Title	Constructor-based Institutions
Author(s)	Gaina, Daniel; Futatsugi, Kokichi; Ogata, Kazuhiro
Citation	Research report (School of Information Science, Japan Advanced Institute of Science and Technology), IS-RR-2009-002: 1-19
Issue Date	2009-05-29
Type	Technical Report
Text version	publisher
URL	http://hdl.handle.net/10119/8177
Rights	
Description	リサーチレポート（北陸先端科学技術大学院大学情報科学研究科）

Constructor-based Institutions

Daniel Găină, Kokichi Futatsugi, and Kazuhiro Ogata

May 29, 2009

IS–RR–2009–002

Constructor-based Institutions

Daniel Găină, Kokichi Futatsugi, and Kazuhiro Ogata

Graduate School of Information Science
Japan Advanced Institute of Science and Technology (JAIST)
{daniel,kokichi,ogata}@jaist.ac.jp

Abstract. Many computer science applications concern properties which are true of a restricted class of models. We present a couple of constructor-based institutions defined on top of some base institutions by restricting the class of models. We define the proof rules for these logics formalized as institutions, and prove their completeness in the abstract framework of institutions.

1 Introduction

Equational specification and programming constitute the bases of the modern algebraic specification languages (like CafeOBJ [9], CASL [2] and Maude [6]), the other features being somehow built on top of it. In 1935 Birkhoff first proved a completeness result for equational logic, in the unsorted case. Goguen and Meseguer extended the result to the many-sorted case, providing a full algebraisation of finitary many-sorted equational deduction. In [7] the result is cast in the framework of institutions capturing both the finitary and the infinitary cases. Here we present an institution-independent completeness result for constructor-based logics [4] obtained from a base logic, basically by restricting the class of models.

Applications concern properties which are true of a restricted class of models. In most of the cases the models of interest include the initial model(s) of a set of axioms. Some approaches consider the initial semantics and reason about properties which are true of initial model [15]. Our work takes into account the generation aspects of software systems by considering the constructor-based institutions. The signatures of these institutions consist of a signature in the base institution and a distinguished set of operation symbols called *constructors*. The result sorts of the constructors are called *constrained* and a sort which is not constrained it is *loose*. The constructors determine the class of *reachable* models which are of interest from the user point of view. Intuitively the carrier sets of such models consist of constructor-generated elements. The sentences and the satisfaction condition are preserved from the base institution. In order to obtain a constructor-based institution the signature morphisms of the base institution are restricted such that the *reducts* of the reachable models along the signature morphisms are also reachable. In the examples presented here it is simply required that constructors are preserved by signature morphisms, and no "new" constructors are introduced for "old" constrained sorts (for sorts being in the image of some constrained sorts of the source signature).

We provide proof rules for the constructor-based institutions and we prove a completeness result using institution independent techniques. However the completeness is relative to a family of *sufficient complete* basic specifications (Σ, Γ) with signature Σ and set Γ of sentences. Intuitively (Σ, Γ) is sufficient complete when any term constructed with symbols from Σ and variables of sort loose can be reduced to a term formed with constructors and variables of sort loose using the equations from Γ.

For lack of space, we omit most of the proofs of our theorems, propositions and lemmas. However, interested readers can find these proofs in the extended version of this paper [10].

In Section 2 we present the notions of institution and entailment system; these will constitute the base for expressing the soundness and completeness properties for a logic, i.e. the semantic deduction coincides with the syntactic provability. After presenting some well known examples of institutions we give the definitions of some important concepts that are used in this paper. Section 3 introduces the abstract concept of universal institution and reachable universal weak entailment system which is proved sound and complete with respect to a class of reachable models, under conditions which are also investigated. Section 4 contains the main result. The entailment system developed in the previous section is borrowed by constructor-based institutions through institution morphisms. Soundness is preserved, and completeness is relative to a family of sets of sentences. Section 5 concludes the paper and discusses the future work.

2 Institutions

Institutions were introduced in [12] with the original goal of providing an abstract framework for algebraic specifications of computer science systems.

Definition 1. *An institution consists of*

- *a category* $\mathbb{S}ig$*, whose objects are called* signatures.

- *a functor* $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$*, providing for each signature a set whose elements are called* (Σ-)sentences.

- *a functor* $\mathbb{M}od : \mathbb{S}ig^{op} \to Cat$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)morphisms.

- *a relation* $\models_\Sigma \subseteq |\mathbb{M}od(\Sigma)| \times \mathbb{S}en(\Sigma)$ *for each* $\Sigma \in |\mathbb{S}ig|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \to \Sigma'$ *in* $\mathbb{S}ig$*, the following* satisfaction condition *holds:* $M' \models_{\Sigma'} \mathbb{S}en(\varphi)(e)$ *iff* $\mathbb{M}od(\varphi)(M') \models_\Sigma e$*, for all* $M' \in |\mathbb{M}od(\Sigma')|$ *and* $e \in \mathbb{S}en(\Sigma)$*.*

Following the usual notational conventions, we sometimes let $\_ \upharpoonright_\phi$ denote the reduct functor $\mathbb{M}od(\varphi)$ and let φ denote the sentence translation $\mathbb{S}en(\varphi)$. When $M = M' \upharpoonright_\varphi$ we say that M' is a *φ-expansion* of M, and that M is the *φ-reduct* of M'.

*Example 1 (First-order logic (***FOL***) [12]).* Signatures are first-order many-sorted signatures (with sort names, operation names and predicate names); sentences are the usual closed formulae of first-order logic built over atomic formulae given either as equalities or atomic predicate formulae; models are the usual first-order structures; satisfaction of a formula in a structure is defined in the standard way.

*Example 2 (Constructor-based first-order logic (***CFOL***)).* The signatures (S, F, F^c, P) consist of a first-order signature (S, F, P), and a distinguished set of *constructors* $F^c \subseteq F$. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w \to s}$. We call the sorts in $S^l = S - S^c$ *loose*. The (S, F, F^c, P)-sentences are the *universal constrained first-order sentences* of the form $(\forall X)\rho$, where X is a finite set of constrained variables [1], and ρ is a first-order formula formed over the atoms by applying Boolean connectives and quantifications over finite sets of loose variables [2]. The (S, F, F^c, P)-models are the usual first-order structures M with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a set Y of variables of loose sorts, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^\#_s : (T_{F^c}(Y))_s \to M_s$ is a surjection, where $f^\#$ is the unique extension of f to a (S, F^c, P)-morphism. A constructor-based first-order signature morphisms $\varphi : (S, F, F^c, P) \to (S_1, F_1, F^c_1, P_1)$ is a first-order signature morphism $\varphi : (S, F, P) \to (S_1, F_1, P_1)$ such that the constructors are preserved along the signature morphisms, and no "new" constructors are introduced for "old" constrained sorts: if $\sigma \in F^c$ then $\varphi(\sigma) \in F^c_1$, and if $\sigma_1 \in (F^c_1)_{w_1 \to s_1}$ and $s_1 \in \varphi(S^c)$ then there exists $\sigma \in F^c$ such that $\varphi(\sigma) = \sigma_1$. Variants of constructer-based first-order logic were studied in [4] and [3].

*Example 3 (Constructor-based Horn clause logic (***CHCL***)).* This institution is obtained from **CFOL** by restricting the sentences to *universal Horn sentences* of the form $(\forall X)(\forall Y) \wedge H \Rightarrow C$, where X is a finite set of constrained variables, Y is a finite set of loose variables, H is a finite set of atoms, and C is an atom. $\mathbf{CHCL}_\infty$ is the infinitary extension of **CHCL** obtained by allowing *infinitary universal Horn sentences* $(\forall X)(\forall Y) \wedge H \Rightarrow C$ where the sets X, Y and H may be infinite.

*Example 4 (Order-sorted algebra (***OSA***) [14]).* An order-sorted signature $(S, \leq, F)$ consists of an algebraic signature (S, F), with a partial ordering $(S, \leq)$ such that the following *monotonicity condition* is satisfied $\sigma \in F_{w_1 \to s_1} \cap F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $s_1 \leq s_2$. A morphism of **OSA** signatures $\varphi : (S, \leq, F) \to (S', \leq', F')$ is just a morphism of algebraic signatures $(S, F) \to (S', F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$. Given an order-sorted signature $(S, \leq, F)$, an order-sorted $(S, \leq, F)$-algebra is a (S, F)-algebra M such that $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and $\sigma \in F_{w_1 \to s_1} \cup F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $M^{w_1,s_1}_\sigma = M^{w_2,s_2}_\sigma$ on M_{w_1}. Given order-sorted $(S, \leq, F)$-algebras

[1] $X = (X_s)_{s \in S}$ is a set of constrained variables if $X_s = \emptyset$ for all $s \in S^l$

[2] $Y = (Y_s)_{s \in S}$ is a set of loose variables if $Y_s = \emptyset$ for all $s \in S^c$.

M and N, an order-sorted $(S, \leq, F)$-homomorphism $h : M \to N$ is a (S, F)-homomorphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}$ on M_{s_1}.

Let $(S, \leq, F)$ be an order-sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S iff $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. An **OSA** signature $(S, \leq, F)$ is *regular* iff for each $\sigma \in F_{w_1 \to s_1}$ and each $w_0 \leq w_1$ there is a unique least element in the set $\{(w, s) \mid \sigma \in F_{w \to s} \text{ and } w_0 \leq w\}$. For regular signatures $(S, \leq, F)$, any $(S, \leq, F)$-term has a least sort and the initial $(S, \leq, F)$-algebra can be defined as a term algebra, cf. [14]. A partial ordering $(S, \leq)$ is *filtered* iff for all $s_1, s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* iff every connected component of it is filtered. An order-sorted signature $(S, \leq, F)$ is *locally filtered* iff $(S, \leq)$ is locally filtered, and it is *coherent* iff it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent.

