Title	Hilbert style proof system for BPL
Author(s)	Suzuki, Yasuhito; Ono, Hiroakira
Citation	Research report (School of Information Science, Japan Advanced Institute of Science and Technology), IS-RR-97-0040F: 1-8
Issue Date	1997-10-07
Type	Technical Report
Text version	publisher
URL	http://hdl.handle.net/10119/8377
Rights	
Description	リサーチレポート（北陸先端科学技術大学院大学情報科学研究科）

Hilbert style proof system for BPL

Yasuhito Suzuki and Hiroakira Ono
October 7, 1997
IS-RR-97-0040F

School of Inforamatio Science,

JAIST,

Tatsunokuchi, Ishikawa 923–12,

Japan.

x-suzuki@jaist.ac.jp, ono@jaist.ac.jp

ISSN 0918-7553

Hilbert style proof system for BPL.

Yasuhito Suzuki and Hiroakira Ono
School of Information Science,
JAIST,
Tatsunokuchi, Ishikawa 923-12,
Japan
(x-suzuki@jaist.ac.jp, ono@jaist.ac.jp)

October 7, 1997

Abstract

The contents of this report are based on [2]. In [3], we mentioned properties of Visser's Basic Propositional Logic(BPL) and it's extentions in the semantical point of view, and we remained giving BPL's Hilbert type proof system as an open problem. Here, we will introduce a Hilbert style proof system for BPL. This is an extension of Corsi's system in [1].

1 Hilbert proof system for BPL

We will introduce the following axiom scheme and inference rule as the proof system for BPL.

1. $A \to A$
2. $(A \to B) \wedge (B \to C) \to (A \to C)$
3. $A \wedge B \to A$
4. $A \wedge B \to B$
5. $(C \to A) \wedge (C \to B) \to (C \to A \wedge B)$
6. $A \to A \vee B$
7. $B \to A \vee B$
8. $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
9. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

10. $\bot \rightarrow A$

11. $A \rightarrow (B \rightarrow A)$

12. $A \rightarrow (B \rightarrow (A \wedge B))$

$$\text{MP}\ \frac{A \quad A \rightarrow B}{B}$$

We denote that formula A is derivable in a proof system S as $\vdash_{\text{S}} A$.

Axioms, from axiom scheme 1 to axiom scheme 10, are included axiom schemes of Corsi's proof system F in [1]. Our system has just only one inference rule MP, while F has two inference rules MP and AF(A Fortiori). And MP in [1] differs from ours. To distinguish which MP we mention in contexts, we denote that MP in Corsi's system as MP_c. Inference rule MP, MP_c and AF are as follows;

- MP : $\vdash_{\text{S}} A$ and $\vdash_{\text{S}} A \rightarrow B$ imply $\vdash_{\text{S}} B$.
- MP_c: $\vdash_{\text{S}} A_1$, ... ,$\vdash_{\text{S}} A_n$ and $\vdash_{\text{S}} A_1 \wedge \ldots \wedge A_n \rightarrow B$ imply $\vdash_{\text{S}} B$.
- AF: $\vdash_{\text{S}} B$ implies $\vdash_{\text{S}} A \rightarrow B$.

Then,

Theorem 1 *MP_c and AF are derivable in BPL.*

Proof In BPL, the adjunction rule ($\vdash_{\text{BPL}} A_1$ and $\vdash_{\text{BPL}} A_2$ imply $\vdash_{\text{BPL}} A_1 \wedge A_2$) is also derivable via axiom scheme 12. Then, $\vdash_{\text{BPL}} A_1$, $\vdash_{\text{BPL}} A_2$ and $\vdash_{\text{BPL}} A_1 \wedge A_2 \rightarrow B$ imply $\vdash_{\text{BPL}} B$ by MP. For AF, AF is obivously derivable via the following proof figure (by using the assumption $\vdash_{\text{BPL}} B$ and axiom scheme 11);

$$\frac{B \quad B \rightarrow (A \rightarrow B)}{A \rightarrow B} : \text{MP}$$

□

Corollary 2 *Suppose* F *is Corsi's proof system in [1]. For any formula A, $\vdash_{\text{F}} A$ implies $\vdash_{\text{BPL}} A$.*

Using above results, we can say that BPL is an extension of F. That is, properties of F also hold on BPL. In particluar, the following results in [1] are useful to prove completeness.

