Uniqueness of normal proofs in $\{\rightarrow, \wedge\}$-fragment of NJ

Takahito Aoto and Hiroakira Ono
August 11, 1994
IS-RR-94-0024F

北陸先端科学技術大学院大学
情報科学研究科
〒923-12 石川県能美郡辰口町旭台 15
aoto@jaist.ac.jp, ono@jaist.ac.jp

ISSN 0918-7553

Uniqueness of normal proofs in $\{\to, \wedge\}$-fragment of NJ

Takahito Aoto
aoto@jaist.ac.jp

Hiroakira Ono
ono@jaist.ac.jp

School of Information Science
Japan Advanced Institute of Science and Technology, Hokuriku
15 Asahidai Tatsunokuchi, Ishikawa 923-12
Japan

Abstract

It is known that in $\{\to, \wedge\}$-fragment of the natural deduction system of intuitionistic propositional logic NJ balanced formulas have unique $\beta\eta$-normal proofs. A balanced formula is a formula such that

1. every propositional variable has at most one negative occurrence and
2. every propositional variable has at most one positive occurrence.

It will be shown here that the above condition 1 is sufficient for formulas to have unique $\beta\eta$-normal proofs. This can be obtained by a close examination of Mints' proof, whose details are given in the present report.

1 Introduction

Recently analogies between proofs/normalization steps of natural deduction system of intuitionistic logic and terms/reductions of typed lambda calculus are widely accepted. Since computational aspects of logic attract our large interest both in logic and in theoretical computer science, this analogies lead us to study the properties of formal proofs rather than usual provability of propositions. We study here relations between formulas and their proofs, especially about conditions for uniqueness of normal proofs with respect to formulas.

Originally, an interest in uniqueness problem arose from coherence problem in category theory. In the late-70s, G.E.Mints proved that proofs of balanced formulas are $\beta\eta$-congruent in $\{\to\}$-fragment of NJ ([4]). A balanced formula is a formula such that

1. every propositional variable has at most one negative occurrence and
2. every propositional variable has at most one positive occurrence.

Successively, A.A.Babaev and S.V.Solovjev proved it in $\{\to, \wedge\}$-fragment ([1]). Later, G.E.Mints gave a simpler proof of it using the idea of V.Orekov's depth-reducing transformations of formulas ([5]).

In the mid-80s, an interest in uniqueness problem arose again from Y.Komori's conjecture that normal proofs of minimal formulas are unique in $\{\to\}$-fragment of NBCK and NJ. NBCK is the natural deduction system of BCK, i.e. the intuitionistic propositional logic without a contraction rule. A minimal formula is a provable formula which is not a non-trivial substitution instance of other provable formulas. It is already known that above conjecture is true for NBCK but false for NJ ([3],[7]). Moreover implicational minimal formulas in BCK is balanced ([2]). So the balancedness condition is a one of the main conditions which come out when we think of the uniqueness problem.

So far, it is still a fascinating question whether there is yet an other condition to have unique normal proofs. Recently, G.E.Mints' proof became to available ([6]). The main result of the present report is to show that the balancedness condition can be weakened by a close examination of his proof. More precisely, the above condition 1 is sufficient for formulas to have unique $\beta\eta$-normal proofs. We will present here a detailed proof of it, supplementing and modifying Mints' proof in [6].

2 Preliminaries

2.1 natural deduction system

We work on the $\{\rightarrow, \wedge\}$-fragment of intuitionistic propositional logic. We use $P, Q, R, S, T, \ldots$ for propositional variables and $A, B, C, D, \ldots$ for arbitrary formulas.

We use $\Gamma, \Sigma, \Delta, \Pi, \ldots$ for finite sets of formulas. Set-union is denoted by $\cup$ and set-difference is by $\setminus$.

We use $\{\ldots, A^0, \ldots\}$ to denote A possibly does not exists in the set. For example, $\{A^0, B, C\}$ might be $\{A, B, C\}$ or $\{B, C\}$. (This notation is used to combine the way of assumption-cancelling in natural deduction systems to set-notation of antecedents of sequents. We can deduce $B, C \vdash A \rightarrow D$ not only from $A, B, C \vdash D$ but also from $B, C \vdash D$. Using our notation, we can say that the subproof over a proof of $B, C \vdash A \rightarrow D$ is a proof of $A^0, B, C \vdash D$. See definitions below and the proof of THEOREM.3.10, Case 3-3.)

We use $\equiv$ for syntactical equality.

DEFINITION. 2.1 [sequent]

Any expression of the form

$$\Gamma \vdash A$$

is called a ***sequent.*** Here Γ is called the ***antecedent*** of the sequent and A the ***succedent.***

We use the natural deduction system NJ which consists of the following inference rules.

1. $\rightarrow$ introduction

$$\frac{\begin{array}{c}[A]^{(1)}\\ \mathcal{D}\\ B\end{array}}{A \rightarrow B}\ (1)$$

Here $[A]^{(1)}$ means that assumptions $[A]$ in $\mathcal{D}$ is often cancelled along with the application of this inference.

2. $\rightarrow$ elimination

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A \rightarrow B & A\end{array}}{B}$$

3. $\wedge$ introduction

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{array}}{A \wedge B}$$

4. $\wedge$ elimination

$$\frac{\begin{array}{c}\mathcal{D} \\ A \wedge B\end{array}}{A} \qquad\qquad \frac{\begin{array}{c}\mathcal{D} \\ A \wedge B\end{array}}{B}$$

Formulas just over an inference are called the ***premises*** of the inference and a formula which is just below an inference is called the ***conclusion.*** In $\rightarrow$ elimination rule, the left premise is called the ***major premise*** and the right premise is called the ***minor premise.***

The $\rightarrow$ introduction rule and $\wedge$ introduction rule are called ***introduction rules.*** The $\rightarrow$ elimination rule and $\wedge$ elimination rule are called ***elimination rules.***

The set of ***assumptions*** of a proof $\mathcal{D}$ is the set of uncancelled assumptions which appear in $\mathcal{D}$. The ***conclusion of a proof*** $\mathcal{D}$ is a conclusion of the last inference. A ***proof*** $\mathcal{D}$ ***of a sequent*** $\Gamma \vdash A$ is a proof which has Γ as the set of assumptions and A as the conclusion (written as $\mathcal{D} : \Gamma \vdash A$). A sequent $\Gamma \vdash A$ is said to be ***derivable*** iff there exists a proof $\mathcal{D}$ of $\Gamma \vdash A$. The outermost $\{$ and $\}$ of Γ are often omitted as usual.

2.2 reductions

We consider following reduction rules.