The atoms of the signature $(S, \leq, F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are closed formulas built by application of boolean connectives and quantification to the equational atoms. Order-sorted algebras were extensively studied in [13] and [14].

*Example 5 (Constructor-based order-sorted logic (***COSA***)).* This institution is defined on top of **OSA** similarly as **CFOL** is defined on top of **FOL**. The constructor-based order-sorted signatures $(S, \leq, F, F^c)$ consists of an order-sorted signature $(S, \leq, F)$, and a distinguished set of operational symbols $F^c \subseteq F$, called *constructors*, such that $(S, \leq, F^c)$ is an order-sorted signature (the monotonicity and coherence conditions are satisfied). As in the first-order case the constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w \to s}$ with the result sort s. We call the sorts in $S^l = S - S^c$ *loose*. The $(S, \leq, F, F^c)$-sentences are the *universal constrained order-sorted sentences* of the form $(\forall X)\rho$, where X is finite set of constrained variables, and ρ is a formula formed over the atoms by applying Boolean connectives and quantifications over finite sets of loose variables. The $(S, \leq, F, F^c)$-models are the usual order-sorted $(S, \leq, F)$-algebras with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a set of loose variables Y, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^{\#}_s : (T_{F^c}(Y))_s \to M_s$ is a surjection, where $f^{\#}$ is the unique extension of f to a $(S, \leq, F^c)$-morphism. A signature morphism $\varphi : (S, \leq, F, F^c) \to (S_1, \leq_1, F_1, F^c_1)$ is an order-sorted signature morphism such that the constructors are preserved along the signature morphisms, no "new" constructors are introduced for "old" constrained sorts, and if $s'_1 \leq_1 s''_1$ and there exists $s'' \in S^c$ such that $s''_1 = \varphi(s'')$ then there exists $s' \in S^c$ such that $s'_1 = \varphi(s')$.

Constructor-based Horn order-sorted algebra (**CHOSA**) is obtained by restricting the sentences of **COSA** to universal Horn sentences. Its infinitary variant $\mathbf{CHOSA}_\infty$ is obtained by allowing infinitary universal Horn sentences.

We introduce the following institutions for technical reasons.

*Example 6 (Generalized first-order logic (**GFOL**)).* Its signatures (S, S^c, F, P) consist of a first-order signature (S, F, P) and a distinguished set of constrained sorts $S^c \subseteq S$. A sort which is not constrained is loose. A *generalized first-order signature morphism* between (S, S^c, F, P) and (S_1, S_1^c, F_1, P_1) is a simple signature morphism between (S, F, P) and $(S_1, F_1 + T_{F_1}, P_1)$, i.e. constants can be mapped to terms. The sentences are the universal constrained first-order sentences and the models are the usual first-order structures.

Generalized Horn clause logic (**GHCL**) is the restriction of **GFOL** to universal Horn sentences, and $\mathbf{GHCL}_\infty$ is extending **GHCL** by allowing infinitary universal Horn sentences.

*Example 7 (Generalized order-sorted algebra (**GOSA**)).* This institution is a variation of **OSA** similarly as **GFOL** is a variation of **FOL**. Its signatures distinguish a subset of constrained sorts and the signature morphisms allow mappings of constants to terms. The sentences are the universal constrained order-sorted sentences. **GHOSA** and $\mathbf{GHOSA}_\infty$ are defined in the obvious way.

Entailment systems. A *sentence system* $(\mathbb{S}ig, \mathbb{S}en)$ consists of a category of signatures $\mathbb{S}ig$ and a sentence functor $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$. An *entailment system* $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ consists of a sentence system $(\mathbb{S}ig, \mathbb{S}en)$ and a family of entailment relations $\vdash = \{\vdash_\Sigma\}_{\Sigma \in |\mathbb{S}ig|}$ between sets of sentences with the following properties:

($Anti$-$monotonicity$) $E_1 \vdash_\Sigma E_2$ if $E_2 \subseteq E_1$,

($Transitivity$) $E_1 \vdash_\Sigma E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_2 \vdash_\Sigma E_3$, and

($Unions$) $E_1 \vdash_\Sigma E_2 \cup E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_1 \vdash_\Sigma E_3$.

($Translation$) $E \vdash_\Sigma E'$ implies $\varphi(E) \vdash_{\Sigma'} \varphi(E')$ for all $\varphi : \Sigma \to \Sigma'$

When we allow infinite unions, i.e. $E \vdash_\Sigma \bigcup_{i \in J} E_i$ if $E \vdash_\Sigma E_i$ for all $i \in J$, we call the entailment system *infinitary*. We say that $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ is a *weak entailment system* (abbreviated *WES*) when it does not satisfy the $Translation$ property. *The semantic entailment system* of an institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ consists of $(\mathbb{S}ig, \mathbb{S}en, \models)$. When there is no danger of confusion we may omit the subscript Σ from $\vdash_\Sigma$ and for every signature morphism $\varphi \in \mathbb{S}ig$, we sometimes let φ denote the sentence translation $\mathbb{S}en(\varphi)$.

Remark 1. The weak entailment system of an institution is an entailment system whenever is sound and complete.

Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\mathcal{M} = \{\mathcal{M}_\Sigma\}_{\Sigma \in |Sig|}$ a family of classes of models, where $\mathcal{M}_\Sigma \subseteq \mathbb{M}od(\Sigma)$ for all signatures $\Sigma \in |\mathbb{S}ig|$. A WES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ of the institution $\mathcal{I}$ is *sound* (resp. *complete*) *with respect to* $\mathcal{M}$ when $E \vdash e$ implies $M \models (E \Rightarrow e)$ [3] (resp. $M \models (E \Rightarrow e)$ implies $E \vdash e$) for all sets of sentences $E \subseteq \mathbb{S}en(\Sigma)$, sentences $e \in \mathbb{S}en(\Sigma)$ and models $M \in \mathcal{M}_\Sigma$. We say that $\mathcal{E}$ is *sound* (resp. *complete*) when $\mathcal{M}_\Sigma = |\mathbb{M}od(\Sigma)|$ for all signatures Σ.

We call the WES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ *compact* whenever: if $\Gamma \vdash E_f$ and $E_f \subseteq \mathbb{S}en(\Sigma)$ is finite then there exists $\Gamma_f \subset \Gamma$ finite such that $\Gamma_f \vdash E_f$.

[3] $M \models (E \Rightarrow e)$ iff $M \models E$ implies $M \models e$

For each WES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ one can easily construct the *compact sub-WES* $\mathcal{E}^c = (\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ by defining the entailment relation $\vdash^c$ as follows: $\Gamma \vdash^c E$ iff for each $E_f \subseteq E$ finite there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \vdash E_f$.

Remark 2. $(\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ is a WES.

Proof. We need to show that $\mathcal{E}^c = (\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ satisfies

1. *Anti-monotonicity*: assuming $E_2 \subseteq E_1$ we prove $E_1 \vdash^c E_2$. For any finite set $E_2' \subseteq E_2$ there exists a finite set $E_1'(= E_2') \subseteq E_1$ such that $E_1' \vdash E_2'$ which implies $E_1 \vdash^c E_2$.
2. *Transitivity*: assuming that $E_1 \vdash^c E_2$ and $E_2 \vdash^c E_3$ we prove $E_1 \vdash^c E_3$. Let $E_3' \subseteq E_3$ finite, since $E_2 \vdash^c E_3$ there exists $E_2' \subseteq E_2$ finite such that $E_2' \vdash E_3'$. Because $E_1 \vdash E_2$ there is a finite set $E_1' \subseteq E_1$ such that $E_1' \vdash E_2'$. By the *Transitivity* of $\mathcal{E}$ we obtain $E_1' \vdash E_3'$ which implies $E_1 \vdash_c E_3$.
3. *Unions*: assuming that $E_1 \vdash^c E_2$ and $E_1 \vdash^c E_3$ we prove $E_1 \vdash^c E_2 \cup E_3$. Let $E \subseteq E_2 \cup E_3$ finite; there exists finite sets $E_2' \subseteq E_2$ and $E_3' \subseteq E_3$ such that $E_2' \cup E_3' = E$; because $E_1 \vdash^c E_2$ and $E_1 \vdash^c E_3$ there is finite sets $E', E'' \subseteq E_1$ such that $E' \vdash E_2'$ and $E'' \vdash E_3'$, respectively; by *Anti-monotonicity* and *Transitivity* property we have $E_1' = E' \cup E'' \vdash E_2'$ and $E_1' \vdash E_3'$ and by *Unions* we obtain $E_1' \vdash E_2' \cup E_3' = E$. Because E was arbitrary we get $E_1 \vdash^c E_2 \cup E_3$.

Basic sentences. A set of sentences $E \subseteq Sen(\Sigma)$ is called *basic* [8] if there exists a Σ-model M_E such that, for all Σ-models M, $M \models E$ iff there exists a morphism $M_E \to M$.