Lemma 3 *The following results are hold:*

1. $\vdash_{\text{BPL}} (A \rightarrow B) \wedge ((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow C)$,
2. $\vdash_{\text{BPL}} (A \wedge B) \rightarrow C$ *and* $\vdash_{\text{BPL}} B$ *imply* $\vdash_{\text{BPL}} A \rightarrow C$,
3. $\vdash_{\text{BPL}} A \rightarrow B$ *implies* $\vdash_{\text{BPL}} (B \rightarrow C) \rightarrow (A \rightarrow C)$,

4. $\vdash_{\text{BPL}} A \to B$ *implies* $\vdash_{\text{BPL}} (C \to A) \to (C \to B)$,

5. $\vdash_{\text{BPL}} (A \to B) \wedge (C \to D) \to (A \wedge C \to B \wedge D)$,

6. $\vdash_{\text{BPL}} (A \to B) \wedge (C \to D) \to (A \vee C \to B \vee D)$,

7. $\vdash_{\text{BPL}} (A \wedge C \to B \vee D) \wedge (E \to C) \wedge (F \to D) \to (A \wedge E \to B \vee F)$,

8. $\vdash_{\text{BPL}} A \wedge B \to C$ *implies* $\vdash_{\text{BPL}} (D \to A) \to ((D \wedge B) \to C)$.

2 Soundness for BPL

Let suppose $\mathcal{L}$ is a set of propositional formulas, and Var$\mathcal{L}$ is the set of propositional variables of $\mathcal{L}$.

We think (transitive) frame is a structure $\langle W, R, P \rangle$, where W is a non-empty set, R is a (transitive) relation on W and P is a subset of the power set of W such that closed under set intersection, set union and the following operation $\to$;

$$X \to Y = \{x \in W : \forall y(xRy \wedge y \in X \Rightarrow y \in Y)\} \tag{1}$$

for all $X, Y \in P$. We denote this frame structure as $\mathfrak{F}$, and a class of frames as $\mathfrak{C}$.

Let suppose frame $\mathfrak{F}(= \langle W, R, P \rangle)$ is given, and suppose $\mathfrak{V}$ is an upward (closed) valuation from Var$\mathcal{L}$ to P. $\mathfrak{V}$ is upward closed means that $x \in \mathfrak{V}(p)$ and xRy imply $y \in \mathfrak{V}(p)$. We call a structure $\langle \mathfrak{F}, \mathfrak{V} \rangle$ (or $\langle W, R, P, \mathfrak{V} \rangle$) a model which is based on $\mathfrak{F}$, and denote it as $\mathfrak{M}$.

Satisfiability relation $\models$ is defined as follows;

$$\begin{array}{rll}
\mathfrak{M}, x \not\models \bot & & \\
\mathfrak{M}, x \models p & \text{iff} & x \in \mathfrak{V}(p) \text{ for all } p \in \text{Var}\mathcal{L}, \\
\mathfrak{M}, x \models A \wedge B & \text{iff} & \mathfrak{M}, x \models A \text{ and } \mathfrak{M}, x \models B, \\
\mathfrak{M}, x \models A \to B & \text{iff} & \forall y \in W.(xRy \text{ and } \mathfrak{M}, y \models A \text{ imply } \mathfrak{M}, y \models B), \\
\mathfrak{M} \models A & \text{iff} & \forall x \in W.(\mathfrak{M}, x \models A), \\
\mathfrak{F} \models A & \text{iff} & \forall \mathfrak{V}.(\langle \mathfrak{F}, \mathfrak{V} \rangle \models A), \\
\mathfrak{C} \models A & \text{iff} & \forall \mathfrak{F} \in \mathfrak{C}.(\mathfrak{F} \models A).
\end{array}$$