1. $\beta_{\rightarrow}$-contraction

$$\frac{\dfrac{\begin{array}{c}[A]^{(1)} \\ \mathcal{D} \\ B\end{array}}{A \rightarrow B}\,(1) \quad \begin{array}{c}\mathcal{D}' \\ A\end{array}}{B} \quad \rightarrow \quad \begin{array}{c}\mathcal{D}' \\ A \\ \mathcal{D} \\ B\end{array}$$

2. $\beta_{\wedge}$-contractions

$$\frac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{array}}{A \wedge B}}{A} \quad \rightarrow \quad \begin{array}{c}\mathcal{D}_1 \\ A\end{array}$$

$$\frac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{array}}{A \wedge B}}{B} \quad \rightarrow \quad \begin{array}{c}\mathcal{D}_2 \\ B\end{array}$$

3. Restricted $\eta_{\rightarrow}$-expansion

$$\begin{array}{c}\mathcal{D}\\ A\rightarrow B\end{array}\qquad\longrightarrow\qquad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A\rightarrow B\end{array}\quad [A]^{(1)}}{B}}{A\rightarrow B}\,(1)$$

Here we assume that the reduction can be made only when this occurrence of $A \rightarrow B$ does not appear in $\mathcal{D}$ neither as a conclusion of $\rightarrow$ introduction nor as a major premise of $\rightarrow$ elimination.

4. Restricted $\eta_{\wedge}$-expansion

$$\begin{array}{c}\mathcal{D}\\ A\wedge B\end{array}\qquad\longrightarrow\qquad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A\wedge B\end{array}}{A}\quad \cfrac{\begin{array}{c}\mathcal{D}\\ A\wedge B\end{array}}{B}}{A\wedge B}$$

Here we assume that the reduction can be made only when this occurrence of $A \wedge B$ does not appear in $\mathcal{D}$ neither as a conclusion of $\wedge$ introduction nor as a premise of $\wedge$ eliminations.

$\beta_{\rightarrow}$-contraction and $\beta_{\wedge}$-contraction are called ***β-reductions.*** Restricted $\eta_{\rightarrow}$-expansion and restricted $\eta_{\wedge}$-expansion are called ***η^{-1}-reductions.***

A proof which is irreducible by $\beta\eta^{-1}$-reductions is said to be ***$\beta\eta^{-1}$-normal.*** A proof which is reduced by a reduction is called ***reduct*** of the reduction. A reduction from $\mathcal{D}_1$ to $\mathcal{D}_2$ is written as $\mathcal{D}_1 \rightarrow \mathcal{D}_2$. A ***$\beta\eta^{-1}$-reduction sequence*** is a sequence of form $\mathcal{D}_1 \rightarrow \mathcal{D}_2 \rightarrow \cdots$. A $\beta\eta^{-1}$-reduction sequence is said to be ***terminating*** if it ends in a finitely many steps with a normal proof. If $\mathcal{D}_1 \rightarrow \mathcal{D}_2 \rightarrow \cdots \rightarrow \mathcal{D}_n$ and $\mathcal{D}_n$ is normal, then $\mathcal{D}_n$ is called a ***normal form*** of $\mathcal{D}_1$.

The following three results on reductions of natural deduction systems are well-known. (Proofs are all omitted.)

THEOREM. 2.2 [strongly normalizing property w.r.t. $\beta\eta^{-1}$-reduction]
Given a proof of $\Gamma \vdash A$, there is no infinite $\beta\eta^{-1}$-reduction sequences starting with it.

THEOREM. 2.3 [Church-Rosser property w.r.t. $\beta\eta^{-1}$-reduction]
Given a proof of $\Gamma \vdash A$, $\beta\eta^{-1}$-reduction sequences which start with it are confluent.

COROLLARY. 2.4
Given a proof of $\Gamma \vdash A$, it has the one and only $\beta\eta^{-1}$-normal form of it.

2.3 negative and positive occurrence

DEFINITION. 2.5 [negative, positive, strictly positive]
A ***negative***, ***positive*** and ***strictly positive*** occurrence of a formula is defined by simultaneous induction as follows:

1. A has a (strictly) positive occurrence in A;

2. A has a (strictly) positive occurrence in $B \to C$ if A has a (resp. strictly) positive occurrence in C or has a negative occurrence in B;

3. A has a negative occurrence in $B \to C$ if A has a negative occurrence in C or has a positive occurrence in B;

4. A has a (strictly) positive occurrence in $B \wedge C$ if A has a (resp. strictly) positive occurrence in B or in C;

5. A has a negative occurrence in $B \wedge C$ if A has a negative occurrence in B or in C;

6. A has a (strictly) positive occurrence in Γ if A has a (resp. strictly) positive occurrence in C for some $C \in \Gamma$;

7. A has a negative occurrence in Γ if A has a negative occurrence in C for some $C \in \Gamma$;

8. A has a (strictly) positive occurrence in $\Gamma \vdash C$ if A has a (resp. strictly) positive occurrence in C or has a negative occurrence in Γ;

9. A has a negative occurrence in $\Gamma \vdash C$ if A has a negative occurrence in C or has a positive occurrence in Γ.

The set of propositional variables which have a positive (negative, strictly positive) occurrence in A is written as $\mathsf{Pos}(A)$ (resp. $\mathsf{Neg}(A)$. $\mathsf{Spos}(A)$).

3 Uniqueness for 2-sequents

3.1 2-sequents

DEFINITION. 3.1 [2-formula]

A ***2-formula*** is a formula which has one of the following forms:

$$Q, Q \to R, (Q \to R) \to S, Q \to (R \to S), (Q \wedge R) \to S, Q \to (R \wedge S),$$

where Q, R and S are mutually distinct.

DEFINITION. 3.2 [2-sequent]

A ***2-sequent*** is a sequent of the form

$$\Gamma \vdash P$$

where A is a 2-formula for all $A \in \Gamma$.

PROPOSITION. 3.3

Any $\beta\eta^{-1}$-normal proof of 2-sequents can be described inductively by following cases:

1.
$$[P];$$

2a.

$$\frac{[Q \to P] \quad \begin{array}{c}\Sigma\\ \mathcal{D}\\ Q\end{array}}{P}\;;$$

2b.

$$\frac{\dfrac{[R \to (Q \to P)] \quad \begin{array}{c}\Sigma_1\\ \mathcal{D}_1\\ R\end{array}}{Q \to P} \quad \begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ Q\end{array}}{P}\;;$$

3.

$$\frac{[(R \to Q) \to P] \quad \dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ Q\end{array}}{R \to Q}}{P}\;;$$

4a.

$$\frac{\dfrac{[R \to (P \land Q)] \quad \begin{array}{c}\Sigma\\ \mathcal{D}\\ R\end{array}}{P \land Q}}{P}\;;$$

4b.

$$\frac{\dfrac{[R \to (Q \land P)] \quad \begin{array}{c}\Sigma\\ \mathcal{D}\\ R\end{array}}{Q \land P}}{P}\;;$$

5.

$$\frac{[(R \land Q) \to P] \quad \dfrac{\begin{array}{c}\Sigma_1\\ \mathcal{D}_1\\ R\end{array} \quad \begin{array}{c}\Sigma_1\\ \mathcal{D}_2\\ Q\end{array}}{R \land Q}}{P}\;.$$

Proof.

