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Hitoshi KIHARA and Hiroakira ONO

ALGEBRAIC CHARACTERIZATIONS OF VARIABLE SEPARATION PROPERTIES

A b s t r a c t. This paper gives algebraic characterizations of Halldén completeness (HC), and of Maksimova's variable separation property (MVP) and its deductive form. Though algebraic characterizations of these properties have been already studied for modal and superintuitionistic logics, e.g. in Wroński [12], Maksimova [7], [9], a deeper analysis of these properties and non-trivial modifications of these results are needed to extend them to those for substructural logics, because of the lack of some structural rules in them. The first attempt in this direction was made in the dissertation [4] of the first author. Results of this paper are partly announced (sometimes in their weaker form) also in Chapter 5 of the book [2].

1. Preliminaries

In the following, we assume a certain familiarity with definitions and results introduced in [3]. They are also discussed in detail in [2]. Here we will

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briefly describe some of them to make our paper self-contained.

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and the monoid operation $\cdot$ is *residuated* with respect to the order by both the left- and right-division operations $\backslash, /$, i.e., for all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

An $\mathbf{FL}$-*algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ with a residuated lattice $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ and an arbitrary element 0 of A. An $\mathbf{FL}$-algebra is an $\mathbf{FL_e}$-*algebra* if the monoid operation $\cdot$ is commutative. An $\mathbf{FL}$-algebra is an $\mathbf{FL_{ew}}$-*algebra* if it is an $\mathbf{FL_e}$-algebra satisfying that $0 \leq x \leq 1$ for each element x. It is easy to show that in any $\mathbf{FL}$-algebra the commutativity of the monoid operation is equivalent to $x\backslash y = y/x$. In this case we sometimes denote $x\backslash y$ by $x \to y$. It is easy to see that the class $\mathcal{FL}$ of $\mathbf{FL}$-algebras forms a variety. We denote the subvariety lattice of $\mathcal{FL}$ by $\mathbf{S}(\mathcal{FL})$.

We adopt the convention that the monoid operation has priority over the division operations, which have priority over the lattice operations. So, for example, we write $x/yz \wedge u\backslash v$ for $[x/(y \cdot z)] \wedge (u\backslash v)$.

The class of $\mathbf{FL}$-algebras provides algebraic semantics for the substructural logic $\mathbf{FL}$, called the full Lambek calculus. For the precise definition of the sequent calculus $\mathbf{FL}$, see [3]. By a substructural logic (over $\mathbf{FL}$), we mean an axiomatic extension of $\mathbf{FL}$. Here, a sequent calculus is an *axiomatic extension* of $\mathbf{FL}$ with axioms $\{\alpha_j : j \in J\}$ if it is obtained from $\mathbf{FL}$ by adding each sequent of the form $\Rightarrow \varphi$ as a new initial sequent, where φ is a *substitution instance* of some axiom α_j. When a substructural logic is obtained from $\mathbf{L}$ by adding axioms $\{\beta_k : k \in K\}$, it is denoted by $\mathbf{L} + \{\beta_k : k \in K\}$. As usual, we identify a given substructural logic $\mathbf{L}$ with the set of formulas provable in it.

The substructural logic $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$) is usually introduced as a sequent calculus obtained from $\mathbf{FL}$ by adding the exchange rule (the exchange rule, and left- and right-weakening rules, respectively). It can be easily seen that both of them are in fact axiomatic extensions of $\mathbf{FL}$. Obviously, the set of all substructural logics over $\mathbf{FL}$ (as sets of formulas) forms a lattice $\mathbf{SL}$.

For an arbitrary class $\mathcal{K}$ of $\mathbf{FL}$-algebras, let $\mathbf{L}(\mathcal{K})$ be the set of formulas that are *valid* in all $\mathbf{FL}$-algebras in $\mathcal{K}$. Then, we can show that $\mathbf{L}(\mathcal{K})$ is a substructural logic for any $\mathcal{K}$. When $\mathcal{K}$ consists of a single algebra $\mathbf{A}$, we denote $\mathbf{L}(\mathcal{K})$ by $\mathbf{L}(\mathbf{A})$. Conversely, for a given substructural logic $\mathbf{L}$, let $V(\mathbf{L})$ be the class of all **FL**-algebras in which every inequation $1 \leq \varphi$ holds for $\varphi \in \mathbf{L}$. Then $V(\mathbf{L})$ belongs to $\mathbf{S}(\mathcal{FL})$. Moreover, we have the following.

Proposition 1.1. *The maps* $\mathbf{L} : \mathbf{S}(\mathcal{FL}) \to \mathbf{SL}$ *and* $\mathcal{V} : \mathbf{SL} \to \mathbf{S}(\mathcal{FL})$ *are mutually inverse, dual lattice isomorphisms.*

For a set of formulas Γ and a formula ψ, we say that ψ is *deducible* from Γ in **FL** ($\Gamma \vdash_{\mathbf{FL}} \psi$, in symbol), when there is a proof of $\Rightarrow \psi$ in the calculus **FL** adding new initial sequents of the form $\Rightarrow \gamma$ for each $\gamma \in \Gamma$. Different from the definition of an axiomatic extension, here we cannot use a sequent $\Rightarrow \delta$ as an initial sequent when δ is a substitution instance of a formula in Γ, exept the case where δ itself belongs to Γ.

The deducibility relation is naturally extended to each substructural logic $\mathbf{L}$ in the following way. For a set of formulas $\Gamma \cup \{\psi\}$, we write $\Gamma \vdash_{\mathbf{L}} \psi$ when $\Gamma \cup \mathbf{L} \vdash_{\mathbf{FL}} \psi$. Then we can show that the relation $\vdash_{\mathbf{L}}$ is a finitary, substitution invariant consequence relation (in the sense of abstract algebraic logic). See [3] for the details.

For formulas α, φ, the *left conjugate* $\lambda_\alpha(\varphi)$ and the *right conjugate* $\rho_\alpha(\varphi)$ of φ (with respect to α) are formulas $(\alpha\backslash\varphi\alpha)\wedge 1$ and $(\alpha\varphi/\alpha)\wedge 1$, respectively. An *iterated conjugate* γ of φ is a composition of left- and right- conjugate of the form $\delta_{\alpha_1}(\delta_{\alpha_2}(\cdots\delta_{\alpha_m}(\varphi)\cdots))$ for some formulas $\alpha_1, \ldots, \alpha_m$ (called *parameters*), where each δ_{α_i} is either left or right conjugate. The following theorem, called the *parameterized local deduction theorem*, is shown in [3]. Here, Π means a finite product of formulas by the fusion.

Proposition 1.2. *If* $\Gamma \cup \Sigma \cup \{\psi\}$ *is a set of formulas and* $\mathbf{L}$ *is a logic over* **FL** *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n}\gamma_i(\varphi_i))\backslash\psi,$$

for some n*, some iterated conjugates* γ_i *of a formula* $\varphi_i \in \Sigma$ *for each* $i < n$*. In particular, if* $\mathbf{L}$ *is over* $\mathbf{FL_e}$ *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n}(\varphi_i \wedge 1)) \to \psi,$$

for some n *and some* $\varphi_i \in \Sigma$ *for each* $i < n$*. Moreover, if* $\mathbf{L}$ *is a logic over* $\mathbf{FL_{ew}}$ *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n}\varphi_i) \to \psi,$$

for some n *and some* $\varphi_i \in \Sigma$ *for each* $i < n$*.*

In the above, iterated conjugates, and therefore parameters do not appear in the second and the third results. Hence, they are called simply the *local deduction theorem*. They are derived from the first one by using the fact that for every formula φ, φ follows from each of left and right conjugate of φ with respect to 1, and also that φ implies both $(\alpha\backslash\varphi\alpha)$ and $(\alpha\varphi/\alpha)$ for any formula α in $\mathbf{FL_e}$.