Lemma 1. *Any set of atomic sentences in* **GHCL** *and* **GHOSA** *is basic.*

Proof. In **GHCL**, for a set E of atomic (S, F, P)-sentences there exists a basic model M_E. Actually it is the initial model for E. This is constructed a follows: on the quotient $(T_F)_{=_E}$ of the term model T_F by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{=_E}, \ldots, t_n/_{=_E}) \mid \pi(t_1, \ldots, t_n) \in E\}$. A similar argument as the preceding holds for **GHOSA** too.

Internal logic. The following institutional notions dealing with logical connectives and quantifiers were defined in [16].

Let Σ be a signature of an institution,

- a Σ-sentence $\neg e$ is a *(semantic) negation* of the Σ-sentence e when for every Σ-model M we have $M \models_\Sigma \neg e$ iff $M \not\models_\Sigma e$,

- a Σ-sentence $e_1 \wedge e_2$ is a *(semantic) conjunction* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \wedge e_2$ iff $M \models_\Sigma e_1$ and $M \models_\Sigma e_2$, and

- a Σ-sentence $(\forall \chi)e'$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathbb{S}ig$ and $e' \in \mathbb{S}en(\Sigma')$, is a *(semantic) universal χ-quantification* of e' when for every Σ-model M we have $M \models_\Sigma (\forall \chi)e'$ iff $M' \models_{\Sigma'} e'$ for all χ-expansions M' of M.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. For a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, we say that the institution has universal $\mathcal{D}$-quantifications when for each $\chi : \Sigma \to \Sigma'$ in $\mathcal{D}$, each Σ'-sentence has a universal χ-quantification.

Reachable models. Consider two signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ of an institution. A signature morphisms $\theta : \Sigma_1 \to \Sigma_2$ such that $\chi_1; \theta = \chi_2$ is called a *substitution* between χ_1 and χ_2.

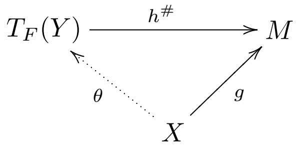
Definition 2. *Let $\mathcal{D}$ be a broad subcategory of signature morphisms of an institution. We say that a Σ-model M is $\mathcal{D}$-reachable if for each span of signature morphisms $\Sigma_1 \xleftarrow{\chi_1} \Sigma_0 \xrightarrow{\chi} \Sigma$ in $\mathcal{D}$, each χ_1-expansion M_1 of $M \upharpoonright_\chi$ determines a substitution $\theta : \chi_1 \to \chi$ such that $M \upharpoonright_\theta = M_1$.*

Proposition 1. *In* **GHCL***, assume that $\mathcal{D}$ is the class of signature extensions with (possibly infinite number of) constants. A model M is $\mathcal{D}$-reachable iff its elements are exactly the interpretations of terms.*

Proof. For every inclusion $\Sigma \hookrightarrow \Sigma(Z)$ in $\mathcal{D}$, where $\Sigma = (S, S^c, F, P)$ and $\Sigma(Z) = (S, S^c, F \cup Z, P)$, the $\Sigma(Z)$-models can be represented as pairs (A, a), where A is a Σ-model and $a : Z \to A$ is a function.

Let $\Sigma = (S, S^c, F, P)$ be a signature and assume a Σ-model M which is $\mathcal{D}$-reachable. We prove that $T_F \to M$ is surjective, i.e. for every $m \in M$ there exists $t \in T_F$ such that $M_t = m$. Let $m \in M_s$ be an arbitrary element of M. Consider a variable x of sort s and let N be an expansion of M along $\Sigma \hookrightarrow \Sigma(\{x\})$ which interprets the constant symbol x as m. Since M is $\mathcal{D}$-reachable there exists a substitution $\theta : \{x\} \to T_F$ such that $M \upharpoonright_\theta = N$. Take $t = \theta(x)$ and we have $M_t = M_{\theta(x)} = (M \upharpoonright_\theta)_x = N_x = m$.

For the converse implication let $\Sigma = (S, S^c, F, P)$ be a signature, X and Y two disjoint sets of constants with elements which are different from the symbols in Σ, and (M, h) a $\Sigma(Y)$-model with elements which are interpretation of terms, i.e. the unique extension $h^\# : T_F(Y) \to M$ of h to a Σ-morphism is surjective. Then for every $\Sigma(X)$-model (M, g) there exists a function $\theta : X \to T_F(Y)$ such that $\theta; h^\# = g$.



We straightforwardly extend θ to a signature morphism $\theta' : \Sigma(X) \to \Sigma(Y)$ such that θ' is

- equal to θ on X, and
- the identity on Σ.

Note that for any $x \in X$ we have $((M, h) \upharpoonright_{\theta'})_x = h^\#(\theta(x)) = g(x) = (M, g)_x$. Hence, $(M, h) \upharpoonright_{\theta'} = (M, g)$.

One can replicate the above proposition for **GHOSA** too. Note that for each set E of atomic sentences in **GHCL** or **GHOSA**, the model M_E defining E as basic set of sentences is $\mathcal{D}$-reachable, where $\mathcal{D}$ is the class of signature extensions with constants.

Definition 3. *We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ is finitary if it is finitely presented* [4] *in the comma category $\Sigma/\mathbb{S}ig$* [5]*.*

In typical institutions the extensions of signatures with finite number of symbols are finitary.

Definition 4. *Let $\mathcal{D}^c$ and $\mathcal{D}^l$ be two broad sub-categories of signature morphisms. We say that a Σ-model M is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable if for every signature morphism $\chi : \Sigma \to \Sigma'$ in $\mathcal{D}^c$ and each χ-expansion M' of M there exists a signature morphism $\varphi : \Sigma \to \Sigma''$ in $\mathcal{D}^l$, a substitution $\theta : \chi \to \varphi$, and a Σ''-model M'' such that $M'' \upharpoonright_\theta = M'$.*

The two notions of reachability, apparently different, are closely related.

Proposition 2. *Let D^c, $\mathcal{D}^l$ and $\mathcal{D}$ be three broad sub-categories of signature morphisms such that $\mathcal{D}^c, \mathcal{D}^l \subseteq \mathcal{D}$. A Σ-model M is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable if there exists a signature morphism $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{D}$ and a φ-expansion M' of M such that*

1. *M' is $\mathcal{D}$-reachable, and*
2. *whether*
   (a) *$\varphi \in \mathcal{D}^l$, or*
   (b) *every signature morphism in $\mathcal{D}^c$ is finitary and φ is the vertex of a directed co-limit $(\varphi_i \xrightarrow{u_i} \varphi)_{i \in J}$ of a directed diagram $(\varphi_i \xrightarrow{u_{i,j}} \varphi_j)_{(i \leq j) \in (J, \leq)}$ in $\Sigma/\mathbb{S}ig$, and $\varphi_i \in \mathcal{D}^l$ for all $i \in J$.*

Proof. The case when $\varphi \in \mathcal{D}^l$ is straightforward. We focus on the second condition. Assume a signature morphism $\Sigma \xrightarrow{\chi} \Sigma_1 \in \mathcal{D}^c$ and a χ-expansion N of M. Since M' is $\mathcal{D}$-reachable, there exists a substitution $\theta : \chi \to \varphi$ such that $M' \upharpoonright_\theta = N$. Because χ is finitely presented in the category $\Sigma/\mathbb{S}ig$, there exists $i \in J$ and $\theta_i : \chi \to \varphi_i$ such that $\theta_i; u_i = \theta$. Note that $M_i = M' \upharpoonright_{u_i}$ is a φ_i-expansion of M such that $M_i \upharpoonright_{\theta_i} = N$.

Note that the above proposition comes in two variants: infinitary and finitary. The infinitary variant corresponds to the first condition ($\varphi \in \mathcal{D}^l$) and is applicable to infinitary institutions, such as $\mathbf{GHCL}_\infty$ and $\mathbf{GHOSA}_\infty$ while the

[4] An object A in a category $\mathcal{C}$ is called *finitely presented* ([1]) if
- for each directed diagram $D : (J, \leq) \to \mathcal{C}$ with co-limit $\{Di \xrightarrow{\mu_i} B\}_{i \in J}$, and for each morphism $A \xrightarrow{g} B$, there exists $i \in J$ and $A \xrightarrow{g_i} Di$ such that $g_i; \mu_j = g$,
- for any two arrows g_i and g_j as above, there exists $i \leq k, j \leq k \in J$ such that $g_i; D(i \leq k) = g_j; D(j \leq k) = g$.

[5] The objects of $\Sigma/\mathbb{S}ig$ are signature morphisms $\Sigma \xrightarrow{\chi} \Sigma'$ with the source Σ, and the arrows between objects $\Sigma \xrightarrow{\chi_1} \Sigma_1$ and $\Sigma \xrightarrow{\chi_2} \Sigma_2$ are also signature morphisms $\varphi : \Sigma_1 \to \Sigma_2$ such that $\chi_1; \varphi = \chi_2$.

finitary variant is applicable to **GHCL** and **GHOSA**. Throughout this paper we implicitly assume that in **GHCL**, **GHOSA**, $\mathbf{GHCL}_\infty$ and $\mathbf{GHOSA}_\infty$, $\mathcal{D}$ represents the subcategory of signature morphisms which consists of signature extensions with constants; $\mathcal{D}^c$ represents the subcategory of signature morphisms which consists of signature extensions with constants of constrained sorts; $\mathcal{D}^l$ represents the subcategory of signature morphisms which consists of signature extensions with constants of loose sorts. In the finitary cases, such as **GHCL** and **GHOSA**, we assume that the signature morphisms in $\mathcal{D}^c$ and $\mathcal{D}^l$ are finitary.