Because of β-normality, assumptions at major premises of inferences of $\to$ introduction can not be cancelled. Hence they must be 2-formulas. Because of η^{-1}-normality, tracing up minor premises of inferences of $\to$ elimination and premises of inferences of $\land$ elimination rule, we must encounter a propositional variable before inferences of introduction rule. This ensures that proofs are described inductively. □

Remark.

Many of our proofs described below proceed by induction on the structure of $\beta\eta^{-1}$-normal proofs, making use of this proposition.

3.2 properties of normal proofs of 2-sequents

PROPOSITION. 3.4
Let $\Delta \vdash P$ be a 2-sequent. Then $P \in \mathsf{Pos}(\Delta)$.

Proof.
This is an immediate corollary of COROLLARY.2.4 and PROPOSITION.3.3. □

PROPOSITION. 3.5
Let $\Delta \vdash P$ be a 2-sequent. If $T \in \mathsf{Neg}(\Delta)$ then $T \in \mathsf{Pos}(\Delta)$.

Proof.
Our proof proceeds by induction on the structure of $\beta\eta^{-1}$-normal proofs of $\Delta \vdash P$.

Case 1. Let $\Delta \vdash P$ be derived as

$$[P].$$

Since $\mathsf{Neg}(P) = \emptyset$, the proposition follows immediately.

Case 2a. Let $\Delta \vdash P$ be derived as

$$\frac{[Q \to P] \quad \begin{matrix}\Sigma \\ \mathcal{D} \\ Q\end{matrix}}{P} .$$

If $T \in \mathsf{Neg}(\Sigma)$, then $T \in \mathsf{Pos}(\Sigma)$ by induction hypothesis. If $T \notin \Sigma$, then $T \equiv Q$. But $Q \in \mathsf{Pos}(\Sigma)$ by applying PROPOSITION.3.4 to $\mathcal{D}$. So, the proposition follows.
Cases 2b~5 follow similarly. □

3.3 properties of suc-sequents

DEFINITION. 3.6 [suc-sequent]
Let $\Gamma \vdash P$ be a 2-sequent. A ***suc-sequent*** of $\Gamma \vdash P$ is a derivable sequent

$$\Gamma', Q_1, Q_2, \ldots, Q_n \vdash R$$

where $\Gamma' \subseteq \Gamma$, $R \in \mathsf{Neg}(\Gamma)$ and for all Q_i there exists Q'_i, Q''_i such that $((Q_i \to Q'_i) \to Q''_i) \in \Gamma$.

Remark.
Suc-sequents are again 2-sequents.

PROPOSITION. 3.7
Let $\Gamma \vdash P$ be a 2-sequent and $\Delta \vdash R$ a suc-sequent of $\Gamma \vdash P$. If $\Delta' \vdash R'$ is a derivable sequent where $\Delta' \subseteq \Delta$ and $R' \in \mathsf{Neg}(\Delta)$, then it is again a suc-sequent of $\Gamma \vdash P$.

Proof.
This is immediate by the definition of suc-sequents. □

LEMMA. 3.8

Let $\Gamma \vdash P$ be a 2-sequent in which P has at most one negative occurrence. If $A \in \Gamma$ and $P \in \mathsf{Spos}(A)$, then all positive occurrences of P in antecedents of suc-sequents of $\Gamma \vdash P$ are in A.

Proof.

Let

$$\Gamma', Q_1, Q_2, \ldots, Q_n \vdash R$$

be a suc-sequent as is in the definition of suc-sequents (DEFINITION.3.6). By our assumptions, $A \in \Gamma$ and $P \in \mathsf{Spos}(A)$.

Now, assume that there is some another formula $B \in (\Gamma' \cup \{Q_1, Q_2, \ldots, Q_n\})$ which contains at least one positive occurrence of P.

1. Case where $B \in \Gamma'$. There are at least two negative occurrences of P in $\Gamma \vdash P$ as in A and in B. This is a contradiction.
2. Case where $B \notin \Gamma'$ (i.e. $B \equiv Q_i$ for some i). Then $P \equiv Q_i$ for some i and from this follows $((P \to Q_i') \to Q_i'') \in \Gamma$ for some Q_i' and Q_i'' by the definition of suc-sequents.
   (a) Subcase where $A \equiv ((P \to Q_i') \to Q_i'')$. Then $Q_i'' \equiv P$ since $P \in \mathsf{Spos}(A)$. So there are at least two negative occurrences of P in $\Gamma \vdash P$ as in $A \equiv ((P \to Q_i') \to P)$. This is a contradiction.
   (b) Subcase where $A \not\equiv ((P \to Q_i') \to Q_i'')$. Then there are at least two negative occurrences of P in $\Gamma \vdash P$ as in A and in $(P \to Q_i') \to Q_i''$. This is a contradiction.

As a consequence, P has no other occurrences in antecedents of suc-sequents of $\Gamma \vdash P$ than in A. □

PROPOSITION. 3.9

Let $\Gamma \vdash P$ be a 2-sequent in which every propositional variable has at most one negative occurrences and $\Delta \vdash R$ be a suc-sequent of $\Gamma \vdash P$. Then $R \notin \mathsf{Neg}(\Delta)$.

Proof.

It suffices to show that there does not occur such A in $\beta\eta^{-1}$-normal proofs. Let $\Delta \vdash R$ be

$$\Gamma', Q_1, Q_2, \ldots, Q_n \vdash R$$

as is in the definition of suc-sequents (DEFINITION.3.6).

Our proof proceeds by induction on the structure of $\beta\eta^{-1}$-normal proofs of $\Delta \vdash R$.

Case 1. Let $\Delta \vdash R$ be derived as

$$[R].$$

Then $R \notin \mathsf{Neg}(\{R\})$ since $\mathsf{Neg}(\{R\}) = \emptyset$.