The following proposition, called the *algebraization theorem*, is fundamental when we consider relations between logic and algebra, though we omit the detailed explanation here. Note that Proposition 1.1 follows from the following proposition. For further information, consult [3].

Proposition 1.3. *For every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL}$*, the deducibility relation* $\vdash_{\mathbf{L}}$ *is algebraizable with defining equation* $1 = x \wedge 1$ *and equivalence formula* $x\backslash y \wedge y\backslash x$*. An equivalent algebraic semantics for* $\vdash_{\mathbf{L}}$ *is the variety* $V(\mathbf{L})$*.*

Let $\mathbf{A}$ be an $\mathbf{FL}$-algebra. Then, a subset F of A is *deductive filter* or simply *filter*, if it satisfies the following;

1. $1 \leq x$ implies $x \in F$,
2. $x, x\backslash y \in F$ implies $y \in F$,
3. $x \in F$ implies $x \wedge 1 \in F$,
4. $x \in F$ implies $a\backslash xa, ax/a \in F$ for any a.

For a subset S of A, let $\mathrm{Fg}^{\mathbf{A}}(S)$ be the filter generated by S, i.e. the smallest filter containing S. Because of close resemblances between the deducibility and filter generation which comes from the algebraization theorem, we have the following (cf. [3]). Here, we use algebraic analogue of the notion of *conjugates*. For an $\mathbf{FL}$-algebra $\mathbf{A}$ and $a, x \in A$, the left conjugate $\lambda_a(x)$ of x with respect to a is the element $(a\backslash xa) \wedge 1$. Right conjugates and iterated conjugates are defined in the similar way.

Lemma 1.4. *Let* $\mathbf{A}$ *be an* $\mathbf{FL}$*-algebra and* S *a subset of* A*. Then*

$$Fg^{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n}\gamma_i(s_i) \leq x \textit{ for some } n, \textit{ for some } s_i \in S, \textit{ and some iterated conjugates } \gamma_i \textit{ with parameters from } A\}.$$

In particular, if $\mathbf{A}$ *is an* $\mathbf{FL_e}$*-algebra then*

$$Fg^{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n}(s_i \wedge 1) \leq x \text{ for some } n \text{ and for some } s_i \in S\}.$$

Also, if $\mathbf{A}$ *is an* $\mathbf{FL_{ew}}$*-algebra then*

$$Fg^{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n} s_i \leq x \text{ for some } n \text{ and for some } s_i \in S\}.$$

It is easy to show that all filters of an $\mathbf{FL}$-algebra $\mathbf{A}$ form a lattice denoted by $\mathbf{Fil}(\mathbf{A})$. Let $\mathbf{Con}(\mathbf{A})$ be the congruence lattice of $\mathbf{A}$. Then the following holds (see [3]).

Lemma 1.5. *Let* $\mathbf{A}$ *be an* $\mathbf{FL}$*-algebra. Then, for* $F \in \mathbf{Fil}(\mathbf{A})$ *and* $\theta \in \mathbf{Con}(\mathbf{A})$*, the maps* $F \mapsto \Theta_F = \{(a,b) \in A^2 \mid a\backslash b \wedge b\backslash a \in F\}$ *and* $\theta \mapsto F_\theta = \{a \in A \mid (a \wedge 1, 1) \in \theta\}$ *are mutually inverse lattice isomorphisms between* $\mathbf{Fil}(\mathbf{A})$ *and* $\mathbf{Con}(\mathbf{A})$.

By the definition of subdirect irreducibility, a non-trivial algebra is *subdirectly irreducible* iff it has the second smallest congruence. Thus, by Lemma 1.5, this condition is equivalent to the condition that it has the second smallest filter. Clearly, the smallest filter is the filter generated by the unit 1, which is equal to the set $I = \{x \in A : 1 \leq x\}$. Then, the second smallest filter, if exists, must be generated by a single element, say a, such that $a \notin I$, and every filter which includes properly I must contain a. Thus, by using Lemma 1.4, we have the following.

Corollary 1.6. *A non-trivial* $\mathbf{FL}$*-algebra is subdirectly irreducible iff*

$$\exists a \not\geq 1, \forall x \not\geq 1, \exists n \in \omega \text{ and iterated conjugates } \gamma_i \text{ such that } \Pi_{i=1}^{n}\gamma_i(x) \leq a.$$

This condition is simplified for an $\mathbf{FL_e}$*-algebra as;*

$$\exists a \not\geq 1, \forall x \not\geq 1, \exists n \in \omega \text{ such that } (x \wedge 1)^n \leq a.$$

and also for an $\mathbf{FL_{ew}}$*-algebra as*

$$\exists a \not\geq 1, \forall x \not\geq 1, \exists n \in \omega \text{ such that } x^n \leq a.$$

2. Halldén Completeness of logics over $\mathbf{FL_{ew}}$

A substructural logic **L** is *Halldén complete* (HC), if for all formulas φ and ψ which have no variables in common, if $\varphi \vee \psi$ is in **L** then either φ or ψ is in **L**. Obviously the disjunction property implies the Halldén completeness, while it is shown in [1] that the converse does not hold for uncountably many superintuitionistic logics.

For superintuitionistic logics, both Lemmon [6] and Wroński [12] gave different characterization results on the Halldén completeness. In this section, they can be extended to those for substructural logics over $\mathbf{FL_{ew}}$. On the other hand, these proofs do not work well for substructural logics in general. In the next section, we show an algebraic characterization of the Maksimova's variable separation property for substructural logics over **FL**, and as its special case, an alternative way of a characterization of the Halldén completeness will be obtained for substructural logics over **FL**.

Here we show some technical lemmas. The first one is on an axiomatization of the meet of logics over $\mathbf{FL_{ew}}$. Suppose that two logics $\mathbf{L}_1$ and $\mathbf{L}_2$ are logics obtained from a logic **L** by adding axioms φ_1 and φ_2, respectively. Then, we can assume that axioms φ_1 and φ_2 have no variables in common, by renaming propositional variables if necessary.

Lemma 2.1. *Suppose that* **L** *is a substructural logic over* $\mathbf{FL_{ew}}$ *and that both* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are logics obtained from* **L** *by adding axioms* φ_1 *and* φ_2*, respectively, such that they have no variables in common. Then, the meet* $\mathbf{L}_1 \cap \mathbf{L}_2$ *is axiomatized over* **L** *by the formula* $\varphi_1 \vee \varphi_2$.