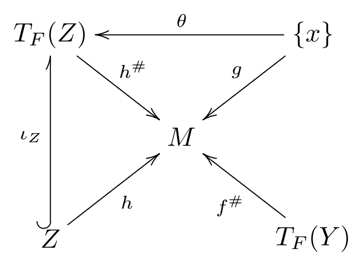
The following is a corollary of Proposition 2.

Corollary 1. *In* **GHCL** *and* $\mathbf{GHCL}_\infty$*, a* Σ*-model* M*, where* $\Sigma = (S, S^c, F, P)$*, is* $(\mathcal{D}^c, \mathcal{D}^l)$*-reachable iff there exists a set of loose variables* Y *and a function* $f : Y \to M$ *such that for every constrained sort* $s \in S^c$ *the function* $f^\#_s : (T_F(Y))_s \to M_s$ *is surjective, where* $f^\#$ *is the unique extension of* f *to a* (S, F, P)*-morphism.*

Proof. The case of $\mathbf{GHCL}_\infty$ is simpler than the case of **GHCL**. We prove this corollary only for **GHCL**.

The implication from right to left is a direct consequence of Proposition 2. Let $\Sigma \overset{\varphi}{\hookrightarrow} \Sigma(Y)$ (where $\Sigma = (S, S^c, F, P)$ and $\Sigma(Y) = (S, S^c, F \cup Y, P)$) be the vertex of the directed co-limit $((\Sigma \overset{\varphi_i}{\hookrightarrow} \Sigma(Y_i)) \overset{u_i}{\hookrightarrow} (\Sigma \overset{\varphi}{\hookrightarrow} \Sigma(Y)))_{Y_i \subseteq Y finite}$ of the directed diagram $((\Sigma \overset{\varphi_i}{\hookrightarrow} \Sigma(Y_i)) \overset{u_{i,j}}{\hookrightarrow} (\Sigma \overset{\varphi_j}{\hookrightarrow} \Sigma(Y_j)))_{Y_i \subseteq Y_j \subseteq Y finite}$. By Proposition 2, M is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable.

For the converse implication we define the set of (loose) variables Y as follows: $Y_s = \emptyset$ for all $s \in S^c$ and Y_s is a renaming of the elements M_s for all $s \in S^l$ such that $Y_s \cap Y_{s'}$ whenever $s \neq s'$. So, there exists a surjective function $f : Y \to M$ (in this case f is a bijection; but the proof works similarly for **GHOSA** and in that case f is surjective only). We prove that for every constraint sort $s' \in S^c$ and element $m \in M_{s'}$ there exists a term $t \in T_F(Y)$ such that $f^\#(t) = m$, where $f^\#$ is the unique extension of f to a Σ-morphism. Let $m \in M_{s'}$ with $s' \in S^c$. Let x be a variable and (M, g) be a $\Sigma(\{x\})$-algebra such that $g(x) = m$. By hypothesis there exists a finite set Z of loose variables, a $\Sigma(Z)$-algebra (M, h) and a substitution $\theta : \{x\} \to T_F(Z)$ such that $\theta; h^\# = g$, where $h^\#$ is the unique extension of h to a Σ-morphism.



Let $t' = \theta(x)$ and $t = t'(z_1 \leftarrow y_1, \ldots, z_n \leftarrow y_n)$, where $t'(z_1 \leftarrow y_1, \ldots, z_n \leftarrow y_n)$ is the term obtained by substituting the variables y_i for z_i, and $y_i \in f^{-1}(h(z_i))$, for all $i \in \{1, \ldots, n\}$. Note that $f^{\#}(t) = M_t(f(y_1), \ldots, f(y_n)) = M_t(h(z_1), \ldots, h(z_n)) = M_{t'}(h(z_1), \ldots, h(z_n)) = h^{\#}(t') = h^{\#}(\theta(x)) = g(x) = m$. [6]

One can replicate the above corollary for **GHOSA** and **GHOSA**$_\infty$ too. Since the three sub-categories of signature morphisms $\mathcal{D}$, $\mathcal{D}^c$ and $\mathcal{D}$ are fixed in concrete institutions, we will refer to $\mathcal{D}$-reachable model(s) as ground reachable model(s), and to $(D^c, \mathcal{D}^l)$-reachable model(s) as reachable model(s) [3].

3 Universal Institutions

Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ be a broad subcategory of signature morphisms, and $\mathbb{S}en^\bullet$ be a sub-functor of $\mathbb{S}en$ (i.e. $\mathbb{S}en^\bullet : \mathbb{S}ig \to \mathbb{S}et$ such that $\mathbb{S}en^\bullet(\Sigma) \subseteq \mathbb{S}en(\Sigma)$ and $\varphi(\mathbb{S}en^\bullet(\Sigma)) \subseteq \mathbb{S}en^\bullet(\Sigma')$, for each signature morphism $\varphi : \Sigma \to \Sigma'$). We denote by $\mathcal{I}^\bullet$ the institution $(\mathbb{S}ig, \mathbb{S}en^\bullet, \mathbb{M}od, \models)$. We say that $\mathcal{I}$ is a *$\mathcal{D}$-universal institution over $\mathcal{I}^\bullet$* [7] when

- $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$ for all signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and sentences $\rho \in \mathbb{S}en^\bullet(\Sigma')$, and

- any sentence of $\mathcal{I}$ is of the form $(\forall\chi)\rho$ as above.

The completeness results below comes both in a finite and an infinite variant, the finite one being obtained by adding (to the hypotheses of the infinite one) all the finiteness hypotheses marked in the brackets.

The *reachable universal weak entailment system* (*RUWES*) developed in this section consists of four layers: the "atomic" layer which in abstract settings is assumed but is developed in concrete examples, the layer of the *weak entailment system with implications* (*IWES*), the layer of the *generic universal weak entailment system* (*GUWES*) and the upmost layer of the RUWES of $\mathcal{I}$. The soundness and the completeness at each layer is obtained relatively to the soundness and completeness of the layer immediately below.

Reachable universal weak entailment systems (RUWES). Let us assume a $\mathcal{D}^c$-universal institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $I_2 = (\mathbb{S}ig, \mathbb{S}en_2, \mathbb{M}od, \models)$ such that I_2 has $\mathcal{D}^l$-quantifications for a broad subcategory $\mathcal{D}^l \subseteq \mathbb{S}ig$ of signature morphisms. We define the following properties, i.e. proof rules, for the WES of $\mathcal{I}$.

(*Case splitting*) $\Gamma \vdash_\Sigma (\forall\chi)\rho$ if $\Gamma \vdash_\Sigma (\forall\varphi)\theta(\rho)$ for all sentences $(\forall\varphi)\theta(\rho)$ such that $\varphi \in \mathcal{D}^l$ and $\theta : \chi \to \varphi$ is a substitution, where $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and $(\forall\chi)e_1 \in \mathbb{S}en(\Sigma)$.

(*Substitutivity*) $(\forall\chi)\rho \vdash_\Sigma (\forall\varphi)\theta(\rho)$, for every sentence $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$ and any substitution $\theta : \chi \to \varphi$.

In **GHCL**, assume a set Γ of Σ-sentences and a Σ-sentence $(\forall x)\rho$ such that x is a constrained variable. In this case, *Case splitting* says that if for any term t formed with loose variables and operation symbols from Σ, we have

[6] For every term $t \in (T_F(\{z_1 : s_1, \ldots, z_m : s_n\}))_s$ we denote by $M_t : M_{s_1} \times, \ldots, \times M_{s_m} \to M_s$ the derived operation defined by $M_t(m_1, \ldots, m_n) = a^{\#}(t)$, where $a : \{z_1 : s_1, \ldots, z_m : s_n\} \to M$, $a(z_i) = m_i$ for all $i \in \{1, \ldots, n\}$, and $a^{\#}$ is the unique extension of a to a morphism.

$\Gamma \vdash (\forall Y)\rho(x \leftarrow t)$ [7], where Y are all (loose) variables which occur in t, then we have proved $\Gamma \vdash (\forall x)\rho$. In most of the cases the set of terms t formed with loose variables and operation symbols from a given signature [8] is infinite which implies that the premises of *Case splitting* are infinite, and thus, the corresponding entailment system is not compact.

Given a compact WES $\mathcal{E}_2 = (\mathbb{S}ig, \mathbb{S}en_2, \vdash^2)$ for I_2, the RUWES of $\mathcal{I}$ consists of the least WES over $\mathcal{E}_2$ closed under *Substitutivity* and *Case splitting*. This is the finitary version of the RUWES, and is applicable to **GHCL** and **GHOSA**. Note that the resulting entailment system is not compact (even if $\mathcal{E}_2$ is compact) since *Case splitting* is an infinitary rule. The infinitary variant is obtained by dropping the compactness condition, and by considering the infinitary WES for $\mathcal{I}$, and is applicable to $\mathbf{GHCL}_\infty$ and $\mathbf{GHOSA}_\infty$.

