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Non-modal propositional languages on transitive frames and their embeddings.

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Abstract

This thesis reveals that many properties of the non-modal propositional language $\mathcal{L}$ and logics on $\mathcal{L}$ which is interpreted on transitive frames.

We can find that there are many unexpected phenomena in $\mathcal{L}$ and these logics on transitive frames as compared with them on quasi-ordered frames. There are no differences as for the properties of their model theory, for instance, the duality theorem holds, generated subframes and homomorphisms preserve validity from their original structures. However, by lacking reflexivity, it is showed that the expressive powers of $\mathcal{L}$ are weaker than that of modal propositional language, and an induced conseqeuce relation of some logics (for instnace, basic propositional logic **BPL**) does not satisfy the deduction theorem, etc. We gave one reason to define extensions on **BPL** not only as a formula-extension but also as a rule-extension, and discussed their model theory. We also indicate that these differences disappear by adding a new implication to $\mathcal{L}$.

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Chapter 1

Introduction

This thesis is a report on extensions of the non-modal propositional logic which is characterized by the class of transitive frames with an intuitionistic interpretation. We show that there exist unexpected differences between those extensions and propositional intermediate logics and we introduce a way to remove these differences.

1.1 Historical background

There exists a semantics called *frame semantics* in the logic. A *frame* $\mathfrak{F} = \langle W, R \rangle$ is a pair of a non-empty set W which elements are called *possible world* and a relation R on W. A frame semantics treats this frame structure. This frame semantics is well-known in the research into the artificial intelligence, and the frame structures which relation is transitive are treated very often in the computer and informaiton science. Our interests are this transitive frame semantics and the propositional language which express these structures under the intuitionistic interpretation.

It is well-known that some logic is complete with respect to some class of frames. In particular, the *intuitionistic logic* **Int** and the modal logic **S4** are complete with respect to the class of quasi-ordered frames, and the modal logic **K4** is complete with respect to the class of transitive frames.

There are many way to investigate properties of different logics which have similar models. One way of investigation is an *embedding*, and it is as follows: Suppose that two logics L_1 and L_2 are given. If there exists a function f from the language of L_1 to the language of L_2 such that it maps

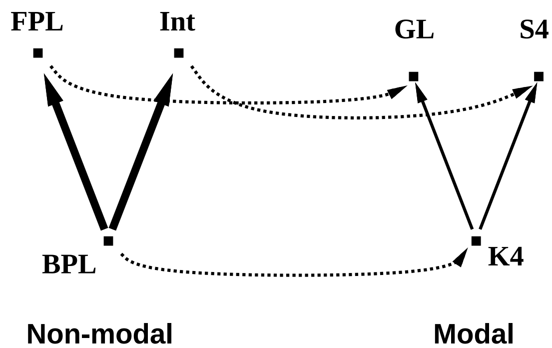


Figure 1.1: The relationship with propositional logics

all theorems and non-theorems of L_1 to theorems and non-theorems of L_2, respectively, we call this function an *embedding* of L_1 into L_2, and we say that L_1 is *embeddable* into L_2.

One of well-known embeddings is Gödel translation ([Göd33]). Throughout the thesis, let T stand for Gödel translation. T is also known as McKinsey-Tarski translation, ([MT48]), and S. Maehara obtained the same result independently ([Mae54]). T is an embedding of **Int** into **S4**. We can think of the modal operator $\Box$ of **S4** as expressing informal provability, whereas, the modal operator $\Box$ of Gödel-Löb logic **GL** expresses the provability of Peano arithmetic. **GL** enjoys a fixed point theorem, and we can also prove Gödel's incompleteness theorem from this fact. Modal logic **K4** is sometimes called *basic modal logic*. Both **S4** and **GL** are normal extensions of **K4**. It is well-known that there are no normal extensions of **S4** which are consistent normal extensions of **GL**.

A. Visser looked for a propositional logic that represents formal provability and enjoys a fixed point theorem. In 1981, he found such a logic system and introduced it as a natural deduction proof system ([Vis81]). He called it *Formal Propositional Logic* **FPL**, and showed that **FPL** is embeddable into **GL** by T. He also introduced *Basic Propositional Logic* **BPL** embeddable into **K4** by T and complete with respect to transitive frames again, as a natural deduction proof system. The propositional logic which is characterized by the class of linear transitive frames is denoted **BPLL**. **Int** and **FPL** (and also **BPLL**) are extensions of **BPL**. Visser pointed out that there are no extensions of **Int** which are consistent extensions of **FPL** simultaneously.

The relationship with these propositional logics is shown in the figure 1.1 . The three arrows "$\dashrightarrow$", "$\boldsymbol{\longrightarrow}$" and "$\longrightarrow$" denote respectively, embeddablity by T, extension by adding rules and extension by adding axioms.

The main results of this thesis are included in [SWZ97], [SO97] and [SWZ98].

We started our investigation by considering these notions introduced by Visser. Since any possible world of transitive frame is not need to be reflexive on any possible worlds, we can easily find models in which all formulas which the outer most connective is implication, for instance, $\neg\top$, are true.

Throughout this thesis, we will divide the provability into a *consequence relation* and a *provability of theorem*. Let a system L be given. When a formula φ is derived from a set , of formulas in this system L, this denoted by , $\vdash_L \varphi$, and we call $\vdash_L$ a *consequence relation*. This consequence relation is different from a *provablity of theorem* which is a possibilty of deducing a formula φ without any assumption in L $\emptyset \vdash_L \varphi$ This division of the provabilities yields that a division of modus ponens. Of course we explain this in the corresponding chapter, however we will also explain it here to emphasize this difference of modus ponens. Usually, modus ponens is expressed as the following figure:

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \quad .$$

The modus ponens of consequence relation means that

$$, \ \vdash_L \varphi \to \psi, \Delta \vdash_L \varphi \text{ implies } , , \Delta \vdash_L \psi,$$

while, the modus ponens of provability means that

$$\vdash_L \varphi, \vdash_L \varphi \to \psi \text{ implies} \vdash_L \psi.$$

In this thesis, we call the later as *modus ponens*. The former is called *implication elimination* as like an inference rule of natural deduction. The former property is known as the deduction theorem. It is well-known that the deduction theorem holds in **Int**. We proved that the deduction theorem does not hold on the consequence relation of **BPL** not only on implication connective but also on abstract implication connectives. In 1995, M. Ardeshir and W. Ruitenburg developed Gentzen style propositional sequent calculus for **BPL** ([AR95]). M. Ardeshir, in his doctoral thesis, introduced Gentzen type sequent calculus of *Basic Predicate Calculus* **GBQC** and proved that cut-elimination theorem holds ([Ard95]). In 1996, M. Ardeshir and W. Ruitenburg described fundamental properties of Gentzen style predicate calculi of

BQL and **BPL** ([AR96]). In 1997, M. Ardeshir investigated strong persistence of **BPL** ([Rui97]). We can introduce a Hilbert-style system which yields the provability of theorem by the axioms proposed by H. Ono ([Ono97], [SO97]). K. Sasaki also introduced a Hilbert-style system using the same axioms ([Sas98]). In particular, he proved that the consequence relation of **BPL** cannot be introduced by using weak modus ponens. In this thesis, we quoted his results to clarify the provability of theorem in **BPL**.

It is useful that treating semantics when we discuss general properties of extensions of logic. We adopt the general frame semantics as Visser adopted ([Vis81]) and the algebraic semantics by Ardeshir and Ruitenburg ([AR95]). The similar results of **Int**'s case hold on semantic structures and their substructures, and the duality between frames and algebras also holds.

There exists a question that what kind of extensions we ought to define. For the case of **Int**, it is usual that any extension is defined by adding formulas as axioms to **Int**. However, for the case of **BPL**, some problems occur by using such a extending way, and it is revealed by a notion of expressive powers. For quasi-ordered frame semantics, it is showed that expressive powers of the non-modal propositional language is same to that of the modal propositinoal language. It is proved, for instance, by using the embedding T. While, for transitive frame semantics, it is revealed that expressive powers of the non-modal propositional language is weaker than that of the modal propositional language, and the class of quasi-ordered frames cannot be axiomatizable. But, Visser showed that the class of all quasi-ordered frames is axiomatizable by adding inference rules ([Vis81]). We also adopt Visser's way, that is, an extension of **BPL** is defined by adding inference rules. We showed that there exists a logic characterized by finite frames which is not Kripke complete, and there are not isomorphism between extensions of **FPL** and normal extensions of **GL** like the Blok-Esakia theorem.

These differences disappear, however, when we add a new implication connective.

1.2 Organization of the thesis

Our goal is to clarify different properties of non-modal propositional logics, which are characterized by a class of quasi-ordered frames and a class of transitive frames. It means that we will advance our research by a comparison with results that hold on intermediate logics and normal extensions of **S4**. In

Chapter 2, we will review those results, which are related to our investigation, of intermediate logics, modal logics and their semantics.

In Chapter 3, we will introduce proof systems of **BPL**. Visser introduced **BPL** because it is embeddable in **K4** ([Vis81]). He seems to define his natural deduction proof system of **BPL** from transitive frame semantics. We will present it in Section 3.1 . Then we will briefly mention Visser's natural deduction proof system of **BPL**. In his system, no implication elimination rule is introduced. It leads to the fact that , , $\Delta \vdash \psi$ is, in general, not derivable from , $\vdash \varphi \to \psi$ and $\Delta \vdash \psi$. Indeed, in Section 3.2, we will prove that deduction theorem hold neither for the implication $\to$ nor any formulas whatsoever. This result is forcing us to consider the consequence relation "$\vdash$" when we investigate an extension of **BPL**. The question of a Hilbert style calculus for **BPL**, which had been an open problem, is taken up in Section 3.3 . We introduce a system **HB** which is identical to **BPL** ([SO97]) with respect to the provability of theorems. In system **HB**, we showed that deduction theorem does not hold not only on usual implication but also on another complex connectives. In Section 3.4, we will address a result of Sasaki's related to **HB** ([Sas98]). Sasaki indicates that we can define the consequence relation of **BPL** by axioms of **HB** and modus ponens without any sets of assumptions. He also stated that we cannot get the consequence relation of **BPL** by implication elimination rule alone. The results in Chapter 3 are also important when we discuss the extensions of **BPL** in Chapter 5.

In Chapter 4, after introducing an algebraic semantics of **BPL**, we will analyse the relationship between transitive frames and algebraic structures. We will proved that these are similar to the case of **Int**. One of main results in Chapter 4 is a duality theorem, that is showed in Section 4.2 . Generated subframes and p-morphisms are discussed in Section 4.3 and 4.4, respectively. We need them in Chapter 5.

Then, we turn to the relationship between **BPL** and modal logic. It is well-known that there is a one-to-one correspondence between extensions of **Int** (intermediate logics) and normal extensions of Grzegorczyk logic **Grz**. This result is called Blok–Esakia theorem ([Esa79a, Esa79b, Blo76]). The modal operator $\Box$ of **Grz** corresponds to the provability in a set theory. We will discuss whether or not a similar relationship holds between **FPL** and **GL**, and we will clarify relations between extensions of **BPL** and normal extensions of **K4** in general. However, since in **BPL**, implication elimination does not hold, some problem occurs. First we will introduce a notion of *global expressive power* to clarify the relationship between **BPL** and **K4**. By

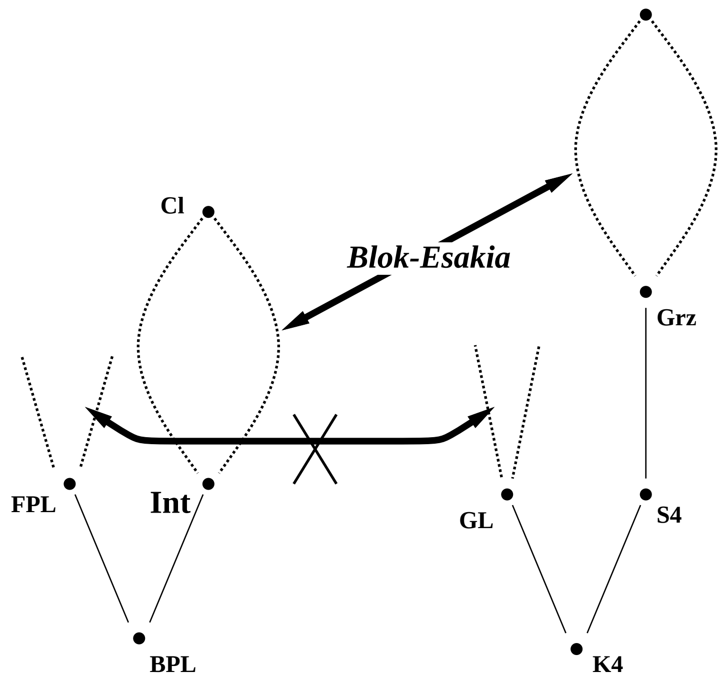


Figure 1.2: Lattice structures of extensions

this notion, it is proved that the frames of all quasi-ordered frames is not obtained by adding any number of axioms to **BPL**. This result leads us that not only axioms but also inference rules are good to define an extension of **BPL**. Therefore, consequence relations play main role whenever we consider extensions of **BPL**. These results are introduced in Section 5.1 and 5.2 . In the case of **Int** and its extensions, we define their semantics by using a validity on a class of frames (and, on a class of algebras). On the other hand, extensions of **BPL** will be defined by a consequence relation. Thus we have to introduce consequence relations in terms of semantics. They are discussed in Section 5.3 . Consequence relations of **BPL** and its extensions, and also consequence relations of semantic structures are defined, and Kripke completeness on some consequence relations is discussed in Section 5.4 . It is completely different from intermediate logics that there exists an extension, which is characterized by a finite general frame but Kripke incomplete. One object of this research is to clarify whether Blok-Esakia type theorem holds or not on extensions of **FPL** and normal extensions of **GL**. In Section 5.5, using these consequence relations and some caliculations, we show that there

	Int	**BPL**
deduction theorem	$\bigcirc$	$\times'$
duality theorem	$\bigcirc$	$\bigcirc'$
generation theorem	$\bigcirc$	$\bigcirc'$
substructures and homomorphisms	$\bigcirc$	$\bigcirc'$
Blok-Esakia theorem	$\bigcirc$	$\times'$
Kripke completeness with finite general frames	$\bigcirc$	$\times'$
	quasi-ordered frame	transitive frame
global expressiveness	$\bigcirc'$	$\times'$
local expressiveness	$\bigcirc'$	$\times'$

Table 1.1: Results of this thesis. (The symbols with $'$ are our results.)

is no isomorphism between extensions of **FPL** and normal extensions of **GL**. This result can be illustrated by Figure 1.2.

In Chapter 6, it is proved that some properties which don't hold in **BPL** but hold in **Int**, hold by introducing another new implication "$\hookrightarrow$" to the propositional language. In Section 5.1, to clarify what **BPL** lacks in comparison with **Int**, we will introduce a notion that *local expressive power* and we will show that a relationship between local expressiveness of quasi-ordered frames and local exrpressiveness of transitive frames. Conspicuous differences occur because "$\rightarrow$" in transitive frames is not able to talk about reflexivity. Thus, our solution is simple. We will introduce a new implication which can talk about the reflexivity and will show that the differences among **BPL** and **Int** are recovered by the help of this implication. The logic obtained from **Int** by adding axioms for $\hookrightarrow$ is denoted by **BiPL**. It is proved that **BiPL** can be translated into an intuitionistic modal logic since the behavior of "$\hookrightarrow$" is the same of the implication of **Int**. In Section 6.3, many properties hold on **BiPL** as similar to **Int**. For instance, it is proved that the Blok-Esakia theorem holds on transitive frames.

In Chapter 7, we will discuss existing representative researches of **BPL**, its related logics, and further works. This attractive subject related to **BPL** is not studied enough. The contents of the researches are classified into topics in terms of syntax and semantics.

Before closing Chapter 1, we put our results of this thesis into Table 1.1.

In the above table, our results is added a prime symbol.

Throughout this thesis, "Theorem" means a new result obtained in our thesis, whlie "Proposition" means a result which is shown already.

Chapter 2

Intuitionistic logic and basic modal logic

In this chapter, we will give a short survey on *intuitionistic propositional logic* **Int**, *basic modal logic* **K4** and their extensions. Here, we will explain the terminology and notions which are used in this thesis. We will also recall basic properties in order to compare them with *basic propositional logic* **BPL** discussed in the present thesis. For the detail we refer here the reader e.g., to [CZ97] or [MR74].

We will start with **Int**. After introducing the Hilbert style proof system, we will explain both frame semantics and algebraic semantics for **Int**. We will also give the notion of consistent extensions (*intermediate logics*) of **Int**. We will show that there exists a direct correspondence between frame semantics and algebraic sematnics.

Sometimes, modal logic **K4** is called basic modal logic, since some important modal logics are extensions of **K4**. Modal logic **K4** is itself an extension of **K**. Thus, we will first give the Hilbert style proof system of **K**. In extensions of modal logics, some complex problems occur in treating the modal operator $\Box$. Here, we will discuss only *normal extension.* After introducing the Hilbert style proof system of **K4** and **S4**, we will mention a completeness theorem of **K4** and **S4**, with respect to transitive frame semantics and quasi-ordered frame semantics, respectively.

Modal logics **S4** and *Gödel-Löb Logic* (**GL**) are normal extensions of **K4**. An *embedding* of a logic L into another logic L' is a function translating theorems (and non-theorems) of L into theorems (and non-theorems) of another logic. One of well-known embeddings is the *Gödel translation* T. T embeds

Int into **S4**, but it can also be regarded as an embedding of each extension of **Int** into a normal extension of **S4**. The study of this translation T is interesting, since some properties can be preserved by T. Also, we will explain the Blok-Esakia theorem without proofs.

2.1 Syntax of Int

In the present thesis, we will fix a (*non-modal*) *propositional language* $\mathcal{L}$. Let Prop be the set of *propositional variables*. Letters $p, q, r, \ldots$ will denote propositional variables. For$\mathcal{L}$ denotes the set of formulas constructed from Prop, $\wedge, \vee, \rightarrow$ and $\perp$ in the usual way. Greek letters $\varphi, \psi, \chi, \ldots$ will denote formulas. When we want to emphasize that a given formula is an element of For$\mathcal{L}$, we will call it an *$\mathcal{L}$-formula*. We consider $\top$ and the negation $\neg\varphi$ of a formula φ as abbreviations of $\perp \rightarrow \perp$ and $\varphi \rightarrow \perp$, respectively.

The *Hilbert style proof system* of **Int** consists of the following axiom schemes and inference rules:

Axiom schemes:

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,

(A2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$,

(A3) $\varphi \wedge \psi \rightarrow \varphi$,

(A4) $\varphi \wedge \psi \rightarrow \psi$,

(A5) $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$,

(A6) $\varphi \rightarrow \varphi \vee \psi$,

(A7) $\psi \rightarrow \varphi \vee \psi$,

(A8) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$,

(A9) $\perp \rightarrow \varphi$;

Inference rule:

- *Modus ponens* (MP): from φ and $\varphi \rightarrow \psi$, infer ψ.

A *derivability* $\vdash_{\mathbf{Int}}$ of this system is defined in the usual way. We identify a system L with the set of theorems of L in this chapter. Thus, **Int** means the set $\{\varphi \in \mathrm{For}\mathcal{L} : \vdash_{\mathbf{Int}} \varphi\}$ of theorems of **Int**.

As for the consequence relation $\vdash_{\mathbf{Int}}$, the following theorem is well-known:

Proposition 2.1 (deduction theorem) *Suppose* Γ *is a set of formulas, then*

$$\Gamma, \varphi \vdash_{\mathbf{Int}} \psi \text{ if and only if } \Gamma \vdash_{\mathbf{Int}} \varphi \to \psi.$$

2.2 Semantics of Int

We will refer two kinds of sematics. In this section, we will introduce a frame semantics of **Int**, and then an algebraic semantics of **Int**. At the end of this section, we will show relationships between frame semantics and algebraic semantics.

2.2.1 Frame semantics of Int

Let W be a non-empty set called a set of *possible worlds*, and R be a *quasi-order* (i.e., a reflexive and transitive relation) on W. A subset of X of W is an *R-cone* when $x \in X$ and xRy imply $y \in X$ for any x, y in W. We denote the set of all R-cones in W by $\mathrm{Up}W$. Note that the empty set $\emptyset$ is a member of $\mathrm{Up}W$.

Next, we will define the following binary operation $\supset$ on W, and call it *set implication*:

$$X \supset Y = \{x \in W : \forall y(xRy \wedge y \in X \Rightarrow y \in Y)\}. \tag{2.1}$$

It is easy to see that $\emptyset \supset \emptyset = W$, and $X \supset Y$ is an R-cone if both X and Y are so.

An *intuitionistic quasi-ordered frame* $\mathfrak{F}$ is a triple $\langle W, R, P\rangle$ such that W is a non-empty set of possible worlds, R a quasi-order on W, and P is a set of R-cones which contains the empty set $\emptyset$ and which is closed under the set union, set intersection and set implication $\supset$. Sometimes, we use the word an *intuitionistic frame* to denote a frame $\mathfrak{F}(= \langle W, R, P\rangle)$ where P is a subset of the set of R-cones. For a given frame $\langle W, R, P\rangle$, if R is a partial order (i.e., a reflexive, transitive and anti-symmetric relation) on W, we call it *intuitionistic partially ordered frame*.

If a given intuitionistic frame $\mathfrak{F}$ is of the form $\langle W, R, \mathrm{Up}W\rangle$, we call it an *intuitionistic Kripke frame* and denote it by $\langle W, R\rangle$.

We often omit the prefix word "intuitionistic". However, in the later sections, we will treat a frame structure for modal logic. To distinguish frames of non-modal propositional logic and modal propositional logic, we add the prefix word "intuitionistic" ("modal") to the former (later) structures.

A *valuation* $\mathfrak{V}$ on $\mathfrak{F}(=\langle W, R, P\rangle)$ is any function from Prop to P. A *model* $\mathfrak{M}$ based on $\mathfrak{F}$ is defined to be a pair $\langle \mathfrak{F}, \mathfrak{V}\rangle$ of a frame $\mathfrak{F}$ and a valuation $\mathfrak{V}$ on it.

The *truth relation* $\models$ is defined as follows:

$$(\mathfrak{M}, x) \not\models \bot, \tag{2.2}$$

$$(\mathfrak{M}, x) \models p \quad \text{iff} \quad x \in \mathfrak{V}(p), \tag{2.3}$$

$$(\mathfrak{M}, x) \models \varphi \wedge \psi \quad \text{iff} \quad (\mathfrak{M}, x) \models \varphi \text{ and } (\mathfrak{M}, x) \models \psi, \tag{2.4}$$

$$(\mathfrak{M}, x) \models \varphi \vee \psi \quad \text{iff} \quad (\mathfrak{M}, x) \models \varphi \text{ or } (\mathfrak{M}, x) \models \psi, \tag{2.5}$$

$$(\mathfrak{M}, x) \models \varphi \to \psi \quad \text{iff} \quad \forall y \in W(xRy \text{ and } (\mathfrak{M}, y) \models \varphi \text{ imply } (\mathfrak{M}, y) \models \psi). \tag{2.6}$$

A formula φ is *true at x in $\mathfrak{M}$* if $(\mathfrak{M}, x) \models \varphi$. A formula φ is *false at x in $\mathfrak{M}$* if φ is not true at x in $\mathfrak{M}$. If $(\mathfrak{M}, x) \models \varphi$ for any $x \in W$, we say φ is *true in* $\mathfrak{M}$ and denote it by $\mathfrak{M} \models \varphi$. If $\mathfrak{M} \models \varphi$ holds for all model $\mathfrak{M}$ based on $\mathfrak{F}$, we say φ is *valid* in $\mathfrak{F}$ and denote it by $\mathfrak{F} \models \varphi$. For a given $x \in W$, if $(\mathfrak{M}, x) \models \varphi$ for all models $\mathfrak{M}$ based on $\mathfrak{F}$, we say φ is *valid at x* in $\mathfrak{F}$ and we denote it by $(\mathfrak{F}, x) \models \varphi$. Let $\mathfrak{C}$ be a class of frames. If φ is valid in all frames of $\mathfrak{C}$, we say that φ is *valid* in $\mathfrak{C}$ and denote it by $\mathfrak{C} \models \varphi$. The above notions of truth and validity can be extended to any set , of formulas. That is, if all formulas of , are true (valid) on a semantic structure, then , is *true* (*valid*) on that semantic structure. For instance, , is *true at x in $\mathfrak{M}$* if $(\mathfrak{M}, x) \models \varphi$ for any formula φ of , , and we will denote it by $(\mathfrak{M}, x) \models$, . A set , of formulas is *valid* in $\mathfrak{C}$ if $\mathfrak{C} \models \varphi$ for any formula φ of , , and is denoted by $\mathfrak{C} \models$, . For a set , of formulas and a class $\mathfrak{C}$ of frames, if , is equal to the set $\{\varphi : \mathfrak{C} \models \varphi\}$, then we say , is *characterized* (or *determined*) by $\mathfrak{C}$. For any intuitionistic frame $\mathfrak{F}$, any set , of formulas and any formula φ, , $\models_{\mathfrak{F}} \varphi$ denotes that for any model $\mathfrak{M}$ based on $\mathfrak{F}$ and any possible world x of $\mathfrak{F}$, if $(\mathfrak{M}, x) \models$, holds then $(\mathfrak{M}, x) \models \varphi$ holds.