Proof. It is clear that the formula $\varphi_1 \vee \varphi_2$ belongs to the meet $\mathbf{L}_1 \cap \mathbf{L}_2$. Thus it suffices to show that for any formula $\psi \in \mathbf{L}_1 \cap \mathbf{L}_2$, ψ follows in **L** from some substitution instances of $\varphi_1 \vee \varphi_2$. Since $\psi \in \mathbf{L}_1 \cap \mathbf{L}_2$, by using the local deduction theorem for $\mathbf{FL_{ew}}$ (see Proposition 1.2) there are substitution instances δ_i with $i = 1, \cdots, n$ of φ_1, and σ_j with $j = 1, \cdots, m$ of φ_2, respectively, such that both formulas

$$\prod_{i=1}^n \delta_i \to \psi \quad \text{and} \quad \prod_{j=1}^m \sigma_j \to \psi$$

are in **L**. Then the formula

$$\left(\prod_{i=1}^n \delta_i \vee \prod_{j=1}^m \sigma_j\right) \to \psi$$

is also in it. On the other hand, using the distributivity of fusion $\cdot$ over disjunction,

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(\delta_i \vee \sigma_j) \to \left(\prod_{i=1}^{n}\delta_i \vee \prod_{j=1}^{m}\sigma_j\right)$$

is always provable in $\mathbf{FL_{ew}}$. Therefore

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(\delta_i \vee \sigma_j) \to \psi$$

is also in $\mathbf{L}$. Since φ_1 and φ_2 have no variables in common, each formula $\delta_i \vee \sigma_j$ is a substitution instance of $\varphi_1 \vee \varphi_2$. Thus $\psi \in \mathbf{L} + \{\varphi_1 \vee \varphi_2\}$. □

We say that an $\mathbf{FL}$-algebra $\mathbf{A}$ is *well-connected* if for any x and y in A, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$. When 1 is the greatest as is in the case for $\mathbf{FL_{ew}}$-algebras, this condition can be replaced obviously by the condition that if both $x < 1$ and $y < 1$ then $x \vee y < 1$.

Lemma 2.2. *Every subdirectly irreducible* $\mathbf{FL_{ew}}$*-algebra is well-connected.*

Proof. Suppose that an $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ is subdirectly irreducible and both $x, y \in A$ is smaller than 1. Then, for some $z < 1$ there exist natural numbers m and n such that $x^m \leq z$ and $y^n \leq z$, using Corollary 1.6. Let $t = m + n - 1$. Then, by the distributivity of $\cdot$ over $\vee$, $(x \vee y)^t = \bigvee_{i=0}^{t} x^i \cdot y^{t-i}$. If $i \geq m$ then $x^i \cdot y^{t-i} \leq x^i \leq x^m \leq z$. Similarly, if $t - i \geq n$, $x^i \cdot y^{t-i} \leq y^{t-i} \leq y^n \leq z$. But, for each i either $i \geq m$ or $t - i \geq n$ holds, and hence $(x \vee y)^t \leq z$. Thus, $x \vee y < 1$. □

The next lemma is on the existence of prime filters of $\mathbf{FL_{ew}}$-algebras. Recall that a filter F of an $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ is *prime* if for all $x, y \in A$, $x \vee y \in F$ implies either $x \in F$ or $y \in F$.

Lemma 2.3. *Let G be a proper filter of an* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$ *such that $a \notin G$. Then there exists a prime filter F_a of* $\mathbf{A}$ *such that $a \notin F_a$ and $G \subseteq F_a$.*

Proof. The proof goes essentially in the same way as the corresponding result on distributive lattices. Let K be the set of all filters F of $\mathbf{A}$ such that $a \notin F$ and $G \subseteq F$. By Zorn's lemma, there exists a maximal element F_a in K. We show that F_a is prime. Assume that both $x \notin F_a$ and $y \notin F_a$. By the maximality of F_a in K, there exist some natural numbers m and n, and $u, v \in F_a$ such that $x^m \cdot u \leq a$ and $y^n \cdot v \leq a$. Let $t = m+n-1$. Then,

$$(x \vee y)^t \cdot u \cdot v = \bigvee_{i=0}^{t} x^i \cdot y^{t-i} \cdot u \cdot v.$$

By a similar argument, if $i \geq m$ then $x^i \cdot y^{t-i} \cdot u \cdot v \leq x^m \cdot u \leq a$, and also if $t-i \geq n$, $x^i \cdot y^{t-i} \cdot u \leq a$. Since either $i \geq m$ or $t-i \geq n$ holds for each i, $(x \vee y)^t \cdot u \cdot v \leq a$. Hence $x \vee y \notin F_a$. This means that F_a is a prime filter. □

Corollary 2.4. *For each element a of a given $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$, if $a < 1$ then there exists a prime filter F of $\mathbf{A}$ such that $a \notin F$ and the quotient algebra $\mathbf{A}/F$ is subdirectly irreducible.*

Proof. Taking the singleton set $\{1\}$ for G in the proof of Lemma 2.3, there exists a prime filter F which is maximal among filters to which a does not belong. Thus every filter of $\mathbf{A}$ which properly includes F contains the filter $\mathrm{Fg}^{\mathbf{A}}(F \cup \{a\})$ generated by the set $F \cup \{a\}$. Since the congruence lattice $\mathbf{Con}(\mathbf{A}/F)$ is isomorphic to the lattice of filters of $\mathbf{A}$ including F and the latter has the second smallest filter $\mathrm{Fg}^{\mathbf{A}}(F \cup \{a\})$, the quotient algebra $\mathbf{A}/F$ is subdirectly irreducible. □

The following theorem is obtained by extending results on the Halldén completeness for superintuitionistic logics to that for substructural logics over $\mathbf{FL_{ew}}$. The equivalence of (1) to (3) is proved by Lemmon [6], and that to (2) by Wroński [12]. We say a logic $\mathbf{L}$ over $\mathbf{FL_{ew}}$ is *meet irreducible* (in the lattice of all substructural logics) if it is not an intersection of two incomparable logics, or equivalently, if it is not a finite meet of strictly bigger logics.

Theorem 2.5. *The following conditions are equivalent for every substructural logic $\mathbf{L}$ over $\mathbf{FL_{ew}}$.*

(1) $\mathbf{L}$ is Halldén complete,

(2) $\mathbf{L} = \mathbf{L}(\mathbf{A})$ for some well-connected $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$,

(3) $\mathbf{L}$ *is meet irreducible.*

Proof. First we show that (2) implies (3). Let $\mathbf{A}$ be a well-connected $\mathbf{FL_{ew}}$-algebra and $\mathbf{L} = \mathbf{L}(\mathbf{A})$. Suppose that $\mathbf{L} = \mathbf{L}_1 \cap \mathbf{L}_2$ for some incomparable logics $\mathbf{L}_1$ and $\mathbf{L}_2$. Then there exist some formulas φ and ψ such that $\varphi \in \mathbf{L}_1 \setminus \mathbf{L}_2$ and $\psi \in \mathbf{L}_2 \setminus \mathbf{L}_1$. Obviously, neither of them belong to $\mathbf{L}$. We can assume here that φ and ψ have no variables in common, since we can replace variables in ψ by distinct variables which do not appear in φ. Thus there exists an assignment f on $\mathbf{A}$ such that $f(\varphi) < 1$ and $f(\psi) < 1$. Since we assume that $\mathbf{A}$ is well-connected, $f(\varphi \vee \psi) < 1$. On the other hand, as $\varphi \vee \psi$ belongs both to $\mathbf{L}_1$ and $\mathbf{L}_2$, it must belong to $\mathbf{L}$. This is a contradiction. Thus, $\mathbf{L}$ is meet irreducible.

Next, we show that (3) implies (1) by taking contraposition. Suppose that (1) does not hold. Then, there exist formulas φ_1 and φ_2 with no variables in common such that $\varphi_1 \vee \varphi_2$ is in $\mathbf{L}$ while neither φ_1 nor φ_2 is in it. Define logics $\mathbf{L}_i = \mathbf{L} + \{\varphi_i\}$ for $i = 1, 2$. Clearly, both $\mathbf{L}_1$ and $\mathbf{L}_2$ are strictly stronger than $\mathbf{L}$. By Lemma 2.1, $\mathbf{L}_1 \cap \mathbf{L}_2$ is axiomatized over $\mathbf{L}$ by the formula $\varphi_1 \vee \varphi_2$. But $\varphi_1 \vee \varphi_2$ is in $\mathbf{L}$ by our assumption, and hence $\mathbf{L}_1 \cap \mathbf{L}_2 = \mathbf{L}$. Thus, $\mathbf{L}$ is not meet irreducible.