Proposition 3. *The RUWES of $\mathcal{I}$ is sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models if the WES of $\mathcal{I}_2$ is sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models.*

Proof. We prove that

1. *Case splitting* is sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models, i.e. for every set Γ of sentences and any sentence $(\forall\chi)\rho$ we have:
   $(M \models (\Gamma \Rightarrow (\forall\varphi)\theta(\rho))$, for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models M, and all sentences $(\forall\varphi)\theta(\rho)$, where $\theta : \chi \to \varphi$ is a substitution and $\varphi \in \mathcal{D}^l)$ implies $(M \models (\Gamma \Rightarrow (\forall\chi)\rho)$ for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models $M)$.
   Let Γ be a set of Σ-sentences and $(\forall\chi)\rho$ a Σ-sentence, where $(\Sigma \xrightarrow{\chi} \Sigma') \in \mathcal{D}^c$, and assume that for every $(\mathcal{D}^c, \mathcal{D}^l)$-reachable model M we have $M \models (\Gamma \Rightarrow (\forall\varphi)\theta(\rho))$, for all substitutions $\theta : \chi \to \varphi$ with $\varphi \in \mathcal{D}^l$. Let M be a $(\mathcal{D}^c, \mathcal{D}^l)$-reachable Σ-model and let M' be an χ-expansion of M. Since M is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable there exists a signature morphism $(\varphi : \Sigma \to \Sigma'') \in \mathcal{D}^l$, a substitution $\theta : \chi \to \varphi$, and an φ-expansion M'' of M such that $M'' \restriction_\theta = M'$. We have $M'' \models \theta(\rho)$ and by the satisfaction condition $M' \models \rho$.
2. *Substitutivity* is sound with respect to all models (in particular to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models), i.e. for any sentence $(\forall\chi)\rho$ we have $(\forall\chi)\rho \models (\forall\varphi)\theta(\rho)$, where $\theta : \chi \to \varphi$ is any substitution. Let M be a Σ-model such that $M \models (\forall\chi)\rho$. Assume a substitution $\theta : \chi \to \varphi$ and let M_2 be any φ-expansion of M. Because $M_2 \restriction_\theta$ is a χ-expansion of M (since $(M_2 \restriction_\theta) \restriction_\chi = M_2 \restriction_\varphi$) which by hypothesis satisfies $(\forall\chi)\rho$, we have $M_2 \restriction_\theta \models \rho$. By the satisfaction condition, we obtain that $M_2 \models \theta(\rho)$. Since M_2 was an arbitrary expansion of M, we have thus proved $M \models (\forall\varphi)\theta(\rho)$.

Since $\mathcal{E}_2$, *Case splitting* and *Substitutivity* are sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models, the least WES over $\mathcal{I}_2$ closed under *Case splitting* and *Substitutivity* (which is the RUWES of $\mathcal{I}$) is also sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models.

Note that *Case splitting* is sound with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models while *Substitutivity* is sound to all models.

[7] $\rho(x \leftarrow t)$ is the formula obtained from ρ by substituting t for x.
[8] We consider terms modulo renaming variables.

Theorem 1 (Reachable universal completeness). *The RUWES of $\mathcal{I}$ is complete with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models if*

1. *the WES of $\mathcal{I}_2$ is complete with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models (and compact), and*
2. *for each set of sentences $E \subseteq \mathbb{S}en_2(\Sigma)$ and each sentence $e \in \mathbb{S}en_2(\Sigma)$, we have $E \models e$ iff $M \models (E \Rightarrow e)$ for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models M.*

Proof. Assume that for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models M we have $M \models \Gamma \Rightarrow (\forall \chi)e'$, where $(\Sigma \xrightarrow{\chi} \Sigma') \in \mathcal{D}^c$. We want $\Gamma \vdash (\forall \chi)e'$. Suppose towards a contradiction that $\Gamma \nvdash (\forall \chi)e'$. Then there exists a signature morphism $\varphi : \Sigma \to \Sigma''$ in $\mathcal{D}^l$ and a substitution $\theta : \chi \to \varphi$ such that $\Gamma \nvdash (\forall \varphi)\theta(e')$.

We define the set of Σ-sentences $\Gamma_2 = \{\rho \in \mathbb{S}en_2(\Sigma) \mid \Gamma \vdash \rho\}$.

We show that $\Gamma_2 \nvdash^2 (\forall \varphi)\theta(e')$. Assume that $\Gamma_2 \vdash^2 (\forall \varphi)\theta(e')$. For the infinitary case take $\Gamma' = \Gamma_2$. For the finitary case, since the WES of $\mathcal{I}_2$ is compact, there exists a finite $\Gamma' \subseteq \Gamma_2$ such that $\Gamma' \vdash^2 (\forall \varphi)\theta(e')$ which implies $\Gamma' \vdash (\forall \varphi)\theta(e')$. Since $\Gamma \vdash \rho$ for all $\rho \in \Gamma'$ we have $\Gamma \vdash \Gamma'$. Hence, $\Gamma \vdash (\forall \varphi)e'$ which is a contradiction with our assumption.

We have $\Gamma_2 \nvdash^2 (\forall \varphi)\theta(e')$, and by completeness of $\mathcal{I}_2$ we obtain $\Gamma_2 \not\models (\forall \varphi)\theta(e')$. There exists a $(\mathcal{D}^c, \mathcal{D}^l)$-reachable model such that $M \models \Gamma_2$ and $M \not\models (\forall \varphi)\theta(e')$. Note that $M \not\models (\forall \varphi)\theta(e')$ implies $M \not\models (\forall \chi)e'$. If we have proved that $M \models \Gamma$ we have reached a contradiction with $\Gamma \models (\forall \chi)e'$.

Let $(\forall \chi_1)e_1 \in \Gamma$, where $(\Sigma \xrightarrow{\chi_1} \Sigma_1) \in \mathcal{D}^c$, and let N be any χ_1-expansion of M. Since M is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable there exists a signature morphism $\varphi_1 : \Sigma \to \Sigma_1'$ in $\mathcal{D}^l$, a substitution $\psi : \chi_1 \to \varphi_1$, and φ-expansion N' of M such that $N' \upharpoonright_\theta = N$. By *Substitutivity* $(\forall \varphi_1)\psi(e_1) \in \Gamma_2$ which implies $M \models (\forall \varphi_1)\psi(e_1)$. Since N' is φ_1-expansion of M we have $N' \models \psi(e_1)$ and by satisfaction condition $N' \upharpoonright_\psi = N \models e_1$.

Generic universal weak entailment systems (GUWES). Let us assume a $\mathcal{D}^l$-universal institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $\mathcal{I}_1$ with $\mathbb{S}en_1$ the subfunctor of $\mathbb{S}en$. We define the following property, for the WES of $\mathcal{I}$.
(*Generalization*) $\Gamma \vdash_\Sigma (\forall \varphi)e'$ iff $\varphi(\Gamma) \vdash_{\Sigma'} e'$, for every set $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and any sentence $(\forall \varphi)e' \in \mathbb{S}en(\Sigma)$, where $\varphi : \Sigma \to \Sigma'$.

Given a compact WES $\mathcal{E}_1 = (\mathbb{S}ig, \mathbb{S}en_1, \vdash^1)$ for $\mathcal{I}_1$, the GUWES of $\mathcal{I}$ consists of the least WES over $\mathcal{E}_1$, closed under *Substitutivity* and *Generalization*. This is the finitary version of the GUWES, and is applicable to the restriction of **GHCL** and **GHOSA** to the sentences quantified over finite sets of variables of loose sorts. Its infinitary variant is obtained by dropping the compactness condition, and by considering the infinitary WES of $\mathcal{I}$; it is applicable to the restriction of $\mathbf{GHCL}_\infty$ and $\mathbf{GHOSA}_\infty$ to the sentences quantified over sets (possible infinite) of loose variables.

Proposition 4. *The GUWES of $\mathcal{I}$ is sound whenever the WES of $\mathcal{I}_1$ is sound.*

Proof. Note that *Generalization* is sound with respect to all models, i.e. for every set Γ of Σ-sentences and each Σ-sentence $(\forall \varphi)e'$ (where $\varphi : \Sigma \to \Sigma'$) we have $\Gamma \models (\forall\varphi)e'$ iff $\varphi(\Gamma) \models e'$. Since *Substitutivity* (see the proof of Proposition 3) and $\mathcal{E}_1$ are also sound, the least WES over $\mathcal{E}_1$ closed under *Substitutivity* and *Generalization* (which is the GUWES of $\mathcal{I}$) is also sound.

Theorem 2 (Generic universal completeness). *Let $\mathcal{D}$ be a broad subcategory of signature morphisms such that $\mathcal{D}^l \subseteq \mathcal{D}$. Assume that*

1. *the WES of $\mathcal{I}_1$ is complete (and compact), and*
2. *for each set of sentences $E \subseteq \mathbb{S}en_1(\Sigma)$ and each sentence $e \in \mathbb{S}en_1(\Sigma)$, we have $E \models_\Sigma e$ iff $M \models_\Sigma (E \Rightarrow e)$ for all $\mathcal{D}$-reachable models M.*

Then we have

1. *the GUWES of $\mathcal{I}$ is complete (and compact), and*
2. *$\Gamma \models_\Sigma (\forall\varphi)e'$, where $(\Sigma \xrightarrow{\varphi} \Sigma') \in \mathcal{D}^l$, iff $M' \models_{\Sigma'} (\varphi(\Gamma) \Rightarrow e')$ for all $\mathcal{D}$-reachable models M'.*