By usual way, we can get a extended valuation of $\mathfrak{V}$ by extending its domain from Var$\mathcal{L}$ to $\mathcal{L}$. To denote this extension, we will use same symbol to the given valuation. We can show, under this extended valuation $\mathfrak{V}$,

$$x \in \mathfrak{V}(A) \text{ iff } \mathfrak{M}, x \models A.$$

Corsi's system is a proof system corresponds to any frame structures with any relations and any valuations. But, BPL and the all extended system of BPL which are mentioned in [3] correspond to frame structures with transitive relations and its valuation is required having upward closed property.

We will show the soundness theorem between BPL and the class of all transitive frames with upward closed valuations.

Lemma 4 *Let suppose $\mathfrak{C}$ is the class of transitive frames with upward closed valuations. For any axiom scheme A of BPL, $\mathfrak{C} \models A$ holds.*

Proof We will just show the cases of axiom scheme 11 and 12. As for the other cases, from axiom scheme 1 to axiom scheme 10, proofs are similar to [1].

At first, we will show the case of axiom scheme 11. Suppose $\mathfrak{C}$ is the class of any frames with upward closed valuations, and $\mathfrak{C} \not\models A \to (B \to A)$. Then,

$$\mathfrak{C} \not\models A \to (B \to A) \quad \text{iff} \quad \exists \mathfrak{F} \in \mathfrak{C}.(\mathfrak{F} \not\models A \to (B \to A)).$$

That is, there exists a valuation $\mathfrak{V}$, an element $x \in W$ and

$$\langle \mathfrak{F}, \mathfrak{V} \rangle, x \not\models A \to (B \to A).$$

By the definition,

$$\begin{aligned} \langle \mathfrak{F}, \mathfrak{V} \rangle, x \not\models A \to (B \to A) \quad &\text{iff} \quad \exists y.(xRy \text{ and } \langle \mathfrak{F}, \mathfrak{V} \rangle, y \models A \text{ and} \\ &\qquad \langle \mathfrak{F}, \mathfrak{V} \rangle, y \not\models B \to A), \\ \langle \mathfrak{F}, \mathfrak{V} \rangle, y \not\models B \to A \quad &\text{iff} \quad \exists z.(yRz \text{ and } \langle \mathfrak{F}, \mathfrak{V} \rangle, z \models B \text{ and} \\ &\qquad \langle \mathfrak{F}, \mathfrak{V} \rangle, z \not\models A). \end{aligned}$$

Then, $\mathfrak{M}, y \models A$ and yRz deduces $\mathfrak{M}, z \models A$ since $\mathfrak{V}$ is an upward closed valuation. But this is contradict to $\mathfrak{M}, z \not\models A$.

Next, we will show about the axiom scheme 12. Suppose $\mathfrak{C} \not\models A \to (B \to (A \wedge B))$. That is, for some frame $\mathfrak{F} \in \mathfrak{C}$ and valuation $\mathfrak{V}$,

$$\langle \mathfrak{F}, \mathfrak{V} \rangle, x \not\models A \to (B \to (A \wedge B)).$$

Using R is transitive and $\mathfrak{V}$ is upward closed, we can easily show that xRy and $\mathfrak{M}, x \models A$ imply $\mathfrak{M}, y \models B$. Then, from these facts, $\langle \mathfrak{F}, \mathfrak{V} \rangle, x \not\models A \to (B \to (A \wedge B))$ derives contradiction. ❑

At last of this section, we will show the soundness theorem holds on BPL.

Theorem 5 *Let suppose $\mathfrak{C}$ is the class of transitive frames with upward closed valuations. Then,$\vdash_{\mathrm{BPL}} A$ implies $\mathfrak{C} \models A$.*

Proof Prove by the mathematical induction on the length of a derivation. The basis step had been already proved in the lemma 4.