Case 2a. Let $\Delta \vdash R$ be derived as

$$\dfrac{[S \to R] \quad \begin{array}{c}\Sigma\\ \mathcal{D}\\ S\end{array}}{R}\ .$$

Firstly, $\Sigma \vdash S$ is a suc-sequent of $\Gamma \vdash P$ by PROPOSITION.3.7 since $\Sigma \subseteq \Delta$ and $S \in \mathsf{Neg}(\Delta)$. So, we can apply induction hypothesis to $\Sigma \vdash S$ and get $S \notin \mathsf{Neg}(\Sigma)$. Hence $(S \to R) \notin \Sigma$.

Now, positive R occurs as $S \to R$ in antecedent of suc-sequents of $\Sigma \cup \{S \to R\} \vdash R$ by LEMMA.3.8 since

$$\begin{cases} (S \to R) \in \Sigma \cup \{S \to R\} \text{ and} \\ R \in \mathsf{Spos}(S \to R). \end{cases}$$

So positive R occurs as $S \to R$ in Σ. Hence $R \notin \mathsf{Pos}(\Sigma)$ since $(S \to R) \notin \Sigma$. From this follows $R \notin \mathsf{Neg}(\Sigma)$ by PROPOSITION.3.5.

Moreover $R \notin \mathsf{Neg}(S \to R)$ by the definition of 2-sequents, because $R \not\equiv S$. Therefore $R \notin \mathsf{Neg}(\Delta)$.

Cases 2b~5 follow similarly. □

3.4 uniqueness theorem for 2-sequents

THEOREM. 3.10

Let $\mathcal{D} : \Delta \vdash P$ and $\mathcal{D}' : \Delta' \vdash P$ be $\beta\eta^{-1}$-normal proofs. If every propositional variable in $\Delta \cup \Delta' \vdash P$ has at most one negative occurrence, then $\mathcal{D} \equiv \mathcal{D}'$.

Proof.

Our proof proceeds by induction on the structure of $\beta\eta^{-1}$-normal proofs $\mathcal{D}$ and $\mathcal{D}'$. We present only several cases here but remaining cases follow similarly.

Case 1-1. Let $\mathcal{D}$ and $\mathcal{D}'$ be proofs of forms

$$[P] \qquad \text{and} \qquad [P],$$

respectively. Then $\mathcal{D} \equiv \mathcal{D}' \equiv [P]$.

Case 1-2a. Let $\mathcal{D}$ and $\mathcal{D}'$ be proofs of forms

$$[P] \qquad \text{and} \qquad \dfrac{[Q' \to P] \quad \begin{array}{c}\mathcal{D}'_1\\ Q'\end{array}}{P}\ ,$$

respectively. Since $P, Q' \to P \in \Delta \cup \Delta'$, there are at least two negative occurrences of P in $\Delta \cup \Delta'$. So, this is not the case.

Case 2a-2a. Let $\mathcal{D}$ and $\mathcal{D}'$ be proofs of forms

$$\dfrac{[Q \to P] \quad \begin{matrix}\Sigma \\ \mathcal{D}_1 \\ Q\end{matrix}}{P} \quad \text{and} \quad \dfrac{[Q' \to P] \quad \begin{matrix}\Sigma' \\ \mathcal{D}_1' \\ Q'\end{matrix}}{P},$$

respectively.

(a) Subcase where $Q \not\equiv Q'$. Since $Q \to P, Q' \to P \in \Delta \cup \Delta'$, there are at least two negative occurrences of P in $\Delta \cup \Delta'$. So, this is not the case.

(b) Subcase where $Q \equiv Q'$. By our assumption, every propositional variable in $\Sigma \cup \Sigma' \cup \{Q \to P\} \vdash P$ has at most one negative occurrence. So, every propositional variable in $\Sigma \cup \Sigma' \vdash Q$ has at most one negative occurrence since no negative occurrence is added. So $\mathcal{D}_1 \equiv \mathcal{D}_1{}'$ by induction hypothesis. Therefore $\mathcal{D} \equiv \mathcal{D}'$.

Case 3-3. Let $\mathcal{D}$ and $\mathcal{D}'$ be proofs of forms

$$\dfrac{[(R \to Q) \to P] \quad \dfrac{\begin{matrix}\Sigma \\ \mathcal{D}_1 \\ Q\end{matrix}}{R \to Q}}{P} \quad \text{and} \quad \dfrac{[(R' \to Q') \to P] \quad \dfrac{\begin{matrix}\Sigma' \\ \mathcal{D}_1' \\ Q'\end{matrix}}{R' \to Q'}}{P},$$

respectively.

(a) Subcase where $Q \not\equiv Q'$ or $R \not\equiv R'$. Since $(R \to Q) \to P, (R' \to Q') \to P \in \Delta \cup \Delta'$, there are at least two negative occurrences of P in $\Delta \cup \Delta'$. So, this is not the case.

(b) Subcase where $Q \equiv Q'$ and $R \equiv R'$. By our assumption, every propositional variable in $(\Sigma \backslash \{R^0\}) \cup (\Sigma' \backslash \{R^0\}) \cup \{(R \to Q) \to P\} \vdash P$ has at most one negative occurrence. Now $(\Sigma \backslash \{R^0\}) \cup (\Sigma' \backslash \{R^0\}) \vdash Q$ is a suc-sequent of it. From this follows $(R \to Q) \to P \notin \Sigma \backslash \{R^0\}, \Sigma' \backslash \{R^0\}$ by PROPOSITION.3.9 since $Q \in \mathsf{Neg}((R \to Q) \to P)$. So, every propositional variable in $\Sigma \cup \Sigma' \vdash Q$ has at most one negative occurrence since no negative occurrence is added. (R already had it's negative occurrence as in $(R \to Q) \to P$.) So $\mathcal{D}_1 \equiv \mathcal{D}_1{}'$ by induction hypothesis. Therefore $\mathcal{D} \equiv \mathcal{D}'$. □

4 Extension to general cases

4.1 reduction to 2-sequents

For a given $\beta\eta^{-1}$-normal proof of $\Gamma \vdash A$, we define reduction to some $\beta\eta^{-1}$-normal proof of a 2-sequent with a variable table.

CONVENTION. 4.1

At each reduction, we need to supply new propositional variables, that is, propositional variables unused in the proof to reduce. We call such variables ***dummy propositional variables***. We use letters $p, q, r, s, t, \ldots$ to denote these propositional variables.

DEFINITION. 4.2

A ***variable table*** is a set of forms $x := A$ where x is a propositional variable and A is a formula.