Proof. 1. Assume that $\Gamma \models_\Sigma (\forall\varphi)e'$ where $(\varphi : \Sigma \to \Sigma') \in \mathcal{D}$. We want to show that $\Gamma \vdash_\Sigma (\forall\varphi)e'$. Suppose towards a contradiction that $\Gamma \not\vdash_\Sigma (\forall\varphi)e'$. We define the set of Σ'-sentences $\Gamma_1^\varphi = \{\rho' \in \mathbb{S}en_1(\Sigma') | \Gamma \vdash_\Sigma (\forall\varphi)\rho'\}$.
Suppose $\Gamma_1^\varphi \vdash^1_{\Sigma'} e'$. For the infinitary case we take $\Gamma' = \Gamma_1^\varphi$. For the finitary case, since the WES of $\mathcal{I}_1$ is compact, there exists a finite $\Gamma' \subseteq \Gamma_1^\varphi$ such that $\Gamma' \vdash^1 e'$. By *Generalization* $\varphi(\Gamma) \vdash_{\Sigma'} \rho'$ for all $\rho' \in \Gamma'$, which implies $\varphi(\Gamma) \vdash_{\Sigma'} \Gamma'$. Since $\Gamma_1^\varphi \vdash^1_{\Sigma'} e'$ implies $\Gamma_1^\varphi \vdash_{\Sigma'} e'$, we obtain $\varphi(\Gamma) \vdash_{\Sigma'} e'$ and again by *Generalization* $\Gamma \vdash_\Sigma (\forall\varphi)e'$, which contradicts our assumption. Hence, $\Gamma_1^\varphi \not\vdash^1_{\Sigma'} e'$.
By completeness of $\mathcal{I}_1$ $\Gamma_1^\varphi \not\models e'$. There exists a $\mathcal{D}$-reachable model M such that $M \models \Gamma_1^\varphi$ but $M \not\models e'$. This implies $M \upharpoonright_\varphi \not\models (\forall\varphi)e'$. If we proved that $M \upharpoonright_\varphi \models \Gamma$ we reached a contradiction with $\Gamma \models (\forall\varphi)e'$. We will therefore focus on proving that $M \upharpoonright_\varphi \models \Gamma$.
Let $(\forall\varphi_1)e_1 \in \Gamma$, where $(\varphi_1 : \Sigma \to \Sigma_1) \in \mathcal{D}$, and let N be any φ_1-expansion of $M \upharpoonright_\varphi$. We have to show that $N \models e_1$. Since M is $\mathcal{D}$-reachable there exists a substitution $\theta : \varphi_1 \to \varphi$ such that $M \upharpoonright_\theta = N$. By *Substitutivity* we obtain $\Gamma \vdash (\forall\varphi)\theta(e_1)$ which implies $\theta(e_1) \in \Gamma_1^\varphi$. Since $M \models \Gamma_1^\varphi$ we have $M \models \theta(\rho)$ and by the satisfaction condition $M \upharpoonright_\theta = N \models e_1$.
For the compactness of the GUWES of $\mathcal{I}$ consider the compact sub-WES $\mathcal{E}^c = (\mathbb{S}ig, Sen, \vdash^c)$ of $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$. It contains $\mathcal{E}_1$ because $\mathcal{E}_1$ is compact. Note that $\mathcal{E}^c$ satisfies *Substitutivity*. If we proved that $\mathcal{E}^c$ satisfies *Generalization* then because $\mathcal{E}$ is the least WES over $\mathcal{E}_1$ satisfying the rules of *Substitutivity* and *Generalization* we obtain $\mathcal{E}^c = \mathcal{E}$.
If $\Gamma \vdash^c (\forall\varphi)e'$ then there exists $\Gamma' \subseteq \Gamma$ finite such that $\Gamma' \vdash (\forall\varphi)e'$. By *Generalization* $\varphi(\Gamma') \vdash e'$ which means $\varphi(\Gamma) \vdash^c e'$. Now if $\varphi(\Gamma) \vdash^c e'$ then there is $\Gamma' \subseteq \Gamma$ finite such that $\varphi(\Gamma') \vdash e'$. Using the *Generalization* again we get $\Gamma' \vdash (\forall\varphi)e'$ which means $\Gamma \vdash^c (\forall\varphi)e'$.

2. The non-trivial implication is from right to left. Assume that $\Gamma \not\models_\Sigma (\forall\varphi)e'$, where $(\varphi : \Sigma \to \Sigma') \in \mathcal{D}^l$, then by soundness of the WES of $\mathcal{I}$ we have $\Gamma \nvdash (\forall\varphi)e'$. Using the first part of the proof we get a $\mathcal{D}$-reachable Σ'-model M such that $M \models \varphi(\Gamma)$ and $M \not\models e'$. Therefore there exists a $\mathcal{D}$-reachable model M such that $M \not\models \varphi(\Gamma) \Rightarrow e'$.

We have proved that the GUWES of $\mathcal{I}$ is complete which implies that it is complete with respect to all $(\mathcal{D}^c, \mathcal{D}^l)$-models, for any subcategory $\mathcal{D}^c \subseteq \mathcal{D}$ of signature morphisms. Therefore, the first condition of Theorem 1 is fulfilled. The following remark addresses the second condition of Theorem 1.

Remark 3. Under the assumption of Theorem 2, for any subcategory $\mathcal{D}^c \subseteq \mathcal{D}$ of signature morphisms, we have $\Gamma \models_\Sigma (\forall\varphi)e'$ iff $M \models_\Sigma (\Gamma \Rightarrow (\forall\varphi)e')$ for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable models M.

Proof. The non-trivial implication is from right to left. Assume that $M \models_\Sigma \Gamma \Rightarrow (\forall\varphi)e'$ for all $(\mathcal{D}^c, \mathcal{D}^l)$-reachable Σ-models M. Note that for each $\mathcal{D}$-reachable Σ'-model M', the model $M' \upharpoonright_\varphi$ is $(\mathcal{D}^c, \mathcal{D}^l)$-reachable which implies $M' \upharpoonright_\varphi \models_\Sigma (\Gamma \Rightarrow (\forall\varphi)e')$ for all $\mathcal{D}$-reachable Σ'-models M' and by Theorem 2 we obtain $\Gamma \models_\Sigma (\forall\varphi)e'$.

Weak entailment systems with implications (IWES). Assume an institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, a sub-functor $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ of $\mathbb{S}en$ such that

- $(\bigwedge H \Rightarrow C) \in \mathbb{S}en(\Sigma)$, for all (finite) sets of sentences $H \subseteq \mathbb{S}en_0(\Sigma)$ and any sentence $C \in \mathbb{S}en_0(\Sigma)$, and
- any sentence in $\mathcal{I}$ is of the form $(\bigwedge H \Rightarrow C)$ as above.

We denote the institution $(\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ by $\mathcal{I}_0$. We define the following proof rules for the IWES of $\mathcal{I}$.
$(Implications)$ $\Gamma \vdash_\Sigma (\bigwedge H \Rightarrow C)$ iff $\Gamma \cup H \vdash_\Sigma C$, for every set $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and any sentence $\bigwedge H \Rightarrow C \in \mathbb{S}en(\Sigma)$.

Given a compact WES $\mathcal{E}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \vdash^0)$ for $\mathcal{I}_0$, the IWES of $\mathcal{I}$ consists of the least WES over $\mathcal{E}_0$, closed under the rules of $Implications$. This is the finitary version of the IWES for $\mathcal{I}$, and is applicable to the restriction of **GHCL** and **GHOSA** to the sentences formed without quantifiers. Its infinitary variant is obtained by dropping the compactness condition and by considering the infinitary WES for $\mathcal{I}$; it is applicable to the restriction of $\mathbf{GHCL}_\infty$ and $\mathbf{GHOSA}_\infty$ to the quantifier-free sentences.

Proposition 5. *[7] Let us assume that*

1. *the WES of $\mathcal{I}_0$ is sound, complete (and compact),*
2. *every set of sentences in $\mathcal{I}_0$ is basic, and*
3. *there exits a broad subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ such that for each set $B \subseteq \mathbb{S}en_0(\Sigma)$ there is a $\mathcal{D}$-reachable model M_B defining B as basic set of sentences.*

Then we have

1. *the IWES of $\mathcal{I}$ is sound, complete (and compact), and*
2. *$E \models e$ iff $M \models (E \Rightarrow e)$ for all $\mathcal{D}$-reachable models M.*

Atomic weak entailment systems (AWES). In order to develop concrete sound and complete universal WES we need to define sound and complete WES for the "atomic" layer of each institution.

Proposition 6. *[7] Let* $\mathbf{GHCL}_0$ *be the restriction of* **GHCL** *to the atomic sentences. The WES of* $\mathbf{GHCL}_0$ *generated by the rules below is sound, complete and compact.*
$(Reflexivity)$ $\emptyset \vdash t = t$*, where* t *is a term.*
$(Symmetry)$ $t = t' \vdash t' = t$ *, where* t, t' *are terms.*
$(Transitivity)$ $\{t = t', t' = t''\} \vdash t = t''$*, where* t, t', t'' *are terms.*
$(Congruence)$ $\{t_i = t'_i | 1 \le i \le n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t'_1, ..., t'_n)$*, where* $t_i, t'_i \in T_F$ *are terms and* σ *is an operation symbol.*
$(PCongruence)$ $\{t_i = t'_i | 1 \le i \le n\} \cup \{\pi(t_1, ..., t_n)\} \vdash \pi(t'_1, ..., t'_n)$*, where* t_i, t'_i *are terms and* π *is a predicate symbol.*

One can define a sound, complete and compact AWES for the atomic part of **GHOSA** by considering all the proof rules from Proposition 6, except the last one which deals with predicates. The following is a corollary of Theorem 1.