Lastly, we show that (1) implies (2). Take any $\mathbf{FL_{ew}}$-algebra $\mathbf{C}$ such that $\mathbf{L} = \mathbf{L}(\mathbf{C})$. (For instance, take the Lindenbaum-Tarski algebra of $\mathbf{L}$ for $\mathbf{C}$.) From this $\mathbf{C}$, we will construct a well-connected $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ such that $\mathbf{L} = \mathbf{L}(\mathbf{A})$ as follows.

Let $\{G_i : i \in I\}$ be an enumeration of all prime filters of $\mathbf{C}$ such that the quotient algebra $\mathbf{C}/G_j$ is subdirectly irreducible. For each formula φ, define a subset $R(\varphi)$ of I by

$$R(\varphi) = \{j \in I : \mathbf{C}/G_j \not\models \varphi\}.$$

Note that each algebra $\mathbf{C}/G_i$ is well-connected. When φ is not provable in $\mathbf{L}$, $R(\varphi)$ is nonempty. In fact, if φ is not provable in $\mathbf{L}$, $f(\varphi) < 1$ for an assignment f in C. Then there exists a prime filter G_k of $\mathbf{C}$ such that (a) $f(\varphi) \notin G_k$ by Corollary 2.4 and that (b) $\mathbf{C}/G_k$ is subdirectly irreducible. Define an assignment f^* in $\mathbf{C}/G_k$ by $f^*(p) = h(f(p))$ for each propositional variable p, where h is a natural homomorphism induced by the congruence determined by the prime filter G_k. Then, $f^*(\varphi) < 1$ in $\mathbf{C}/G_k$. Thus, $k \in R(\varphi)$.

We show next that the set $E = \{R(\varphi) | \varphi \notin \mathbf{L}\}$ has the finite intersection property, i.e. every intersection of finitely many members from E is nonempty. To see this, it suffices to show that for each $R(\varphi)$ and $R(\psi)$ in E there exists a formula γ such that $R(\gamma) \subseteq R(\varphi) \cap R(\psi)$ and $R(\gamma) \in E$. It may happen that φ and ψ have some variables in common. Then we take a formula φ' which is obtained from φ by *renaming* propositional variables so that φ' and ψ have no variables in common. Clearly, $\varphi' \notin \mathbf{L}$ and $R(\varphi') = R(\varphi)$. So we can assume from the beginning that φ and ψ have no variables in common. Since we assume that $\mathbf{L}$ is Halldén complete, $R(\varphi \vee \psi) \in E$. Now, we show that $R(\varphi \vee \psi) \subseteq R(\varphi) \cap R(\psi)$. Suppose that $j \in R(\varphi \vee \psi)$. Then $\mathbf{C}/G_j \not\models \varphi \vee \psi$, and hence for some assignment g in $\mathbf{C}/G_j$, $g(\varphi \vee \psi) = g(\varphi) \vee g(\psi) < 1$. Therefore, $\mathbf{C}/G_j \not\models \varphi$ and $\mathbf{C}/G_j \not\models \psi$. Hence $j \in R(\varphi) \cap R(\psi)$.

By the finite intersection property of E, there exists a ultrafilter U over I such that $E \subseteq U$. Let $\mathbf{A}$ be the ultraproduct $(\prod_{i \in I}(\mathbf{C}/G_i))/U$. Since each algebra $\mathbf{C}/G_i$ is well-connected and moreover the well-connectedness can be expressed by a first-order sentence, $\mathbf{A}$ is also well-connected. Obviously, $\mathbf{A} \in V(\mathbf{L})$. If a formula φ is not provable in $\mathbf{L}$ then $R(\varphi) \in U$ and hence $\mathbf{A} \not\models \varphi$. Thus, $\mathbf{L} = \mathbf{L}(\mathbf{A})$. □

The *strict meet-irreducibility* is a stronger form of the meet-irreducibility. That is, a logic $\mathbf{L}$ is strictly meet-irreducible if whenever it is an intersection of logics $\{\mathbf{K}_i : i \in I\}$ then $\mathbf{L} = \mathbf{K}_j$ for some $j \in I$. For more information on strict meet-irreduciblity, see Kracht [5]. We can easily show the following.

Lemma 2.6. *(1) For each logic* $\mathbf{L}$ *over* $\mathbf{FL}$*, if* $\mathbf{L}$ *is strictly meet-irreducible then* $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some subdirectly irreducible* $\mathbf{FL}$*-algebra* $\mathbf{A}$*,*

(2) For each logic $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$*, if* $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some subdirectly irreducible* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$ *then* $\mathbf{L}$ *is meet-irreducible.*

Proof. For (1), suppose that $\mathbf{L} = \mathbf{L}(\mathbf{C})$ for some $\mathbf{FL}$-algebra $\mathbf{C}$. By Birkhoff's theorem, $\mathbf{C}$ has a subdirect representation $(\mathbf{C}_i : i \in I)$ with subdirectly irreducible factors $\mathbf{C}_i$ for $i \in I$. Then, $\mathbf{L}(\mathbf{C}) = \bigcap_i \mathbf{L}(\mathbf{C}_i)$ holds in this case. Since $\mathbf{L}$ is strictly meet-irreducible, $\mathbf{L} = \mathbf{L}(\mathbf{C}_i)$ with a subdirectly irreducible $\mathbf{C}_i$ for some $i \in I$.

To show (2), suppose that $\mathbf{L} = \mathbf{L}(\mathbf{A})$ for some subdirectly irreducible $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ then by Lemma 2.2 $\mathbf{A}$ is well-connected. Thus, $\mathbf{L}$ is meet-irreducible by Theorem 2.5. □

In [12], Wroński proved that the converse of the above (2) holds always for superintuitionistic logics. That is, a superintuitionistic logic $\mathbf{L}$ is meet-irreducible, or equivalently, it is Halldén complete iff

$\mathbf{L} = \mathbf{L}(\mathbf{A})$ for a *subdirectly irreducible* Heyting algebra $\mathbf{A}$.

This result follows from the proof of our Theorem 2.5, since every algebra $\mathbf{C}/G_i$ in the proof is subdirectly irreducible and the subdirect irreducibility of Heyting algebras can be expressed by the following first-order sentence:

$$\exists z < 1 \forall x < 1 \ x \leq z.$$

On the other hand, the subdirect irreducibility of $\mathbf{FL_{ew}}$-algebras is written as:

$$\exists z < 1 \forall x < 1 \text{ for some } n \in \mathbb{N} \ x^n \leq z,$$

which is not a first-order sentence. Note that in the proof of Theorem 2.5, it can take also an enumeration of all prime filters of $\mathbf{C}$ for $\{G_i : i \in I\}$. On the other hand, we have chosen another set of prime filters to clarify these differences. When a logic over $\mathbf{FL_{ew}}$ satisfies the axiom of m-potency, i.e., $\alpha^m \to \alpha^{m+1}$, then we can take a fixed number m for n in the above statement and hence it becomes a first-order sentence. Thus, we have the following.

Corollary 2.7. *The following two conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$ *satisfying the axiom of m-potency for some m.*

(1) $\mathbf{L}$ *is Halldén complete,*

(2) $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some subdirectly irreducible* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$.