Here we will show the induction step. That is, suppose MP does not hold. This is same to

$$\mathfrak{C} \models A, \mathfrak{C} \models A \to B, \mathfrak{C} \not\models B.$$

$\mathfrak{C} \not\models B$ means that there exists a models $\mathfrak{M}$ based on $\mathfrak{F} \in \mathfrak{C}$ such that $\mathfrak{M}, x \not\models B$, $\mathfrak{M}, x \models A$ and $\mathfrak{M}, x \models A \to B$, where $\mathfrak{M} = \langle W, R, P, \mathfrak{V} \rangle$. Define W_0, R_0 and P_0 as $W \cup \{x_0\}$, the transitive closure of $R \cup \{(x_0, x)\}$ and $P \cup \{W_0\}$, respectively. Then, we get a new frame $\mathfrak{F}_0$ as $\langle W_0, R_0, P_0 \rangle$. Putting $\mathfrak{V}_0(p) = \mathfrak{V}(p)$, we can get a upward closed valuation $\mathfrak{V}_0$ such that $\langle \mathfrak{F}_0, \mathfrak{V}_0 \rangle, x \not\models B$, $\langle \mathfrak{F}_0, \mathfrak{V}_0 \rangle, x \models A$, and $\langle \mathfrak{F}_0, \mathfrak{V}_0 \rangle, x_0 \models A \to B$, since obivously $\mathfrak{F}_0$ is an element of $\mathfrak{C}$. But, by the definition of $\models$, we can get $\langle \mathfrak{F}_0, \mathfrak{V}_0 \rangle, x \models B$. This is contradicion. ❑

3 Completeness for BPL

Using the same way which is mentioned in [1], we will show the completeness theorem for BPL. So, here, to prove the completeness thereom, we will copy almost all of notions and propositions from [1].

Let suppose Λ and Γ are subset of $\mathcal{L}$, and Σ is a non-empty subset of $\mathcal{L}$.

Definition 6 *Γ is Σ-consistent, iff, for any finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of Γ and finite subset $\{\sigma_1, \ldots, \sigma_m\}$ of Σ,*

$$\nvdash_{\mathrm{BPL}} \gamma_1 \wedge \ldots \wedge \gamma_n \to \sigma_1 \vee \ldots \vee \sigma_m.$$

Definition 7 *Γ is Σ-maximal, iff, for any formula α, which is not an element of Γ, there exists a finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of Γ and a finite subset $\{\sigma_1, \ldots, \sigma_m\}$ such that*

$$\vdash_{\mathrm{BPL}} \gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \to \sigma_1 \vee \ldots \vee \sigma_m.$$

Remark that, in general, Σ-maximal set Γ is not Σ-consistent. For instance, suppose the case Γ is $\{\bot\}$.

Lemma 8 *Let suppose Σ-consistent Λ is given. Then, there exists a set Γ such that*

1. *$\Lambda \subseteq \Gamma$,*
2. *Γ is Σ-consistent,*
3. *Γ is Σ-maximal.*

Proof Show [1]. ❑

In the following lemmas, we treat Γ as Σ-consistent and Σ-maximal set. We need lemma 3 to prove the following lemmas. Concreate proofs, show [1].

Lemma 9 *$\vdash_{\mathrm{BPL}} \alpha$ implies $\alpha \in \Gamma$.*

Lemma 10 *$\alpha \in \Gamma$ and $\vdash_{\mathrm{BPL}} \alpha \to \beta$ imply $\beta \in \Gamma$.*

Lemma 11 *($\alpha \in \Gamma$ and $\beta \in \Gamma$) iff $\alpha \wedge \beta \in \Gamma$.*

Lemma 12 *($\alpha \in \Gamma$ or $\beta \in \Gamma$) iff $\alpha \vee \beta \in \Gamma$.*

Lemma 13 *$\alpha \to \beta \in \Gamma$ and $\beta \to \delta \in \Gamma$ imply $\alpha \to \delta \in \Gamma$.*