We can show the following completeness theorem.

Proposition 2.2 *Let $\mathfrak{C}$ be the class of intuitionistic partially ordered (or quasi-ordered) frames. Suppose that , is a set of formulas and φ is a formula. , $\vdash_{\mathbf{Int}} \varphi$ if and only if for any model $\mathfrak{M}$ and any x, $(\mathfrak{M}, x) \models$, implies $(\mathfrak{M}, x) \models \varphi$.*

Suppose $\mathfrak{F}(= \langle W, R, P\rangle)$ and $\mathfrak{G}(= \langle V, S, Q\rangle)$ are given. A *homomorphism* f from $\mathfrak{F}$ to $\mathfrak{G}$ is a map from W to V which satifies the following three conditions:

1. $xRy \Rightarrow f(x)Sf(y)$,
2. $f(x)Sz \Rightarrow \exists y \in W(xRy \wedge f(y) = z)$,
3. $X \in Q \Rightarrow f^{-1}(X) \in P$ where $f^{-1}(X) = \{x \in W : f(x) \in X\}$.

When a homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$ is surjective, we call it a *reduction* (or a *p-morphism*). For given frames $\mathfrak{F}$ and $\mathfrak{G}$, if there exists a homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$, we say that $\mathfrak{F}$ is *reducible* to $\mathfrak{G}$. If a homomorphism f from $\mathfrak{F}$ to $\mathfrak{G}$ is a bijective map and f^{-1} is a homomorphism from $\mathfrak{G}$ to $\mathfrak{F}$, we call f an *isomorphism* from $\mathfrak{F}$ to $\mathfrak{G}$. If there exists an isomorphism from $\mathfrak{F}$ to $\mathfrak{G}$, we denote this as $\mathfrak{F} \simeq \mathfrak{G}$, and say that $\mathfrak{F}$ and $\mathfrak{G}$ are *isomorphic*. A reduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called a *reduction* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ if $\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p))$ for every $p \in$ Prop, i.e.,

$$(\mathfrak{M}, x) \models p \text{ iff } (\mathfrak{N}, f(x)) \models p.$$

Models $\mathfrak{M}(= \langle \mathfrak{F}, \mathfrak{V}\rangle)$ and $\mathfrak{N}(= \langle \mathfrak{G}, \mathfrak{U}\rangle)$ are *isomorphic* if there is an isomorphism f from $\mathfrak{F}$ to $\mathfrak{G}$ such that $\mathfrak{U}(p) = f(\mathfrak{V}(p))$ for every $p \in$ Prop, i.e., for every $x \in W$,

$$(\mathfrak{M}, x) \models p \text{ iff } (\mathfrak{N}, f(x)) \models p.$$

We denote it by $\mathfrak{M} \simeq \mathfrak{N}$.

Proposition 2.3 *Let f be a reduction of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$. Then, for any point x of $\mathfrak{F}$ and any formula φ, $(\mathfrak{M}, x) \models \varphi$ holds if and only if $(\mathfrak{N}, f(x)) \models \varphi$ holds. If $\mathfrak{F}$ is reducible to $\mathfrak{G}$, then for every formula φ, $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$.*

Another structure-preserving notion is that of a substructure. The following is a substructure in the context of frame semantics. A *generated subframe* $\mathfrak{G}$ of a given frame $\mathfrak{F} = \langle W, R, P\rangle$ is a structure $\langle V, S, Q\rangle$ such that V is an R-cone, S is the restriction of R to V and Q is the set $\{X \cap V : X \in P\}$. It is easy to show that any generated subframe of intuitionistic quasi-ordered (partially ordered) frame is an intuitionisitc quasi-ordered (partially ordered) frame.

Proposition 2.4 *Let $\mathfrak{G}(=\langle V,S,Q\rangle)$ be a generated subframe of $\mathfrak{F}$. Suppose that $\mathfrak{M}$ is $\langle\mathfrak{F},\mathfrak{V}\rangle$ and $\mathfrak{N}$ is $\langle\mathfrak{G},\mathfrak{U}\rangle$, where $\mathfrak{U}(p)=\mathfrak{V}(p)\cap V$ for every propositional variables p. Then, for any point x of V and any formula φ, $(\mathfrak{M},x)\models\varphi$ holds if and only if $(\mathfrak{N},x)\models\varphi$ holds. Thus, $\mathfrak{F}\models\varphi$ implies $\mathfrak{G}\models\varphi$.*

2.2.2 Algebraic semantics of Int

Suppose $\langle A,\wedge,\vee,0,1\rangle$ is a bounded distributive lattice. For elements a,b of A, the greatest element of the set $\{x\in A : a\wedge x\leq b\}$, if it exists, is called the *relatively pseudo-complement* of a with respect to b, and write it as $a\rightarrow b$. A *Heyting algebra* $\mathfrak{A}$ is a structure $\langle A,\wedge,\vee,\rightarrow,0,1\rangle$ where $\langle A,\wedge,\vee,0,1\rangle$ is a bounded distributive lattice and $\rightarrow$ is the relative pseudo-complement.

A *valuation* $\mathfrak{V}$ on $\mathfrak{A}$ is any function from Prop to A. For a given valuation $\mathfrak{V}$, we will associate any formulas to algebraic terms by putting $\mathfrak{V}(\bot)=0$ and $\mathfrak{V}(\varphi\odot\psi)=\mathfrak{V}(\varphi)\odot\mathfrak{V}(\psi)$ where $\odot\in\{\wedge,\vee,\rightarrow\}$.

A *truth relation* $\models$ on $\mathfrak{A}$ is defined as follows: For any valuation $\mathfrak{V}$ on $\mathfrak{A}$, if $\mathfrak{V}(\varphi)=1$, we say that φ is *valid* in $\mathfrak{A}$ and write $\mathfrak{A}\models\varphi$. If $\mathfrak{A}\models\varphi$ for all $\varphi\in$, , then we write it as $\mathfrak{A}\models$, . For a class $\mathfrak{C}$ of Heyting algebras, if $\mathfrak{A}\models$, holds for any $\mathfrak{A}\in\mathfrak{C}$, then we write $\mathfrak{C}\models$, .

Suppose , is a set of formulas. If , is finite, $\bigwedge$, denotes the conjunction of all elements of , . For any valuation $\mathfrak{V}$ on $\mathfrak{A}$, if there exists a finite subset , $'$ of , such that $\mathfrak{V}(\varphi\wedge\bigwedge,')=\mathfrak{V}(\bigwedge,')$ (that is, $\mathfrak{V}(\bigwedge,')\leq\mathfrak{V}(\varphi)$), we denote it by , $\models_{\mathfrak{A}}\varphi$. If , $\models_{\mathfrak{A}}\varphi$ for any $\mathfrak{A}$ in a class $\mathfrak{C}$ of Heyting algebras, then we denote this as , $\models_{\mathfrak{C}}\varphi$.

For any $\mathfrak{A}$, it is trivial that $\mathfrak{A}\models\varphi$ implies , $\models_{\mathfrak{A}}\varphi$ for every , and φ, since $\mathfrak{V}(\varphi)=1$ holds for any valuaion $\mathfrak{V}$ on $\mathfrak{A}$.

Suppose formulas φ and ψ are given. We call the form $\varphi=\psi$ *equation* of formulas φ and ψ. Let , be a set of equation of formulas. In this set , , we think of $\varphi\leq\psi$ is the abbreviation for $\varphi\wedge\psi=\varphi$. We write $\mathfrak{A}\models$, , if $\mathfrak{V}(\varphi)=\mathfrak{V}(\psi)$ holds for any equation $\varphi=\psi$ of , and any valuation $\mathfrak{V}$ on $\mathfrak{A}$, and we say , is *valid* on $\mathfrak{A}$.

The following completeness theorem holds:

Proposition 2.5 *Let $\mathfrak{C}$ be the class of Heyting algebras. Suppose , is a set of formulas and φ is any formula. Then , $\vdash_{\mathbf{Int}}\varphi$ holds if and only if , $\models_{\mathfrak{C}}\varphi$ holds.*

Suppose $\mathfrak{A}(=\langle A,\wedge_A,\vee_A,\rightarrow_A,0_A,1_A\rangle)$ and $\mathfrak{B}(=\langle B,\wedge_B,\vee_B,\rightarrow_B,0_B,1_B\rangle)$ are given. A *homomorphism h from $\mathfrak{A}$ into $\mathfrak{B}$* is a map from A to B which preserves each operation: that is, $h(a \odot_A b) = h(a) \odot_B h(b)$ for each $\odot \in \{\wedge, \vee, \rightarrow\}$, $h(0_A) = 0_B$ and $h(1_A) = 1_B$. If a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ is injective, it is called an *embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$. If a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ is surjective, it is called an *epimorphism*, and $\mathfrak{B}$ is said to be *homomorphic image* of $\mathfrak{A}$. If an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ is onto, we call it an *isomorphism*. If there exists an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, we say $\mathfrak{A}$ is *isomorphic* to $\mathfrak{B}$ and denote it by $\mathfrak{A} \simeq \mathfrak{B}$.

Proposition 2.6 *Let $\mathfrak{A}$ be a Heyting algebra, and $\mathfrak{B}$ a homomorphic image of $\mathfrak{A}$. Then, for any formula φ, $\mathfrak{A} \models \varphi$ implies $\mathfrak{B} \models \varphi$.*

A *subalgebra* $\mathfrak{B}$ of $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a structure $\langle B, \wedge, \vee, \rightarrow, 0, 1\rangle$ where B is a subset of A such that B is closed under each operation of $\mathfrak{A}$. It is easy to see that any subalgebra of Heyting algberba is Heyitng algebra.

Proposition 2.7 *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. Then, for any formula φ, $\mathfrak{A} \models \varphi$ implies $\mathfrak{B} \models \varphi$.*

Suppose a Heyting algebra $\mathfrak{A}(= \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle)$ is given. A *filter* ∇ in $\mathfrak{A}$ is a subset of A which satisfies
1) $a, b \in \nabla$ implies $a \wedge b \in \nabla$,
2) $a \in \nabla$ and $a \leq b$ imply $b \in \nabla$.
A *proper filter* ∇ is a filter in $\mathfrak{A}$ which satisfies
3) $\nabla \neq A$.
A *prime filter* ∇ is a proper filter such that
4) $a \vee b \in \nabla$ implies $a \in \nabla$ and $b \in \nabla$.
The dual notions of the above filters are *ideal*, *proper ideal* and *prime ideal*. That is, an ideal Δ in $\mathfrak{A}$ is a subset of A which satisfies
1') $a, b \in \Delta$ implies $a \vee b \in \Delta$,
2') $a \in \Delta$ and $b \leq a$ imply $b \in \Delta$.
A proper ideal Δ in $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ which satisfies
3') $\Delta \neq A$,
A prime ideal Δ in $\mathfrak{A}$ is a proper ideal in $\mathfrak{A}$ such that
4') $a \wedge b \in \Delta$ implies $a \in \Delta$ or $b \in \Delta$.
When we discuss a representation of Heyting algebras, ideals and prime filters play the main role. We will show this in the next section.

2.2.3 Relations between the two semantics

By the completeness theorem of **Int** with respect to both semantics, we can show that φ is valid in any inutionisitic partially ordered frame if and only if φ is valid in any Heyting algebra. However, this result can be deduced in a more direct way. In the following, we will give a way of constructing a Heyting algebra from an intuitionistic partially ordered frame, and vice versa. Then, we will show that the validity of each formula is preserved by these constructions.

Suppose that an intuitionistic frame $\mathfrak{F}(=\langle W, R, P\rangle)$ is given. The *dual* of $\mathfrak{F}^+$ is a structure $\langle P, \cap, \cup, \supset, \emptyset, W\rangle$, where $\cap$, $\cup$ and $\supset$ is the set-intersection, set-union and set-implication which is defined by (2.1), respectively. The following is well-known.

Proposition 2.8 *For any intuitionistic partial-ordered frame $\mathfrak{F}$, $\mathfrak{F}^+$ is a Heyting algebra. Moreover, for any formula φ, $\mathfrak{F} \models \varphi$ if and only if $\mathfrak{F}^+ \models \varphi$.*

Conversely, suppose that a Heyting algebra $\mathfrak{A}(=\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle)$ is given. Let $W_{\mathfrak{A}}$ be the set of all prime filters, $\mathbf{p}$ be a map from A to the power set of $W_{\mathfrak{A}}$ such that $\mathbf{p}(a) = \{\nabla \in W_{\mathfrak{A}} : a \in \nabla\}$ and $P_{\mathfrak{A}}$ be the set $\{\mathbf{p}(a) : a \in \mathfrak{A}\}$. We will define a relation $R_{\mathfrak{A}}$ on $W_{\mathfrak{A}}$ as follows:

$$\nabla_0 R_{\mathfrak{A}} \nabla_1 \text{ iff } \forall a, b \in \mathfrak{A}(a \rightarrow b \in \nabla_0 \text{ and } a \in \nabla_1 \text{ imply } b \in \nabla_1).$$

The *dual* $\mathfrak{A}_+$ of $\mathfrak{A}$ is a structure $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}}\rangle$. The following result holds.

Proposition 2.9 *For any Heyting algebra $\mathfrak{A}$, $\mathfrak{A}$ is an intuitionistic partially ordered frame. Moreover, for any formula φ, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}_+ \models \varphi$.*

For a given intuitionistic frame $\mathfrak{F}$, the intuitionistic frame $(\mathfrak{F}^+)_+$ is called the *bidual* of $\mathfrak{F}$. Similarly, for a given Heyting algebra $\mathfrak{A}$, the Heyting algebra $(\mathfrak{A}_+)^+$ is called the *bidual* of $\mathfrak{A}$. Both kinds of biduals are useful when we try to construct a counter model for a given formula. When we consider the bidual $(\mathfrak{A}_+)^+$ of a given algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ where its universe A is finite, we may consider a possibility that $P_{\mathfrak{A}}$ becomes uncountable. However, for Heyting algebras, the following representation result holds. We will call the following *dualtiy theorem on Heyting algebras.*

Proposition 2.10 (duality) *For any Heyting algebra $\mathfrak{A}$, $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$.*

In the case of general frames an analogous result does not hold in general as the following example shows.

Example 2.11 *Let $\mathfrak{F}$ be $\langle W, R, \{W, \emptyset\}\rangle$ where W is the set $\{a, b, c\}$ and R is the reflexive and transitive closure of $\{\langle a, b\rangle, \langle b, c\rangle\}$. In this case, $\mathfrak{F}^+$ is a 2-valued Boolean algebra. Therefore, $(\mathfrak{F}^+)_+$ consists of a single reflexive point.*

We need the following conditons for a given bidual of a frame to be isomorphic to an original frame. Let a set X be given. Suppose Y be a subset of the power set of X. Then, Y has the *finite intersection property* if for any finite subset Y' of Y, $\bigcap Y' \neq \emptyset$.

Definition 2.12 *$\mathfrak{F} = \langle W, R, P\rangle$ is* descriptive *if*
1) $x = y$ iff for any $X \in P(x \in X \Leftrightarrow y \in X)$,
2) xRy iff for any $X, Y \in P(x \in X \supset Y$ and $y \in X \Rightarrow y \in Y)$,
3) $\langle W, P\rangle$ is compact, i.e., for all $\mathcal{X} \subseteq P$ and all $\mathcal{Y} \subseteq \{W \Leftrightarrow \mathcal{X} : X \in P\}$ if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\cap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

Then, we have the following *duality theorem on frames.*

Proposition 2.13 (duality) *For any frame $\mathfrak{F}$, $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$ if and only if $\mathfrak{F}$ is descriptive.*

In fact, there exist the following relations between dual structures and homomorphisms.

Proposition 2.14 (i) *If $\mathfrak{G} = \langle V, S, Q\rangle$ is a generated subframe of $\mathfrak{F} = \langle W, R, P\rangle$ then the map f from P to Q defined by $f(X) = X \cap V$ for $X \in P$ is a homomorphism from $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If f is a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ then f_+ from $W_\mathfrak{B}$ to $W_\mathfrak{A}$ defined by $f_+(\nabla) = f^{-1}(\nabla)$ for a prime filter ∇ of $\mathfrak{B}$, is an isomorphism from $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

(iii) *If h is a reduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G} = \langle V, S, Q\rangle$ then the map h^+ from P to Q defined by $h^+(X) = h^{-1}(X)$ for $X \in Q$, is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(iv) *If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ then the map h defined by $h(\nabla) = \nabla \cap B$, ∇ a prime filter in $\mathfrak{A}$ and B the universe of $\mathfrak{B}$, is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

2.3 Intermediate logics

In this section, we will consider extensions of **Int** which are usually called *superintuitionistic logic.* Let us strat with the syntactic side. Any map s from Prop to For$\mathcal{L}$ is called a *substitution.* For any formula φ and any substitution s, φs is defined inductively as follows:
1) $ps = s(p)$ for any $p \in$ Prop,
2) $\bot s = \bot$,

3) $(\varphi \odot \psi)s = \varphi s \odot \psi s$ for $\odot \in \{\wedge, \vee, \rightarrow\}$.
A set L of formulas is called a *superintuitionistic logic (si-logic)*, if it satisfies the following three conditions:

- $\mathbf{Int} \subseteq L$,
- L is closed under modus ponens (MP),
- L is closed under substitution (Subst):
  for any substitution s, $\varphi \in L$ implies $\varphi s \in L$.

For any set , of formulas, s, is $\{\varphi s : \varphi \in , \}$, and $S(,)$ is the substitution closure of , , i.e., $S(,) = \{\varphi s : s$ is any substitution $\}$. A si-logic L is called *intermediate logic* if it is consistent (i.e., $\perp \notin L$). We write Ext**Int** for the class of intermediate logics.

It is well-known that Ext**Int** forms a lattice by using some operators. We define $+$ as a binary operator on For$\mathcal{L}$ such that $,_1 + ,_2$ is the smallest set containing the union $,_1 \cup ,_2$ which is closed under both MP and Subst. Since $L = \mathbf{Int} + L$ always holds for any si-logic L, any si-logic L can be represented in the form $\mathbf{Int} + ,$ for some set , of fomulas. For instance, the classical propositional logic **Cl** has the form $\mathbf{Int} + p \vee (p \rightarrow \perp)$. Let $L_i = \mathbf{Int} + ,_i$ for $i = 1, 2$. Then $L_1 + L_2$ is equal to $\mathbf{Int} + (,_1 \cup ,_2)$.

Let L be $\mathbf{Int} + ,$. For any Σ and φ, we write $\Sigma \vdash_L \varphi$ if $S(, \cup \Sigma) \vdash_{\mathbf{Int}} \varphi$. This is well-known that for all intermediate logics L_1 and L_2, both $L_1 + L_2$ and the intersection $L_1 \cap L_2$ become intermediate logics. Moreover the following holds.

Proposition 2.15 $\langle \mathrm{Ext}\mathbf{Int}, \cap, +, \mathbf{Int}, \mathbf{Cl} \rangle$ *forms a bounded distribuitive lattice.*

Next, we will discuss semantics for extensions of **Int**. Let Log$\mathfrak{C}$ be the set of formulas which are valid in $\mathfrak{C}$, i.e., $\mathrm{Log}\mathfrak{C} = \{\varphi : \mathfrak{C} \models \varphi\}$. An intermediate logic is *Kripke complete* if it is charcterized by some class of intuitionistic Kripke frames. An intermediate logic L is *strongly characterized* by $\mathfrak{C}$ if for any set , of formulas and any formula φ,

$$, \vdash_L \varphi \text{ iff } , \models_{\mathfrak{F}} \varphi, \text{ for every } \mathfrak{F} \text{ in } \mathfrak{C}.$$

Proposition 2.16 *Every intermediate logic is characterized by a class of Heyting algebras. Hence, every intermediate logic is characterized by a class of intuitionistic general frames.*

L is *strongly Kripke complete* if L is strongly characterized by some class of Kripke frames. A logic L is said to be *finitely approximable* (or to have a *finite model property*) if L is characterized by a class of finite frames. It is known that there exists an intermediate logic L which is not Kripke complete or which is not finitely approximable. As for **Int**, we have the following.

Proposition 2.17 **Int** *is strongly Kripke complete and finitely approximable.*

2.4 Basic modal logic K4 and its extensions

Sometimes, modal logic **K4** is called *basic modal logic* since some of important modal logics are extensions of **K4**. We will often use some properties which are associated with modal logics **K4**, **S4**, **Grz**, **GL** and their extensions. In this section, we will recall an introductory information about these logics. We will begin with the Hilbert-type proof system of **K**, and its frame semantics. Here, we will consider only *normal extensions* of **K4**. Then, we will introduce the syntax and frame semantics of **K4** and **S4**. From the fact that each extension of **S4** has a quasi-ordered frame semantics, there will be a relation between intermediate logics and extensions of **S4**. This topic will be discussed in the next section.

2.4.1 Modal logic K

First, we will fix a *modal propositional language* $\mathcal{ML}$. The set For$\mathcal{ML}$ is constructed from Prop, $\wedge$, $\vee$, $\rightarrow$, $\Box$ and $\bot$ in the usual induction. We call each element of For$\mathcal{ML}$ a *modal formula* or a $\mathcal{ML}$*-formula*. We take $\Diamond\varphi$ and $\Box^{+}\varphi$ to be the abbreviation of $\neg\Box\neg\varphi$ and $\Box\varphi and \varphi$, respectively, and $\Box^{n}\varphi$ to denote the following formula:
i) $\Box^{0}\varphi = \varphi$,
ii) $\Box^{n+1}\varphi = \Box\Box^{n}\varphi$.

The *Hilbert-style proof system* of **K** consists of the following axiom schemes and inference rules:

Axiom schemes:

(**Int**) Axiom schemes (A1), . . . , (A9) of **Int**,

(**Cl**) $\varphi \vee (\varphi \rightarrow \bot)$, (dist) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

Inference rules:

- MP,
- *Necesitation* (RN) : from a formula φ, infer $\Box\varphi$.

As usual, $\vdash_{\mathbf{K}} \varphi$ means [1] that φ is derived from the above axiom schemes and inference rules.

Then, we will introduce *frame semantics* for $\mathcal{ML}$. Let W be a non-empty set of possible worlds and R a relation on W. The box operator $\Box$ on 2^W (the power set of W) is defined as follows. For any subset X of W,

$$\Box X = \{x \in W : \forall y(xRy \Rightarrow y \in X)\}.$$

A *modal frame* is a triple $\langle W, R, P\rangle$ where P is a subset of 2^W which contains both W and $\emptyset$ and is closed under the set-difference, set-intersection, set-union and $\Box$ on 2^W. If R is a transitive relation (or a quasi-order) on W, a frame $\langle W, R, P\rangle$ is called a *modal transitive frame* (or a *modal quasi-ordred frame*). A *modal Kripke frame* is a frame of the form $\langle W, R, 2^W\rangle$, and we denote it by $\langle W, R\rangle$. A *valuation* $\mathfrak{V}$ in a modal frame $\mathfrak{F}(= \langle W, R, P\rangle)$ is any function from Prop to P. A *model* based on $\mathfrak{F}$ is defined to be a pair $\langle \mathfrak{F}, \mathfrak{V}\rangle$ of a frame $\mathfrak{F}$ and a valuation $\mathfrak{V}$ in it.

For a given valuation $\mathfrak{V}$ in a given frame $\mathfrak{F}$, the *truth relation* $\models$ determined by $\mathfrak{V}$ is defined as follows:

$$(\mathfrak{M}, x) \not\models \bot, \tag{2.7}$$
$$(\mathfrak{M}, x) \models p \quad \text{iff} \quad x \in \mathfrak{V}(p), \tag{2.8}$$
$$(\mathfrak{M}, x) \models \varphi \wedge \psi \quad \text{iff} \quad (\mathfrak{M}, x) \models \varphi \text{ and } (\mathfrak{M}, x) \models \psi, \tag{2.9}$$
$$(\mathfrak{M}, x) \models \varphi \vee \psi \quad \text{iff} \quad (\mathfrak{M}, x) \models \varphi \text{ or } (\mathfrak{M}, x) \models \psi, \tag{2.10}$$
$$(\mathfrak{M}, x) \models \varphi \to \psi \quad \text{iff} \quad (\mathfrak{M}, x) \not\models \varphi \text{ or } (\mathfrak{M}, x) \models \psi, \tag{2.11}$$
$$(\mathfrak{M}, x) \models \Box\varphi \quad \text{iff} \quad \forall y \in W(xRy \text{ implies}(\mathfrak{M}, y) \models \varphi). \tag{2.12}$$

Similar to Section 2.2, validity and all associated notions, in a given modal frame can be defined in a usual way. The following completeness theorem is known.