We do not know whether the converses of the statements in Lemma 2.6 hold.

A similar result for logics over $\mathbf{FL_e}$ can be shown, but some modifications are necessary, since the unit 1 is not always equal to the greatest element.

Theorem 2.8. *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$.

(1) $\mathbf{L}$ *is weakly Halldén complete, i.e. for all formulas* φ *and* ψ *which have no variables in common, if* $(\varphi \wedge 1) \vee (\psi \wedge 1)$ *is in* $\mathbf{L}$ *then either* φ *or* ψ *is in* $\mathbf{L}$,

(2) $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some* $\mathbf{FL_e}$*-algebra* $\mathbf{A}$ *satisfying that* $x \vee y = 1$ *implies* $x = 1$ *or* $y = 1$ *for all* $x, y \in A$,

(3) $\mathbf{L}$ *is meet irreducible.*

This theorem gives us a characterization rather of the meet- irreducibility but not of the Halldén completeness for logics over $\mathbf{FL_e}$. To discuss the latter in a more proper way, we give algebraic characterizations of Maksimova's variable separation properties in the next section, from which an algebraic condition for Halldén completeness will follow as a particular case.

3. Maksimova's variable separation property

In this section, we consider algebraic characterizations of two forms of Maksimova's variable separation property. A substructural logic $\mathbf{L}$ is said to have the *Maksimova's variable separation property* (MVP), when for all formulas $\alpha_1 \backslash \alpha_2$ and $\beta_1 \backslash \beta_2$ that have no propositional variables in common if a formula $(\alpha_1 \wedge \beta_1) \backslash (\alpha_2 \vee \beta_2)$ is provable in $\mathbf{L}$, then either $\alpha_1 \backslash \alpha_2$ or $\beta_1 \backslash \beta_2$ is provable in it. Note that Halldén completeness follows from the MVP, by taking the constant 1 for both α_1 and β_1 in the definition of the MVP.

A substructural logic $\mathbf{L}$ has the *deductive Maksimova's variable separation property* (DMVP), when for all formulas $\alpha_1 \backslash \alpha_2$ and $\beta_1 \backslash \beta_2$ that have no variables in common, $\alpha_1 \wedge \beta_1 \vdash_{\mathbf{L}} \alpha_2 \vee \beta_2$ implies either $\alpha_1 \vdash_{\mathbf{L}} \alpha_2$ or $\beta_1 \vdash_{\mathbf{L}} \beta_2$. Since for arbitrary formulas γ and σ, the condition $\gamma, \sigma \vdash_{\mathbf{L}} \psi$ is equivalent to the condition $\gamma \wedge \sigma \vdash_{\mathbf{L}} \psi$, and the compactness of the deducibility relation $\vdash_{\mathbf{L}}$, we may state the definition of the DMVP as follows.

> A substructural logic $\mathbf{L}$ has the DMVP, if for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \vee \psi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$ or $\Sigma \vdash_{\mathbf{L}} \psi$.

Subalgebras **B** and **C** of an **FL**-algebra **A** form a *strongly well-connected pair* if for all elements $b_1, b_2 \in B$ and $c_1, c_2 \in C$, $b_1 \wedge c_1 \leq b_2 \vee c_2$ implies $b_1 \leq b_2$ or $c_1 \leq c_2$. When $b_1 = c_1 = 1$, the above becomes the condition that for all elements $b \in B$ and $c \in C$, $1 \leq b \vee c$ implies $1 \leq b$ or $1 \leq c$. In this case, we say that **B** and **C** form a *well-connected pair* of **A**. Thus, an algebra **A** is well-connected iff **A** with itself form a well-connected pair, and also iff every pair of subalgebras of **A** form a well-connected pair.

Theorem 3.1. *Let* **L** *be a logic over* **FL**. *Then the following two conditions are equivalent;*

(1) **L** *has the MVP,*

(2) for every two non-degenerate **FL***-algebras* **A**, **B** *in* $V(\mathbf{L})$*, there exist an* **FL***-algebra* **C** *and subalgebras* $\mathbf{C}_1, \mathbf{C}_2$ *of* **C** *in* $V(\mathbf{L})$ *such that* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *form a strongly well-connected pair, and moreover that* **A** *and* **B** *are homomorphic images of* $\mathbf{C}_1$ *and* $\mathbf{C}_2$*, respectively.*

Proof. Suppose first that **L** has the MVP, and let **A** and **B** be non-degenerate **FL**-algebras in $V(\mathbf{L})$. Take disjoint sets of variables Y and Z that are enough big to ensure the existence of surjective maps from Y to A and Z to B, respectively. Let X be the union of Y and Z, and let **C**, $\mathbf{C}_1$ and $\mathbf{C}_2$ be free algebras in $V(\mathbf{L})$, generated by X, Y and Z, respectively. By the universal mapping property, **A** and **B** are homomorphic images of $\mathbf{C}_1$ and $\mathbf{C}_2$. Also both $\mathbf{C}_1$ and $\mathbf{C}_2$ are regarded as subalgebras of **C**. So, it remains to show that $\mathbf{C}_1$ and $\mathbf{C}_2$ form a strongly well-connected pair.

Take arbitrary elements $a_1, a_2 \in C_1$ and $b_1, b_2 \in C_2$. Then there exist terms s_1, s_2 over the set Y and terms t_1, t_2 over Z such that

$$a_1 = s_1/{\equiv_{\mathbf{L}}},\ a_2 = s_2/{\equiv_{\mathbf{L}}},\ b_1 = t_1/{\equiv_{\mathbf{L}}} \text{ and } b_2 = t_2/{\equiv_{\mathbf{L}}}.$$

Here, the binary relation $\equiv_{\mathbf{L}}$ is a congruence on the set of terms over the set X defined by:

for all terms s and t, $s \equiv_{\mathbf{L}} t$ iff $(s\backslash t) \wedge (t\backslash s)$ is provable in **L**.

Now suppose that $a_1 \not\leq a_2$ and $b_1 \not\leq b_2$. Then, neither $s_1\backslash s_2$ nor $t_1\backslash t_2$ are provable in **L**. Since $s_1\backslash s_2$ and $t_1\backslash t_2$ have no variables in common, $(s_1 \wedge t_1)\backslash(s_2 \vee t_2)$ is neither provable in **L** by our assumption that **L** has the MVP. This means that $a_1 \wedge b_1 \not\leq a_2 \vee b_2$. Thus, $\mathbf{C}_1$ and $\mathbf{C}_2$ form a strongly well-connected pair.