DEFINITION. 4.3 [succedent reduction]

Given a pair of a proof of $\Gamma \vdash A$ where A is not dummy propositional variable and a variable table θ, we reduce it to a proof of $\Gamma, A \to p \vdash p$ with θ as

$$\begin{array}{c}\mathcal{D}\\ A\end{array} \quad \mapsto \quad \dfrac{[A \to p] \quad \begin{array}{c}\mathcal{D}\\ A\end{array}}{p}$$

We call this reduction ***succedent reduction.***

DEFINITION. 4.4 [antecedent reduction rule]

Given a pair of a proof of $\beta\eta^{-1}$-normal proof of $\Gamma \vdash A$ and a variable table θ, we define ***antecedent reduction rules*** as below.

Any rule in the following is applied to proofs if they contain assumptions which are not 2-formulas. We will not mention about the variable table θ when it is not changed by its reduction rule.

1. Case for assumption of form $A \to B$ where either
   - neither A nor B is a propositional variable or
   - A and B are the same propositional variable.

$$\begin{array}{c}\dfrac{[A \to B] \quad \begin{array}{c}\mathcal{D}_1\\ A\end{array}}{B}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[p \to B] \quad \dfrac{[A \to p] \quad \begin{array}{c}\mathcal{D}_1\\ A\end{array}}{p}}{B}\\ \mathcal{D}.\end{array}$$

2. Case for assumption of form $A \to P$ where A is not a propositional variable.
   (a) Subcase for assumption of form $(C \to Q) \to P$ where either C is not a propositional variable or $C \equiv Q$ or $C \equiv P$.

$$\begin{array}{c}\dfrac{[(C \to Q) \to P] \quad \dfrac{\begin{array}{c}[C]^{(1)}\\ \mathcal{D}_1\\ Q\end{array}}{C \to Q}\,(1)}{P}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[(r \to Q) \to P] \quad \dfrac{\begin{array}{c}\dfrac{[r \to C] \quad [r]^{(1)}}{C}\\ \mathcal{D}_1\\ Q\end{array}}{r \to Q}\,(1)}{P}\\ \mathcal{D}.\end{array}$$

If the cancelled assumption $[C]^{(1)}$ does not appear in the proof, $r := C$ is added to the variable table.

(b) Subcase for assumption of form $(R \to D) \to P$ where either D is not a propositional variable or $D \equiv P$.

$$\begin{array}{c}\dfrac{[(R \to D) \to P] \quad \dfrac{\begin{array}{c}[R]^{(1)}\\ \mathcal{D}_1\\ D\end{array}}{R \to D}\,(1)}{P}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[(R \to q) \to P] \quad \dfrac{\dfrac{[D \to q] \quad \begin{array}{c}[R]^{(1)}\\ \mathcal{D}_1\\ D\end{array}}{q}}{R \to q}\,(1)}{P}\\ \mathcal{D}.\end{array}$$

(c) Subcase for assumption of form $(C \to D) \to P$ where neither C nor D is a propositional variable.

$$\begin{array}{c}\dfrac{[(C \to D) \to P] \quad \dfrac{\begin{array}{c}[C]^{(1)}\\ \mathcal{D}_1\\ D\end{array}}{C \to D}\,(1)}{P}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[(r \to q) \to P] \quad \dfrac{\dfrac{[D \to q] \quad \begin{array}{c}\dfrac{[r \to C] \quad [r]^{(1)}}{C}\\ \mathcal{D}_1\\ D\end{array}}{q}}{r \to q}\,(1)}{P}\\ \mathcal{D}.\end{array}$$

If the cancelled assumption $[C]^{(1)}$ does not appear in the proof, $r := C$ is added to the variable table.

(d) Subcase for assumption of form $(C \wedge Q) \to P$ where C is not a propositional variable or $C \equiv Q$ or $C \equiv P$.

$$\begin{array}{c}\dfrac{[(C \wedge Q) \to P] \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ Q\end{array}}{C \wedge Q}}{P}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[(r \wedge Q) \to P] \quad \dfrac{\dfrac{[C \to r] \quad \begin{array}{c}\mathcal{D}_1\\ C\end{array}}{r} \quad \begin{array}{c}\mathcal{D}_2\\ Q\end{array}}{r \wedge Q}}{P}\\ \mathcal{D}.\end{array}$$

(e) Subcase for assumption of form $(R \wedge D) \to P$ where either D is not a propositional variable or $D \equiv P$.

$$\begin{array}{c}\dfrac{[(R \wedge D) \to P] \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ R\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ D\end{array}}{R \wedge D}}{P}\\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{[(R \wedge q) \to P] \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ R\end{array} \quad \dfrac{[D \to q] \quad \begin{array}{c}\mathcal{D}_2\\ D\end{array}}{q}}{R \wedge q}}{P}\\ \mathcal{D}.\end{array}$$

(f) Subcase for assumption of form $(C \wedge D) \to P$ where neither C nor D is a propositional variable.

$$\dfrac{[(C \wedge D) \to P] \qquad \dfrac{\overset{\mathcal{D}_1}{C} \quad \overset{\mathcal{D}_2}{D}}{C \wedge D}}{\begin{array}{c} P \\ \mathcal{D} \end{array}} \qquad \rhd \qquad \dfrac{[(r \wedge q) \to P] \qquad \dfrac{\dfrac{[C \to r] \quad \overset{\mathcal{D}_1}{C}}{r} \quad \dfrac{[D \to q] \quad \overset{\mathcal{D}_2}{D}}{q}}{r \wedge q}}{\begin{array}{c} P \\ \mathcal{D}. \end{array}}$$

3. Case for assumption of form $P \to B$ where B is not a propositional variable.

(a) Subcase for assumption of form $P \to (C \to R)$ where either C is not a propositional variable or $C \equiv R$ or $C \equiv P$.

$$\dfrac{\dfrac{[P \to (C \to R)] \quad \overset{\mathcal{D}_1}{P}}{C \to R} \qquad \overset{\mathcal{D}_2}{C}}{\begin{array}{c} R \\ \mathcal{D} \end{array}} \qquad \rhd \qquad \dfrac{\dfrac{[P \to (q \to R)] \quad \overset{\mathcal{D}_1}{P}}{q \to R} \qquad \dfrac{[C \to q] \quad \overset{\mathcal{D}_2}{C}}{q}}{\begin{array}{c} R \\ \mathcal{D}. \end{array}}$$

(b) Subcase for assumption of form $P \to (Q \to D)$ where either D is not a propositional variable or $D \equiv P$.

$$\dfrac{\dfrac{[P \to (Q \to D)] \quad \overset{\mathcal{D}_1}{P}}{Q \to D} \qquad \overset{\mathcal{D}_2}{Q}}{\begin{array}{c} D \\ \mathcal{D} \end{array}} \qquad \rhd \qquad \dfrac{[r \to D] \qquad \dfrac{\dfrac{[P \to (Q \to r)] \quad \overset{\mathcal{D}_1}{P}}{Q \to r} \qquad \overset{\mathcal{D}_2}{Q}}{r}}{\begin{array}{c} D \\ \mathcal{D}. \end{array}}$$