[1] Here, we will treat not only a consequence relation of **K** but also the provability of theorems of **K**.

Proposition 2.18 *Let $\mathfrak{C}$ be the class of modal frames. Then, for any modal formula φ, $\models_{\mathfrak{C}} \varphi$ holds if and only if $\vdash_{\mathbf{K}} \varphi$ holds.*

When we try to define a consequence relation for a modal logic L, we meet the following problem, connected with RN. Let us consider the formula $p \to \Box p$. Then, we can easily show that there exists a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that it gives $\mathfrak{F} \not\models p \to \Box p$. An example of such a model is given by $W = \{a, b\}$, $R = \{(a, b)\}$ and $\mathfrak{V}(p) = \{a\}$. If we take the definition of the consequence relation $\vdash^*_{\mathbf{K}}$ of modal logic **K** in the same way as the intuitionistic case, we can derive $p \vdash^*_{\mathbf{K}} \Box p$ as we can get $\Box p$ by applying RN to p. If moreover the deduction theorem holds, we get $\vdash^*_{\mathbf{K}} p \to \Box p$.

We want to define a consequence relation of modal logic L which would have "the same meaning" as $\models$. For that reason, we will allow to apply RN only to axioms. More precisely, we will define a consequence relation $\vdash_L$ for a modal logic L as follows: Let a derivation $\varphi_1, \ldots, \varphi_n$ be given. In this derivation, we will express that φ_k *depends* on φ_i if either $k = i$ or φ_k is obtained by applying MP or RN to formulas, at least one of which depends on φ_i. Then, , $\vdash_L \varphi$ means that there exists a derivation where RN is applied only to formulas which depend on axioms, but not on other assumptions.

The following is the deduction theorem for modal logic **K**:

Proposition 2.19 *Suppose , , $\varphi \vdash_{\mathbf{K}} \psi$ and there exists a derivation which derives ψ from the assumption , $\cup \{\varphi\}$ by applying RN n-times to formulas which depend on φ. Then , $\vdash_{\mathbf{K}} \Box^0 \varphi \wedge \ldots \wedge \Box^n \varphi \to \psi$.*

2.4.2 Modal logic K4 and its normal extensions

Modal logic **K4** is defined by adding the following axiom scheme 4 to **K**:

(4) $\Box \varphi \to \Box \Box \varphi$;

and modal logic **S4** is defined by adding the following axiom scheme T to **K4**:

(T) $\Box \varphi \to \varphi$.

A set L of $\mathcal{ML}$-formulas is a *normal extension*[2] of modal logic **K** if L satisfies the following four conditions:

[2] There are other types of extensions of a modal logic. For instance, a *quasi-normal extension* is obtained when we drop the condition iv). However, in this thesis, our attention is only on normal extensions.

i) $\mathbf{K} \subseteq L$,
ii) L is closed under modus ponens (MP),
iii) L is closed under substituition (Subst),
iv) L is closed under necessitation (RN).
Sometimes, a normal extension L of $\mathbf{K}$ is called a *normal modal logic*. When a modal logic L which is defined by adding an axiom scheme ψ to $\mathbf{K}$ it is denoted by $\mathbf{K} \oplus \psi$. That is, $\mathbf{K} \oplus \psi$ means that we can apply RN to ψ directly. It is clear this modal logic L is a normal extension of $\mathbf{K}$, and $\oplus$ denotes an operator which "takes a MP, Subst and RN closure of the union of axiom scheme $\mathbf{K}$ and $\{\psi\}$". By this notation, we can express $\mathbf{K4}$ as $\mathbf{K} \oplus (\Box\varphi \rightarrow \Box\Box\varphi)$. Sometimes, if a given axiom scheme has a name (for instance, $\Box\varphi \rightarrow \Box\Box\varphi$ has a name 4), we will use a name of an axiom scheme instead of this given axiom scheme in this notation. We can extend this normal extension from $\mathbf{K}$ to all normal extensions of $\mathbf{K}$. For instance, $\mathbf{S4}$ is a normal extension of $\mathbf{K4}$, and $\mathbf{S4}$ is expressed by $\mathbf{K4} \oplus \mathrm{T}$. The class of all normal extensions of L is denoted by $\mathrm{NExt}L$. The following completeness results are well-known.

Proposition 2.20 *Let $\mathfrak{C}$ be the class of all modal transitive frames be given. Then, for any formula φ, $\vdash_{\mathbf{K4}} \varphi$ holds, if and only if, $\mathfrak{C} \models \varphi$ holds.*

Proposition 2.21 *Let $\mathfrak{C}$ be the class of all modal quasi-ordered frames. Then, for any modal formula φ, $\vdash_{\mathbf{S4}} \varphi$ holds, if and only if, $\mathfrak{C} \models \varphi$ holds.*

A consequence relation $\vdash_{\mathbf{K4}}$ (and $\vdash_{\mathbf{S4}}$) for $\mathbf{K4}$ (and $\mathbf{S4}$) is defined as same as the consequence relation $\vdash_{\mathbf{K}}$ for modal logic $\mathbf{K}$. Since $\Box\varphi \leftrightarrow \Box\Box\varphi$ is provable in $\mathbf{S4}$, we have the following deduction theorem for $\mathbf{S4}$.

Proposition 2.22 *For any set , of $\mathcal{ML}$-formulas and for any modal formulas φ and ψ, , $,\varphi \vdash_{\mathbf{S4}} \psi$ holds if and only if , $\vdash_{\mathbf{S4}} \Box\varphi \rightarrow \psi$.*

Let M be a normal modal logic and L a normal extension of M. Then, we can express L in the form $M \oplus$, for some set , of modal formulas. For any normal extensions L_1 and L_2 of M, $L_1 \oplus L_2$ is defined as $M \oplus (,_1 \cup ,_2)$ where $L_i = M \oplus ,_i$ for $i = 1, 2$.

Proposition 2.23 *Every consistent normal extension L of* $\mathbf{K4}$ *is charcterized by some class of modal transitive frames.*

Similar to Section 2.3, we consider $\mathrm{Log}\mathfrak{C}$ denote the set of modal formulas which are characterized by a given class $\mathfrak{C}$ of modal frames. Then,

Proposition 2.24 *Let $\mathfrak{C}$ be any class of transitive frames. Then,* $\mathrm{Log}\mathfrak{C}$ *is a normal extension of* **K4***.*

One of important normal extensions of **K4** is *Gödel-Löb logic* (**GL**). This is defined by adding the following axiom scheme la to **K4**:

(la) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

That is, the modal logic **GL** is defined as $\mathbf{K4} \oplus \Box(\Box\varphi \to \varphi) \to \Box\varphi$. This modal formula $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ is known as *Löb-formula.* The modal operator $\Box$ of **GL** denotes the provability of formal Peano arithmetic (**PA**). We will explain this in the following. Let φ be a formula of **PA**. We suppose that $\ulcorner\varphi\urcorner$ denotes the *Gödel number* of a formula φ. Moreover, we suppose $\overline{n}$ denotes the term of **PA** which represents the natural number n. Gödel introduced the predicate $\mathrm{Pr}(x)$ in **PA** which satisfies the following:

$$\vdash_{\mathbf{PA}} \mathrm{Pr}(\overline{n}) \text{ iff for some sentence } \varphi, \overline{n} = \ulcorner\varphi\urcorner \text{ and } \vdash_{\mathbf{PA}} \varphi.$$

We will call a map $*$ from $\mathrm{For}\mathcal{ML}$ to the set of arithmetic sentences an *arithmetic interpretation* if it satisfies the following three conditions:

- $\bot^*$ is $\overline{0} = \overline{1}$,
- $(\varphi \odot \psi)^* = \varphi^* \odot \psi^*$, for $\odot \in \{\wedge, \vee, \to\}$,
- $(\Box\varphi)^* = \mathrm{Pr}(\ulcorner\varphi^*\urcorner)$.

For any propositional variable, $*$ can be corresponded to any sentence, i.e., there are many kinds of arithemetic interpretation which depend on correspondece between propositional variables and arithemetic sentences. Solovay ([Sol76]) showed the following.

$$\vdash_{\mathbf{GL}} \varphi \text{ iff } \vdash_{\mathbf{PA}} \varphi^* \text{ for all arithemetic interpretations } * .$$

Therefore, via these arithmetic interpretations, we can view the modal operator $\Box$ of **GL** as expressing the provability notion of **PA**. It is easy to see that **GL** is not a normal extension of **S4** since modal logic $(\mathbf{K} \oplus \mathrm{T}) \oplus \mathrm{la}$ is inconsistent. Because,
Step i) apply RN to T $\Box\varphi \to \varphi$, then we have $\Box(\Box\varphi \to \varphi)$;
Step ii) apply MP to la and $\Box(\Box\varphi \to \varphi)$, we get $\Box\varphi$;
Step iii) apply MP to T and $\Box\varphi$, we obtain φ.
We will call a frame *Noetherian* if it does not contain any infinite strictly ascending chain. Then, we have the following.

Proposition 2.25 *Let $\mathfrak{C}$ be the class of Noetherian strict partial-ordered frame. Then, for any modal formula φ, $\vdash_{\mathbf{GL}} \varphi$ holds, if and only if, $\mathfrak{C} \models \varphi$ holds.*

The proof of the previous proposition is introduced, for instance, in [CZ97] by the mothod which is called selective filtration.

The *Grzegorczyk logic* (**Grz**) is a normal extension of **S4** such that the modal operator of **Grz** denotes the provability of ZF-set theory. The logic **Grz** is obtained from **S4** by adding the following axiom scheme grz:

(grz) $\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$.

Thus, $\mathbf{S4} \oplus \mathrm{grz}$ is equal to **Grz**. It is known that **Grz** is also equal to $\mathbf{K} \oplus \mathrm{grz}$. The following completeness result holds for **Grz**.

Proposition 2.26 *Let $\mathfrak{C}$ be the class of Noetherian partially ordered frame. Then, for any modal formula φ, $\vdash_{\mathbf{Grz}} \varphi$ holds if and only if $\mathfrak{C} \models \varphi$ holds.*

We can show also the following.

Proposition 2.27 *The lattice $\langle \mathrm{NExt}\mathbf{K}, \cap, \oplus \rangle$ of all normal extensions of* **K** *is distributive. Thus, all its sublattices – in particular* $\mathrm{NExt}\mathbf{S4}$ *and* $\mathrm{NExt}\mathbf{Grz}$ *are distributive as well.*

2.5 Gödel translation as an embedding

An *embedding* f of a given logic L_1 into another logic L_2 is a function translating formulas of L_1 into formulas of L_2 in such a way that a formula is a theorem of L_1 if and only if the translated formula is a theorem of L_2. We will call this situation as L_1 is *embedded* into L_2 by f. In 1933, Gödel showed in [Göd33] that the following function T is an embedding of **Int** into **S4**:

$$
\begin{aligned}
\mathsf{T}(\bot) &= \Box\bot, \\
\mathsf{T}(p) &= \Box p, \text{ for any } p \in \mathrm{Prop}, \\
\mathsf{T}(\varphi \wedge \psi) &= \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi), \\
\mathsf{T}(\varphi \vee \psi) &= \mathsf{T}(\varphi) \vee \mathsf{T}(\psi), \\
\mathsf{T}(\varphi \to \psi) &= \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi)).
\end{aligned}
$$

In this section we will present a proof of the fact that T is an embedding of **Int** not only into **S4** but also into **Grz**.

Proposition 2.28 *Let φ be any $\mathcal{L}$-formula. Then, the following three conditions are equivalent: 1) $\varphi \in \mathbf{Int}$, 2) $\mathsf{T}(\varphi) \in \mathbf{S4}$, and 3) $\mathsf{T}(\varphi) \in \mathbf{Grz}$.*

There are many ways of proving this proposition. Here we will give a sketch of a proof using frame semantics following [CZ97].

For any given modal (or intuitionistic) frame $\langle W, R, P\rangle$, define the *cluster* $C(x)$ of $x(\in W)$ as the set $\{y : xRy \text{ and } yRx\} \cup \{x\}$. We will denote the class of intuitionistic partially ordered frames and the class of modal quasi-ordered frames, by $\mathfrak{C}_{\mathbf{Int}}$ and $\mathfrak{C}_{\mathbf{S4}}$, respectively. Then, we define a mapping ρ from $\mathfrak{C}_{\mathbf{S4}}$ to $\mathfrak{C}_{\mathbf{Int}}$ as follows: Suppose a modal (or intuitionistic) quasi-ordered frame $\mathfrak{F}(= \langle W, R, P\rangle)$ is given. For any subset X of W, let $\rho(X)$ be the set of clusters of X, i.e., $\rho(X) = \{C(x) : x \in X\}$. Then, the *skeleton* $\rho\mathfrak{F}$ of $\mathfrak{F}$ is a triple $\langle \rho W, \rho R, \rho P\rangle$ such that

- $\rho W = \{C(x) : x \in W\}$,
- $C(x)\rho RC(y)$ iff xRy,
- $\rho P = \{\rho(X) : X \in P \text{ and } X = X\!\uparrow\}$,

where $X\!\uparrow$ is the set $\{y \in W : xRy \text{ for some } x \in X\}$. Note that the condition $X = X\!\uparrow$ means that X is an upward closed set (with respect to R). It is easy to show that ρR is well-defined. If $\mathfrak{F}$ is a modal quasi-ordered frame, $\rho\mathfrak{F}$ is an intuitionistic partially ordered frame, since $\rho(X\!\uparrow) = (\rho(X))\!\uparrow$. Next suppose a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is given. We define a map $\rho\mathfrak{V}$ from Prop to $2^{\rho W}$ by putting $\rho\mathfrak{V}(p) = \rho(\mathfrak{V}(p))$ for any $p \in$ Prop. It is also trivial that $\rho\mathfrak{V}$ is well-defined. We call the pair $\langle \rho\mathfrak{F}, \rho\mathfrak{V}\rangle$ the *skeleton* $\rho\mathfrak{M}$ of $\mathfrak{M}$. Suppose that $\mathfrak{N}$ is an intuitionistic model $\langle \rho\mathfrak{F}, \mathfrak{U}\rangle$ based on the skeleton of a modal quasi-ordered frame $\mathfrak{F} = \langle W, R, P\rangle$ where $\mathfrak{U}$ is an arbitrary valuation. Define the valuation $\mathfrak{V}$ on $\mathfrak{F}$ by the condition that for every $p \in$ Prop

$$\mathfrak{V}(p) = \{x \in W : C(x) \in \mathfrak{U}(p)\}.$$

Then we can show that the skelton of a modal model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is isomorphic to $\mathfrak{N}$.

Conversely, we can define a modal frame from a given intuitionistic frame: Suppose an intuitionistic frame $\mathfrak{F}(= \langle W, R, P\rangle)$ is given. An operator σ on $2^{\mathrm{Up}W}$ is obtained by taking, for any subset A of $\mathrm{Up}W$, σA to be the Boolean closure of A, i.e., the set-difference, set-union and set-intersection closure of A. For this operator and in this thesis, the following results are useful:

Lemma 2.29 *For every $X \subseteq W$, X is in σP if and only if*

$$X = (\Leftrightarrow X_1 \cup Y_1) \cap \ldots \cap (\Leftrightarrow X_n \cup Y_n)$$

for some $X_1, Y_1, \ldots, X_n, Y_n \in P$ and $n \geq 1$.

Lemma 2.30 *Suppose that $X \in \sigma P$ is represented as in Lemma 2.29. Then,*

$$\Box X = (X_1 \supset Y_1) \cap \ldots \cap (X_n \supset Y_n) \in P \subseteq \sigma P.$$

Proof See, e.g. Lemma 8.32 and Lemma 8.33 [CZ97]. ❑

A mapping σ from $\mathfrak{C}_{\mathbf{Int}}$ to $\mathfrak{C}_{\mathbf{S4}}$ is defined by $\sigma\mathfrak{F} = \langle W, R, \sigma P\rangle$ for every intuitionistic frame $\mathfrak{F}(= \langle W, R, P\rangle)$. By Lemma 2.30, it is obivously true that $\sigma\mathfrak{F}$ is a modal frame and $\rho\sigma\mathfrak{F} \simeq \mathfrak{F}$ for any intuitionistic frame $\mathfrak{F}$.

To prove Proposition 2.28, we need the following lemma.

Lemma 2.31 (skeleton lemma) *For every modal model $\mathfrak{M}$ based on a modal quasi-ordered frame $\mathfrak{F}$, every $\mathcal{L}$-formula φ and every possible world x in $\mathfrak{F}$,*

$$(\rho\mathfrak{M}, C(x)) \models \varphi \text{ iff } (\mathfrak{M}, x) \models \mathsf{T}(\varphi),$$

and therefore

$$\rho\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F} \models \mathsf{T}(\varphi).$$

Proof Our lemma is proved by induction on the complexity of φ. Suppose that $\mathfrak{M}$ is a modal model based on a modal quasi-ordered frame $\mathfrak{F}(= \langle W, R, P\rangle)$. We will show the cases where φ is a propositional variable p or a formula of the form $\psi \to \chi$. Let φ be p. By the definition of ρP, $\rho\mathfrak{V}(p) \in \rho P$ implies $\mathfrak{V}(p) = \mathfrak{V}(p)\!\uparrow$. Then, $x \in \mathfrak{V}(p)$ yields $(\mathfrak{M}, x) \models \Box p$. That is, $\mathfrak{V}(p) \subseteq \mathfrak{V}(\Box p)$. Since R is reflexive, $\mathfrak{V}(\Box p) \subseteq \mathfrak{V}(p)$ holds. Thus $\mathfrak{V}(p) = \mathfrak{V}(\Box p)$.

$$\begin{aligned}(\rho\mathfrak{M}, C(x)) \models p &\Leftrightarrow C(x) \in \rho\mathfrak{V}(p)\\ &\Leftrightarrow x \in \mathfrak{V}(p)\\ &\Leftrightarrow x \in \mathfrak{V}(\Box p)\\ &\Leftrightarrow (\mathfrak{M}, x) \models \Box p.\end{aligned}$$

Next, consider the case where φ is $\psi \to \chi$. Then,

$$\begin{aligned}(\rho\mathfrak{M}, C(x)) \not\models \psi \to \chi \quad &\Leftrightarrow \quad \exists C(y)(C(x)\rho R C(y), (\rho\mathfrak{M}, C(y)) \models \psi \text{ and} \\ &\qquad (\rho\mathfrak{M}, C(y)) \not\models \chi) \\ &\Leftrightarrow \quad \exists y(xRy, (\mathfrak{M}, y) \models \mathsf{T}(\psi) \text{ and } (\mathfrak{M}, y) \not\models \mathsf{T}(\chi)) \\ &\Leftrightarrow \quad (\mathfrak{M}, x) \not\models \Box(\mathsf{T}(\psi) \to \mathsf{T}(\chi)) \\ &\Leftrightarrow \quad (\mathfrak{M}, x) \not\models \mathsf{T}(\psi \to \chi).\end{aligned}$$

It is easy to see that $\rho\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \mathsf{T}(\varphi)$. ❑

Now, we turn to Proposition 2.28.

Proof (of Proposition 2.28). Suppose $\mathsf{T}(\varphi) \notin$ **S4**. Then, there exists a model $\mathfrak{M}$ such that $(\mathfrak{M}, x) \not\models \mathsf{T}(\varphi)$. By Lemma 2.31, there exist a model $\rho\mathfrak{M}$ and a point $C(x)$ such that $(\rho\mathfrak{M}, C(x)) \not\models \varphi$. That is $\varphi \notin$ **Int**. Since **Grz** is a normal extension of **S4**, a modal model of **Grz** is also a modal model of **S4**. Thus, we can derive that $\varphi \notin$ **Int** from $\mathsf{T}(\varphi) \notin$ **Grz**. For the converse direction, suppose $\varphi \notin$ **Int**. By Proposition 2.17, there exists a finite model $\mathfrak{M}(= \langle \mathfrak{F}, \mathfrak{V} \rangle)$ such that $(\mathfrak{M}, x) \not\models \varphi$. By the fact that $\rho\sigma\mathfrak{F} \simeq \mathfrak{F}$, we have an intuitionistic model $\langle \rho\sigma\mathfrak{F}, \mathfrak{V}' \rangle$ isomorphic to $\mathfrak{M}$. As we saw in the above, the skelton of a finite modal model $\mathfrak{N} = \langle \sigma\mathfrak{F}, \mathfrak{U} \rangle$ is isomorphic to $\mathfrak{M}$. Then, by Lemma 2.31, we have $\mathsf{T}(\varphi) \notin$ **S4**. Since every finite partially ordered frame is a Noetherian partially ordered frame, we also have $\mathsf{T}(\varphi) \notin$ **Grz**. ❑

2.5.1 Blok-Esakia theorem

Blok ([Blo76]) and Esakia ([Esa79a, Esa79b]) showed that there is a one-to-one and onto correspondence between Ext**Int** and NExt**Grz**. We call this result Blok-Esakia Theorem. Here, we will introduce this theorem without any proofs.

For any $M \in$ NExt**S4** and $L \in$ Ext**Int**, we define the following three maps:

$$\rho M = \{\varphi \in \mathrm{For}\mathcal{L} : \mathsf{T}(\varphi) \in M\}, \tag{2.13}$$
$$\tau L = \mathbf{S4} \oplus \mathsf{T}(L), \tag{2.14}$$
$$\sigma L = \tau L \oplus (\mathrm{grz}). \tag{2.15}$$

The map ρ is a map from NExt**S4** to Ext**Int**, and both τ and σ are maps from Ext**Int** to NExt**S4**. Here we use the same symbols ρ and τ as maps

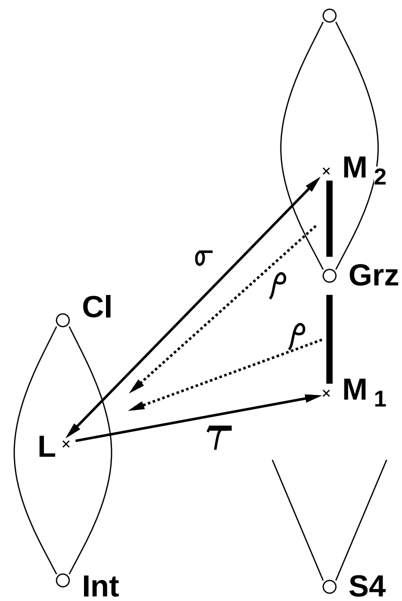


Figure 2.1: the Blok-Esakia theorem

among the class of frames treated in the previous section. When we try to show some properties of ρ (or τ) on logics, properties of ρ (or τ) on frames are used. For instance, if we discuss about ρM for any M of NExt**S4**, it is easy to find a frame by ρ which determines a logic in Ext**Int**. This is the reason why we use the same symbols.

Proposition 2.32 *Let ρ, τ and σ be maps defined by (2.13), (2.14) and (2.15), respectively. Then,*

1. *ρ is a surjective homomorphism of* $\langle \mathrm{NExt}\mathbf{S4}, \cap, \oplus \rangle$ *to* $\langle \mathrm{Ext}\mathbf{Int}, \cap, +, \mathbf{Int}, \mathbf{Cl} \rangle$,
2. *τ is an isomorphism of* $\langle \mathrm{Ext}\mathbf{Int}, \cap, +, \mathbf{Int}, \mathbf{Cl} \rangle$ *into* $\langle \mathrm{NExt}\mathbf{S4}, \cap, \oplus \rangle$,
3. (**The Blok-Esakia theorem**) *σ is an isomorphism of* $\langle \mathrm{Ext}\mathbf{Int}, \cap, +, \mathbf{Int}, \mathbf{Cl} \rangle$ *onto* $\langle \mathrm{NExt}\mathbf{Grz}, \cap, \oplus \rangle$.

The Blok-Esakia theorem is illustrated in above Figure 2.1 . Classes Ext**Int**, NExt**S4** and NExt**Grz** form bounded distribuitive lattices, and maps ρ, τ, σ are defined between Ext**Int** and NExt**Grz**. In this figure, $M_1 = \tau L$ and $M_2 =$

σL. The arrows $\longrightarrow$ and $\longleftrightarrow$ denote τ and σ, respectively. Proposition 2.32 asserts that for any intermediate logic L,

$$\rho^{-1}(L) = \{M \in \mathrm{NExt}\mathbf{S4} : \tau L \subseteq M \subseteq \sigma L\}.$$

In Figure 2.1, the line ▬▬ denotes the range of $\rho^{-1}(L)$, and, ⋯⋯▸ denotes the map ρ.