Lemma 14 *$\alpha \wedge \beta \to \delta \in \Gamma$ and $\vdash_{\mathrm{BPL}} \beta$ imply $\alpha \to \delta \in \Gamma$.*

Lemma 15 *$\vdash_{\mathrm{BPL}} \alpha \wedge \eta \to \beta \vee \gamma$, $\delta \to \eta \in \Gamma$ and $\gamma \to \theta \in \Gamma$ imply $\alpha \wedge \delta \to \beta \vee \delta \in \Gamma$.*

Lemma 16 *$\alpha \to \delta \in \Gamma$ and $\alpha \wedge \delta \to \beta \in \Gamma$ imply $\alpha \to \beta \in \Gamma$.*

Lemma 17 *Let suppose Λ is Σ-consistent and Σ-maximal set such that $\alpha \rightarrow \beta \notin \Lambda$, and Σ' is the set $\{\varphi : \varphi \rightarrow \beta \in \Lambda\}$. Then, $\{\alpha\}$ is Σ'-consistent.*

Using the lemmas from 8 to 17, we can get the following results.

Lemma 18 *Let Λ be Σ-consistent and Σ-maximal set such that $\alpha \rightarrow \beta \notin \Lambda$, and Σ' be the set $\{\varphi : \varphi \rightarrow \beta \in \Lambda\}$. Then, there exists a Σ'-consistent and Σ'-maximal set Γ such that $\alpha \in \Gamma$, $\beta \notin \Gamma$ and ($\delta \rightarrow \eta \in \Lambda$ and $\delta \in \Gamma$ imply $\eta \in \Gamma$).*

Here, we will introduce the formal definition of a model of BPL, which is based on the soundness theorem. In the following definition, by the soundness theorem, we can say that a model of BPL has transitive relation and upward closed valuation.

Definition 19 *$\mathfrak{M}$ is a model for BPL if $\mathfrak{M} = \langle W, R, P, \mathfrak{V}\rangle$, W is non-emptyset, $R \subseteq W \times W$, $\mathfrak{V}$ is a valuation function from* Var$\mathcal{L}$ *to P, P has $\emptyset$ and W as elements and P is closed under the set union, set intersection and the operation which is defined by (1), and $\vdash_{\mathrm{BPL}} \alpha$ implies $\mathfrak{M} \models \alpha$.*

Definition 20 *$\mathfrak{M}$ is a canonical model for BPL iff $\mathfrak{M} = \langle W, R, P, \mathfrak{V}\rangle$ where*

1. *W is the class of all sets of $\mathcal{L}$ which are Σ-consistent and Σ-maximal for some set $\Sigma \neq \emptyset$,*
2. *for all $w, w' \in W$, wRw' iff ($\alpha \rightarrow \beta \in w$ and $\alpha \in w'$ imply $\beta \in w'$),*
3. *$\mathfrak{V}$ is a valuation function such that for all propositional variables p of $\mathcal{L}$, $\mathfrak{V}(p) = \{w : p \in w\}$,*
4. *P is a set $\{\mathfrak{V}'(A) : A \in \mathcal{L}\}$, where $\mathfrak{V}'$ is, for all $A \in \mathcal{L}$, $\mathfrak{V}'(A) = \{w : A \in w\}$.*

Lemma 21 *Let $\mathfrak{M} = \langle W, R, P, \mathfrak{V}\rangle$ be the canonical model for BPL. For all $w \in W$ and $\alpha \in \mathcal{L}$,*

$$\alpha \in w \text{ iff } \mathfrak{M}, w \models \alpha.$$

Proof Prove by the structual induction of a formula α.

- α is propositional variable.

  Trivial.

- α is $\bot$.

  For any w is Σ-consistent. So $\bot \notin w$ holds.

- α is $\beta \wedge \gamma$.