Corollary 2. *[Completeness of the* **GHCL***] The RUWES of* **GHCL** *generated by the rules of* $Case\ splitting$*,* $Substitutivity$*,* $Generalization$*,* $Implications$*,* $Reflexivity$*,* $Symmetry$*,* $Transitivity$*,* $Congruence$ *and* $PCongruence$ *is sound and complete with respect to all reachable models.*

4 Borrowing Completeness

Let $\mathcal{I}' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ and $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be two institutions. An *institution morphism* $(\phi, \alpha, \beta) : \mathcal{I}' \to \mathcal{I}$ consists of

- a functor $\phi : \mathbb{S}ig' \to \mathbb{S}ig$, and
- two natural transformations $\alpha : \phi; \mathbb{S}en \Rightarrow \mathbb{S}en'$ and $\beta : \mathbb{M}od' \Rightarrow \phi^{op}; \mathbb{M}od$

such that the following satisfaction condition for institution morphisms holds: $M' \models'_{\Sigma'} \alpha_{\Sigma'}(e)$ iff $\beta_{\Sigma'}(M') \models_{\phi(\Sigma')} e$, for every signature $\Sigma' \in \mathbb{S}ig'$, each Σ'-model M', and any $\phi(\Sigma')$-sentence e.

Definition 5. *We say that a WES* $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ *of an institution* $\mathcal{I} = (\mathbb{S}ig, Sen, \mathbb{M}od, \models)$ *is* Ω*-complete, where* $\Omega = (\Omega_\Sigma)_{\Sigma \in |\mathbb{S}ig|}$ *is a family of sets of sentences (*$\Omega_\Sigma \subseteq \mathcal{P}(\mathbb{S}en(\Sigma))$ *for all signatures* Σ*) iff* $\Gamma \models_\Sigma E$ *implies* $\Gamma \vdash_\Sigma E$ *for all* $\Gamma \in \Omega_\Sigma$*.*

Remark 4. Let $(\phi, \alpha, \beta) : \mathcal{I}' \to \mathcal{I}$ be an institution morphism (where $I' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ and $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$). Every WES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ for $\mathcal{I}$ generates freely a WES $\mathcal{E}' = (\mathbb{S}ig', \mathbb{S}en', \vdash')$ for $\mathcal{I}'$, where $\mathcal{E}'$ is the least WES closed under the rules $\alpha_{\Sigma'}(\Gamma) \vdash'_{\Sigma'} \alpha_{\Sigma'}(E)$, where Σ' is a signature in $\mathcal{I}'$ and $\Gamma \vdash_{\phi(\Sigma')} E$ is a deduction in $\mathcal{E}$.

Theorem 3. *Consider*

1. *an institution morphism* $(\phi, \alpha, \beta) : \mathcal{I}' \to \mathcal{I}$ *(where* $\mathcal{I}' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ *and* $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$*) such that* $\alpha_{\Sigma'}$ *is surjective for all* $\Sigma' \in |\mathbb{S}ig'|$*,*
2. *a class of models* $\mathcal{M} = (\mathcal{M}_\Sigma)_{\Sigma \in |\mathbb{S}ig|}$ *(in* $\mathcal{I}$*) such that* $\beta_{\Sigma'}(|\mathbb{M}od'(\Sigma')|) \subseteq \mathcal{M}_{\phi(\Sigma')}$ *for all signatures* $\Sigma' \in |\mathbb{S}ig|$*, and*

3. a WES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ for $\mathcal{I}$ which is sound and complete with respect to $\mathcal{M}$.

Then the entailment system $\mathcal{E}' = (\mathbb{S}ig', \mathbb{S}en', \vdash')$ of $\mathcal{I}'$ determined by $\mathcal{E}$ is sound and Ω-complete, where for every signature $\Sigma' \in |\mathbb{S}ig'|$ we have $\Gamma' \in \Omega_{\Sigma'}$ iff $\Gamma = \alpha_{\Sigma'}^{-1}(\Gamma')$ has the following property: $M \models_{\phi(\Sigma')} \Gamma$ implies $M \in \beta_{\Sigma'}(|\mathbb{M}od'(\Sigma')|)$, for any $M \in \mathcal{M}_{\phi(\Sigma')}$.

Proof. Since $\alpha_{\Sigma'}$ is surjective, for all signatures $\Sigma' \in |\mathbb{S}ig'|$, $\mathcal{E}' = (\mathbb{S}ig', \mathbb{S}en', \vdash')$ with $\vdash'_{\Sigma'} = \alpha_{\Sigma'}(\vdash_{\phi(\Sigma')})$, for all signatures $\Sigma' \in |\mathbb{S}ig'|$, is the WES of $\mathcal{I}'$ determined by the institution morphism (ϕ, α, β).

1. Suppose that $\Gamma' \vdash'_{\Sigma'} E'$ and let M' be a Σ'-model such that $M' \models' \Gamma'$. By the definition of $\mathcal{E}'$ there exists $\Gamma \vdash_{\phi(\Sigma')} E$ such that $\alpha_{\Sigma'}(\Gamma) = \Gamma'$ and $\alpha_{\Sigma'}(E) = E'$. By the satisfaction condition for the institution morphisms we have $\beta_{\Sigma'}(M') \models_{\phi(\Sigma')} \Gamma$. Since $\mathcal{E}$ is sound with respect to $\mathcal{M}$ we have $M \models_{\phi(\Sigma')} (\Gamma \Rightarrow E)$ for all models $M \in \mathcal{M}_{\phi(\Sigma')}$. Because $\beta_{\Sigma'}(M') \in \mathcal{M}_{\phi(\Sigma')}$ we have that $\beta_{\Sigma'}(M') \models_{\phi(\Sigma')} (\Gamma \Rightarrow E)$ which implies $\beta_{\Sigma'}(M') \models_{\phi(\Sigma')} E$. By the satisfaction condition for institution morphisms we get $M' \models'_{\Sigma'} \alpha_{\Sigma'}(E)$. Hence $M' \models'_{\Sigma'} E'$.

2. Assume $\Gamma' \models_{\Sigma'} E'$, where $\Gamma' \in \Omega$, and let $\Gamma = \alpha_{\Sigma'}^{-1}(\Gamma')$ and $E = \alpha_{\Sigma'}^{-1}(E')$. Note that $M \models (\Gamma \Rightarrow E)$ for all $M \in \mathcal{M}_{\Sigma}$. Indeed for any $M \in \mathcal{M}_{\Sigma}$ we have: $M \models_{\phi(\Sigma')} \Gamma$ implies $M \in \beta_{\Sigma'}(|\mathbb{M}od'(\Sigma')|)$; so, there exists a Σ'-model M' such that $\beta_{\Sigma'}(M') = M$ and by satisfaction condition for institution morphisms $M' \models' \Gamma'$ which implies $M' \models' E'$; applying again satisfaction condition we obtain $M \models E$. Since $\mathcal{I}$ is complete with respect to $\mathcal{M}$ we have $\Gamma \vdash E$ which implies $\Gamma' \vdash' E'$.

In order to develop sound and complete WES for the constructor-based institutions we need to set the parameters of Theorem 3. We define the institution morphism $\Delta_{\mathbf{HCL}} = (\phi, \alpha, \beta) : \mathbf{CHCL} \rightarrow \mathbf{GHCL}$ such that

1. the functor ϕ maps
   - every **CHCL** signature (S, F, F^c, P) to a **GHCL** signature (S, S^c, F, P), where S^c is the set of constrained sorts determined by F^c, and
   - every **CHCL** signature morphism $(\varphi^{sort}, \varphi^{op}, \varphi^{pred})$ to the **GHCL** signature morphism $(\varphi^{sort}, \varphi^{op}, \varphi^{pred})$;
2. α is the identity natural transformation (recall that $\mathbb{S}en(S, F, F^c, P) = \mathbb{S}en(S, S^c, F, P)$, where S^c is a the set of constrained sorts determined by the constructors in F^c), for every **CHCL** signature (S, F, F^c, P) we have $\alpha_{(S,F,F^c,P)} = 1_{\mathbb{S}en(S,F,F^c,P)}$;
3. β is the inclusion natural transformation (note that every (S, F, F^c, P)-model M is also a (S, S^c, F, P)-model; indeed if there exists a set of loose variables Y and a function $f : Y \rightarrow M$ such that for every constrained sort $s \in S^c$ the function $f_s^{\#} : (T_{F^c}(Y))_s \rightarrow M_s$ is a surjection, where $f^{\#}$ is the unique extension of f to a (S, F^c, P)-morphism, then for every constrained sort $s \in S^c$ the function $\overline{f}_s : (T_F(Y))_s \rightarrow M_s$ is a surjection too, where $\overline{f}$ is the unique extension of f to a (S, F, P)-morphism), for every **CHCL** signature (S, F, F^c, P) the functor $\beta_{(S,F,F^c,P)} : \mathbb{M}od(S, F, F^c, P) \rightarrow \mathbb{M}od(S, S^c, F, P)$

is defined by $\beta_{(S,F,F^c,P)}(M) = M$ for all models $M \in |\mathbb{M}od(S,F,F^c,P)|$ and $\beta_{(S,F,F^c,P)}(h) = h$ for all morphism $h \in \mathbb{M}od(S,F,F^c,P)$.

Remark 5. A (S,S^c,F,P)-model M in **GHCL** is reachable iff there exists a set of loose variables Y and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^{\#}_s : (T_{F^{cons}}(Y))_s \to M_s$ is surjective, where F^{cons} is the set operations with constrained resulting sorts, and $f^{\#}$ is the unique extension of f to a (S, F^{cons}, P)-morphism.