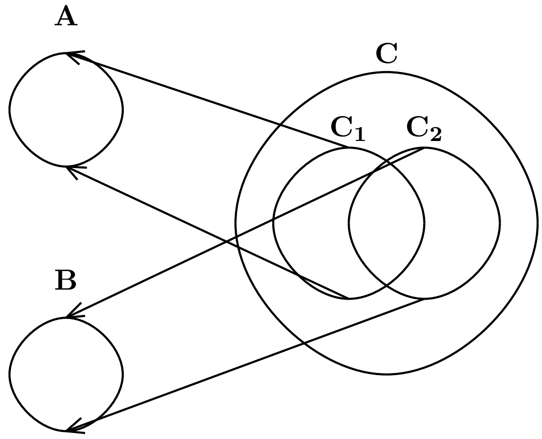

Figure 1: An algebraic characterization of the MVP

We show next that the condition (2) implies the MVP of a logic **L**. Suppose that for given formulas $\varphi_1\backslash\varphi_2$ and $\psi_1\backslash\psi_2$ which have no variables in common, neither $\varphi_1\backslash\varphi_2$ nor $\psi_1\backslash\psi_2$ are provable in **L**. Let Y and Z be the sets of variables appearing in $\varphi_1\backslash\varphi_2$ and $\psi_1\backslash\psi_2$, respectively. Then there exist non-degenerate **FL**-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathbf{L})$ and valuations f on $\mathbf{A}$ and g on $\mathbf{B}$ such that

$$f(\varphi_1) \not\leq_{\mathbf{A}} f(\varphi_2) \text{ and } g(\psi_1) \not\leq_{\mathbf{B}} g(\psi_2).$$

By our assumption, for some **FL**-algebra $\mathbf{C}$ and some subalgebras $\mathbf{C}_1, \mathbf{C}_2$ of $\mathbf{C}$ in $V(\mathbf{L})$, there exist surjective homomorphisms h from $\mathbf{C}_1$ to $\mathbf{A}$ and j from $\mathbf{C}_2$ to $\mathbf{B}$ such that $\mathbf{C}_1$ and $\mathbf{C}_2$ form a strongly well-connected pair. We define a valuation k in $\mathbf{C}$ for formulas over the set of variables $Y \cup Z$ so that $k(p) \in h^{-1} \circ f(p)$ for each $p \in Y$ and $k(q) \in j^{-1} \circ g(q)$ for each $q \in Z$. Such a map k exists since both h and j are surjective, and is well-defined since Y and Z are disjoint. Then, for each formula φ over Y, $f(\varphi) = h \circ k(\varphi)$ holds and similarly for each formula ψ over Z, $g(\psi) = j \circ k(\psi)$ holds. Therefere,

$$h \circ k(\varphi_1) \not\leq_{\mathbf{A}} h \circ k(\varphi_2) \text{ and } j \circ k(\psi_1) \not\leq_{\mathbf{B}} j \circ k(\psi_2).$$

Hence,

$$k(\varphi_1) \not\leq k(\varphi_2) \text{ and } k(\psi_1) \not\leq k(\psi_2).$$

Since $k(\varphi_1), k(\varphi_2) \in C_1$ and $k(\psi_1), k(\psi_2) \in C_2$,

$$k(\varphi_1 \wedge \psi_1) = k(\varphi_1) \wedge k(\psi_1) \not\leq k(\varphi_2) \vee k(\psi_2) = k(\varphi_2 \vee \psi_2)$$

by the strong well-connectedness of $\mathbf{C}_1$ and $\mathbf{C}_2$. Thus, $(\varphi_1 \wedge \psi_1)\backslash(\varphi_2 \vee \psi_2)$ is not provable in $\mathbf{L}$. Hence, $\mathbf{L}$ has the MVP. □

We assume that both $\mathbf{A}$ and $\mathbf{B}$ are *non-degenerate* in the condition (2) of Theorem 3.1. By checking the above proof carefully, it can be shown that this assumption is replaced by a stronger assumption that *both* $\mathbf{A}$ *and* $\mathbf{B}$ *are subdirectly irreducible.* Also, it is not hard to show that this assumption can be replaced by a weaker form that *both* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary algebras in* $V(\mathbf{L})$.

As we mentioned before, Halldén completeness is a special case of the MVP. Thus, by replacing *strong well-connectedness* by *well-connectedness* in Theorem 3.1, we get another characterization of the Halldén completeness.

Theorem 3.2. *Let* $\mathbf{L}$ *be a logic over* $\mathbf{FL}$*. Then the following two conditions are equivalent;*

(1) $\mathbf{L}$ *is Halldén complete,*

(2) for every two non-degenerate $\mathbf{FL}$*-algebras* $\mathbf{A}, \mathbf{B}$ *in* $V(\mathbf{L})$*, there exist an* $\mathbf{FL}$*-algebra* $\mathbf{C}$ *and subalgebras* $\mathbf{C}_1, \mathbf{C}_2$ *of* $\mathbf{C}$ *in* $V(\mathbf{L})$ *such that* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *form a well-connected pair and moreover that* $\mathbf{A}$ *and* $\mathbf{B}$ *are homomorphic images of* $\mathbf{C}_1$ *and* $\mathbf{C}_2$*, respectively.*

Again, it is shown that in the condition (2), the assumption for algebras $\mathbf{A}$ and $\mathbf{B}$ can be replaced by the condition that they are arbitrary algebras in $V(\mathbf{L})$.

It will be interesting to compare Theorem 3.2 with an algebraic characterization of the disjunction property (DP), since from a syntactic point of view the Halldén completeness follows immediately from the disjunction property. D. Souma [11] pointed out that Maksimova's characterization of the DP given in [8] holds for all substructural logics over $\mathbf{FL}$. That is,

Proposition 3.3. *Let* $\mathbf{L}$ *be a logic over* $\mathbf{FL}$*. Then the following two conditions are equivalent;*

(1) $\mathbf{L}$ *has the disjunction property,*

(2) for all $\mathbf{A}, \mathbf{B}$ *in* $V(\mathbf{L})$ *there exist a well-connected algebra* $\mathbf{C} \in V(\mathbf{L})$ *and a surjective homomorphism* h *from* $\mathbf{C}$ *to the direct product* $\mathbf{A} \times \mathbf{B}$ *of* $\mathbf{A}$ *and* $\mathbf{B}$*.*

In fact, if there exists such a well-connected algebra $\mathbf{C}$ and a surjective homomorphism h as mentioned in Proposition 3.3, $\mathbf{C}$ with itself form a well-connected pair, and moreover the composition of h and each projection map gives a surjective homomorphism from $\mathbf{C}$ to $\mathbf{A}$ or from $\mathbf{C}$ to $\mathbf{B}$.

The next theorem says that an algebraic characterization of the deductive Maksimova's variable separation property (DMVP) for substructural logics can be obtained from the algebraic characterization of the MVP in Theorem 3.1 by replacing the strongly well-connectedness and the existence of homomorphisms by the *well-connectedness* and the existence of *isomorphisms*, respectively.

Theorem 3.4. *Let* $\mathbf{L}$ *be a logic over* $\mathbf{FL}$*. Then the following two conditions are equivalent;*

(1) $\mathbf{L}$ *has the DMVP,*

(2) for every two non-degenerate $\mathbf{FL}$*-algebras* $\mathbf{A}, \mathbf{B}$ *in* $V(\mathbf{L})$*, there exist an* $\mathbf{FL}$*-algebra* $\mathbf{D}$ *and subalgebras* $\mathbf{D}_1, \mathbf{D}_2$ *of* $\mathbf{D}$ *in* $V(\mathbf{L})$ *such that* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *form a well-connected pair and moreover that* $\mathbf{A}$ *and* $\mathbf{B}$ *are isomorphic to* $\mathbf{D}_1$ *and* $\mathbf{D}_2$*, respectively. In other words, every two non-degenerate* $\mathbf{FL}$*-algebras* $\mathbf{A}, \mathbf{B}$ *in* $V(\mathbf{L})$ *can be jointly embedded into an* $\mathbf{FL}$*-algebra* $\mathbf{D} \in V(\mathbf{L})$ *and their images form a well-connected pair.*