(c) Subcase for assumption of form $P \to (C \to D)$ where neither C nor D is a propositional variable.

$$\dfrac{\dfrac{[P \to (C \to D)] \quad \overset{\mathcal{D}_1}{P}}{C \to D} \qquad \overset{\mathcal{D}_2}{C}}{\begin{array}{c} D \\ \mathcal{D} \end{array}} \qquad \rhd \qquad \dfrac{[r \to D] \qquad \dfrac{\dfrac{[P \to (q \to r)] \quad \overset{\mathcal{D}_1}{P}}{q \to r} \qquad \dfrac{[C \to q] \quad \overset{\mathcal{D}_2}{C}}{q}}{r}}{\begin{array}{c} D \\ \mathcal{D}. \end{array}}$$

(d) Subcase for assumption of form $P \to (C \wedge R)$ where C is not a propositional variable or $C \equiv R$ or $C \equiv P$.

$$\dfrac{\dfrac{[P \to (C \wedge R)] \quad \overset{\mathcal{D}_1}{P}}{C \wedge R}}{\begin{array}{c} C \\ \mathcal{D} \end{array}} \qquad \rhd \qquad \dfrac{[q \to C] \qquad \dfrac{\dfrac{[P \to (q \wedge R)] \quad \overset{\mathcal{D}_1}{P}}{q \wedge R}}{q}}{\begin{array}{c} C \\ \mathcal{D}. \end{array}}$$

$$\begin{array}{c} \dfrac{\dfrac{[P\to(C\wedge R)] \quad \overset{\mathcal{D}_1}{P}}{C\wedge R}}{R} \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\dfrac{[P\to(q\wedge R)] \quad \overset{\mathcal{D}_1}{P}}{q\wedge R}}{R} \\ \mathcal{D},\end{array}$$

and $q := C$ is added to the variable table.

(e) Subcase for assumption of form $P \to (Q \wedge D)$ where either D is not a propositional variable or $D \equiv P$.

$$\begin{array}{c} \dfrac{\dfrac{[P\to(Q\wedge D)] \quad \overset{\mathcal{D}_1}{P}}{Q\wedge D}}{D} \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{[r\to D] \qquad \dfrac{\dfrac{[P\to(Q\wedge r)] \quad \overset{\mathcal{D}_1}{P}}{Q\wedge r}}{r}}{D} \\ \mathcal{D}.\end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{[P\to(Q\wedge D)] \quad \overset{\mathcal{D}_1}{P}}{Q\wedge D}}{Q} \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\dfrac{[P\to(Q\wedge r)] \quad \overset{\mathcal{D}_1}{P}}{Q\wedge r}}{Q} \\ \mathcal{D},\end{array}$$

and $r := D$ is added to the variable table.

(f) Subcase for assumption of form $P \to (C \wedge D)$ where neither C nor D is a propositional variable.

$$\begin{array}{c} \dfrac{\dfrac{[P\to(C\wedge D)] \quad \overset{\mathcal{D}_1}{P}}{C\wedge D}}{C} \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{[q\to C] \qquad \dfrac{\dfrac{[P\to(q\wedge r)] \quad \overset{\mathcal{D}_1}{P}}{q\wedge r}}{q}}{C} \\ \mathcal{D},\end{array}$$

and $r := D$ is added to the variable table.

$$\begin{array}{c} \dfrac{\dfrac{[P\to(C\wedge D)] \quad \overset{\mathcal{D}_1}{P}}{C\wedge D}}{D} \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{[r\to D] \qquad \dfrac{\dfrac{[P\to(q\wedge r)] \quad \overset{\mathcal{D}_1}{P}}{q\wedge r}}{q}}{D} \\ \mathcal{D},\end{array}$$

and $q := C$ is added to the variable table.

4. Case for assumption of form $A \wedge B$.

$$\begin{array}{c} [A\wedge B] \\ \mathcal{D}\end{array} \quad \rhd \quad \begin{array}{c} \dfrac{[p\to A\wedge B] \quad [p]}{A\wedge B} \\ \mathcal{D}.\end{array}$$

An ***antecedent reduction*** consists of parallel applications of a reduction rule to each occurrence of the assumption assuming that dummy propositional variables are newly selected in each reduction. Note that at reductions which are applications of 2(a) or 2(c), $r := C$ is added to the variable table, only when the cancelled assumption $[C]$ does not appear in any occurrence of the redex.

Remark.

By "parallel applications", we mean that given an assumption that is not a 2-formula we rewrite all occurrences of the assumption (and subproofs around it) simultaneously by a reduction rule that is specified by the form of the assumption. This is possible since there is only one open assumption in the left side of each reduction rule and for any assumption there is at most one applicable reduction rule according to the form of the assumption. This emphasis is because of our assumption that dummy propositional variables are newly selected in each reduction. So even if there are many occurrences of the same assumption in a proof, a reduction produces at most three new assumptions. In each reduction, a new propositional variable is added to the proof. We would give an example of the antecedent reduction to supplement these explanations.

A proof of $R, S \to T, (R \land (S \to T)) \to P, P \to R \vdash P$ —

$$\dfrac{[(R\land(S\to T))\to P] \qquad \dfrac{\dfrac{[P\to R] \qquad \dfrac{[(R\land(S\to T))\to P] \qquad \dfrac{[R] \qquad \dfrac{\dfrac{[S\to T][S]^{(1)}}{T}}{S\to T}\,(1)}{R\land(S\to T)}}{P}}{R} \qquad \dfrac{\dfrac{[S\to T][S]^{(2)}}{T}}{S\to T}\,(2)}{R\land(S\to T)}}{P}$$

reduces to a proof of $R, S \to T, (S \to T) \to Q, (R \land Q) \to P, P \to R \vdash P$ —

$$\dfrac{[(R\land Q)\to P] \qquad \dfrac{\dfrac{[P\to R] \qquad \dfrac{[(R\land Q)\to P] \qquad \dfrac{[R] \qquad \overset{\mathcal{D}_1}{Q}}{R\land Q}}{P}}{R} \qquad \overset{\mathcal{D}_2}{Q}}{R\land Q}}{P}$$

where $\mathcal{D}_1$ is

$$\dfrac{[(S\to T)\to Q] \qquad \dfrac{\dfrac{[S\to T][S]^{(1)}}{T}}{S\to T}\,(1)}{Q}$$

and $\mathcal{D}_2$ is

$$\dfrac{[(S \to T) \to Q] \qquad \dfrac{\dfrac{[S \to T][S]^{(2)}}{T}}{S \to T}\,(2)}{Q}\,.$$

Note that the same dummy propositional variable Q is used in both $\mathcal{D}_1$ and $\mathcal{D}_2$. This is because we apply a reduction rule $2(e)$ to two occurrences of redex around the same assumption $[(R \wedge (S \to T)) \to P]$ at the same time.