The following proposition asserts that the operator ρ and σ on frame theory relate the operator ρ and σ on a set of formulas, respectively.

Proposition 2.33 (Lemma 9.67 i) – iii) in [CZ97]) (i) *For every intuitionistic frame $\mathfrak{F}$ and logic $M \in \mathrm{NExt}\mathbf{S4}$,*

$$\mathfrak{F} \models \rho M \text{ iff } \sigma\mathfrak{F} \models M.$$

(ii) *For every intuitionistic frame $\mathfrak{F}$ and logic $L \in \mathrm{Ext}\mathbf{Int}$,*

$$\mathfrak{F} \models L \text{ iff } \sigma\mathfrak{F} \models \sigma L.$$

(iii) *For every quasi-ordered frame $\mathfrak{F}$ and intermediate logic L,*

$$\sigma\mathfrak{F} \models L \text{ iff } \mathfrak{F} \models \tau L.$$

2.6 Notes

In this chapter, we have surveyed several results about intermediate logics and modal logics which are related to the present thesis. These contents are written in most of textbooks on modal logics. Thus any reader will be able to check the details easily. Our survey is mainly based on the textbooks[CZ97] and [Boo93].

We note here some relations between **Grz** and **GL**. It is known that there exists an embedding of **Grz** into **GL**. Let us define a mapping $+$ is a map from For$\mathcal{ML}$ to For$\mathcal{ML}$ defined as follows:

$$\begin{aligned}
\bot^+ &= \bot, \\
p^+ &= p, \text{ for any } p \in \mathrm{Prop}, \\
(\varphi \wedge \psi)^+ &= \varphi^+ \wedge \psi^+, \\
(\varphi \vee \psi)^+ &= \varphi^+ \vee \psi^+, \\
(\Box\varphi)^+ &= \Box\varphi^+ \wedge \varphi^+.
\end{aligned}$$

Then, we can show that for any modal formula φ,

$$\vdash_{\mathbf{Grz}} \varphi \text{ iff } \vdash_{\mathbf{GL}} \varphi^{+}.$$

Thus, we can say that every theorem of **Grz** can be translated into a provable arithmetic sentence in **PA**.

Chapter 3

The basic system BPL

In this chapter, we will study *basic propositional logic* (**BPL**), in particular its syntactic properties.

BPL was introduced by A. Visser ([Vis81]). At first, Visser looked for a propositional logic which would be embeddable into the modal logic **GL** by the translation T. He found such a logic and called it *formal propositional logic* (**FPL**). He also defined another interesting logic called **BPL** which is embedded into **K4** by T. Thus, by the translation T, **Int** and **FPL** correspond to **S4** and **GL**, respectively. On the other hand, both **S4** and **GL** are extensions of **K4**, to which **BPL** corresponds. Therefore, both **FPL** and **Int** turn out to be extensions of **BPL**. In [Vis81], Visser introduced **BPL** as a natural deduction system. He proved also that **BPL** is complete to the class of transitive frames. So, we will discuss transitive frame semantics first.

Next, we will introduce two kinds of proof systems for **BPL**. One is the natural deduction system **NBPL** by Visser ([Vis81]), and the second is a Hilbert-style system **HB** introduced in the author's joint paper with Ono ([SO97]). The completeness theorem of **HB** is proved similarly to [Cor87]. This is answer to a question raised in [SWZ97].

Then, we will discuss Sasaki's results [Sas98]. Sasaki showed that the consequence relation of **BPL** can be defined by the axioms of **HB** and *a weak modus ponens* : $, \vdash \varphi$ and $\vdash \varphi \to \psi$ imply $, \vdash \psi$. He investigated $\mathbf{V}^*$ which resembles **HB**. Using $\mathbf{V}^*$, Sasaki also proved that no consequece relation including implication elimination rule (i.e., modus ponens with assumptions : $, \vdash \varphi \to \psi$ and $\Delta \vdash \varphi$ imply $, , \Delta \vdash \psi$) can not characterize the consequence relation of **BPL**.

3.1 Transitive frame semantics

Visser applied the intuitionistic interpretation to transitive Kripke frames ([Vis81]). Here, we will introduce *(general) transitive frame semantics* and some notations.

An *intuitionistic transitive frame* $\mathfrak{F}$ is a triple $\langle W, R, P\rangle$ such that W is a non-empty set of possible worlds, R a transitive relation on W and P is a set of R-cones which contains both $\emptyset$, and which is closed under the set-union, set-intersection and the set-implication operation $\supset$ defined by (2.1). The notions of R-cone and the set UpW of all R-cones are already introduced in Chapter 2. Sometimes, we call an intuitionistic transitive frame, simply a "transitive frame" (or just a "frame"). If a given intuitionisitic transitive frame $\mathfrak{F}$ is of the form $\langle W, R, \text{Up}W\rangle$, we call it *intuitionistic transitive Kripke frame*, and abbreviate it to $\langle W, R\rangle$. We denote $X \cup X\uparrow$ as $X\underline{\uparrow}$ for any subset X of W. Any intuitionistic transitive frame $\mathfrak{F}(=\langle W, R, P\rangle)$ is called *rooted* if $x\underline{\uparrow} = W$ holds for some element $x \in W$, and x is called the *root* of $\mathfrak{F}$.

A valuation in a frame, a model based on a frame, the truth relation $\models$, the notions of vailidty, homomorphism, etc are defined in the same way as the case of intuitionistic quasi-ordered frames.

We have to pay an attention to treating modus ponens in **BPL**. Since we don't assume the reflexivity in transitive frame semantics, it sometimes shows quite different behaviours. Usually, modus ponens is described as follows:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}.$$

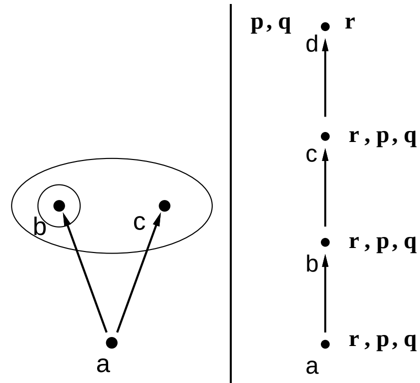
However, the meaning of modus ponens on natural deduction system and on Hilbert-style system differ. The former rule can be applied not only to derived theorems but also assumptions, and the later can be applied only to theorems. That is, their different points are put as follows:

Natural deduction	$\varphi, \varphi \to \psi \vdash \psi$
Hilbert-style	$\vdash \varphi$ and $\vdash \varphi \to \psi$ imply $\vdash \psi$

To distuinguish from modus ponens of Hilbert-style system, we will call the rule in natural deduction system $\to$ E rule (or *implication elimantion rule*).

The following example denotes the difference between $\bot$ and $\neg\top$.

Example 3.1 *In an irreflexivity Kripke frame with a single possible world, every formula of the form $\psi \to \xi$ is valid. In particular $\neg\top(= \top \to \bot)$ is valid in it.*



Example 3.2 | Example 3.3

To picture frames or models, we will sometimes use graph diagrams. In our diagrams, white circles and black circles denote reflexive points and irreflexive points, respectively.

○	reflexive point
●	irreflexive point

Names of nodes are written as in a, b, c, x, y, z, ... Arrows denote the transitive relation. Since relations are transitive, we will omit trivial arrows induced by the transitivity. In a diagram, capital letters $A, B, C, ...$ denote non-atomic formulas, and small letters $p, q, r, ...$ denote atomic formulas (propositional variables). We will sometimes use a realm represented by a thin line (or, a broken line) to denote an element of P. Since P includes W and $\emptyset$, we always omit these trivial elements from a diagram. The following is an example of a diagram for a frame.

Example 3.2 *Suppose* $W = \{a, b, c\}$, $R = \{(a, b), (a, c)\}$ *and* $P = \{\emptyset, \{b\}, \{b, c\}, W\}$. *Then the above figure denotes the intuitionistic transitive frame* $\mathfrak{F} = \langle W, R, P \rangle$.

When a given frame $\mathfrak{F}$ is pictured as a graph and a valuation is given, we will write a formula A at the left-hand side of x if A is true at x in $\mathfrak{M}$, and write A at the right-hand side of x if A is not true. The following is an example of a diagram for a model.

Example 3.3 *We can show that there exists a frame such that* $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ *is not valid. We present such a frame and a* *valuation in the above figure. It is easy to calculate that* $(\mathfrak{M}, a) \not\models (p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

3.2 Natural deduction system of BPL

In [Vis81], A. Visser introduced a natural deduction system **NBPL** of **BPL**, which we present below.

Inference rules:

$$\vee\mathrm{E} : \frac{\varphi \vee \psi \quad \begin{matrix} \not{\varphi} \\ \vdots \\ \chi \end{matrix} \quad \begin{matrix} \not{\psi} \\ \vdots \\ \chi \end{matrix}}{\chi}, \qquad \to\mathrm{I} : \frac{\begin{matrix} \not{\varphi} \\ \vdots \\ \psi \end{matrix}}{\varphi \to \psi}, \qquad \bot : \frac{\bot}{\varphi},$$

$$\wedge\mathrm{I} : \frac{\varphi \quad \psi}{\varphi \wedge \psi}, \qquad \wedge\mathrm{E} : \frac{\varphi \wedge \psi}{\varphi}, \quad \frac{\varphi \wedge \psi}{\psi}, \qquad \vee\mathrm{I} : \frac{\varphi}{\varphi \vee \psi}, \quad \frac{\psi}{\varphi \vee \psi},$$

$$\wedge\mathrm{I\text{-}f} : \frac{\varphi \to \psi \quad \varphi \to \chi}{\varphi \to \psi \wedge \chi}, \qquad \vee\mathrm{E\text{-}f} : \frac{\varphi \to \chi \quad \psi \to \chi}{\varphi \vee \psi \to \chi}, \qquad \mathrm{Tr} : \frac{\varphi \to \psi \quad \psi \to \chi}{\varphi \to \chi}.$$

Natural deduction system of **BPL**.

The system differs from the natural deduction system **NJ** of **Int** in **NBPL** has no $\to$ E rule, but it has $\wedge$ I-f, $\vee$ E-f and Tr, instead. It is easy to see that these three special rules of **NBPL** are derivable from $\to$ E rule. However, as Visser pointed out, $\to$ E rule is not derivable in **NBPL**.

Let , be a set of formulas. , $\vdash_{\mathbf{BPL}} \varphi$ means φ is derived in **NBPL** from , , in other words, , includes the all assumptions of proof of φ. If $\emptyset \vdash_{\mathbf{BPL}} \varphi$ holds we will denote it by $\vdash_{\mathbf{BPL}} \varphi$.

Now, we will show an example of a proof in **NBPL**. The following example 3.4 shows that $\to$ I-f rule $\dfrac{\varphi \wedge \psi \to \chi}{\varphi \to (\psi \to \chi)}$ is derivable in **NBPL**.

Example 3.4 *A proof of* $\varphi \wedge \psi \to \chi \vdash_{\mathbf{BPL}} \varphi \to (\psi \to \chi)$*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\not{\varphi} \quad \not{\psi}}{\varphi \wedge \psi}}{\psi \to \varphi \wedge \psi} \quad \varphi \wedge \psi \to \chi}{\psi \to \chi}}{\varphi \to (\psi \to \chi)}.$$

The completeness theorem of **NBPL** was proved by Visser ([Vis81]).

Proposition 3.5 *Let $\mathfrak{C}$ be the class of (rooted) intuitionistic transitive frames. Suppose that $\mathfrak{M}$ is an arbitrary model which is based on a frame in $\mathfrak{C}$, and x is a possible world of $\mathfrak{M}$. Then,*

$$, \vdash_{\mathbf{BPL}} \varphi \textit{ iff } \forall \mathfrak{M} \forall x((\mathfrak{M}, x) \models ,\ \Rightarrow (\mathfrak{M}, x) \models \varphi).$$

Let $\mathfrak{M}$ be a model based on the single irrflexive Kripke frame in Example 3.1, in which p is true in $\mathfrak{M}$. Let x stand for the point of $\mathfrak{M}$. Then, we can calculate easily that $(\mathfrak{M}, x) \models p$ and $(\mathfrak{M}, x) \models p \to q$ but $(\mathfrak{M}, x) \not\models q$. This means that modus ponens is not a derivable rule in **NBPL**. Even if, for any complex connective C, we put that C *satisfies modus ponens* when $\varphi, C(\varphi, \psi) \vdash_{\mathbf{BPL}} \psi$, modus ponens of this type does not hold either. To show this, we need some terminology. Let a frame $\mathfrak{F}$ and valuations $\mathfrak{V}$ and $\mathfrak{V}'$ on $\mathfrak{F}$ be given. We denote models $\langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\langle \mathfrak{F}, \mathfrak{V}' \rangle$ as $\mathfrak{N}$ and $\mathfrak{N}'$, respectively. For any formula φ and any possible world x of $\mathfrak{F}$, $\varphi^{(\mathfrak{N},x)} \leq \varphi^{(\mathfrak{N}',x)}$ if φ is false at x in $\mathfrak{N}$ or φ is true at x in $\mathfrak{N}'$. Suppose that φ includes at least n-variables $p_1, \ldots, p_n$. We will express this by $\varphi(p_1, \ldots, p_n)$. We call a formula $\varphi(p_1, \ldots, p_n)$ *monotone* at x in $\mathfrak{F}$ if, for any two models $\mathfrak{M}$ and $\mathfrak{N}$ based on $\mathfrak{F}$,

$$[\forall i \in \{1, \ldots, n\}(p_i^{(\mathfrak{M},x)} \leq p_i^{(\mathfrak{N},x)})] \Rightarrow \varphi^{(\mathfrak{M},x)} \leq \varphi^{(\mathfrak{N},x)}.$$

Theorem 3.6 *There exists no formulas $C(p, q)$ such that, for all , , φ, ψ,*

$$, , \varphi \vdash_{\mathbf{BPL}} \psi \textit{ iff } , \vdash_{\mathbf{BPL}} C(\varphi, \psi).$$

Proof Suppose on the contrary that such a formula $C(p, q)$ exists. Then, we have

$$\top \to \bot, \varphi \vdash_{\mathbf{BPL}} \psi \quad \text{iff} \quad \top \to \bot \vdash_{\mathbf{BPL}} C(\varphi, \psi) \tag{3.1}$$

Here, $\top$ is the abbreviation of $\bot \to \bot$. If $\top \to \bot$ holds in a model $\mathfrak{M}$ then $\mathfrak{M}$ is a disjoint union of irrflexive points. Then, all formulas, whose outermost connective is implication, are true, as we saw in Example 3.3 . We can easily show that any formula is monotone at any point in that frame. And, clearly, $(p \wedge q)^{(\mathfrak{M},x)} \leq p^{(\mathfrak{M},x)}$ for any point x. Thus,

$$\top \to \bot, C(p \wedge q, q) \vdash_{\mathbf{BPL}} C(p, q)$$

By (3.1) and $\top \to \bot, p \wedge q \vdash_{\mathbf{BPL}} q$, we can derive that $\top \to \bot \vdash_{\mathbf{BPL}} C(p \wedge q, q)$. Then, we have $\top \to \bot \vdash_{\mathbf{BPL}} C(p, q)$. And, again by (3.1), $\top \to \bot, p \vdash_{\mathbf{BPL}} q$. But, it is clear that $\top \to \bot, p \not\vdash_{\mathbf{BPL}} q$. This is a contradiction. ❑

Remark. By Theorem 3.6, whenever we think of an extension L of **BPL**, we meet the problem that what kind of extensions are worth condsidering. Let us consider the case of **Int**. Suppose L is a superintuitionistic logic which is axiomatizable by a finite number of axioms. Then, by deduction theorem, we are able to translate a theorem of L into a theorem of **Int**. But this translation does not go through for the case of **BPL** by Theorem 3.6. To solve this problem, we will consider not only a theorem of L but also a consequence relation of L. We will discuss this problem in Chapter 5, again.

3.3 Hilbert style proof system of BPL

Next, we will introduce a Hilbert-style system of **BPL**, which was given [SO97]. Throughout this chapter, we will identify the logic **BPL** with the set of theorems of **BPL**. We start with stating some preliminary facts. Firstly, the theorem $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ of **Int** is not provable in **BPL** as we saw in Example 3.3 .

Lemma 3.7 **BPL** *is closed under modus ponens and substituition.*

Proof It is easy to see that **BPL** is closed under substituiton. So we will show that **BPL** is closed under modus ponens. Suppose otherwise. Then, there are formulas φ and ψ such that $\varphi, \varphi \to \psi \in$ **BPL** but $\psi \notin$ **BPL**. This means that there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ in which ψ is refuted at some point y. Add a new root x to $\mathfrak{F}$, and call the resulting frame $\mathfrak{G}$. Let $\mathfrak{U}$ be the valuation in $\mathfrak{G}$ such that $\mathfrak{U}(p) = \mathfrak{V}(p)$ for every variable p, and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$. Clearly, $(\mathfrak{N}, y) \not\models \psi$. On the other hand, we have $(\mathfrak{N}, y) \models \varphi$ and hence $(\mathfrak{N}, x) \not\models \varphi \to \psi$, which contradicts the assumption that we can show that $\varphi \to \psi \in$ **BPL**. Thus **BPL** is closed under modus ponens. ❑

3.3.1 Hilbert style proof system HB of BPL

Our Hilbert-style proof system **HB** of **BPL** is given as follows. This system resembles Corsi's system **F** in [Cor87].

Axiom schemes:

(B1) $\varphi \to \varphi$,

(B2) $(\varphi \to \psi) \wedge (\psi \to \chi) \to (\varphi \to \chi)$, (B3) $\varphi \wedge \psi \to \varphi$,

(B4) $\varphi \wedge \psi \to \psi$,

(B5) $(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$,

(B6) $\varphi \to \varphi \vee \psi$,

(B7) $\psi \to \varphi \vee \psi$,

(B8) $(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$,

(B9) $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$,

(B10) $\bot \to \varphi$,

(B11) $\varphi \to (\psi \to \varphi)$,

(B12) $\varphi \to (\psi \to \varphi \wedge \psi)$;

Inference rules:

- modus ponens.

We will use $\vdash_{\mathbf{HB}}$ to denote the derivability in **HB**.
The soundness theorem is easily shown by induction.

Theorem 3.8 *Let $\mathfrak{C}$ be the class of intuitionistic transitive frames. Then, for any formula φ,*

$$\vdash_{\mathbf{HB}} \varphi \text{ implies } \mathfrak{C} \models \varphi.$$

3.3.2 Some preparations

To prove the completeness theorem of **HB**, we need the following lemmas and notions. A good deal of the proofs go through in the same way as in [Cor87].

Lemma 3.9 *The following hold:*

(F1) *:* $\vdash_{\mathbf{HB}} (\varphi \to \psi) \wedge ((\varphi \wedge \psi) \to \chi) \to (\varphi \to \psi)$,

(F2) *:* $\vdash_{\mathbf{HB}} (\varphi \wedge \psi) \to \chi$ *and* $\vdash_{\mathbf{HB}} \psi$ *imply* $\vdash_{\mathbf{HB}} \varphi \to \chi$, (F3) *:* $\vdash_{\mathbf{HB}} \varphi \rightarrow \psi$ *implies* $\vdash_{\mathbf{HB}} (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$,

(F4) *:* $\vdash_{\mathbf{HB}} \varphi \rightarrow \psi$ *implies* $\vdash_{\mathbf{HB}} (\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)$,

(F5) *:* $\vdash_{\mathbf{HB}} (\varphi \rightarrow \psi) \wedge (\chi \rightarrow \eta) \rightarrow (\varphi \wedge \chi \rightarrow \psi \wedge \eta)$,

(F6) *:* $\vdash_{\mathbf{HB}} (\varphi \rightarrow \psi) \wedge (\chi \rightarrow \eta) \rightarrow (\varphi \vee \chi \rightarrow \psi \vee \eta)$,

(F7) *:* $\vdash_{\mathbf{HB}} (\varphi \wedge \chi \rightarrow \psi \vee \eta) \wedge (\xi \rightarrow \chi) \wedge (\eta \rightarrow \mu) \rightarrow (\varphi \wedge \xi \rightarrow \psi \vee \mu)$,

(F8) *:* $\vdash_{\mathbf{HB}} \varphi \wedge \psi \rightarrow \chi$ *implies* $\vdash_{\mathbf{HB}} (\eta \rightarrow \varphi) \rightarrow ((\eta \wedge \psi) \rightarrow \chi)$.

Definition 3.10 *Let , be a subset of* For$\mathcal{L}$ *and* Σ *be a non-empty subset of* For$\mathcal{L}$*. , is* Σ*-consistent if there exist a finite subset* $\{\gamma_1, \ldots, \gamma_n\}$ *of , and a finite subset* $\{\sigma_1, \ldots, \sigma_m\}$ *of* Σ *such that*

$$\nvdash_{\mathbf{HB}} \gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \sigma_1 \vee \ldots \vee \sigma_m.$$

Moreover, , is Σ*-maximal if for any formula* α *which is not an element of , , there exists a finite subset* $\{\gamma_1, \ldots, \gamma_n\}$ *of , and a finite subset* $\{\sigma_1, \ldots, \sigma_m\}$ *of* Σ *such that*

$$\vdash_{\mathbf{HB}} \alpha \wedge \gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \sigma_1 \vee \ldots \vee \sigma_m.$$

, is Σ*-maximal-consistent if , is both* Σ*-maximal and* Σ*-consistent.*

Lemma 3.11 *Suppose that* Σ*-consistent set* Λ *is given. Then, there exists a set , such that,*

1. $\Lambda \subseteq$, ,
2. *, is* Σ*-maximal-consistent set.*

Proof Enumerate all the elements of For$\mathcal{L}$. Define $(\Lambda_i)_{i \in \mathfrak{N}}$ as follows;

$$\begin{aligned} \Lambda_0 : &= \Lambda \\ \Lambda_{i+1} &= \begin{cases} \Lambda_i \cup \{\varphi_i\} & \text{if } \Lambda \cup \{\varphi_i\} \text{ is } \Sigma\text{-consistent,} \\ \Lambda_i & \text{otherwise.} \end{cases} \end{aligned}$$

Let , be $\bigcup \Lambda_i$. Then, this , satisfies the above conditions. ❑

Lemma 3.12 *Suppose , is* Σ*-maximal-consistent. Then,*

1. $\vdash_{\mathbf{HB}} \alpha$ *implies* $\alpha \in$, ,
2. $\alpha \in$, *and* $\vdash_{\mathbf{HB}} \alpha \rightarrow \beta$ *imply* $\beta \in$, ,
3. ($\alpha \in$, *and* $\beta \in$,) *iff* $\alpha \wedge \beta \in$, ,
4. ($\alpha \in$, *or* $\beta \in$,) *iff* $\alpha \vee \beta \in$, ,
5. $\alpha \rightarrow \beta \in$, *and* $\beta \rightarrow \delta \in$, *imply* $\alpha \rightarrow \delta \in$, ,
6. $\alpha \wedge \beta \rightarrow \delta \in$, *and* $\vdash_{\mathbf{HB}} \beta$ *imply* $\alpha \rightarrow \delta \in$, ,
7. $\vdash_{\mathbf{HB}} \alpha \wedge \eta \rightarrow \beta \vee \gamma$, $\delta \rightarrow \eta \in$, *and* $\gamma \rightarrow \theta \in$, *imply* $\alpha \wedge \delta \rightarrow \beta \vee \delta \in$, ,
8. $\alpha \rightarrow \delta \in$, *and* $\alpha \wedge \delta \rightarrow \beta \in$, *imply* $\alpha \rightarrow \beta \in$, .