Proof. The proof goes similarly to that of Theorem 3.1. Suppose that $\mathbf{L}$ has the DMVP, and let $\mathbf{A}$ and $\mathbf{B}$ be non-degenerate $\mathbf{FL}$-algebras in $V(\mathbf{L})$. Like before, we take disjoint sets of variables Y and Z that are enough big to ensure the existence of surjective maps from Y to A and Z to B, respectively. Let X be the union of Y and Z, and let $\mathbf{C}$, $\mathbf{C}_1$ and $\mathbf{C}_2$ be free algebras in $V(\mathbf{L})$ generated by X, Y and Z, respectively. Obviously, $\mathbf{C}_1$ and $\mathbf{C}_2$ are regarded as subalgebras of $\mathbf{C}$, and by the universal mapping property, there exist surjective homomorphisms $h : \mathbf{C}_1 \to \mathbf{A}$ and $k : \mathbf{C}_2 \to \mathbf{B}$. Let $F_1 = h^{-1}(\uparrow 1_{\mathbf{A}})$ and $F_2 = k^{-1}(\uparrow 1_{\mathbf{B}})$. Here, $\uparrow 1_{\mathbf{A}} = \{a \in A : 1 \leq a\}$, and similarly $\uparrow 1_{\mathbf{B}}$ can be defined. Then both F_1 and F_2 are proper filters of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, as both $\mathbf{A}$ and $\mathbf{B}$ are non-degenerate. By the homomorphism theorem, $\mathbf{A} \cong \mathbf{C}/F_1$ and $\mathbf{B} \cong \mathbf{C}/F_2$. We show that there exists a filter G of $\mathbf{C}$ such that

(a) $F_1 = C_1 \cap G$ and $F_2 = C_2 \cap G$,

(b) for any $b \in C_1$ and any $c \in C_2$, $b \vee c \in G$ implies either $b \in G$ or $c \in G$.

Let G be the filter of $\mathbf{C}$ generated by the set $F_1 \cup F_2$. Obviously, G is written as follows.

> $G = \{x \in C : \Pi_{i=1}^{n}\gamma_i(a_i) \leq x$ for some $a_i \in F_1 \cup F_2$ and some iterate conjugates γ_i on $\mathbf{C}$ with $1 \leq i \leq n\}$.

Now we show that $F_1 = C_1 \cap G$. It is easy to see that $F_1 \subseteq C_1 \cap G$. For the converse direction, suppose that $d \in C_1 \cap G$. Then, there exist some $a_i \in F_1 \cup F_2$ and some iterated conjugates γ_i on $\mathbf{C}$ with $1 \leq i \leq m$ such that $\Pi_{i=1}^{m}\gamma_i(a_i) \leq d$. Since d belongs to C_1, there exists a formula s^* over Y such that $d = s^*/\equiv_{\mathbf{L}}$. Similarly, if a_j belongs to F_1 (F_2) there exists a formula u_j over Y (over Z, respectively) such that $a_j = u_j/\equiv_{\mathbf{L}}$. Then, $\Pi_{i=1}^{m}\gamma_i(a_i) \leq d$ holds in $\mathbf{C}$ iff $1_{\mathbf{C}} \leq (\Pi_{i=1}^{m}\gamma_i(a_i))\backslash d$ holds in $\mathbf{C}$. The latter implies that $(\Pi_{i=1}^{m}\sigma_i(u_i))\backslash s^*$ is provable in $\mathbf{L}$, where each $\sigma_i(u_i)$ is a formula (over X) corresponding to $\gamma_i(a_i)$ for each i. Then, this in turn implies $\{u_i : 1 \leq i \leq m\} \vdash_{\mathbf{L}} s^*$. Now let us take a formula t^* over Z such that $t^*/\equiv_{\mathbf{L}} \notin F_2$, as F_2 is a proper deductive filter. Obviously, $\{u_i : 1 \leq i \leq m\} \vdash_{\mathbf{L}} s^* \vee t^*$. Since Y and Z are disjoint, by our assumption on the DMVP of $\mathbf{L}$ either $\{s_j : j \in J\} \vdash_{\mathbf{L}} s^*$ or $\{t_p : p \in P\} \vdash_{\mathbf{L}} t^*$ holds, where each s_j (and t_p) is a formula in $\{u_i : 1 \leq i \leq m\}$ over Y (and Z, respectively). Suppose that $\{t_p : p \in P\} \vdash_{\mathbf{L}} t^*$. Let k' be a valuation on $\mathbf{B}$ defined by $k'(z) = k(z/\equiv_{\mathbf{L}})$ for $z \in Z$. Then, we have $\mathbf{B}, k' \models t_p$ by the definition of F_2, and also $k(t_p/\equiv_{\mathbf{L}}) \geq 1_{\mathbf{B}}$. Hence, $\mathbf{B}, k' \models t^*$, which implies $k'(t^*) = k(t^*/\equiv_{\mathbf{L}}) \geq 1_{\mathbf{B}}$. Hence, $t^*/\equiv_{\mathbf{L}} \in F_2$. This contradicts to the choice of t^*. Thus, $\{s_j : j \in J\} \vdash_{\mathbf{L}} s^*$ must hold. Using the same argument as the above, this implies that $d = s^*/\equiv_{\mathbf{L}} \in F_1$. Therefore, $F_1 = C_1 \cap G$. Similarly, we can show $F_2 = C_2 \cap G$.

It remains to show that G satisfies the condition (b). Suppose that $u \vee v \in G$ for $u \in C_1$ and $v \in C_2$. Then, by the definition of G, there exist some $a_i \in F_1 \cup F_2$ and iterated conjugates γ_i on $\mathbf{C}$ with $1 \leq i \leq m$ such that $\Pi_{i=1}^{m}\gamma_i(a_i) \leq u \vee v$. Then, there exist formulas s' over Y and

t' over Z such that $u = s'/\equiv_{\mathbf{L}}$ and $v = t'/\equiv_{\mathbf{L}}$. Also, there exists a formula u_j over Y (Z) if a_j belongs to F_1 (F_2, respectively) such that $a_j = u_j/\equiv_{\mathbf{L}}$. Then, $\Pi_{i=1}^m \gamma_i(a_i) \le u \vee v$ iff $1_{\mathbf{C}} \le (\Pi_{i=1}^m \gamma_i(a_i))\backslash(u \vee v)$, which implies $(\Pi_{i=1}^m \sigma_i(\varphi_i))\backslash(s' \vee t')$ is provable in $\mathbf{L}$, where each $\sigma_i(u_i)$ is a suitable formula (over X) corresponding to $\gamma_i(a_i)$ for each i. Thus, we have $\{u_i : 1 \le i \le m\} \vdash_{\mathbf{L}} s' \vee t'$.

By the DMVP of $\mathbf{L}$, either $\{s_j : j \in J\} \vdash_{\mathbf{L}} s'$ or $\{t_p : p \in P\} \vdash_{\mathbf{L}} t'$ holds, where each s_j (and t_p) is a formula in $\{u_i : 1 \le i \le m\}$ over Y (and Z, respectively). Suppose that the latter holds. Taking the same valuation k' on $\mathbf{B}$ introduced in the above, we can show that $v = t'/\equiv_{\mathbf{L}} \in F_2$. Since $F_2 = C_2 \cap G$ holds, we have $v \in G$. Similarly, $\{s_j : j \in J\} \vdash_{\mathbf{L}} s'$ implies $u \in G$. Therefore, either $u \in G$ or $v \in G$.