DEFINITION. 4.5
Antecedent reductions and succedent reduction are called ***transformations***. We say $\mathcal{D} : \Gamma \vdash A$ transforms to $\langle \mathcal{E} : \Delta \vdash P,\ \theta \rangle$ if $\langle \mathcal{D} : \Gamma \vdash A,\ \emptyset \rangle$ transforms to $\langle \mathcal{E} : \Delta \vdash P,\ \theta \rangle$.

PROPOSITION. 4.6
Given a $\beta\eta^{-1}$-normal proof $\mathcal{D}$, there is a transformation sequence that terminates, and the final reduct is a pair of a $\beta\eta^{-1}$-normal proof of a 2-sequent and a variable table θ. Moreover, the final reduct is unique up to renaming of dummy propositional variables regardless of reduction sequences.

Proof.
Succedent reduction can be applied only once. All antecedent reduction except applications of Case 4 reduce the length of the formulas. No reductions produce assumptions of form $A \wedge B$. So strong normalizability follows. Church-Rosser property follows from the fact that redex are independent, so the final reduct is unique up to renaming of dummy propositional variables regardless of reduction sequences. $\beta\eta^{-1}$-normality follows from the definition of transformations. The reduct is a proof of 2-sequent also by the definition of transformations. □

4.2 general uniqueness theorem

DEFINITION. 4.7 [decomposition]
Given a formulas A, we define a ***decomposition*** of A as follows:

1. $\mathsf{decomp}(A \to B) \overset{\text{def}}{\equiv} \mathsf{decomp}(A) \cup \mathsf{decomp}(B)$ where either
   - neither A nor B is a propositional variable or
   - A and B are the same propositional variable;
2. $\mathsf{decomp}(A \to P)$ where A is not a propositional variable;
   (a) $\mathsf{decomp}((R \to Q) \to P) \overset{\text{def}}{\equiv} \{(R \to Q) \to P\}$ where R, Q and P are mutually distinct;
   (b) $\mathsf{decomp}((C \to Q) \to P) \overset{\text{def}}{\equiv} \{(r \to Q) \to P\} \cup \mathsf{decomp}(r \to C)$ where C is not a propositional variable or $C \equiv Q$ or $C \equiv P$;
   (c) $\mathsf{decomp}((R \to D) \to P) \overset{\text{def}}{\equiv} \{(R \to q) \to P\} \cup \mathsf{decomp}(D \to q)$ where D is not a propositional variable or $D \equiv P$;

(d) $\mathsf{decomp}((C \to D) \to P) \stackrel{\text{def}}{\equiv} \{(r \to q) \to P\} \cup \mathsf{decomp}(r \to C) \cup \mathsf{decomp}(D \to q)$ where neither C nor D is not a propositional variable;

(e) $\mathsf{decomp}((R \land Q) \to P) \stackrel{\text{def}}{\equiv} \{(R \land Q) \to P\}$ where R, Q and P are mutually distinct;

(f) $\mathsf{decomp}((C \land Q) \to P) \stackrel{\text{def}}{\equiv} \{(r \land Q) \to P\} \cup \mathsf{decomp}(C \to r)$ where C is not a propositional variable or $C \equiv Q$ or $C \equiv P$;

(g) $\mathsf{decomp}((R \land D) \to P) \stackrel{\text{def}}{\equiv} \{(R \land q) \to P\} \cup \mathsf{decomp}(D \to q)$ where D is not a propositional variable or $D \equiv P$;

(h) $\mathsf{decomp}((C \land D) \to P) \stackrel{\text{def}}{\equiv} \{(r \land q) \to P\} \cup \mathsf{decomp}(r \to C) \cup \mathsf{decomp}(D \to q)$ where neither C nor D is a propositional variable;

3. $\mathsf{decomp}(P \to B)$ where B is not a propositional variable;

(a) $\mathsf{decomp}(P \to (Q \to R)) \stackrel{\text{def}}{\equiv} \{P \to (Q \to R)\}$ where P, Q and R are mutually distinct;

(b) $\mathsf{decomp}(P \to (C \to R)) \stackrel{\text{def}}{\equiv} \{P \to (q \to R)\} \cup \mathsf{decomp}(C \to q)$ where C is not a propositional variable or $C \equiv P$ or $C \equiv R$;

(c) $\mathsf{decomp}(P \to (Q \to D)) \stackrel{\text{def}}{\equiv} \{P \to (Q \to r)\} \cup \mathsf{decomp}(r \to D)$ where D is not a propositional variable or $D \equiv P$;

(d) $\mathsf{decomp}(P \to (C \to D)) \stackrel{\text{def}}{\equiv} \{P \to (q \to r)\} \cup \mathsf{decomp}(C \to q) \cup \mathsf{decomp}(r \to D)$ where neither C nor D is a propositional variable;

(e) $\mathsf{decomp}(P \to (Q \land R)) \stackrel{\text{def}}{\equiv} \{P \to (Q \land R)\}$ where P, Q and R are mutually distinct;

(f) $\mathsf{decomp}(P \to (C \land R)) \stackrel{\text{def}}{\equiv} \{P \to (q \land R)\} \cup \mathsf{decomp}(q \to C)$ where C is not a propositional variable or $C \equiv P$ or $C \equiv R$;

(g) $\mathsf{decomp}(P \to (Q \land D)) \stackrel{\text{def}}{\equiv} \{P \to (Q \land r)\} \cup \mathsf{decomp}(r \to D)$ where D is not a propositional variable or $D \equiv P$;

(h) $\mathsf{decomp}(P \to (C \land D)) \stackrel{\text{def}}{\equiv} \{P \to (q \land r)\} \cup \mathsf{decomp}(C \to q) \cup \mathsf{decomp}(r \to D)$ where neither C nor D is a propositional variable;

4. $\mathsf{decomp}(A \land B) \stackrel{\text{def}}{\equiv} \mathsf{decomp}(p \to (A \land B)) \cup \{p\}$.

Here p, q, r are dummy propositional variables that follows the same convention in the case of definition of transformations. We assume that dummy propositional variables are newly selected in each decomposition step.