Proof Use Lemma 3.9 and the definition of Σ-maximal-consistent set. ❑

Lemma 3.13 *Let* Λ *be* Σ*-maximal-consistent such that* $\alpha \rightarrow \beta \notin \Lambda$*. Then,* $\{\alpha\}$ *is* Σ'*-consistent, where* $\Sigma' = \{\varphi : \varphi \rightarrow \beta \in \Lambda\}$*.*

Proof Suppose otherwise. Then, $\vdash_{\mathbf{HB}} \alpha \rightarrow \sigma_1 \vee \ldots \vee \sigma_m$ for some subset $\{\sigma_1, \ldots, \sigma_m\}$ of Σ'. By the lemma 3.9,

$$\vdash_{\mathbf{HB}} (\sigma_1 \vee \ldots \vee \sigma_m \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta). \tag{3.2}$$

By the axiom scheme B8,

$$\vdash_{\mathbf{HB}} (\sigma_1 \rightarrow \beta) \wedge (\sigma_2 \rightarrow \beta) \rightarrow (\sigma_1 \vee \sigma_2 \rightarrow \beta).$$

Repeating this, we get

$$\vdash_{\mathbf{HB}} (\sigma_1 \rightarrow \beta) \wedge \ldots \wedge (\sigma_m \rightarrow \beta) \rightarrow (\sigma_1 \vee \ldots \vee \sigma_m \rightarrow \beta). \tag{3.3}$$

By the axiom scheme B2, B12, (3.2), (3.3) and MP, we get

$$\vdash_{\mathbf{HB}} (\sigma_1 \rightarrow \beta) \wedge \ldots \wedge (\sigma_m \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta). \tag{3.4}$$

By the assumptions that $\alpha \rightarrow \beta \notin \Lambda$ and Λ is Σ-maximal-consistent,

$$\vdash_{\mathbf{HB}} \lambda_1 \wedge \ldots \wedge \lambda_n \wedge (\alpha \rightarrow \beta) \rightarrow \tau_1 \vee \ldots \vee \tau_t, \tag{3.5}$$

for some $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $\tau_1, \ldots, \tau_t \in \Sigma$. Lets write λ for $\lambda_1 \wedge \ldots \wedge \lambda_n$ and τ for $\tau_1 \vee \ldots \vee \tau_t$. Since from $\vdash_{\mathbf{HB}} A \to B$ we deduce $\vdash_{\mathbf{HB}} C \wedge A \to C \wedge B$, by (3.4), we get:

$$\vdash_{\mathbf{HB}} \lambda \wedge (\sigma_1 \to \beta) \wedge \ldots \wedge (\sigma_m \to \beta) \to \lambda \wedge (\alpha \to \beta). \tag{3.6}$$

And, again by B2, B12, (3.6), (3.5) and MP,

$$\vdash_{\mathbf{HB}} \lambda \wedge (\sigma_1 \to \beta) \wedge \ldots \wedge (\sigma_m \to \beta) \to \tau.$$

This contradicts our assumption that Λ is Σ-consistent. ❑

Lemma 3.14 *Let Λ be Σ-maximal-consistent such that $\alpha \to \beta \notin \Lambda$, and Σ' be the set $\{\varphi : \varphi \to \beta \in \Lambda\}$. Then, there exists a Σ'-maximal-consistent set , where $\alpha \in$, , $\beta \notin$, and for any δ and η, $\delta \to \eta \in \Lambda$ and $\delta \in$, imply $\eta \in$, .*

Proof By Lemma 3.13, $\{\alpha\}$ is Σ'-consistent. By Lemma 3.11, there exist a Σ'-maximal-consistent set , , with $\alpha \in$, . That $\beta \notin$, , and $\delta \to \eta \in \Lambda$, and $\delta \in$, imply $\eta \in$, , can be derived similarly to the proof of Lemma 3.13 . ❑

3.3.3 Completeness theorem

In this section, we will prove the completeness theorem of $\vdash_{\mathbf{HB}}$ by means of its *canonical frame.*

Definition 3.15 *The canonical frame $c\mathfrak{F}$ is a triple $\langle W, R, P\rangle$ where*

1. *W is the class of all subsets of For$\mathcal{L}$ which are Σ-maximal-consistent for some non-empty subset Σ of For$\mathcal{L}$,*
2. *R is a binary relation on W such that for any , , , ′,*
   , R, ′ iff for any $\alpha, \beta \in$, , $\alpha \to \beta \in$, and $\alpha \in$, ′ imply $\beta \in$, ′,
3. *P is the set $\{\mathbf{v}(\alpha) : \alpha \in For\mathcal{L}\}$ where $\mathbf{v}(\alpha) = \{, \in W : \alpha \in , \}$.*

The canonical model $c\mathfrak{M}$ is the pair $\langle c\mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{V}$ is the function such that $\mathfrak{V}(\alpha) = \mathbf{v}(\alpha)$ for any formula $\alpha \in For\mathcal{L}$.

Theorem 3.16 *$c\mathfrak{F}(= \langle W, R, P\rangle)$ is an intuitionistic transitive frame.*

Proof We will show that R is transitive, and that P is a set of R-cones containing $\emptyset$ which is closed under set-intersection $\cap$, set-union $\cup$ and $\supset$. First, we will show that R is transitive. To prove this, it is sufficient to show that $, R,'$ implies $, \subseteq ,'$. Suppose $, R,'$ and $\varphi \in ,$. Then, by Lemma 3.12 and axiom scheme (B11), $\vdash_{\mathbf{HB}} \varphi \rightarrow (\top \rightarrow \varphi)$. By Lemma 3.12, $\top \rightarrow \varphi \in ,$. So, $\varphi \in ,'$ by the definition of R and the fact that $\top \in ,'$, thus $, R,'$ implies $, \subseteq ,'$. Clearly, P is a set of R-cones since $, R,'$ implies $, \subseteq ,'$. It is clear that P contains both W and $\emptyset$. To prove that P is closed under $\cap$, $\cup$ and $\supset$, it is enough to show $\mathbf{v}(\alpha \wedge \beta) = \mathbf{v}(\alpha) \cap \mathbf{v}(\beta)$, $\mathbf{v}(\alpha \vee \beta) = \mathbf{v}(\alpha) \cup \mathbf{v}(\beta)$ and $\mathbf{v}(\alpha \rightarrow \beta) = \mathbf{v}(\alpha) \supset \mathbf{v}(\beta)$. Obviously, $\mathbf{v}(\alpha \wedge \beta) = \mathbf{v}(\alpha) \cap \mathbf{v}(\beta)$ and $\mathbf{v}(\alpha \vee \beta) = \mathbf{v}(\alpha) \cup \mathbf{v}(\beta)$ hold by Lemmas 3.12 and 3.12, respectively. So, it remains to show $\mathbf{v}(\alpha \rightarrow \beta) = \mathbf{v}(\alpha) \supset \mathbf{v}(\beta)$. Suppose $, \in \mathbf{v}(\alpha \rightarrow \beta)$. Then, $\alpha \rightarrow \beta \in ,$. Assume there exists $,'$ such that $, R,'$ and $,' \in \mathbf{v}(\alpha)$. Then, by the definition of R and $\mathbf{v}$, $,' \in \mathbf{v}(\beta)$. And $, \in \mathbf{v}(\alpha) \supset \mathbf{v}(\beta)$ by (2.1). The opposite direction can be derived by Lemma 3.14. ❑

Corollary 3.17 *The canonical model $c\mathfrak{M}$ satisfies*

$$\varphi \in , \quad \textit{iff} \ (c\mathfrak{M}, ,) \models \varphi,$$

for any $, \in W$ and any formula φ.

Now, we will prove the completeness theorem for $\vdash_{\mathbf{HB}}$.

Theorem 3.18 *Let $\mathfrak{C}$ be the class of intuitionistic transitive frames. Then, for any formula φ,*

$$\mathfrak{C} \models \varphi \ \textit{implies} \ \vdash_{\mathbf{HB}} \varphi.$$

Proof Suppose $\not\vdash_{\mathbf{HB}} \varphi$. Let $,$ be $\{\psi :\vdash_{\mathbf{HB}} \psi\}$. It is trivial that $,$ is $\{\varphi\}$-consistent set. By Lemma 3.11, we have a $\{\varphi\}$-maximal-consistent set Λ such that $, \subseteq \Lambda$. Obviously, $\Lambda \in W$ and $\varphi \notin \Lambda$. By Lemma 3.17, $(c\mathfrak{M}, \Lambda) \not\models \varphi$. Hence, $\mathfrak{C} \not\models \varphi$. ❑

3.4 Sasaki's results

After we obtained our system **HB** in [SO97], Sasaki in [Sas98] strengthened our result. In this section, we will give a sketch of Sasaki's results. In [Sas98], he proved that the consequence relation of **BPL** is axiomatizable with the axioms of **HB** and a weak modus ponens.

Sasaki showed that the consequence relation of **BPL** is not axiomatizable by any proof system S which includes *modus ponens without any assumption*, i.e., $\vdash_S \varphi$ and $\vdash_S \varphi \to \psi$ imply $\vdash_S \psi$. He also proved that it is not axiomatizable by any proof system S which includes modus ponens with assumptions, i.e., , $\vdash_S \varphi$ and $\Delta \vdash_S \varphi \to \psi$ imply , $, \Delta \vdash_S \psi$.

This result contrasts especially with Theorem 3.6 . Theorem 3.6 means that deduction theorem does not hold for any type of implication, while Sasaki's result says that modus ponens does not work.

Consequence relation $\vdash_{V^*}$ is defined as follows ([Sas98]): Let Ax be the set of all axiom schemes of $\vdash_{\mathbf{HB}}$ except Axiom (B12). $\vdash_{V^*}$ is the smallest consequence relation which satisfies the following conditions:

1. $\varphi \in$ Ax implies , $\vdash_{V^*} \varphi$,
2. $\varphi \in$, implies , $\vdash_{V^*} \varphi$,
3. , $\vdash_{V^*} \varphi$ and $\vdash_{V^*} \varphi \to \psi$ imply , $\vdash_{V^*} \psi$,
4. , $\vdash_{V^*} \varphi$ and , $\vdash_{V^*} \psi$ imply , $\vdash_{V^*} \varphi \wedge \psi$.

This system has the same deduction power as **NBPL** ([Sas98]). Therefore V^* gives our axiomatization of the consequence relation of **BPL**.

Proposition 3.19 , $\vdash_{V^*} \varphi$ *iff* , $\vdash_{\mathbf{BPL}} \varphi$.

In order to show that no proof systems with modus ponens with any assumption can axiomatize **BPL**, Sasaki introduced an *abstract consequence relation* $\vdash_{\mathrm{S,MP}}$ for a subset S of For$\mathcal{L}$ and a subset MP of For$\mathcal{L} \times$ For$\mathcal{L}$ as follows ([Sas98]):

1. $\varphi \in$ S implies , $\vdash_{\mathrm{S,MP}} \varphi$;
2. $\varphi \in$, implies , $\vdash_{\mathrm{S,MP}} \varphi$;
3. For any $(\varphi, \psi) \in$ MP, , $\vdash_{\mathrm{S,MP}} \varphi$ and , $\vdash_{\mathrm{S,MP}} \varphi \to \psi$ imply , $\vdash_{\mathrm{S,MP}} \psi$.

We will now define a derivation and the length of a derivation for this abstract consequence relation $\vdash_{\mathrm{S,MP}}$: A *derivation* of , $\vdash_{\mathrm{S,MP}} \varphi$ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that for any $i \in \{1, \ldots, n\}$,

1. $\varphi_n = \varphi$,

2. $\varphi_i \in \,, \cup \mathrm{S}$,
3. there exist $j, k < i$ such that $\varphi_k = \varphi_j \rightarrow \varphi_i$, $(\varphi_j, \varphi_i) \in \mathrm{MP}$, $, \vdash_{\mathrm{S,MP}} \varphi_j$ and $, \vdash_{\mathrm{S,MP}} \varphi_k$.

The *length* of a derivation $\varphi_1, \ldots, \varphi_n$ of $, \vdash_{\mathrm{S,MP}} \varphi_n$ is defined to be n. Let

$$\begin{aligned}\mathrm{MP}_{\mathbf{BPL}} &= \{(\alpha, \beta) : \text{for any } , \text{, if } , \vdash_{\mathbf{BPL}} \alpha \text{ and } , \vdash_{\mathbf{BPL}} \alpha \rightarrow \beta, \\ &\qquad \text{then } , \vdash_{\mathbf{BPL}} \beta\}\end{aligned}$$

and consider the consequence relation $\vdash_{\mathbf{BPL},\mathbf{MP}_{\mathbf{BPL}}}$ ([Sas98]). We will use $\vdash'_{\mathrm{BPL}}$ instead of $\vdash_{\mathbf{BPL},\mathbf{MP}_{\mathbf{BPL}}}$.

Then, the following propositions are proved as lemmas in [Sas98].

Proposition 3.20 *If* $\mathrm{MP}_1 \subseteq \mathrm{MP}_2$, $\mathrm{S}_1 \subseteq \mathrm{S}_2$ *and* $, \vdash_{\mathrm{S}_1,\mathrm{MP}_1} \varphi$, *then* $, \vdash_{\mathrm{S}_2,\mathrm{MP}_2} \varphi$.

Proposition 3.21 $, \vdash'_{\mathrm{BPL}} \varphi$ *implies* $, \vdash_{\mathbf{BPL}} \varphi$.

Proposition 3.22 $\top \rightarrow \bot \vdash'_{\mathrm{BPL}} \alpha \rightarrow \beta$.

Proof We can show $\vdash_{\mathbf{BPL}} (\top \rightarrow \bot) \rightarrow (\alpha \rightarrow \beta)$ by Lemma 3.9 and completeness theorem. By the definition of $\vdash'_{\mathrm{BPL}}$, we have $\top \rightarrow \bot \vdash'_{\mathrm{BPL}} \top \rightarrow \bot$, and hence $\top \rightarrow \bot \vdash'_{\mathrm{BPL}} \alpha \rightarrow \beta$. ❑

Proposition 3.23 *If* $\top \rightarrow \bot, \varphi, \chi \vdash'_{\mathrm{BPL}} \psi$, *then,*

$$\textit{either } \top \rightarrow \bot, \varphi \vdash'_{\mathrm{BPL}} \psi \textit{ or } \top \rightarrow \bot, \chi \vdash'_{\mathrm{BPL}} \psi.$$

Proof By induction on the length of the derivation of $\top \rightarrow \bot, \varphi, \chi \vdash'_{\mathrm{BPL}} \psi$. We will denote the length of derivation by $l(\psi)$. If $l(\psi) = 1$, then either $\psi \in \mathbf{BPL}$ or $\psi \in \{\top \rightarrow \bot, \varphi, \chi\}$ holds. The proposition is trivial in either case. Suppose that it holds for any α such that α is derived from $\top \rightarrow \bot, \varphi, \chi$ within n steps. Let $l(\psi) = n+1$. We will show that ψ is derived from $\mathbf{MP}_{\mathbf{BPL}}$. Suppose that $\top \rightarrow \bot, \varphi, \chi \vdash'_{\mathrm{BPL}} \psi$ is derived from $\top \rightarrow \bot, \varphi, \chi \vdash'_{\mathrm{BPL}} \eta$, $\top \rightarrow \bot, \varphi, \chi \vdash'_{\mathrm{BPL}} \eta \rightarrow \psi$ and $(\eta, \psi) \in \mathrm{MP}_{\mathbf{BPL}}$ for some ψ. By the hypothesis of induction, either $\top \rightarrow \bot, \varphi \vdash'_{\mathrm{BPL}} \eta$ or $\top \rightarrow \bot, \chi \vdash'_{\mathrm{BPL}} \eta$. By Proposition 3.22, both $\top \rightarrow \bot, \varphi \vdash'_{\mathrm{BPL}} \eta \rightarrow \psi$ and $\top \rightarrow \bot, \chi \vdash'_{\mathrm{BPL}} \eta \rightarrow \psi$. Then, either $\top \rightarrow \bot, \varphi \vdash'_{\mathrm{BPL}} \psi$ or $\top \rightarrow \bot, \chi \vdash'_{\mathrm{BPL}} \psi$. ❑

Proposition 3.24 *$p, q \vdash'_{\mathrm{BPL}} p \wedge q$ does not hold for any distinct propositional variables p, q.*

Proof Suppose otherwise. Then, clearly,

$$\top \to \bot, p, q \vdash'_{\mathrm{BPL}} p \wedge q.$$

Then, by Proposition 3.23 , we have either $\top \to \bot, p \vdash'_{\mathrm{BPL}} p \wedge q$ or $\top \to \bot, q \vdash'_{\mathrm{BPL}} p \wedge q$. And, by Proposition 3.21, $\top \to \bot, p \vdash_{\mathbf{BPL}} p \wedge q$ or $\top \to \bot, q \vdash_{\mathbf{BPL}} p \wedge q$. But, taking the frame $\langle \{a\}, \emptyset \rangle$, we can easily show that neither of $\top \to \bot, p \vdash_{\mathbf{BPL}} p \wedge q$ and $\top \to \bot, q \vdash_{\mathbf{BPL}} p \wedge q$ holds. ❑

By the above lemmas, we can prove the following.

Proposition 3.25 *There exists no pair* (S, MP) *such that for each α and each , ,*

$$, \vdash_{\mathbf{BPL}} \alpha \textit{ iff } , \vdash_{\mathrm{S,MP}} \alpha.$$

Proof Suppose that for a pair (S, MP), $, \vdash_{\mathbf{BPL}} \alpha$ iff $, \vdash_{\mathrm{S,MP}} \alpha$ for any formula α and , . If S $\not\subseteq$ **BPL**, then there exists a formula $\psi \in \mathrm{S} \cap (\mathcal{L} \Leftrightarrow \mathbf{BPL})$. So, we have $\vdash_{\mathrm{S,MP}} \psi$ but $\not\vdash_{\mathbf{BPL}} \psi$. This is a contradiction. If MP $\not\subseteq \mathrm{MP}_{\mathbf{BPL}}$, then there exists a pair $(\eta, \psi) \in \mathrm{MP} \cap (\mathcal{L}^2 \Leftrightarrow \mathrm{MP}_{\mathbf{BPL}})$. This proves that there exists a set Σ of formulas such that $\Sigma \vdash_{\mathbf{BPL}} \eta$, $\Sigma \vdash_{\mathbf{BPL}} \eta \to \psi$ and $\Sigma \not\vdash_{\mathbf{BPL}} \psi$. Thus $\Sigma, \eta, \eta \to \psi \not\vdash_{\mathbf{BPL}} \psi$. On the other hand, $(\eta, \psi) \in \mathrm{MP}$ implies $\Sigma, \eta, \eta \to \psi \vdash_{\mathrm{S,MP}} \psi$. This is a contradiction. Therefore, S $\subseteq$ **BPL** and MP $\subseteq \mathrm{MP}_{\mathbf{BPL}}$. By Proposition 3.20, $, \vdash'_{\mathrm{BPL}} \psi$ if $, \vdash_{\mathrm{S,MP}} \psi$. From Proposition 3.24, we have $p, q \not\vdash_{\mathrm{S,MP}} p \wedge q$. But $p, q \vdash_{\mathbf{BPL}} p \wedge q$. This is a contradiction. ❑

3.5 Notes

We has discussed both Hilbert-style calculus **HB** of **BPL** and natural deduction system **NBPL** of **BPL**. Then, it will be natural to ask the following question: How about a Gentzen style proof system for **BPL**? A Gentzen type proof system of **BPL** is introduced first by M. Ardeshir and W. Ruitenburg ([AR95]). The cut-elimination theorem proved by M. Ardeshir ([Ard95]). His system **GBPC** without any structual rule is essentailly identical to the following system **GBPL** with structual rules.

Let , , Δ and Π be a (possibly, empty) finite sequences of $\mathcal{L}$-formulas. We call the following forms *sequents* (of **BPL**);

- $, \Rightarrow \varphi,$
- $, \Rightarrow .$

Gentzen style sequent calculus **GBPL** of **BPL** is as follows:

Initial sequents:

i1. $\varphi \Rightarrow \varphi,$

i2. $\bot \Rightarrow .$

Logical rules:

$$\frac{\varphi,, \Rightarrow \Delta}{\varphi \wedge \psi,, \Rightarrow \Delta}(\wedge\text{-left1}), \qquad \frac{\psi,, \Rightarrow \Delta}{\varphi \wedge \psi,, \Rightarrow \Delta}(\wedge\text{-left2}),$$

$$\frac{, \Rightarrow \varphi \quad , \Rightarrow \psi}{, \Rightarrow \varphi \wedge \psi}(\wedge\text{-right}), \qquad \frac{\varphi,, \Rightarrow \Delta \quad \psi,, \Rightarrow \Delta}{\varphi \vee \psi,, \Rightarrow \Delta}(\vee\text{-left}),$$

$$\frac{, \Rightarrow \varphi}{, \Rightarrow \varphi \vee \psi}(\vee\text{-right1}), \qquad \frac{, \Rightarrow \psi}{, \Rightarrow \varphi \vee \psi}(\vee\text{-right2}),$$

$$\frac{\varphi,, \Rightarrow \psi}{, \Rightarrow \varphi \to \psi}(\to\text{-right}), \qquad \frac{, \Rightarrow \varphi \to \psi \quad , \Rightarrow \psi \to \chi}{, \Rightarrow \varphi \to \chi}(\text{Tr}),$$

$$\frac{\varphi \wedge \psi,, \Rightarrow \Delta \quad \varphi \wedge \chi,, \Rightarrow \Delta}{\varphi \wedge (\psi \vee \chi),, \Rightarrow \Delta}(\text{D-left}),$$

$$\frac{, \Rightarrow \varphi \vee \psi \quad , \Rightarrow \varphi \vee \chi}{, \Rightarrow \varphi \vee (\psi \wedge \chi)}(\text{D-right}),$$

$$\frac{, \Rightarrow \varphi \to \psi \quad , \Rightarrow \varphi \to \chi}{, \Rightarrow \varphi \to (\psi \wedge \chi)}(\text{F}\wedge),$$

$$\frac{, \Rightarrow \varphi \to \psi \quad , \Rightarrow \chi \to \psi}{, \Rightarrow (\varphi \vee \chi) \to \psi}(\text{F}\vee).$$

Structual rules:

$$\frac{, \Rightarrow \Delta}{\varphi,, \Rightarrow \Delta}(\text{weakening-left}), \qquad \frac{, \Rightarrow}{, \Rightarrow \varphi}(\text{weakening-right}),$$

$$\frac{\varphi, \varphi,, \Rightarrow \Delta}{\varphi,, \Rightarrow \Delta}(\text{contraction}), \qquad \frac{,, \varphi, \psi, \Pi \Rightarrow \Delta}{,, \psi, \varphi, \Pi \Rightarrow \Delta}(\text{exchange}),$$

$$\frac{, \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Delta}{,, \Pi \Rightarrow \Delta}(\text{cut}).$$

In [Ard95], it is shown that the cut rule is admissible in **GBPL**.

Chapter 4

Frames and algebraic structures

In the previous chapter, we studied the syntactical side of **BPL**. In this chapter, we will discuss its semantics. Firstly, we will present an algebraic semantics of **BPL**. The class of **BPL**-algebras was introduced by M. Ardeshir and W. Ruitenburg ([AR95]). A **BPL**-algebra is a bounded distributive lattice which satisfies distributive law for meet, join and which has the $\rightarrow$ operator. The $\rightarrow$ operator satisfies some condition which is weaker than the conditions for the relative pseudo-complement for Heyting algebra. We will show that a **BPL**-algebra becomes a Heyting algebra if one simple inequality is added. In Section 4.2, we will discuss duality between algebras and frames. Dual structures for semantics of **BPL** will be introduced as same as for **Int**. In Section 4.3, a *generated subframe* of intuitionistic transitive frame is introduced. This is a substructure of frames, and its definition is the same as **Int**. In the last section, we will discuss relationships between semantic structures and their substructures via homomorphisms and p-morphisms.

4.1 Algebraic structures

Here, we will adopt the definition of algebraic semantics for **BPL** by M. Ardeshir and W. Ruitenburg ([AR95]). In [AR95], algberaic semantics for **BPL** is introduced as follows:

Definition 4.1 $\mathfrak{A}(=\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle)$ *is called a* **BPL**-algebra *if*

- $\langle \wedge, \vee, \rightarrow, 0, 1\rangle$ *is a type,*
- $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice,*

- *the operation* $\to$ *satisfies that for any* $a, b, c \in A$,

$$\begin{aligned} a \to (b \wedge c) &= (a \to b) \wedge (a \to c), \\ (b \vee c) \to a &= (b \to a) \wedge (c \to a), \\ (a \to b) \wedge (b \to c) &\leq (a \to c), \\ a \to a = 1 \quad \textit{and} &\quad a \leq 1 \to a, \end{aligned}$$

where $a \leq b$ *means* $a = a \wedge b$ *(equivalently* $b = a \vee b$*).*

Subalgebras, homomorphisms, embeddings and isomorphisms are defined similarly to those of Heyting algebra. For **BPL**-algebras, results similar to Proposition 2.6 and 2.7 also hold. And the completeness theorem for **BPL** is proved by constructing the Lindenbaum algebra for (see [AR95]).

Proposition 4.2 *The logic* **BPL** *is complete with respect to the class of all* **BPL**-*algebras.*

The following properties are often useful.