  Suppose $\beta \wedge \gamma \in w$. Then, by the lemma 11, $\beta, \gamma \in w$. By the induction hypothesis, $\mathfrak{M}, w \models \beta$ and $\mathfrak{M}, w \models \gamma$ hold. Thus, $\mathfrak{M}, w \models \beta \wedge \gamma$. Vice versa.

- α is $\beta \to \gamma$.

  Suppose $\mathfrak{M}, w \not\models \beta \to \gamma$. By the definition of $\models$,

  $$\exists w'.(wRw' \text{ and } \mathfrak{M}, w' \models \beta \text{ and } \mathfrak{M}, w' \not\models \gamma).$$

  By I.H.,

  $$\exists w'.(wRw' \text{ and } \beta \in w' \text{ and } \gamma \notin w').$$

  If we suppose $\beta \to \gamma \in w$, by the definition of R, $\gamma \in w'$ holds. This is contradiction.

  Suppose $\beta \to \gamma \notin w$. By the lemma 18, there exists a set w' such that

  $$\beta \in w' \text{ and } \gamma \notin w' \text{ and } (\delta \to \sigma \in w \text{ and } \delta \in w' \text{ imply } \sigma \in w').$$

  That is,

  $$\exists w'.(\beta \in w' \text{ and } \gamma \notin w' \text{ and } wRw').$$

  By I.H.,

  $$\exists w'.(wRw' \text{ and } \mathfrak{M}, w' \models \beta \text{ and } \mathfrak{M}, w' \not\models \gamma).$$

  Thus, $\mathfrak{M}, w \not\models \beta \to \gamma$.

❑

Corollary 22 *Let $\mathfrak{M} \models \langle W, R, P, \mathfrak{V}\rangle$ be a canonical model for BPL.*

$$\vdash_{\mathrm{BPL}} \alpha \textit{ iff } \mathfrak{M} \models \alpha.$$

Proof Let suppose $\vdash_{\mathrm{BPL}} \alpha$. By the lemma 9, $\alpha \in w$ for any Σ-maximal and Σ-consistent set w. By the lemma 21, $\mathfrak{M} \models \alpha$ holds.

Suppose $\not\vdash_{\mathrm{BPL}} \alpha$. Put Γ as $\{\beta :\vdash_{\mathrm{BPL}} \beta\}$. Γ is trivially $\{\alpha\}$-consistent set. By the lemma 8, there exists a set Λ such that $\Gamma \subset \Lambda$, Λ is $\{\alpha\}$-maximal set and Λ is $\{\alpha\}$-consistent set. Obivously, $\Lambda \in W$ and $\alpha \notin \Lambda$. By the lemma 21, $\mathfrak{M}, \Lambda \not\models \alpha$. ❑

Lemma 23 $\mathfrak{M}$ *is a model for BPL if* $\mathfrak{M}$ *is a canonical model for BPL.*

Proof Suppose $\mathfrak{M} = \langle W, R, P, \mathfrak{V}\rangle$. Here, we will show just that W is not emptyset, P has $\emptyset$ and W, and P is closed under the set union, set intersection and the operation 1. The other conditions are trivial.

$\{\alpha :\vdash_{\mathrm{BPL}} \alpha\}$ is a $\{\bot\}$-consistent. Using lemma 8, we can get $\{\bot\}$-maximal and $\{\bot\}$-consistent set. Thus, $W \neq \emptyset$.

$\bot \in \mathcal{L}$ deduces $\mathfrak{V}'(\bot) \in P$. Suppose $\mathfrak{V}'(\bot) \neq \emptyset$. That is, there exists an element w in W such that $\bot \in w$. But, this is contradiction that w is Σ-consistent. Thus, $\mathfrak{V}'(\bot) = \emptyset$.

$\bot \to \bot \in \mathcal{L}$ implies $\mathfrak{V}'(\bot \to \bot) \in P$. Obivously, $\mathfrak{V}'(\bot \to \bot) \subseteq W$. For any element $w \in W$, $\mathfrak{M}, w \not\models \bot$ holds, deduces $w \in \mathfrak{V}'(\bot \to \bot)$. Thus, $W \subseteq \mathfrak{V}'(\bot \to \bot)$.