Definition 6. *A basic specification (Σ,Γ) in* **CHCL** *is* sufficient complete, *where $\Sigma = (S,F,F^c,P)$, if for every term t formed with operation symbols from F^{cons} (the set of operations with constrained resulting sorts) and loose variables from Y there exists a term t' formed with constructors and loose variables from Y such that $\Gamma \vdash_{(S,S^c,F,P)} (\forall Y)t = t'$ in* **GHCL**.

Since $\models_{(S,S^c,F,P)} \subseteq \vdash_{(S,S^c,F,P)}$ for all **GHCL**-signatures (S,S^c,F,P) (see Corollary 2 for the definition of $\vdash_{(S,S^c,F,P)}$), the condition $\Gamma \vdash_{(S,S^c,F,P)} (\forall Y)t = t'$ in Definition 6 is more general than if we assumed $\Gamma \models_{(S,S^c,F,P)} (\forall Y)t = t'$.

The following is a corollary of Theorem 3.

Corollary 3. *The entailment system of* **CHCL** *generated by the proof rules for* **GHCL** *is sound and Ω-complete, where $\Gamma \in \Omega_{(S,F,F^c,P)}$ iff $((S,S^c,F,P),\Gamma)$ is a sufficient complete* **CHCL***-specification.*

Proof. We set the parameters of Theorem 3. The institution $\mathcal{I}'$ is **CHCL** and the institution $\mathcal{I}$ is **GHCL**. The institution morphism is $\Delta_{\mathbf{HCL}}$ and the entailment system $\mathcal{E}$ of **GHCL** is the least entailment system closed under the rules enumerated in Corollary 2. $\mathcal{M}$ is the class of all reachable models. We need to prove that for every sufficient complete specification $((S,F,F^c,P),\Gamma)$ and any reachable (S,S^c,F,P)-model M (where S^c is the set constrained sorts determined by F^c) we have: $M \models \Gamma$ implies $M \in |\mathbb{M}od(S,F,F^c,P)|$. Because M is reachable by Remark 5 there exists a set Y of loose variables and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^{\#}_s : (T_{F^{cons}}(Y))_s \to M_s$ is a surjection, where $f^{\#}$ is the unique extension of f to a (S,F^{cons},P)-morphism. Because $((S,F,F^c,P),\Gamma)$ is sufficient complete, for every constrained sort $s \in S^c$ the function $\overline{f}_s : (T_{F^c}(Y))_s \to M_s$ is a surjection too, where $\overline{f}$ is the unique extension of f to a (S,F^c,P)-morphism.

Similar results as Corollary 3 can be formulated for **GHOSA**, $\mathbf{GHCL}_\infty$, and $\mathbf{GHOSA}_\infty$.

In general, the proof rules given here for the constructor-based institutions are not complete. Consider the signature (S,F,F^c,P) in **CHCL**, where $S = \{s\}$, $F_{\to s} = \{a,b\}$, $F^c = \{a\}$ and $P = \emptyset$. It is easy to notice that $\models a = b$ but there is no way to prove $\emptyset \vdash a = b$.

Assume that $S = Nat$, $F_{\to Nat} = \{0\}$, $F_{Nat \to Nat} = \{s\}$, $F_{Nat Nat \to Nat} = \{+\}$, $F^c_{\to Nat} = \{0\}$, $F^c_{Nat \to Nat} = \{s\}$ and $P = \emptyset$. Consider the following equations $\rho_1 = (\forall x : Nat)x + 0 = x$ and $\rho_2 = (\forall x : Nat)(\forall x' : Nat)x + (s\ x') = s(x + x')$. Then $((S,S^c,F),\{\rho_1,\rho_2\})$ is a sufficient complete specification. Intuitively, if $\Gamma \in \mathbb{S}en(S, F, F^c, P)$ specify that non-constructor operators are inductively defined with respect to the constructors then $((S, F, F^c, P), \Gamma)$ is a sufficient complete specification.

Structural Induction. In the constructor-based institutions presented here the carrier sets of the models consist of interpretations of terms formed with constructors and elements of loose sort. Thus, *Case Splitting* can be rephrased as follows:

(*Case Splitting*) $\Gamma \vdash (\forall x)\rho$ if $\Gamma \vdash (\forall Y)\rho(x \leftarrow t)$ for all terms t formed with constructors and variables of sort loose, where Γ is a set of sentences, and $(\forall x)\rho$ a sentence such that x is a constrained variable.

In order to prove the premises of *Case splitting*, in many cases, we use induction on the structure of terms. For any t formed with constructors in F^c and loose variables we have
(*Structural induction*) $\Gamma \vdash_{(S,F,F^c)} (\forall V)\rho(x \leftarrow t)$ if

1. (*Induction base*) for all $cons \in F^c_{\to s}$, $\Gamma \vdash_{(S,F,F^c)} \rho(x \leftarrow cons)$,
2. (*Induction step*) for all $\sigma \in F^c_{s_1 \ldots s_n \to s}$, $\Gamma \cup \{\rho(x \leftarrow x') \mid x' \in X\} \vdash_{(S,F \cup C,F^c)} \rho(x \leftarrow \sigma(c_1, \ldots, c_n))$, where
   - $C = \{c_1, \ldots, c_n\}$ is a set of new variables such that c_i has the sort s_i, for all $i \in \{1, \ldots, n\}$, and
   - $X \subseteq C$ is the set of variables with the sort s.

where V are all (loose) variables in t.

5 Conclusions and Future Work

We define the infinitary rules of *Case splitting* and show that the WES of **CHCL** is sound and complete with respect to all sufficient complete specifications. We define the rules of *Structural induction* to deal with the infinitary premises of *Case spliting* but the infinitary rules can not be replaced with the finitary ones in order to obtain a complete and compact WES because the class of sentences true of a class of models for a given constructor-based specification is not in general recursively enumerable. Gödel's famous incompleteness theorem show that this holds even for the specification of natural numbers.

Due the abstract definition of reachable model given here, one can easily define a constructor-based institution on top of some base institution by defining the constructor-based signatures as signature morphisms in the base institution. This construction may be useful when lifting the interpolation and amalgamation properties (necessary for modularization) from the base institution.

The area of applications provided by the general framework of the present work is much wider. For example we may consider partial algebras [5] , preorder algebras [9], or variations of these institutions, such as order-sorted algebra with transitions. The present work is much general than [7]. If $\mathcal{D}^c$ is the broad subcategory consisting of identity morphisms then all models are constrained and we may obtain the result in [7] concerning Horn institutions. It is to investigate the applicability of Theorem 1 to **GFOL** by adapting the completeness of first-order institutions developed in [11]. Then it is straightforward to construct an institution morphism **CFOL** → **GFOL** and obtain an entailment system sound and complete (relatively to a family of sufficient complete basic specifications) for **CFOL**.

References

1. J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Number 189 in London Mathematical Society Lecture Notes. Cambridge University Press, 1994.
2. E. Astesiano, M. Bidoit, H. Kirchner, B. Krieg-Brückner, P. D. Mosses, D. Sannella, and A. Tarlecki. Casl: the common algebraic specification language. *Theor. Comput. Sci.*, 286(2):153–196, 2002.
3. M. Bidoit and R. Hennicker. Constructor-based observational logic. *J. Log. Algebr. Program.*, 67(1-2):3–51, 2006.
4. M. Bidoit, R. Hennicker, and A. Kurz. Observational logic, constructor-based logic, and their duality. *Theor. Comput. Sci.*, 3(298):471–510, 2003.
5. P. Burmeister. *A Model Theoretic Oriented Appraoch to Partial Algebras*. Akademie-Verlag Berlin, 1986.
6. M. Clavel, F. Durán, S. Eker, P. Lincoln, N. Martí-Oliet, J. Meseguer, and C. L. Talcott, editors. *All About Maude - A High-Performance Logical Framework, How to Specify, Program and Verify Systems in Rewriting Logic*, volume 4350 of *Lecture Notes in Computer Science*. Springer, 2007.
7. M. Codescu and D. Gaina. Birkhoff completeness in institutions. *Logica Universalis*, 2(2):277–309, September 2008.
8. R. Diaconescu. Institution-independent ultraproducts. *Fundamenta Informaticae*, 55(3-4):321–348, 2003.
9. R. Diaconescu and K. Futatsugi. Logical foundations of CafeOBJ. *Theor. Comput. Sci.*, 285(2):289–318, 2002.
10. D. Găină, K. Futatsugi, and K. Ogata. Constructor-based Institutions. Technical report, Japan Advanced Institute of Science and Technology, 2009. to appear.
11. D. Găină and M. Petria. Completeness by forcing. submitted.
12. J. A. Goguen and R. Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.
13. J. A. Goguen and R. Diaconescu. An Oxford survey of order sorted algebra. *Mathematical Structures in Computer Science*, 4(3):363–392, 1994.
14. J. A. Goguen and J. Meseguer. Order-sorted algebra I: Equational deduction for multiple inheritance, overloading, exceptions and partial operations. *Theoretical Computer Science*, 105(2):217–273, 1992.
15. J. A. Goguen and J. W. Thatcher. Initial algebra semantics. *Annual Symposium on Switching and Automata Theory*, 0:63–77, 1974.
16. A. Tarlecki. Bits and pieces of the theory of institutions. In D. Pitt, S. Abramsky, A. Poigné, and D. Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming, Lecture Notes in Computer Science*, volume 240, pages 334–360. Springer, 1986.