Lemma 4.3 *Suppose* $\mathfrak{A}$ *is BPL-algebra. For any* $a, b, c \in \mathfrak{A}$,

$$\begin{aligned} &1) \quad a \leq b \text{ implies } b \to c \leq a \to c, \\ &2) \quad a \leq b \text{ implies } c \to a \leq c \to b. \end{aligned}$$

Proof Suppose $a \leq b$. This is equal to $a \vee b = b$. Then,

$$\begin{aligned} b \to c &= (a \vee b) \to c \\ &= (a \to c) \wedge (b \to c). \end{aligned}$$

This means $b \to c \leq a \to c$. The other point is similar. ❑

An example of applying the above lemma is the following:

Theorem 4.4 *Let* $\mathfrak{A}$ *be a* **BPL**-*algebra.* $\mathfrak{A}$ *is a Heyting algbera, if* $\mathfrak{A}$ *satisfies the inequality* $a \geq 1 \to a$ *for any element* a *of* $\mathfrak{A}$.

Proof It is enough to show that $\to$ is the relatively pseudo-complement. We will show that $a = 1 \to a$ holds for any a. Since $a \leq 1 \to a$ holds always by the definition. Let a and b be elements of $\mathfrak{A}$. Suppose $x \leq (a \to b)$ for an element x. Then, $a \wedge x \leq a \wedge (a \to b)$. And, $a \wedge (a \to b) = (1 \to a) \wedge (a \to b) \leq 1 \to b = b$. Thus, $a \wedge x \leq b$ holds. Conversely, suppose $a \wedge x \leq b$. By Lemma 4.3, $a \to (a \wedge x) \leq a \to b$. Then, $a \to (a \wedge x) = (a \to a) \wedge (a \to x) = a \to x$. Again by Lemma 4.3, $a \leq 1$ implies $x = 1 \to x \leq a \to x$. Then, $x \leq a \to b$ holds. ❑

As a **BPL**-algebra is a distributive lattice, it makes sense to speak of its prime filters. The following results are well-known for distributive lattices.

Proposition 4.5 *Let* $\mathfrak{A}$ *be a distributive lattice,* ∇_0 *a filter and* Δ *an ideal of* $\mathfrak{A}$*. If* $\nabla_0 \cap \Delta = \emptyset$ *then there is a prime filter* ∇ *such that* $\nabla_0 \subseteq \nabla$ *and* $\nabla \cap \Delta = \emptyset$*.*

Corollary 4.6 *If* Δ *is an ideal and* $a \in \mathfrak{A} \Leftrightarrow \Delta$ *there is a prime filter* ∇ *such that* $a \in \nabla$ *and* $\nabla \cap \Delta = \emptyset$*.*

Corollary 4.7 *If* $a, b \in \mathfrak{A}$ *are such that* $a \not\leq b$ *there is a prime filter* ∇ *such that* $a \in \nabla$ *and* $b \notin \nabla$*.*

Suppose $\mathfrak{A}$ is a **BPL**-algebra and $W_{\mathfrak{A}}$ is the set of all prime filters on $\mathfrak{A}$. We define a relation R on $W_{\mathfrak{A}}$, as follows;

$$\nabla R \nabla' \text{ iff } \forall a, b \in A (a \to b \in \nabla \wedge a \in \nabla' \Rightarrow b \in \nabla'). \tag{4.1}$$

The next lemma asserts the existence of prime filters satisfying a condition that will prove useful later on.

Lemma 4.8 *Suppose* $\mathfrak{A} = \langle A, \wedge, \vee, \to, 0, 1 \rangle$ *is a BPL-algebra,* ∇ *a prime filter in* $\mathfrak{A}$ *and let* C *and* D *be subsets of* A *such that*

$$\forall c_1, \ldots, c_n \in C \ \forall d_1, \ldots, d_m \in D \ c_1 \wedge \ldots \wedge c_n \to d_1 \vee \ldots \vee d_m \notin \nabla. \tag{4.2}$$

Then these exists a prime filter ∇' *in* $\mathfrak{A}$ *such that* $C \subseteq \nabla'$*,* $\nabla' \cap D = \emptyset$ *and* $\nabla R \nabla'$*.*

Proof By Zorn's lemma, there is a maximal set ∇' which contains C and satisfies (4.2). We show that ∇' is the required prime filter. First, it is easily checked that ∇' is a filter such that for any a, b, $b \in \nabla'$ whenever $a \in \nabla'$ and $a \to b \in \nabla$. Since $a \to a = 1 \in \nabla$ for every $a \in A$, we have $\nabla' \cap D = \emptyset$. So it remains to show that ∇' is prime, i.e., $b_1 \vee b_2 \in \nabla'$ implies $b_1 \in \nabla'$ or $b_2 \in \nabla'$. Suppose $b_1, b_2 \notin \nabla'$. Then there are $a \in \nabla'$ and $d_1, \ldots, d_n \in D$ such that for $i = 1, 2$,

$$a \wedge b_i \to d_1 \vee \ldots \vee d_n \in \nabla.$$

Hence

$$(a \wedge b_1) \vee (a \wedge b_2) \to d_1 \vee \ldots \vee d_n \in \nabla$$

and so, by distributivity,

$$a \wedge (b_1 \vee b_2) \to d_1 \vee \ldots \vee d_n \in \nabla,$$

which means that $b_1 \vee b_2 \notin \nabla'$. This contradicts our assumption. ❑

4.2 Duality theorem

For the case of intuitionistic logic, it is known that a relation called "duality" holds between Heyting algebras and quasi-ordered frames. Here, we will show that this relation also holds between transitive frames and BPL-algebras. The whole terminology for duals, including definitions, is the same for **BPL**-algebras and transitive frames as for Heyting algebras and quasi-ordered frames (see, Chapter 2). Firstly, it is proved that the dual of a transitive frame (and a **BPL**-algebra) becomes a **BPL**-algebra (and a transitive frame). Then, we will prove that the duality theorem for transitive frames (and **BPL**-algebras) holds.

Theorem 4.9 *For any intuitionistic transitive frame $\mathfrak{F}$, $\mathfrak{F}^+$ is a BPL-algebra.*

Proof Suppose $\mathfrak{F}$ is $\langle W, R, P\rangle$. We will show that $\mathfrak{F}^+ = \langle P, \cap, \cup, \supset, \emptyset, W\rangle$ is a BPL-algebra. It is obvious that $\langle P, \cap, \cup, \emptyset, W\rangle$ is a bounded distributive lattice with respect to the set-theoretic operations. We need to prove that this structure satisfies the conditions for BPL-algebras. For any $X, Y, Z \in P$,

$$\begin{aligned} x \in X \supset (Y \cap Z) &\Leftrightarrow \forall y \in W(xRy \wedge y \in X \Rightarrow y \in Y \cap Z), \\ &\Leftrightarrow \forall y \in W(xRy \wedge y \in X \Rightarrow y \in Y) \\ &\quad\ \wedge \forall y \in W(xRy \wedge y \in X \Rightarrow y \in Z), \\ &\Leftrightarrow x \in (X \supset Y) \cap (X \supset Z); \end{aligned}$$

$$\begin{aligned} x \in (Y \cup Z) \supset X &\Leftrightarrow \forall y \in W(xRy \wedge y \in (Y \cup Z) \Rightarrow y \in X), \\ &\Leftrightarrow \forall y \in W(xRy \wedge y \in Y \Rightarrow y \in X) \\ &\quad\ \wedge \forall y \in W(xRy \wedge y \in Z \Rightarrow y \in X), \\ &\Leftrightarrow x \in (Y \supset X) \cap (Z \supset X); \end{aligned}$$

It is easy to see that for any $x \in W$, $x \in X \supset X$. Thus, $W = X \supset X$.

Next, we will show that $X \subseteq (W \supset X)$. Since X is an R-cone, $x \in X$ and xRy imply $y \in X$. Thus, if $x \in X$, and if xRy and $y \in W$ then $y \in X$ for all y.

Lastly, we will show that

$$x \in (X \supset Y) \cap (Y \supset Z) \Rightarrow x \in (X \supset Z).$$

Let $x \in (X \supset Y) \cap (Y \supset Z)$. Suppose xRy and $y \in X$. $y \in Y$ since $x \in (X \supset Y)$. Then, xRy and $y \in Y$ yield $y \in Z$, since $x \in (Y \supset Z)$. Thus, $x \in X \supset Z$. ❑

Clearly, the following result holds:

Theorem 4.10 *For any formula* φ, $\mathfrak{F} \models \varphi \Leftrightarrow \mathfrak{F}^+ \models \varphi$.

Next, we prove that the dual of any **BPL**-algebra is an intuitionistic transitive frame.

Theorem 4.11 *For any BPL-algebra* $\mathfrak{A}$, $\mathfrak{A}_+$ *is an intuitionistic transitive frame.*

Proof Suppose that $\mathfrak{A}$ is $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$. We will show that $\mathfrak{A}_+ = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}}\rangle$ is an intuitionistic transitive frame. To prove $R_{\mathfrak{A}}$ is transitive, it is sufficient to show that $\nabla R_{\mathfrak{A}} \nabla'$ implies $\nabla \subseteq \nabla'$. Let $\nabla R_{\mathfrak{A}} \nabla'$. Suppose $a \in \nabla$. By Lemma 4.3, for any $b \in A$, $b \leq 1$, $a \leq (1 \rightarrow a) \leq (b \rightarrow a)$. Thus, $b \rightarrow a \in \nabla$. Fix the b above as an element of ∇'. Then by the definition of $R_{\mathfrak{A}}$, $a \in \nabla'$. Thus, $\nabla R_{\mathfrak{A}} \nabla'$ yields $\nabla \subseteq \nabla'$. Finally the properties of a set $P_{\mathfrak{A}} (= \{\mathbf{p}(a) : a \in \mathfrak{A}\})$ remain, that is, $P_{\mathfrak{A}}$ is a set of $R_{\mathfrak{A}}$-cones, and is closed under the set operations $\cap$, $\cup$ and $\supset$. That any element $\mathbf{p}(a)$ of $P_{\mathfrak{A}}$ is an $R_{\mathfrak{A}}$-cone, is straightforward, since $\nabla R_{\mathfrak{A}} \nabla'$ implies $\nabla \subseteq \nabla'$, and $\nabla \in \mathbf{p}(a)$ is equivalent to $a \in \nabla$. Thus, it remains to show that $P_{\mathfrak{A}}$ is closed under $\cap$, $\cup$ and $\supset$. As for $\cap$ and $\cup$, this is straightforward by the definition of prime filter:

$$\begin{aligned}
\nabla \in \mathbf{p}(a \wedge b) \;&\Leftrightarrow\; a \wedge b \in \nabla, \\
&\Leftrightarrow\; a \in \nabla \text{ and } b \in \nabla, \\
&\Leftrightarrow\; \nabla \in \mathbf{p}(a) \text{ and } \nabla \in \mathbf{p}(b), \\
&\Leftrightarrow\; \nabla \in \mathbf{p}(a) \cap \mathbf{p}(b);
\end{aligned}$$

$$\begin{aligned}
\nabla \in \mathbf{p}(a \vee b) \;&\Leftrightarrow\; a \vee b \in \nabla, \\
&\Leftrightarrow\; a \in \nabla \text{ or } b \in \nabla, \\
&\Leftrightarrow\; \nabla \in \mathbf{p}(a) \text{ or } \nabla \in \mathbf{p}(b), \\
&\Leftrightarrow\; \nabla \in \mathbf{p}(a) \cup \mathbf{p}(b).
\end{aligned}$$

For $\supset$ the proof is as follows: Suppose $\nabla \in \mathbf{p}(a \rightarrow b)$. By the definition of $\mathbf{p}$, $a \rightarrow b \in \nabla$. The condition that $\nabla \in \mathbf{p}(a) \supset \mathbf{p}(b)$ is equivalent to the condition that $\nabla R_{\mathfrak{A}} \nabla'$ and $\nabla' \in \mathbf{p}(a)$ imply $\nabla' \in \mathbf{p}(b)$ for any prime filter ∇'. Now, suppose that a prime filter ∇' satisfies $\nabla R_{\mathfrak{A}} \nabla'$ and $\nabla' \in \mathbf{p}(a)$. By the definition of $R_{\mathfrak{A}}$ and the fact that $a \rightarrow b \in \nabla$, we obtain that $\nabla' \in \mathbf{p}(b)$

follows from $\nabla' \in \mathbf{p}(a)$. Thus, $\mathbf{p}(a \to b) \subseteq \mathbf{p}(a) \supset \mathbf{p}(b)$. The opposite direction is derived immediately from Lemma 4.8 by putting $C = \{a\}$ and $D = \{b\}$. ❑

Now, we will prove our two duality theorems.

Theorem 4.12 (duality) *For any BPL-algebra* $\mathfrak{A}$, $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$.

Proof Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$. Then, $(\mathfrak{A}_+)^+$ is $\langle P_{\mathfrak{A}}, \cap, \cup, \supset, \emptyset, W_{\mathfrak{A}}\rangle$. Define a function h from A to $P_{\mathfrak{A}}$ by $h(a) = \mathbf{p}(a)$ for any $a \in A$. Clearly, h is a surjective homomorphism of $\mathfrak{A}$ in $(\mathfrak{A}_+)^+$ by the definition of $\mathbf{p}$. It remains to show that $\mathbf{p}$ is one-to-one. Suppose that $a \neq b$. Then, either $a \not\leq b$ or $b \not\leq a$ holds. By Proposition 4.7, $\mathbf{p}(a) \neq \mathbf{p}(b)$. ❑

As we saw in Section 2.2, isomorphism of frames does not hold similary to the case of algebras. For frames, we need the descriptivity condition again (see Definition 2.12).

Theorem 4.13 (duality) *An intuitionistic transitive frame* $\mathfrak{F} = \langle W, R, P\rangle$ *is isomorphic to* $(\mathfrak{F}^+)_+$ *iff* $\mathfrak{F}$ *is descriptive.*

Proof Suppose $\mathfrak{F}$ is descriptive. Define the map h from W to 2^{2^W} by $h(x) = \{X \in P : x \in X\}$ for any $x \in W$. It is trivial that $h(x)$ is a prime filter. Thus, h from W to $W_{\mathfrak{F}^+}$. We will show that h is an isomorphism from $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$.

- h is an injection.

  Suppose $h(x) = h(y)$. By the definition of h, $x \in X \Leftrightarrow y \in X$ for any $X \in P$. By Condition 1 of Definition 2.12, $x = y$.

- h is a surjection.

  We will show that for any prime filter ∇, there exists an element x of W such that $h(x) = \nabla$. Let $\Delta = \{W \Leftrightarrow Z : Z \in P \text{ and } Z \notin \nabla\}$. For any $X_1, \ldots, X_n \in \nabla$ and $Y_1, \ldots, Y_m \in \Delta$,

$$\begin{aligned} & (X_1 \cap \ldots \cap X_n) \cap (Y_1 \cap \ldots \cap Y_m) \\ = \; & (X_1 \cap \ldots \cap X_n) \cap (W \Leftrightarrow (Z_1 \cup \ldots \cup Z_m)) \\ = \; & (X_1 \cap \ldots \cap X_n) \Leftrightarrow (Z_1 \cup \ldots \cup Z_m). \end{aligned}$$

If $(X_1 \cap \ldots \cap X_n) \cap (Y_1 \cap \ldots \cap Y_m) = \emptyset$, then $X_1 \cap \ldots \cap X_n \subseteq Z_1 \ldots Z_m$. Since ∇ is a prime filter, $Z_i \in \nabla$ for some i. But this is a contradiction. Thus, $(X_1 \cap \ldots \cap X_n) \cap (Y_1 \cap \ldots \cap Y_m) \neq \emptyset$. That is, $\bigcap(\nabla \cup \Delta)$ has the finite intersection property. By Condition 3 of Definition 2.12, $\bigcap(\nabla \cup \Delta) \neq \emptyset$. That is,

$$\exists x \in W \ (\forall X \in \nabla(x \in X)) \wedge (\forall Y \in \Delta(x \in Y)). \tag{4.3}$$

If there exists an element X of P such that $X \in h(x)$ and $X \notin \nabla$, both $x \in X$ and $x \in W \Leftrightarrow X$ hold by (4.3). But this is a contradiction again. Thus, $h(x) \subseteq \nabla$. The converse direction follows easily by (4.3).

- $h(x) R_{\mathfrak{F}^+} h(y) \Rightarrow xRy$;

  This can be shown in the following way, applying Condition 2 of Definition 2.12:

$$\begin{aligned} h(x)R_{\mathfrak{F}^+}h(y) &\Leftrightarrow \forall X, Y \in P[(X \supset Y) \in h(x) \wedge X \in h(y) \\ &\qquad\qquad \Rightarrow Y \in h(y)] \\ &\Leftrightarrow \forall X, Y \in P[x \in (X \supset Y) \wedge y \in X \Rightarrow y \in Y] \\ &\Leftrightarrow xRy. \end{aligned}$$

- $xRz \Rightarrow \exists \nabla \in W_{\mathfrak{F}^+}[h(x) R_{\mathfrak{F}^+} \nabla \wedge \nabla = h(z)]$;

  Since h is a bijective mapping, there exists ∇ such that $h(z) = \nabla$. It remains to show that $xRz \Rightarrow h(x)R_{\mathfrak{F}^+}h(y)$, but we have already done it above.

- $X \in P \Rightarrow h(X) \in P_{\mathfrak{F}^+}$;

  Follows by the equalities below:

$$h(X) = \{h(x) : x \in X\} = \{h(x) \in W_{\mathfrak{F}^+} : X \in h(x)\} = \mathbf{p}(X).$$

Next, suppose that $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$. Then,

1. $x = y$ iff $\forall X \in P(x \in X \Leftrightarrow y \in X)$.

   The 'only if' part is trivial. The 'if' part is as follows. Without loss of generality, we can concentrate on the bidual $(\mathfrak{F}^+)_+$ of $\mathfrak{F}$, by the assumption. Suppose that, for $\mathbf{p}(X) \in P_{\mathfrak{F}^+}$, we have $\nabla \in \mathbf{p}(X) \Leftrightarrow \nabla' \in \mathbf{p}(X)$. By the definition of $\mathbf{p}$, $X \in \nabla \Leftrightarrow X \in \nabla'$ for all $X \in P$, and then, $\nabla = \nabla'$.

2. xRy iff $\forall X, Y \in P(x \in X \supset Y \wedge y \in X \Rightarrow y \in Y)$.

   Suppose xRy, $x \in X \supset Y$ and $y \in X$. Then, by the definition of $\supset$, $y \in Y$. For the other direction, we concentrate on the bidual of $\mathfrak{F}$. For any $\mathbf{p}(X), \mathbf{p}(Y) \in P_{\mathfrak{F}^+}$, suppose $\nabla \in \mathbf{p}(X \supset Y) \wedge \nabla' \in \mathbf{p}(X) \Rightarrow \nabla' \in \mathbf{p}(Y)$. By the definition of $\mathbf{p}$, $X \supset Y \in \nabla \wedge X \in \nabla' \Rightarrow Y \in \nabla'$ for all $X, Y \in P$. Then, $\nabla R_{\mathfrak{F}^+} \nabla'$.

3. $\langle W, P \rangle$ is compact.

   Let $\mathcal{X} \subseteq P$, $\mathcal{Y} \subseteq \{W \Leftrightarrow X : X \in P\}$, and let $\mathcal{X} \cup \mathcal{Y}$ have the finite intersection property, i.e., for any finite subsets $\mathcal{X}'$ of $\mathcal{X}$ and $\mathcal{Y}'$ of $\mathcal{Y}$, $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$. In $\mathfrak{F}^+$, we take the filter ∇_0 and the ideal Δ, which are generated by $\mathcal{X}$ and $\{W \Leftrightarrow Y : Y \in \mathcal{Y}\}$, respectively, namely:

$$\begin{aligned} \nabla_0 &= \{Z : \exists X \in \mathcal{X}(X \subseteq Z)\}, \\ \Delta &= \{Z : \exists Y \in \mathcal{Y}(Z \subseteq W \Leftrightarrow Y)\}. \end{aligned}$$

   Suppose $\nabla_0 \cap \Delta \neq \emptyset$. Then, for some finite subsets $\mathcal{X}'$ of $\mathcal{X}$ and $\mathcal{Y}'$ of $\mathcal{Y}$,we have $X \subseteq W \Leftrightarrow Y$, where $X = \bigcap \mathcal{X}'$ and $Y = \bigcap \mathcal{Y}'$. It follows that $X \cap Y = \emptyset$, but this contradicts the finite intersection property. So, $\nabla_0 \cap \Delta = \emptyset$. By Proposition 4.5, there exists a prime filter ∇ such that $\nabla_0 \subseteq \nabla$ and $\nabla \cap \Delta = \emptyset$. As there exists an isomorphism h from $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$, we can assume that our prime filter ∇ has $h(x) = \nabla$, for some $x \in W$. We can take h to be the isomorphism which is defined in the 'if' part of Proposition 4.5, that is $\nabla = \{X \in P : x \in X\}$. Then, $x \in X$ for any $X \in \nabla_0$ and $x \notin Y$ for any $Y \in \Delta$. Thus, $x \in X$ and $x \in Y$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and so $\bigcap(\mathcal{X} \cap \mathcal{Y}) \neq \emptyset$.

❑

4.3 Generated subframes

Generated subframes can be defined also for intuitionistic transitive frames. Then, the following holds.

Theorem 4.14 *Any generated subframe of a given intuitionistic transitive frames is an intuitionistic transitive frame.*

Proof Suppose $\mathfrak{G} = \langle V, S, T\rangle$ is a generated subframe of $\langle W, R, P\rangle$. It is enough to show that $T(=\{X \cap V : X \in P\})$ is closed under the operation $\supset$. Let X, Y be elements of P. Suppose $a \in (X \supset Y) \cap V$. That is, $a \in (X \supset Y)$ and $a \in V$. Then, for any b, aRb and $b \in X$ imply $b \in Y$. So, for any b, aSb and $b \in X \cap V$ imply $b \in Y \cap V$, i.e., $a \in (X \cap V) \supset (Y \cap V)$. The opposite direction is shown as follows: Suppose $a \in (X \cap V) \supset (Y \cap V)$. That is, for any b, aSb and $b \in (X \cap V)$ imply $b \in (Y \cap V)$. Clearly a is an element of V because aSb holds. Then, an element b which enjoys aRb is an element of V since V is an R-cone. That is, for any b, aRb and $b \in X$ imply both aSb and $b \in (X \cap V)$. So, by the hypothesis, $a \in (X \cap V) \supset (Y \cap V)$, $b \in (Y \cap V)$, inparticular, $b \in Y$. Thus, $a \in X \supset Y$ and so $a \in (X \supset Y) \cap V$. Using this fact, we can easily derive that T is closed under $\supset$. ❑

The proof of the next lemma is the same as tha of Proposition 2.4:

Lemma 4.15 *Let $\mathfrak{G}$ be a generated subframe of $\mathfrak{F}$. Then, for any $\mathcal{L}$-formula φ, $\mathfrak{F} \models \varphi$ implies $\mathfrak{G} \models \varphi$.*

4.4 Homomorphisms and isomorphisms

Many properties of semantics of **Int** hold also for **BPL**. For instance, as shown in Proposition 2.4 and Lemma 4.15, validity of formulas is inherited from a given original frame to its generated subframes. It is also easily shown that results analogous Proposition 2.3, 2.6 and 2.7 hold for semantics of **BPL**.

And, the followings hold as well.