Let suppose $X, Y \in P$. That is there exist $A, B \in \mathcal{L}$ such that $\mathfrak{V}'(A) = X$ and $\mathfrak{V}'(B) = Y$. We will show that $\mathfrak{V}'(A \wedge B) = \mathfrak{V}'(A) \cap \mathfrak{V}'(B)$, $\mathfrak{V}'(A \vee B) = \mathfrak{V}'(A) \cup \mathfrak{V}'(B)$ and $\mathfrak{V}'(A \to B) = \mathfrak{V}'(A) \to \mathfrak{V}'(B)$.

- $\mathfrak{V}'(A \wedge B) = \mathfrak{V}'(A) \cap \mathfrak{V}'(B)$

  Trivial by lemma 11.

- $\mathfrak{V}'(A \vee B) = \mathfrak{V}'(A) \cup \mathfrak{V}'(B)$

  Trivial by lemma 12.

- $\mathfrak{V}'(A \to B) = \mathfrak{V}'(A) \to \mathfrak{V}'(B)$

  Let suppose $w \in \mathfrak{V}'(A \to B)$. That is, let suppose $A \to B \in w$. We want to show $w \in \mathfrak{V}'(A) \to \mathfrak{V}'(B)$. This is equal to, for all w', wRw' and $w' \in \mathfrak{V}'(A)$ imply $w' \in \mathfrak{V}'(B)$. But, wRw' is holding $\alpha \to \beta \in w$ and $\alpha \in w'$ imply $\beta \in w'$, and $w' \in \mathfrak{V}'(\delta)$ is holding $\delta \in w'$. Thus, $w \in \mathfrak{V}'(A) \to \mathfrak{V}'(B)$ holds obivously.

  The other direction holds by the lemma 18.

❑

Lemma 24 *Let $\mathfrak{M} = \langle W, R, P, \mathfrak{V} \rangle$ be the canonical model for BPL. Then R is transitive, and $\mathfrak{V}$ is upward closed.*

Proof At first, we will show that $\mathfrak{V}$ is upward closed. Suppose $x \in \mathfrak{V}(p)$ and xRy hold for any $x, y \in W$ and propositional variable p. By the definition of canonical model, they are same to $p \in x$, and $\alpha \to \beta \in x$ and $\alpha \in y$ imply $\beta \in y$, respectively. By the lemma 10 and $\vdash_{\text{BPL}} p \to (\top \to p)$, $\top \to p \in x$ holds. Then $p \in y$ holds. Next, we will show that R is transitive. To prove R is transitive, it is sufficient to show that xRy implies $x \subseteq y$. Suppoe $\varphi \in x$. Holding $\vdash_{\text{BPL}} \varphi \to (\top \to \varphi)$ and lemma 10, $\top \to \varphi \in x$ hold. Then, by xRy, $\varphi \in y$ hold. ❑

Theorem 25 *Let suppose $\mathfrak{C}$ be the class of transitive frames with upward closed valuation. Then,*

$$\mathfrak{C} \models \alpha \textit{ implies } \vdash_{\text{BPL}} \alpha.$$

Proof Suppose $\not\vdash_{\text{BPL}} \alpha$. By the corollary 22, there exists a canonical model for BPL $\mathfrak{M}$ such that $\mathfrak{M} \not\models \alpha$. By the lemma 23 and 24, $\mathfrak{M} \in \mathfrak{C}$. That is $\mathfrak{C} \not\models \alpha$.

❑

References

[1] G.Corsi, Weak logics with strict implication. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 33:389–406, 1987.

[2] H.Ono, A note on Visser's basic propositional logic. manuscript, 1997.

[3] Y.Suzuki, F.Wolter and M.Zakharyaschev, Speaking about transitive frames in propositional languages. (to appear in) *the Journal of Logic, Language and Information*.