.77	power-law
AS05	19846	40485	4.0799	0.249	3.79	power-law
AS06	22456	45050	4.0123	0.219	3.87	power-law

Table 1

Summary of the topological characteristics: the network size  $N$ , the total number of links  $M$ , the average degree  $\langle k \rangle$ , the average clustering coefficient  $\langle C \rangle$ , the average path length  $\langle L \rangle$  based on the minimum number of hops, and the types of degree distribution  $P(k)$ .

## Acknowledgment

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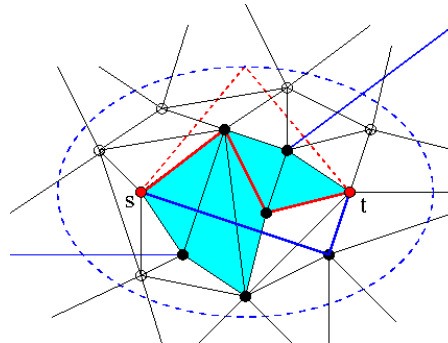


Fig. 9. (Color online) Illustration of the extended routing. The (red) thick route on the edges of (cyan) shaded faces is the shortest path  $l_s$  whose distance is the same as the dashed (red) chord of the ellipsoid. The (blue) thick route is the optimal path including a shortcut.

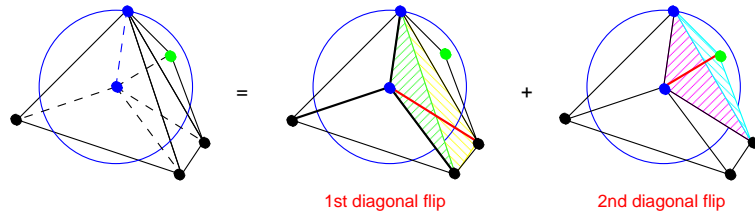


Fig. 10. (Color online) Linking procedures in a Delaunay-like SF network. The long-range links (black solid lines in the left) are exchanged to red ones in the shaded triangles by diagonal flips in the middle and right. The dashed lines are new links from the barycenter, and form new five triangles with contours in the left (The two black solid lines crossed with dashed lines are removed after the second diagonal flip).

## Appendix 1

For adding shortcuts, the efficient routing algorithm [22] on a planar network can be extended as follows (see Figure 9) in the ellipsoid whose chord is defined by the distance of the shortest path  $l_s$  on the edges of faces that intersect the straight line between the source and terminal as the two foci. We describe the outline of procedures .

- Find the shortest path  $l_s$  on the original planar network without shortcuts.
- Then search shorter one including shortcuts in the ellipsoid.
- Through backtrackings from the terminal to the source in the above process, prune the nodes that located out of the ellipsoid or on longer paths than  $l_s$  by using the positions.

We expect the additional steps for searching are not so much as visiting almost all nodes, when the rate of shortcuts is low. Moreover, even in this case, the robustness of connectivity can be considerably improved.

## Appendix 2

The geographical networks are constructed as follows.

### Planar networks [15]

**Step 0:** Set an initial planar triangulation on a space.

**Step 1:** At each time step, select a triangle at random and add a new node at the barycenter. For each model, different linking processes are applied.

**RA:** Then, connect the new node to its three nodes as the subdivision of triangle.

**DLSF:** Moreover, by iteratively applying diagonal flips [20], connect it to

the nearest node within a radius defined by the distance between the new node and the nearest node of the chosen triangle, as shown in Fig. 10.

If there is no nearest node within the radius, this flipping is skipped, therefore the new node is connected to the three nodes.

**DT:** After the subdivision of the chosen triangle, diagonal flips are globally applied to a pair of triangles until the minimum angle is not increased by any exchange of diagonal links in a quadrilateral.

**Step 2:** The above process is repeated until the required size  $N$  is reached.

### LESF networks [10]

**Step 0:** To each node on the lattice, assign a random degree  $k$  taken from the distribution  $P(k) = Ck^{-\gamma}$ ,  $m \leq k \leq K$ ,  $\gamma > 2$ , where  $C \approx (\gamma - 1)m^{\gamma-1}$  is the normalization constant for large  $K$ .

**Step 1:** Select a node  $i$  at random, and connect it to its nearest neighbors until its connectivity  $k_i$  is realized, or until all nodes up to a distance  $r(k_i) = Ak_i^{1/d}$  have been explored: The connectivity quota  $k_j$  of the target node  $j$  is already filled in saturation. Here  $A > 0$  is a constant.

**Step 2:** The above process is repeated for all nodes.

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