PROPOSITION. 4.8

Assume that $\mathcal{D} : \Gamma \vdash A$ reduces to $\langle\ \mathcal{E} : \Delta \vdash P,\ \theta\rangle$, where $\Delta \vdash P$ is 2-sequent and θ is a variable table. Then by suitable renaming of dummy propositional variables, $\Delta \subseteq \mathsf{decomp}(A \to P) \cup \mathsf{decomp}(\Gamma)$.

Proof.
This is immediate by the definition of transformations and decomposition. □

PROPOSITION. 4.9
Assume that $\mathcal{D}, \mathcal{D}' : \Gamma \vdash A$ transforms to $\langle \mathcal{E} : \Delta \vdash P,\ \theta \rangle$, $\langle \mathcal{E}' : \Delta' \vdash P,\ \theta' \rangle$ where $\Delta \vdash P, \Delta' \vdash P$ are 2-sequents and θ, θ' are variable tables. If $\mathcal{D} \not\equiv \mathcal{D}'$, then $\mathcal{E} \not\equiv \mathcal{E}'$ or $\theta \not\equiv \theta'$ for any renaming of dummy propositional variables.

Proof.
This is immediate by the definition of transformations. □

PROPOSITION. 4.10
Assume that $\mathcal{D} : \Gamma \vdash A$ transforms to $\langle \mathcal{E} : \Delta \vdash P,\ \theta \rangle$, where $\Delta \vdash P$ is a 2-sequent and θ is a variable table.

1. If every propositional variable in $\Gamma \vdash A$ has at most one positive occurrence, then so is in $\Delta \vdash P$.
2. If every propositional variable in $\Gamma \vdash A$ has at most one negative occurrence, then so is in $\Delta \vdash P$.

Proof.
This is immediate by the definition of transformations. □

THEOREM. 4.11 [general uniqueness theorem]
Let $\mathcal{D}, \mathcal{D}'$ be a $\beta\eta^{-1}$-normal proof of $\Gamma \vdash A$. If every propositional variable in $\Gamma \vdash A$ has at most one negative occurrence, then $\mathcal{D} \equiv \mathcal{D}'$.

Proof.
Assume that they transform to $\langle \mathcal{E} : \Delta \vdash P,\ \theta \rangle$ and $\langle \mathcal{E}' : \Delta' \vdash P,\ \theta' \rangle$ respectively. Given an assumption $A \in \Gamma$, it's decomposition by transformations is defined unique up to renaming of dummy variables. So, only difference between Δ and Δ' is that decomposition of formulas that appear in the right of $:=$ in $\theta \backslash \theta'$ appear in Δ' and decomposition of formulas that appear in the right of $:=$ in $\theta' \backslash \theta$ appear in Δ. By PROPOSITION.4.10, every propositional variable in $\Delta \vdash P$ and $\Delta' \vdash P$ has at most one negative occurrence. So, by suitable renaming of dummy variables, every propositional variable in $\Delta \cup \Delta' \vdash P$ become to have at most one negative occurrence. So, $\mathcal{E} \equiv \mathcal{E}'$ by THEOREM.3.10, and $\Delta \equiv \Delta'$ for such renaming. From this also follows $\theta \equiv \theta'$ for such renaming. Therefore $\mathcal{D} \equiv \mathcal{D}'$ by PROPOSITION.4.9. □

COROLLARY. 4.12
Let $\mathcal{D}, \mathcal{D}'$ be a $\beta\eta$-normal proof of $\Gamma \vdash A$. If every propositional variable in $\Gamma \vdash A$ has at most one negative occurrence, then $\mathcal{D} \equiv \mathcal{D}'$.

Remark.
This is because η^{-1}-reduction can change proofs only locally and η-normal proof and η^{-1}-normal proof is in one-one correspondence.

5 Conclusion

We have proved that for any provable formulas in which every propositional variable has at most one negative occurrence has unique $\beta\eta$-normal proofs. This extends the result on balanced formulas by Mints [6].

The following are examples of unique $\beta\eta$-normal proofs of formulas in which every propositional variable has at most one negative occurrence, but which are not balanced.

Examples.

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{[((P \to R) \to Q) \to S]^{(3)} \qquad \dfrac{\dfrac{[P \to Q]^{(2)} \quad [P]^{(1)}}{Q}}{(P \to R) \to Q}}{S}}{P \to S}\,(1)}{(P \to Q) \to (P \to S)}\,(2)}{(((P \to R) \to Q) \to S) \to (P \to Q) \to (P \to S)}\,(3)$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P \to Q \to R]^{(3)} \quad [P]^{(1)}}{Q \to R} \qquad \dfrac{[P \to Q]^{(2)} \quad [P]^{(1)}}{Q}}{R}}{P \to R}\,(1)}{(P \to Q) \to (P \to R)}\,(2)}{(P \to Q \to R) \to (P \to Q) \to (P \to R)}\,(3)$$

Acknowledgements

The first author would like to express his thanks to Prof. Yoshihito Toyama whose question gave him a push to this direction of the research. He also thanks Keisuke Tanaka for encouragements on early period of this topic of the research. The authors would like to thank Prof. Hajime Ishihara and Hiroyuki Shirasu for their careful comments and their continual encouragements during the research.

References

[1] Babaev, A. A. and Solovjev, S. V., A coherence thorem for canonical morphism in cartesian closed categories. *Journal of Soviet Mathematics*, Vol. 20, pp. 2263–2279, 1982. Translated from: Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V.A.Steklova An SSSR, Vol. 88, pp. 3–29, 1979.

[2] Hirokawa, S., Balanced formulas, minimal formulas and their proofs. Technical Report RIFIS-TR-CS-49, Kyushu University, Research Institute of Fundamental Information Science, 1991.

[3] Hirokawa, S., Principal Types of BCK-lambda-terms. *Theoretical Computer Science*, Vol. 107, pp. 253–276, 1993.

[4] Mints, G. E., A coherence theorem for cartesian closed categories (abstract). *Journal of Symbolic Logic*, Vol. 44, No. 3, pp. 453–454, 1979.

[5] Mints, G. E., Simple proof of the coherence theorem for cartesian closed categories (abstract). *Journal of Symbolic Logic*, Vol. 49, No. 2, p. 701, 1984.

[6] Mints, G. E., A simple proof of the coherence thoerem for cartesian closed categories. In Mints, G. E. (ed.), *Selected Papers in Proof Theory*, pp. 213–220. Bibliopolis, Napoli, 1992.

[7] Ono, H., Structural rules and a logical hierarchy. In Petkov, P. P. (ed.), *Mathmatical Logic*, pp. 95–104. Plenum Press, New York and London, 1990. Heyting'88 Summer School and Conference on Mathematical Logic, September 13-23, 1988 in Chaika, Bulgaria.