Theorem 4.16 (i) *If $\mathfrak{G} = \langle V, S, Q\rangle$ is a generated subframe of $\mathfrak{F} = \langle W, R, P\rangle$ then the map $f : P \to Q$ defined by $f(X) = X \cap V$ for $X \in P$, is a homomorphism from $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If f is a homomorphism from a BPL-algebra $\mathfrak{A}$ onto a BPL-algebra $\mathfrak{B}$ then $f_+ : W_{\mathfrak{B}} \to W_{\mathfrak{A}}$ defined by $f_+(\nabla) = f^{-1}(\nabla)$ for a prime filter ∇ of $\mathfrak{B}$, is an isomorphism from $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

(iii) *If h is a reduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G} = \langle V, S, Q\rangle$ then $h^+ : P \to Q$ defined by $h^+(X) = h^{-1}(X)$ for $X \in Q$, is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(iv) *If $\mathfrak{B}(= \langle B, \wedge, \vee, \to, 0, 1\rangle)$ is a subalgebra of a* **BPL**-*algebra $\mathfrak{A}(= \langle A, \wedge, \vee, \to, 0, 1\rangle)$ then the map $h : W_{\mathfrak{A}} \to W_{\mathfrak{B}}$ defined by $h(\nabla) = \nabla \cap B$ for a prime filter ∇ in $\mathfrak{A}$, is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

Proof We show only (ii), other points can be proved analogously. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ and $\langle B, \wedge, \vee, \rightarrow, 0, 1\rangle$, respectively. It is trivial that f_+ is well-defined, an injection and that $f_+(\nabla)$ is a prime filter in $\mathfrak{A}$, for every $\nabla \in W_{\mathfrak{B}}$. It is also straightforward to see that for every $\nabla, \nabla' \in W_{\mathfrak{B}}$, we have $\nabla R_{\mathfrak{B}} \nabla'$ iff $f_+(\nabla) R_{\mathfrak{A}} f_+(\nabla')$. It is not difficult to show that for every $b \in B$, there exists $a \in A$ such that $f(a) = b$ and $f_+(\mathbf{p}(b)) = \mathbf{p}(a) \cap f_+(W_{\mathfrak{B}})$. Then, to prove this proposition, it is enough to show that $f_+(W_{\mathfrak{B}})$ is $R_{\mathfrak{A}}$-cone. Suppose that for every $\nabla, \nabla' \in W_{\mathfrak{A}}$, we have $\nabla R_{\mathfrak{A}} \nabla'$ and $\nabla \in f_+(W_{\mathfrak{B}})$. Then the condition (4.2) of Lemma 4.8 holds for $C = \nabla'$ and $D = A \Leftrightarrow \nabla'$. So there exists a prime filter $\nabla_{\mathfrak{B}}$ such that $\nabla' = f_+(\nabla_{\mathfrak{B}})$. Then $\nabla' \in f_+(W_{\mathfrak{B}})$. ❑

Chapter 5

Extensions of BPL

In this chapter, extensions of **BPL** are introduced, and relations between the extensions of **BPL** and modal logics are discussed. In 1970's, Blok and Esakia proved that there is a lattice isormorphism from Ext**Int** onto NExt**Grz** ([Blo76], [Esa79a, Esa79b]). This result is known as the Blok-Esakia theorem. As we mentioned in Chapter 2, the modal operator $\Box$ of **Grz** denotes the provability of ZF set-theory, while the modal operator of **GL** denotes the formal provability of Peano arithmetic. In 1981, Visser proved that **FPL** which is an extension of **BPL** is embedded into **GL** by using Gödel translation T ([Vis81]). Then, the following question naturally occurs: is there a lattice isomorphism from extensions of **FPL** onto NExt**GL** like Blok-Esakia Theorem? Moreover, what relations hold between **BPL** and **K4**? To answer this question, we need to get an appropriate definition of extensions of **BPL**. However, to define extensions of **BPL**, we do not adopt the similar way intermediate logics. We will try to explain the reason by investigating two kinds of expressive power on non-modal propositional language and modal propositional language, semantically. This result of expressive powers will be proved in the first section of this chapter. We will follow the definition of extensions of **BPL** by Visser in [Vis81], and therefore extensions are obtained from **BPL** by adding not only axioms but also inference rules.

In Section 5.2, some interesting extensions of **BPL** are introduced. In the remaing sections, consequence relations are discussed and Kripke completeness of these consequence relations are proved. In Section 5.5, we will show that there are no lattice isomorphisms from the class of extensions of **FPL** into NExt**GL**.

5.1 Expressive power of $\mathcal{L}$ and $\mathcal{ML}$

We want to know a relation between the propositional language $\mathcal{L}$ and the modal propositional language $\mathcal{ML}$ by a translation. Here, two kinds of expressive power of $\mathcal{L}$ and $\mathcal{ML}$ will be discussed. Differences of expressive power between distinct languages are interesting to investigate itself. First two expressive powers on quasi-ordered frames are studied. It is shown that there are no difffernces of expressive power between $\mathcal{L}$ and $\mathcal{ML}$. Next, we will discuss them on transitive frames. We will show that the expressive power of $\mathcal{L}$ weaker than that of $\mathcal{ML}$ in this case. As a consequence, we can see that the set of $\mathcal{L}$-formulas which is validated only by the class of quasi-ordered frames cannot be obtained by adding formulas as axioms to **BPL**.

5.1.1 Local expressive power on quasi-ordered frame

Let $\mathfrak{F} = \langle W, R\rangle$ be a quasi-ordered Kripke frame. Our first question is the following: Suppose that a valuation function $\mathfrak{V}$ from Prop to 2^W is given for both For$\mathcal{L}$ and For$\mathcal{ML}$. For this valuation $\mathfrak{V}$, each formula φ in $\mathcal{L}$ (and in $\mathcal{ML}$) with n propositional variables can be regarded as a function on UpW (and 2^W). The *local expressive power* of a given language will be measured by the set of all function on UpW or 2^W which are defined by some formulas in that language. Now, we will introduce this precisely. Let a quasi-ordered Kripke frame $\mathfrak{F} = \langle W, R\rangle$ and a $\mathcal{L}$-formula φ be given. We also suppose that all propositional variables of φ occurs in $p_1, \ldots, p_n$, and denote it as $\varphi(p_1, \ldots, p_n)$. For given frame $\mathfrak{F}$ and $\varphi(p_1, \ldots, p_n)$, an operator $\varphi_{\mathfrak{F}}$ from $(\mathrm{Up}W)^n$ to UpW is defined as follows: for any valuation $\mathfrak{V}$,

$$\varphi_{\mathfrak{F}}(\mathfrak{V}(p_1), \ldots, \mathfrak{V}(p_n)) = \mathfrak{V}(\varphi).$$

In the same way, for a quasi-ordered Kripke frame $\mathfrak{F} = \langle W, R\rangle$ and a $\mathcal{ML}$-formula φ, we can define an operator $\varphi^*_{\mathfrak{F}}$ from $(2^W)^n$ to 2^W for any valuation $\mathfrak{V}$,

$$\varphi^*_{\mathfrak{F}}(\mathfrak{V}(p_1), \ldots, \mathfrak{V}(p_n)) = \mathfrak{V}(\varphi).$$

Now, we want to compare the local expressive power of $\mathcal{L}$ with that of $\mathcal{ML}$. But, $\varphi_{\mathfrak{F}}$ is a function on UpW while $\varphi^*_{\mathfrak{F}}$ is a function on 2^W. For our puropose, we will introduce $\varphi^{\triangle}_{\mathfrak{F}}$ from $(\mathrm{Up}W)^n$ to UpW as follows:

$$\varphi^{\triangle}_{\mathfrak{F}} = \begin{cases} \varphi^*_{\mathfrak{F}} & \text{if } \varphi^*_{\mathfrak{F}}(X_1, \ldots, X_n) \in \mathrm{Up}W \text{ for any } X_1, \ldots, X_n \in \mathrm{Up}W \\ \bot^*_{\mathfrak{F}} & \text{otherwise.} \end{cases}$$

Then, the next result guarantees that local expressive power of $\mathcal{L}$ and $\mathcal{ML}$ on quasi-ordere frame is same.

Theorem 5.1 *Let $\mathfrak{F}$ be a quasi-ordered Kripke frame. Then,*

$$\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} = \{\varphi_{\mathfrak{F}}^{\triangle} : \varphi \in \mathcal{ML}\}.$$

Proof To prove $\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} \subseteq \{\varphi_{\mathfrak{F}}^{\triangle} : \varphi \in \mathcal{ML}\}$, it is enough to show $\varphi_{\mathfrak{F}} = (\mathsf{T}\varphi)_{\mathfrak{F}}^{\triangle}$ by structural induction on φ where T is the Gödel translation. So, we will show the opposite direction $\{\varphi_{\mathfrak{F}}^{\triangle} : \varphi \in \mathcal{ML}\} \subseteq \{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\}$. Let $\varphi_{\mathfrak{F}}^{\triangle}$ be an element of $\{\chi_{\mathfrak{F}}^{\triangle} : \chi \in \mathcal{ML}\}$. We have to show that there exists a $\mathcal{L}$-formula ψ such that $\varphi_{\mathfrak{F}}^{\triangle} = \psi_{\mathfrak{F}}$. Suppose $\varphi_{\mathfrak{F}}^{*}(X_1, \ldots, X_n) \notin \mathrm{Up}W$ holds for some $X_1, \ldots, X_n \in \mathrm{Up}W$. Then $\varphi_{\mathfrak{F}}^{\triangle}$ is $\bot_{\mathfrak{F}}^{*}$. The operator $\bot_{\mathfrak{F}}^{*}$ is same to the operator $\bot_{\mathfrak{F}}$. That is, ψ is $\bot (\in \mathcal{L})$ for this case. Let us consider the other case. Namely, for any $X_1, \ldots, X_n \in \mathrm{Up}W$, $\varphi_{\mathfrak{F}}^{*}(X_1, \ldots, X_n) \in \mathrm{Up}W$ holds. In this case, it is easy to show that $\varphi_{\mathfrak{F}}^{\triangle} = (\Box\varphi)_{\mathfrak{F}}^{\triangle}$ holds for any $\mathcal{ML}$-formula φ. By Lemma 2.29, we can easily find a formula ψ in Form$\mathcal{ML}$ which is in the form[1] $\bigwedge_{i=1}^{m}(\neg p_i \vee q_i)$. Then, by Lemma 2.30, there exists a $\mathcal{ML}$-formula ψ such that $\varphi_{\mathfrak{F}}^{\triangle} = \psi_{\mathfrak{F}}$. ❑

5.1.2 Global expressive power on quasi-ordered frame

The local expressive power discussed in the previous section is concerned with a fixed frame. In this section, we will discuss the more global expressivity, that is, an axiomatizablity for a given class of frames.

Suppose that $\mathfrak{C}_{\mathrm{qo}}$ is the class of all quasi-ordered Kripke frames. A class $\mathfrak{C}$ of quasi-ordered frames is said to be $\mathcal{L}$-($\mathcal{ML}$-)*axiomatic* if there exists a set , of $\mathcal{L}$-($\mathcal{ML}$-)formulas such that for every frame $\mathfrak{F}$ of $\mathfrak{C}_{\mathrm{qo}}$,

$$\mathfrak{F} \models , \text{ iff } \mathfrak{F} \in \mathfrak{C}.$$

This axiomatizability for a given class of frames on $\mathcal{L}$ ($\mathcal{ML}$) is called the *global expressive power* on $\mathcal{L}$ ($\mathcal{ML}$). Since truth value of any $\mathcal{L}$-formula is same at any point in one cluster, it is suffices to consider frame classes modulo clusters when axiomatic powers of modal and intuitionistic frames

[1] When applying Lemma 2.29 to this proof, it is easy to understand that index i is added not only to propositional variables p and q, but also to disjunction. That is, Lemma 2.29 states that any $(\varphi_{\mathfrak{F}})^{\triangle}$ can be expressed in an operator whose $\mathcal{ML}$-formula is a conjunctive normal form.

are compared. A class $\mathfrak{C}$ of quasi-ordered frames is *skelton-closed* if for any frame $\mathfrak{F}$ of $\mathfrak{C}$, $\mathfrak{C}$ contains the all quasi-ordered frames whose skelton frames are isomorphic to the skelton of $\mathfrak{F}$.

To prove that global expressive powers of $\mathcal{L}$ and of $\mathcal{ML}$ are same on a skelton-closed class of quasi-ordered Kripke frames, we need the results by A.Chagrov and M. Zakharyachev ([Zak92] and [CZ97]). Here, we will cite their results without proofs. Let $\mathfrak{F}$ be a finite Kripke frame, and $a_0, \dots, a_n$ all its distinct elements, with a_0 being a root. Suppose $\mathfrak{D}$ is a (possibily empty) set of antichains in $\mathfrak{F}$. Then, the following $\mathcal{ML}$-formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is associated with $\mathfrak{F}$ and $\mathfrak{D}$:

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) = (\bigwedge_{a_i R a_j} \varphi_{ij} \wedge \bigwedge_{i=0}^{n} \varphi_i \wedge \bigwedge_{\delta \in \mathfrak{D}} \varphi_\delta \wedge \varphi_\bot) \to p_0,$$

where

$$\begin{aligned}
\varphi_{ij} &= \Box^+(\Box p_j \to p_i),\\
\varphi_i &= \Box^+((\bigwedge_{\neg a_i R a_k} \Box p_k \wedge \bigwedge_{j=0, i\neq 0}^{n} p_j \to p_i) \to p_i),\\
\varphi_\delta &= \Box^+(\bigwedge_{a_i \in W - \delta\underline{\uparrow}} \Box p_i \wedge \bigwedge_{i=0}^{n} p_i \to \bigvee_{a_j \in \delta} \Box p_j),\\
\varphi_\bot &= \Box^+(\bigwedge_{i=0}^{n} \Box^+ p_i \to \bot).
\end{aligned}$$

Meanwhile, with intuitionistic $\mathfrak{F}$ and $\mathfrak{D}$, the following $\mathcal{L}$-formula $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ is associated:

$$\beta(\mathfrak{F}, \mathfrak{D}, \bot) = (\bigwedge_{a_i R a_j} \psi_{ij} \wedge \bigwedge_{\delta \in \mathfrak{D}} \psi_\delta \wedge \psi_\bot) \to p_0,$$

where,

$$\begin{aligned}
\psi_{ij} &= (\bigwedge_{\neg a_j R a_k} p_k \to p_j) \to p_i,\\
\psi_\delta &= \bigwedge_{a_i \in W - \delta\uparrow} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \bigvee_{a_j \in \delta} p_j,\\
\psi_\bot &= \bigwedge_{i=0}^{n} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \bot.
\end{aligned}$$

Between the above two canonical formulas α and β, the following lemma is proved in [CZ97].

Lemma 5.2 *Suppose $\mathfrak{F}$ is a finite rooted intuitionistic quasi-ordered Kripke frame, $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$ and $\mathfrak{G}$ a modal quasi-ordered Kripke frame. Then*

$$\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}, \bot) \text{ iff } \rho\mathfrak{G} \models \beta(\mathfrak{F}, \mathfrak{D}, \bot).$$

Proof See Lemma 9.59 in [CZ97]. ❑

Using the above facts on canonical formulas, the following result, which states global expressive powers of $\mathcal{L}$ and $\mathcal{ML}$ are same, is deduced:

Theorem 5.3 *A skelton-closed class $\mathfrak{C}$ of quasi-orderd Kripke frames is $\mathcal{L}$-axiomatic if and only if it is $\mathcal{ML}$-axiomatic.*

Proof Suppose $\mathfrak{C}$ is $\mathcal{L}$-axiomatic. Then, there exists a set , of $\mathcal{L}$-formulas such that $\mathfrak{C} \models ,$. That is, for any $\mathfrak{F} \in \mathfrak{C}$ and any $\mathcal{L}$-formula $\varphi \in ,$, $\mathfrak{F} \models \varphi$. By the proof of Theorem 5.1, for any $\mathfrak{F} \in \mathfrak{C}$ and for any $\varphi \in ,$, $\varphi_{\mathfrak{F}}$ is identical to $(\mathsf{T}\varphi)^{\triangle}_{\mathfrak{F}}$, and this identity does not depend on a selection of frames in $\mathfrak{C}$. Thus, $\mathfrak{C}$ is $\mathcal{ML}$-axiomatizable by the set $\{\mathsf{T}\varphi : \varphi \in ,\ \}$. Conversely, suppose $\mathfrak{C}$ is $\mathcal{ML}$-axiomatic. Then, by [Zak92], it can be axiomatizable by a set of modal canonical formulas $\{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$ built on quasi-ordered frames. By the criterion for canonical formulas, $\mathfrak{F} \models \alpha(\rho\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ implies $\mathfrak{F} \models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, and $\mathfrak{F} \not\models \alpha(\rho\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, for some quasi-ordered Kripke frame $\mathfrak{G}$ with $\rho\mathfrak{G} \simeq \rho\mathfrak{F}$. Since $\mathfrak{C}$ is skelton closed, it follows that it is axiomatizable by the set $\{\alpha(\rho\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$. By Lemma 5.2, $\mathfrak{C}$ is axiomatizable by the set $\{\beta(\rho\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$. ❑

5.1.3 Expressive powers on transitive frame

In the previous two sections, it is showed that both local and global expressive power of $\mathcal{L}$ and $\mathcal{ML}$ on quasi-ordered Kripke frame are the same. These two notions of expressive powers can be extened to a class of transitive frames by rewriting the term "quasi-ordred" by "transitive" in the definition. However, we will show that both cases, the expressive power of $\mathcal{L}$ is weaker than that of $\mathcal{ML}$ on transitive frame.

Theorem 5.4 *Let $\mathfrak{F}$ be a transitive Kripke frame. Then,*

$$\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} \subseteq \{\varphi^{\triangle}_{\mathfrak{F}} : \varphi \in \mathcal{ML}\}.$$

Proof Let T' be a function from For$\mathcal{L}$ to For$\mathcal{ML}$ which prefixes $\Box$ to every subformula of $\mathcal{L}$-formula φ of the form $\psi \to \chi$. Then, it is not hard to show that $\varphi_{\mathfrak{F}} = (\mathsf{T}'\varphi)^{\triangle}_{\mathfrak{F}}$ by induction on the construction of an $\mathcal{L}$-formula φ. On the other hand, there are no $\mathcal{L}$-formulas φ such that $\varphi_{\mathfrak{F}} = (\Box^{+}\neg p)^{\triangle}_{\mathfrak{F}}$ for some transitive frame $\mathfrak{F}$. For instance, suppose $\mathfrak{F}$ be $\langle\{a,b\},\emptyset\rangle$. Clearly, $(\Box^{+}\neg p_1)^{\triangle}_{\mathfrak{F}}(\{a\},X_2,\ldots,X_n) = \{b\}$. However, for any $\mathcal{L}$-formula $\varphi(p_1,\ldots,p_n)$, $\varphi_{\mathfrak{F}}(\{a\},X_2,\ldots,X_n) \neq \{b\}$ holds, since the intuitionistic transitive Kripke frame $\mathfrak{F}$ always validiates a formula of the form $\psi \to \chi$. ❑

The result for global expressive power on transitive frame deduces an important fact. By Proposition 2.2, the class of intuitionistic quasi-ordered frame is $\mathcal{L}$-axiomatic (in terms of quasi-ordered frame) by all theorems of **Int**. However, the following holds:

Theorem 5.5 *Let $\mathfrak{C}_{qo}$ be the class of all quasi-ordered frames. Then, $\mathfrak{C}_{qo}$ is $\mathcal{ML}$-axiomatic but not $\mathcal{L}$-axiomatic.*

Proof Clearly, $\mathfrak{C}_{qo}$ is skelton-closed. It is well-known result that $\mathfrak{F} \in \mathfrak{C}_{qo}$ holds if and only if $\mathfrak{F} \models \Box p \to p$ holds. Thus, $\mathfrak{C}_{qo}$ is $\mathcal{ML}$-axiomatic.

The frame $\langle\{a\},\emptyset\rangle$, which is a single irreflexive point, is not quasi-ordered frame. We will prove that $\varphi \in \mathbf{Int}$ implies $\langle\{a\},\emptyset\rangle \models \varphi$ by induction on the construction of φ. To show this, we will consider the contrapositive proposition, namely:

$$\langle\{a\},\emptyset\rangle \not\models \varphi \text{ implies } \varphi \notin \mathbf{Int}.$$

If φ is a propositional variable p, $\langle\{a\},\emptyset\rangle \not\models p$ implies $p \notin \mathbf{Int}$ is trivial. Let φ be the form $\alpha \wedge \beta$. Suppose $\langle\{a\},\emptyset\rangle \not\models \alpha\wedge\beta$. Clearly, $\langle\{a\},\emptyset\rangle \not\models \alpha$ or $\langle\{a\},\emptyset\rangle \not\models \beta$. By I.H., $\alpha \notin \mathbf{Int}$ or $\beta \notin \mathbf{Int}$. On the other hand $\alpha\wedge\beta \in \mathbf{Int}$ implies $\alpha,\beta \in \mathbf{Int}$. This is a contradiction. Thus $\alpha\wedge\beta \notin \mathbf{Int}$. Let φ be of the form $\alpha\vee\beta$. Suppose $\langle\{a\},\emptyset\rangle \not\models \alpha\vee\beta$. Then, $\langle\{a\},\emptyset\rangle \not\models \alpha$ and $\langle\{a\},\emptyset\rangle \not\models \beta$. By I.H., we have $\alpha \notin \mathbf{Int}$ and $\beta \notin \mathbf{Int}$. Since **Int** has the disjunction property, it follows that $\alpha\vee\beta \notin \mathbf{Int}$. Let φ be of the form $\alpha \to \beta$. But, for any formula which has the form $\alpha\to\beta$, $\langle\{a\},\emptyset\rangle \models \alpha\to\beta$ holds. Thus, this is trivial.

Let $\mathcal{C}_{qo}$ be the class of quasi-ordered frames. Suppose that $\mathcal{C}_{qo}$ is $\mathcal{L}$-axiomatic, namely, there exists a set , of formulas such that $\mathfrak{F} \models$, if and only if $\mathfrak{F} \in \mathcal{C}_{qo}$, for any quasi-ordered frame $\mathfrak{F}$, and , $\subseteq \mathbf{Int}$ holds since $\mathcal{C}_{qo}$

includes any partially ordered frame. Then $\langle\{a\},\emptyset\rangle \models$, holds. $\mathcal{C}_{qo}$ must include $\langle\{a\},\emptyset\rangle$ since we suppose $\mathcal{C}_{qo}$ is $\mathcal{L}$-axiomatic. This is a contradiction. ❑

The above argument goes through not only for the class of quasi-ordered frames but also for the class of partially ordered frames (and for the class of single reflexive point frames). Namely, we cannot discuss those classes by a (even if, infinite) set of formulas. Those results comes from the fact that intuitionistic implication does not tell a present point of a transitive frame since R does not need to be reflexive. In particular, by the fact that the global expressive power of $\mathcal{L}$ on transitive frame is properly weaker than that of $\mathcal{ML}$, we deduced that the class of quasi-ordered frames is not defined by adding an axioms to **BPL**.

5.2 Extensions of BPL

Now let us think of extensions of **BPL**. The first problem we encouter with is what kinds of extensions are worth considering. Of course, as in the case of intermediate logics, we can define a *formula-extension* L of **BPL** as a set L of formulas L that contains **BPL** and is closed under substituions and $\vdash_{\mathbf{BPL}}$ (in the sense that $\psi \in L$ whenever , $\subseteq L$ and , $\vdash_{\mathbf{BPL}} \psi$). **Int** as well as classical logics are certainly formula-extensions of **BPL**. However, as we obsereved above, the class $\mathcal{C}_{qo}$ of quasi-ordered frames is not $\mathcal{L}$-axiomatic. In other words, there is no formula-extension of **BPL** whose frames are precisely all the quasi-ordered frames. Many other natural classes of frames, e.g., frames with the diagonal accesibilty relations, are not definable by means of formula-extensions.

A possible solution to this problem is to consider extensions not of the set of theorems in **BPL** but of the consequence reltaion $\vdash_{\mathbf{BPL}}$. The most general class of extensions of **BPL** consists of arbitrary *finitary* (i.e., if , $\vdash \varphi$ then $\Delta \vdash \varphi$ for some finite $\Delta \subseteq$,) *structural* (i.e., closed under substituition) consequence relations containing $\vdash_{\mathbf{BPL}}$. In fact, Visser defined extensions of **BPL** by finitary structural consequence relations which are defined by adding inference rules Ξ to natural deduction system **NBPL** ([Vis81]). Here, we will express extension of this type as $\vdash_{\mathbf{BPL}} +\Xi$ where $\vdash_{\mathbf{BPL}}$ denotes a consequence relation of Visser's natural deduction system of **BPL** and Ξ is a set of inference rules, respectively. In [Vis81], Visser treated three extensions of **BPL** — **Int**, **FPL** and **BPLL**.

Intuitionistic propositional logic **Int** is given as follows:

$$\vdash_{\mathbf{Int}} = \vdash_{\mathbf{BPL}} + \frac{p \quad p \to q}{q}(\to \mathrm{E}).$$

As we mentioned in Section 3.2, it is easy to show that $\wedge$ I-f rule, $\vee$ E-f rule and Tr rule are derived from $\to$ E rule. We have already shown that many differences between **Int** and **BPL**.

Formal propositional logic (**FPL**) is first introduced by Visser in [Vis81]. In his paper, his first goal is to find **FPL** which is embeddable into modal logic **GL** by Gödel translation T. **FPL** is defined as follows:

$$\vdash_{\mathbf{FPL}} = \vdash_{\mathbf{BPL}} + \frac{(\top \to p) \to p}{\top \to p}(\text{Löb}).$$

It notes that $\vdash_{\mathbf{FPL}} + \to$ E is inconsistent. In other words, $\vdash_{\mathbf{FPL}}$ and $\vdash_{\mathbf{Int}}$ has no common extension, so that even classical propositional logic $\vdash_{\mathbf{Cl}}$ is not an extension of **FPL**. We have another definition of **FPL**. It is easy to show that $\vdash_{\mathbf{FPL}}$ is identical to $\vdash_{\mathbf{BPL}} + ((\top \to p) \to p) \to (\top \to p)$. Thus, **FPL** is a formula-extension of **BPL**.