We continue the proof. We show next that both $\mathbf{C}_1/F_1$ and $\mathbf{C}_2/F_2$ are embedded into $\mathbf{C}/G$. Define mappings $f : \mathbf{C}_1/F_1 \to \mathbf{C}/G$ and $g : \mathbf{C}_2/F_2 \to \mathbf{C}/G$ by $f(x/F_1) = x/G$ and $g(y/F_2) = y/G$, respectively. Since F_1 and F_2 are subsets of G, these mappings f and g are well-defined homomorphisms. For $x, x' \in C_1$, $f(x/F_1) = f(x'/F_1)$ implies $x/G = x'/G$, and by the property of G shown above, $x/F_1 = x'/F_1$. Thus, f is injective. Similarly, g is also injective. Now, let us denote $\mathbf{C}/G$, $f(\mathbf{C}_1/F_1)$ and $g(\mathbf{C}_2/F_2)$ by $\mathbf{D}$, $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively. Then, $\mathbf{D}_1$ and $\mathbf{D}_2$ are subalgebras of $\mathbf{D}$, and $\mathbf{A}$ and $\mathbf{B}$ are isomorphic to $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively. It remains to show that $\mathbf{D}_1$ and $\mathbf{D}_2$ form a well-connected pair. Suppose that $a \vee b \ge 1_{\mathbf{D}}$ for $a \in D_1$ and $b \in D_2$. Then there exist a formula s over Y and a formula t over Z such that $a = f(s/F_1)$ and $b = g(t/F_2)$. Then, $a \vee b = s/G \vee t/G = (s \vee t)/G \ge 1_{\mathbf{D}}$ and hence, $s \vee t \in G$. By using the property (b) of G shown above, either $s \in G$ or $t \in G$. Thus, either $a = s/G \ge 1$ or $b = t/G \ge 1$ holds. Therefore, $\mathbf{D}_1$ and $\mathbf{D}_2$ form a well-connected pair.

Conversely, we assume the second condition (2) in our theorem and show that $\mathbf{L}$ has the DMVP. Suppose that neither $\varphi_1 \vdash_{\mathbf{L}} \varphi_2$ nor $\psi_1 \vdash_{\mathbf{L}} \psi_2$ hold for formulas $\varphi_1, \varphi_2, \psi_1$ and ψ_2 such that any variable appearing either of φ_1 and φ_2 appears neither of ψ_1 and ψ_2. Then, there are $\mathbf{FL}$-algebras $\mathbf{A}$ and $\mathbf{B}$ in $V(\mathbf{L})$ and valuations f and g on them such that (i) $\mathbf{A}, f \models \varphi_1$ but $\mathbf{A}, f \not\models \varphi_2$, and (ii) $\mathbf{B}, g \models \psi_1$ but $\mathbf{B}, g \not\models \psi_2$. By our assumption, there exist an $\mathbf{FL}$-algebra $\mathbf{D}$ and subalgebras $\mathbf{D}_1, \mathbf{D}_2$ of $\mathbf{D}$ in $V(\mathbf{L})$ such that $\mathbf{D}_1$ and $\mathbf{D}_2$ form a well-connected pair and moreover that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic to $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively. For the sake of simplicity, we identify **A** with $\mathbf{D}_1$ and **B** with $\mathbf{D}_2$, since each of these pairs is isomorphic. Because of the disjointness of variables, we can take a valuation h on **D** such that $h(p) = f(p)$ for each variable p appearing either of φ_1 and φ_2 and $h(q) = g(q)$ for each variable q appearing either of ψ_1 and ψ_2. Then $h(\varphi_i) = f(\varphi_i)$ and $h(\psi_i) = g(\psi_i)$ for $i = 1, 2$. It is clear that $h(\varphi_1 \wedge \psi_1) = f(\varphi_1) \wedge g(\psi_1) \geq 1$. On the other hand, $h(\varphi_2) = f(\varphi_2) \not\geq 1$ and $h(\psi_2) = g(\psi_2) \not\geq 1$. Therefore, $h(\varphi_2 \vee \psi_2) = h(\varphi_2) \vee h(\psi_2) \not\geq 1$, since **A** and **B** form a well-connected pair. That is, $\mathbf{D}, h \models \varphi_1 \wedge \psi_1$ but $\mathbf{D}, h \not\models \varphi_2 \vee \psi_2$. Thus, $\varphi_1 \wedge \psi_1 \vdash_{\mathbf{L}} \varphi_2 \vee \psi_2$ doen't hold. This completes the proof of the DMVP of **L**. □

Similarly to Theorem 3.1, in the condition (2) of Theorem 3.4 the assumption that **A** and **B** are non-degenerate is replaced also by a stronger one that they are subdirectly irreducible. But, we cannot replace it by *arbitrary* algebras. For, if one is degenerate and the other is non-degenerate, they cannot be jointly embedded.

Our characterizations given in the present paper can be summerized as follows. For given two algebras **A** and **B**, each of them says about a condition (a) on two subalgebras of the third algebra and a condition (m) on mappings from these to **A** and **B**.

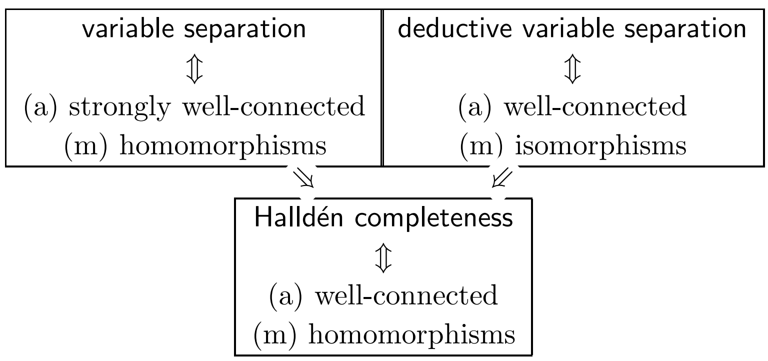

Figure 2: Relations among MVP, DMVP and HC

If we restrict our attention to logics over $\mathbf{FL_{ew}}$, we can give better results on algebraic characterizations. For example, the following is an extension of a result by Maksimova [9] for superintuitionistic logics.

Theorem 3.5. *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$.

(1) $\mathbf{L}$ *has the DMVP,*

(2) *all pairs of subdirectly irreducible algebras in* $V(\mathbf{L})$ *are jointly embeddable into a subdirectly irreducible algebra in* $V(\mathbf{L})$.

However, as it happens in the case of the Halldén completeness (see Theorem 2.5), the characterization of the DMVP using joint embeddability of $V(\mathbf{L})$ which is mentioned in the above Theorem 3.5 will not be properly extendible to an arbitrary logic $\mathbf{L}$ over $\mathbf{FL_e}$. In fact as the following result shows, the condition (1) does not exactly express the DMVP.

Theorem 3.6. *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$.

(1) *for all formulas* $\alpha_1 \to \alpha_2$ *and* $\beta_1 \to \beta_2$ *that have no propositional variables in common,* $\alpha_1 \wedge \beta_1 \vdash_{\mathbf{L}} (\alpha_2 \wedge 1) \vee (\beta_2 \wedge 1)$ *implies either* $\alpha_1 \vdash_{\mathbf{L}} \alpha_2$ *or* $\beta_1 \vdash_{\mathbf{L}} \beta_2$,

(2) *all pairs of subdirectly irreducible algebras in* $V(\mathbf{L})$ *are jointly embeddable into a subdirectly irreducible algebra in* $V(\mathbf{L})$.

Though the DMVP can be characterized by the joint embeddability of given two algebras of a given variety into the third, in our characterization of the MVP we represent these two algebras as homomorphic images of two subalgebras of the third. This idea works quite well also for algebraic characterizations of various types of interpolation properties, which will be discussed in our forthcoming paper.

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School of Information Science
Japan Advanced Institute of Science and Technology
Asahidai, Nomi, Ishikawa, 923-1292, Japan
{h-kihara,ono}@jaist.ac.jp