Proposition 5.6 (Theorem 5.4 in [Vis81]) *For any $\mathcal{L}$-formula φ and any set , of $\mathcal{L}$-formulas, , $\vdash_{\mathbf{FPL}} \varphi$ if and only if T, $\vdash_{\mathbf{GL}} \mathsf{T}\varphi$, where T, is the set $\{\mathsf{T}\varphi : \varphi \in , \}$.*

The following completeness holds:

Proposition 5.7 (Theorem 2.2 in [Vis81]) *Let $\mathfrak{C}$ be the class of all irreflexive Kripke frames without infinite ascending chains. Then, for any φ and , , , $\vdash_{\mathbf{FPL}} \varphi$ if and only if , $\models_{\mathfrak{C}} \varphi$.*

Basic propositional logic for intuitionistic linear frame (**BPLL**) is also introduced in [Vis81] as follows:

$$\vdash_{\mathbf{BPLL}} = \vdash_{\mathbf{BPL}} + (p \to q) \vee ((p \to q) \to p).$$

Intuitionistic transitive frame $\mathfrak{F}(= \langle W, R, P \rangle)$ is called *linear* if xRy or yRx for any different elements x and y of W. For any intuitionistic linear transitive frame $\mathfrak{F} = \langle W, R, P \rangle$, it is not necessary to require that this linear relation R is strict linear relation, since R is transitive. Visser showed that **BPLL** is complete with respect to the class of all intuitionistic linear transitive frames.

Proposition 5.8 (Theorem 4.10 in [Vis81]) *Let $\mathfrak{C}$ be the class of all intuitionistic linear transitive frames. Then, for any φ and , , , $\vdash_{\mathbf{BPLL}} \varphi$ if and only if , $\models_{\mathfrak{C}} \varphi$.*

For the case of **Int**, the linearlity of frames is expressed by the formula $(p \to q) \vee (q \to p)$. Moreover, in **Int**, $\vdash_{\mathbf{Int}} +(p \to q) \vee (q \to p)$ is identical to $\vdash_{\mathbf{Int}} +(p \to q) \vee ((p \to q) \to p)$. On the other hand, in **BPL**, the formula $(p \to q) \vee (q \to p)$ expresses, that a frame has an inverse tree-structure (i.e., xRz_1 and xRz_2 deduce $z_1 = z_2$, $z_1 R z_2$ or $z_2 R z_1$ for elements x, z_1, z_2 in a given frame). Also, while $\vdash_{\mathbf{BPL}} +(p \to q) \vee ((p \to q) \to p)$ is Kripke-complete but $\vdash_{\mathbf{BPL}} +(p \to q) \vee (q \to p)$ is Kripke incomplete, as we will show later.

5.3 Semantic consequence

Any extension of **BPL** is defined as a consequence relation. Thus, we have to define a class of semantic structures by a consequence relation.

Any subset $\vdash$ of $2^{\mathrm{For}\mathcal{L}} \times \mathrm{For}\mathcal{L}$ can be regarded as an abstract consequence relation. A consequence relation $\vdash$ is called **BPL**-*consequence relation* if it is finitary and identical to $\models_{\mathcal{F}}$ characterized by a class $\mathcal{F}$ of intuitionistic transtive frames. The class $\{\mathfrak{F} : \vdash \subseteq \models_{\mathfrak{F}}\}$ of intuitionistic transitive frames for a given $\vdash$ is denoted by $\mathrm{Fr} \vdash$. As shown in the previous section, it is trivial that all of $\vdash_{\mathbf{Int}}$, $\vdash_{\mathbf{FPL}}$ and $\vdash_{\mathbf{BPLL}}$ are **BPL**-consequence relations. Similarly, the class $\{\mathfrak{A} : \vdash \subseteq \models_{\mathfrak{A}}\}$ of **BPL**-algebras for a given $\vdash$ is denoted by $\mathrm{Alg} \vdash$.

Before showing relations between those above consequence relations and classes of semantic structures, we will introduce another notion of *varieties* of algebras. In universal algebra, this is one of most important notions. Let , be a set of equations of formulas. Here, any equation of formulas is identical to a term of algebra. A *variety* Var, of algebras for , is the class of algebras, in which equation of , is valid.

Now we will show that the following properties hold on semantic consequence relations.

Theorem 5.9 (i) *A class of* **BPL**-*algebras is of the form* $\mathrm{Alg} \vdash$ *for a* **BPL**-*consequence* $\vdash$ *iff it is a subvariety of the variety of all* **BPL**-*algebras.*
(ii) *Let $\mathcal{F}$ be a subclass of the class $\mathfrak{C}$ of all intuitionistic trantsitive frames. Then, $\mathcal{F}$ is of the form* $\mathrm{Fr} \vdash$ *for a* **BPL**-*consequence* $\vdash$ *iff it is closed under generated subframes, reductions, disjoint unions, and moreover both $\mathcal{F}$ and its complement $(\mathfrak{C} \Leftrightarrow \mathcal{F})$ are closed under the formation of biduals.*

Proof (i) Let $\mathcal{A}$ and $\vdash$ be a class of **BPL**-algebras and **BPL**-consequence relation, respectively. Suppose $\mathcal{A} = \mathrm{Alg}\vdash$. Then, we define , $\leq \varphi$ to be the set of all equations of the form $\bigwedge$, $' \leq \varphi$ with a finite subset , $'$ of , . Moreover, define a set X of equations as the union of , $\leq \varphi$ such that , $\vdash \varphi$ holds. Any inequality $\psi \leq \chi$ is short for the equation $\psi \wedge \chi = \psi$. Since $\vdash$ is finitary consequence relation, if , $\vdash \varphi$ holds, there exists a finite set , $'$ such that , $' \vdash \varphi$ holds. We will show that $\mathrm{Alg}\vdash = \mathrm{Var}X$ where $X = \{, \leq \varphi : , \vdash \varphi\}$. The following is easily shown since $\vdash$ is a **BPL**-consequence:

$$\begin{aligned} \mathfrak{A} \in \mathrm{Alg}\vdash \;&\Leftrightarrow\; \vdash \subseteq \models_{\mathfrak{A}} \\ &\Leftrightarrow\; \forall, \forall\varphi(, \vdash \varphi \Rightarrow , \models_{\mathfrak{A}} \varphi) \\ &\Leftrightarrow\; \forall, \forall\varphi(, \leq \varphi \in X \Rightarrow , \models_{\mathfrak{A}} \varphi). \end{aligned}$$

Therefore, $\mathrm{Alg}\vdash = \mathrm{Var}X$ holds. Conversely, suppose that $\mathcal{A} = \mathrm{Var}X$ for some set X of equations. We will show that $\mathcal{A} = \mathrm{Alg}\models_{\mathcal{A}}$. Assume $\mathfrak{A} \in \mathcal{A}$. Then, by the definition of $\models_{\mathcal{A}}$, $\models_{\mathcal{A}} \subseteq \models_{\mathfrak{A}}$ holds. Thus, $\mathfrak{A} \in \mathrm{Alg}\models_{\mathcal{A}}$ holds for each $\mathfrak{A} \in \mathcal{A}$, so $\mathcal{A} \subseteq \mathrm{Alg}\models_{\mathcal{A}}$ holds. Now, suppose $\mathcal{A}$ is a proper subset of $\mathrm{Alg}\models_{\mathfrak{A}}$. Then, assume that there exists a **BPL**-algebra $\mathfrak{A}$ such that $\mathfrak{A} \notin \mathcal{A}$ and $\mathfrak{A} \in \mathrm{Alg}\models_{\mathcal{A}}$. Then, by $\mathfrak{A} \notin \mathcal{A}$, there exist formulas φ and ψ such that $\varphi = \psi \in X$ and $\mathfrak{A} \not\models \varphi = \psi$. It is easily shown that $\alpha = \beta \in X$ implies $\alpha \models_{\mathcal{A}} \beta$ and $\beta \models_{\mathcal{A}} \alpha$ for any formulas α and β. Since $\mathfrak{A} \in \mathrm{Alg}\models_{\mathcal{A}}$, both $\varphi \models_{\mathfrak{A}} \psi$ and $\psi \models_{\mathfrak{A}} \varphi$ must hold, but this is a contradiction.

(ii) Let $\mathcal{F}$ and $\vdash$ be a class of intuitionistic transitive frames and a **BPL**-consequence relation, respectively. Suppose $\mathcal{F} = \mathrm{Fr}\vdash$. Then, $\mathfrak{F} \in \mathcal{F}$ holds if and only if $\vdash \subseteq \models_{\mathfrak{F}}$ holds. The closure conditions are proved easily. For instance, by Theorem 4.10, both $\mathcal{F}$ and its complement are closed under the formation of biduals. The proof that $\mathcal{F}$ is closed under reduction goes as follows: Let $\mathfrak{F}$ be an element of $\mathcal{F}$ which is reducible to $\mathfrak{G}$ by a reduction f. Assume , $\not\models_{\mathfrak{G}} \varphi$ for some , and φ. That is, there exists a valuation $\mathfrak{U}$ such that $(\mathfrak{N}, x) \models$, and $(\mathfrak{N}, x) \not\models \varphi$ where $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$. Then, by taking a valuation $\mathfrak{V}$ on $\mathfrak{F}$ where $\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p))$, $(\mathfrak{M}, x) \models$, and $(\mathfrak{N}, x) \not\models \varphi$ hold where $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. Therefore, , $\not\models_{\mathfrak{F}} \varphi$ holds. Thus, $\models_{\mathfrak{F}} \subseteq \models_{\mathfrak{G}}$. Remaining conditions are proved similary. Next, we will show the converse direction. Clearly, $\models_{\mathcal{F}}$ is a **BPL**-consequence, which is finitary and is charaterized by the class $\mathcal{F}$ itself. So it remains to show that $\mathcal{F} = \mathrm{Fr}\models_{\mathcal{F}}$. By Proof (i), $\mathcal{F}^+ = \mathrm{Alg}\models_{\mathcal{F}^+}$ is trivial where $\mathcal{F}^+ = \{\mathfrak{A} : \mathfrak{F} \in \mathcal{F}, \mathfrak{F}^+ \simeq \mathfrak{A}\}$. Also $\mathrm{Alg}\models_{\mathcal{F}} = \mathrm{Alg}\models_{\mathcal{F}^+}$ holds by Theorem 4.10 . Then, for any frame $\mathfrak{F}$,

$$\mathfrak{F} \in \mathrm{Fr}\models_{\mathcal{F}} \;\Leftrightarrow\; \forall, \forall\varphi(, \models_{\mathcal{F}} \Rightarrow , \models_{\mathfrak{F}} \varphi)$$

$$\begin{aligned} &\Leftrightarrow \ \forall ,\forall\varphi(,\ \models_{\mathcal{F}^+} \varphi \Rightarrow ,\ \models_{\mathfrak{F}^+} \varphi) \\ &\Leftrightarrow \ \mathfrak{F}^+ \in \mathrm{Alg} \models_{\mathcal{F}^+} \\ &\Leftrightarrow \ \mathfrak{F}^+ \in \mathcal{F}^+ \\ &\Leftrightarrow \ \mathfrak{F} \in \mathcal{F}. \end{aligned}$$

Thus, $\mathcal{F} = \mathrm{Fr} \models_{\mathcal{F}}$ holds. ❑

Using these classes of semantic structures, we will investigate extensions of **BPL**. For instance, suppose that $\vdash_L$ is a consistent **BPL**-consequence and that $\mathfrak{A}_L$ is a Lindenbaum algebra for L. Hence, it is proved by standard method that $\vdash_L = \models_{\mathfrak{A}_L}$. That is, any consistent **BPL**-consequence relation is complete. Also the following holds:

Theorem 5.10 *Let* $\vdash$ *be a* **BPL**-*consequence relation. Suppose that* $D(\mathrm{Fr} \vdash)$ *is the class of all descriptive frames of* $\mathrm{Fr} \vdash$. *Then,* $\models_{\mathrm{Fr}\vdash} = \models_{D(\mathrm{Fr}\vdash)} = \models_{\mathrm{Alg}\vdash}$.

5.4 Kripke completeness

In this thesis, frame semantics using general frames is adopted instead of standard frame semantics using Kripke frames. Therefore, whether a given logic (here, a consequence relation) is complete with respect to a class of Kripke frames or not will be an interesting problem. A **BPL**-consequence relation $\vdash$ is *Kripke-complete* with respect to a class $\mathfrak{C}$ of Kripke frames if for any *finite* set , of formulas and formula φ, , $\vdash \varphi$ holds if and only if , $\models_{\mathfrak{F}} \varphi$ holds for every Kripke frame $\mathfrak{F}$ in $\mathfrak{C}$. If this condition holds for any set , , it is called *strongly Kripke-complete*.

In this section, it is proved firstly that **Int** and **BPLL** are strongly Kripke complete. Then, it is shown that **FPL** is Kripke complete but not strongly Kripke complete.

One notion which is important in discussions of the Kripke completeness is *d-persistent*. For any intuitionistic transitive frame $\mathfrak{F}(= \langle W, R, P\rangle)$, *underlying Kripke frame* $K(\langle W, R, P\rangle)$ of $\mathfrak{F}$ means the Kripke frame $\langle W, R\rangle$. A set , of formulas is *d-persistent* if for any **BPL**-algebra $\mathfrak{A}$,

$$\mathfrak{A} \models ,\ \text{ implies } K(\mathfrak{A}_+) \models , .$$

Then, it is easy to show that the following:

Lemma 5.11 *Let $\mathfrak{C}$ be the class of intuitionistic transitive frames which validate all formulas of a given d-persistent set. Then, any* **BPL***-consequence relation* $\models_{\mathfrak{C}}$ *is strongly Kripke complete.*

Proof Let , be a d-persistent set and $\mathfrak{C}$ be a class of all intuitionistice transitive frames which validiate all formulas of , . Suppose that $K(\mathfrak{C})$ is the set $\{K(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{C}\}$, and $\mathfrak{G}$ is the class of all algebras which validiate all formulas of , . By Theorem 5.10, $\models_{\mathfrak{C}}=\models_{\mathfrak{G}}$ hold. We will show that $\models_{\mathfrak{C}}=\models_{K(\mathfrak{C})}$. In general, $\models_{K(\mathfrak{C})}\subseteq\models_{\mathfrak{C}}$ holds. Suppose $(\Xi, \varphi) \notin \models_{\mathfrak{C}}$. Then, in $\mathfrak{G}$, there exists a **BPL**-algebra $\mathfrak{A}$ such that $\Xi \not\models_{\mathfrak{A}} \varphi$. In other words, $\mathfrak{A} \models$, and $\Xi \not\models_{\mathfrak{A}} \varphi$ hold. Since , is d-persistent, $K(\mathfrak{A}_+) \models$, and $\Xi \not\models_{K(\mathfrak{A}_+)} \varphi$ hold. Then, $(\Xi, \varphi) \notin K(\mathfrak{C})$ holds. ❑

Using the above lemma, the next theorem is deduced.

Theorem 5.12 $\vdash_{\mathbf{Int}}$ *is strongly Kripke complete.*

Proof It is sufficient to show that the set **I** of all theorems of **Int** is d-persistent, however, it is trivial. Because, if a **BPL**-algebra $\mathfrak{A}$ validates **I** then $\mathfrak{A}$ is a Heyting algebra and $K(\mathfrak{A}_+)$ is an intuitionistic quasi-ordered Kripke frame. ❑

For **BPLL**, the strong Kripke-completeness is proved as follows:

Theorem 5.13 $\vdash_{\mathbf{BPLL}}$ *is strongly Kripke complete.*

Proof Suppose , $\not\vdash_{\mathbf{BPLL}} \varphi$. Then, there exists an intuitionistic linear transitive frame $\mathfrak{F} = \langle W, R, P\rangle$, a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ and $x_0 \in W$ such that $(\mathfrak{M}, x_0) \models$, and $(\mathfrak{M}, x_0) \not\models \varphi$. In $\mathfrak{F}$, it is easily found that the maximum element element y_0 of W such that $x_0 \in y_0\underline{\uparrow}$ and $y_0\underline{\uparrow} \in P$. Put $V = \{x \in W : x\underline{\uparrow} \in P\}$, $S = R \cap (V \times V)$, $Q = \{X \cap V : X \in P\}$ and $\mathfrak{U}(p) = \mathfrak{V}(p) \cap V$. It is easily shown that $\mathfrak{U}$ is well-defined, $Q = \mathrm{Up}Q$, and $\mathfrak{G} = \langle V, S, Q\rangle$ is an intuitionistic linear transitive Kripke frame. Put $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$, and then we can show that $(\mathfrak{N}, y_0) \models$, and $(\mathfrak{N}, y_0) \not\models \varphi$ hold. That is, , $\not\models_{\mathfrak{G}} \varphi$. So, $\vdash_{\mathbf{BPLL}}$ is strongly Kripke complete. ❑

As for **FPL**, the finitary condition of the **BPL**-consequence derives that $\vdash_{\mathbf{FPL}}$ is not strongly Kripke complete.

Theorem 5.14 $\vdash_{\mathbf{FPL}}$ *is not strongly Kripke complete.*

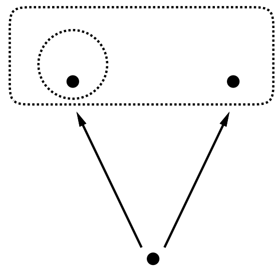


Figure 5.1: frame $\mathfrak{G}$

Proof Suppose d gives the greatest number $d(\mathfrak{F})$ of elements of ascending chain for an given frame $\mathfrak{F}$, if it exists. For any natural number n and any frame $\mathfrak{F}$, $c(\mathfrak{F}, n)$ is short for the following condition:

$$\forall x_1, \ldots, x_n \in \mathfrak{F}(\bigwedge_{i=1}^{n-1} x_i R x_{i+1} \to \neg \exists y \in \mathfrak{F}(yRx_1 \vee (\bigvee_{i=1}^{n-1} x_i Ry \wedge yRx_{i+1}) \vee x_n Ry)).$$

It is easy to check that $d(\mathfrak{F}) \leq n$ if and only if $c(\mathfrak{F}, n)$ holds. For any n, put bd_n inductively as follows[2] : $bd_1 = p_1 \vee (p_1 \to \bot)$, $bd_{n+1} = p_{n+1} \vee (p_{n+1} \to bd_n)$. For any frame $\mathfrak{F}$, $\mathfrak{F}$ validaites bd_n holds if and only if $d(\mathfrak{F}) \leq n$ holds. Put the set , of fomrulas as $\{bd_n : 1 \leq n < \omega\}$. For any frame $\mathfrak{F} \in \mathrm{Fr} \vdash_{\mathbf{FPL}}$, there exists a valuation $\mathfrak{V}$ and possible world x of $\mathfrak{F}$ such that $(\mathfrak{M}, x) \models bd_n$ where $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. This fact deduces that , $\not\vdash_{\mathbf{FPL}} \bot$. Clearly, for any model $\mathfrak{M}$ which is based on a frame in $\mathrm{Fr} \vdash_{\mathbf{FPL}}$ and any possible worlds x of $\mathfrak{M}$, $\mathfrak{M} \not\models$, , since $\mathfrak{M}$ does not have an infinite ascending chain. That is, for all Kripke frame $\mathfrak{F} \in \mathrm{Fr} \vdash_{\mathbf{FPL}}$, , $\models_{\mathfrak{F}} \bot$. ❑

It is well-known that any intermediate logics characterized by a finite frame is Kripke complete. However, there exists an extension of **BPL** which is not Kripke complete although it is characterized by finite frame.

Lemma 5.15 (i) *The consequence relation* $\models_{\mathfrak{G}}$ *is not Kripke complete where* $\mathfrak{G}$ *is depicted in Figure 5.1.*
(ii) $\vdash_{\mathbf{BPL}} + (\varphi \to \psi) \vee (\psi \to \varphi)$ *is not Kripke complete.*

Proof Let φ_1 and φ_2 be $(p \to q) \vee ((p \to q) \to p)$ and $(p \to q) \vee (q \to p)$, respectively. In general, any Kripke frame validates φ_1 (or φ_2) if and only if it is a linear frame. However, the general frame $\mathfrak{G}$ refutes φ_1 but validates φ_2. ❑

[2] This formula bd comes from Proposition 2.38 in [CZ97].

The above lemma states that the axiom $(p \to q) \vee (q \to p)$ which represents linear frames in intermediate logics gives a Kripke incomplete logic. Thus, Lemma 5.15 forms a striking contrast to Theorem 5.13 about the linearlity axiom. By Lemma 5.15, we have also the following:

Theorem 5.16 *There exists an Kripke incomplete extension of* **BPL** *which is characterized by a finite frame.*

5.5 Between FPL and GL

In this section, we will prove the theorem which resembles the Blok-Esakia theorem does not hold for intuitionistic transitive frames.

From Blok-Esakia Theorem, it is natural to conjecture that there exists an isomorphism between Ext $\vdash_{\mathbf{BPL}}$ and NExt$L$ for some $L \in$ NExt**K4**. A natural candidate for L would be the logic **GL** determined by all transitive frames without proper clusters and infinite strictly ascending chains. While for **BPL**, **FPL** is the extension of **BPL** determined by the class of all intuitionistic irrflexive transitive frames without inifinite strictly ascending chains. Thus, we will check a correspondence between Ext $\vdash_{\mathbf{FPL}}$ and NExt**GL**. Here, we will describe a rough sketch of the proof.

Any consequence relation can be considered as a set of pairs of a set of formulas and a formula. Let C be the class of all **BPL**-consequence relations. Then,

Theorem 5.17 *The structure $\langle C, \subseteq \rangle$ forms a complete lattice with respect to the set-inclusion $\subseteq$.*

Proof We will show that both C has the least element and any non-empty subset X of C has the least upper bound. Let $\mathcal{F}$ be the class of all intuitionistic transitive frames. It is trivial that C includes the consequence relation $\models_{\mathcal{F}}$ and $\models_{\mathcal{F}}$ is the least element of C. Suppose X is a nonempty subset of C. Clearly, $\bigcup X$ is the least upper bounds of X, where $\bigcup X = \{(, , \varphi) : \exists \vdash \in C((, , \varphi) \in \vdash)\}$. ❑

When we talk about Ext $\vdash_{\mathbf{FPL}}$, we notice not only the class of all intuitionistic irrfelxive transitive frames witout infinite strictly ascending chains but also its subclass. The following is easily showed:

Lemma 5.18 *Let $\mathcal{F}$ be the class of all intuitionistic irreflexive rooted descriptive frames without infinte strictly ascending chains. Then, $\vdash_{\mathbf{FPL}}=\models_{\mathcal{F}}$ holds.*

In this section, we denote the class refered in the above lemma by $\mathcal{F}$. Then, it is obivous that all extensions of **BPL**-consequce relation including $\vdash_{\mathbf{FPL}}$ is characterized by subclasses of $\mathcal{F}$. Next, we will introduce $\cap$-*irreducible* consequence relation as follows: Any **BPL**-consequence relation $\vdash$ is called $\cap$-*irreducible* if, for any class C of **BPL**-consequence relations, $\vdash = \cap C$ implies $\vdash \in C$ where $\cap C = \cap\{\vdash : \vdash \in C\}$. Then,

Lemma 5.19 *Let $\vdash$ be a* **BPL**-*consequences including $\vdash_{\mathbf{FPL}}$. Then, $\vdash$ is $\cap$-irreducible if and only if $\vdash=\models_{\mathfrak{F}}$ holds for some $\mathfrak{F} \in \mathcal{F}$.*

Proof We can prove by a straightforward but rather tedious proof. The detail is, for example, in [Rau79]. ❑

While, it is easy to show that any normal extension of **GL** is characterized by a Kripke frame. Using these results, the following theorem is proved.

Theorem 5.20 *The lattice of all* **BPL**-*consequence relations containing $\vdash_{\mathbf{FPL}}$ is not isomorphic to the lattice* NExt**GL**.

Proof The *codimension* of an element c in a lattice $\mathfrak{D}$ is the length of a longest chain from c of the top element of $\mathfrak{D}$. Any frame $\mathfrak{F}$ depicted in Figure 5.2 is an element of $\mathcal{F}$, and $\models_{\mathfrak{F}}$ is $\cap$-irreducible by Lemma 5.19 . Each row from the top of Figure 5.2 denote codimensions 2, 3 and 4, respectively. Let $\mathfrak{F}$ and $\mathfrak{G}$ be a frame of codimension n and m, respectively. If $n < m$, $\mathfrak{F}^+$ is a subalgebra of $\mathfrak{G}^+$, thus $\models_{\mathfrak{G}}\subseteq\models_{\mathfrak{F}}$ holds. For NExt**GL**, the corresponding figure is Figure 5.3 . Then, NExt**GL** contains only 5 $\cap$-irreducible consequence relations of codimension 4, while Ext $\vdash_{\mathbf{FPL}}$ has 6. Thus, the lattices under consideration are not isomorphic. ❑

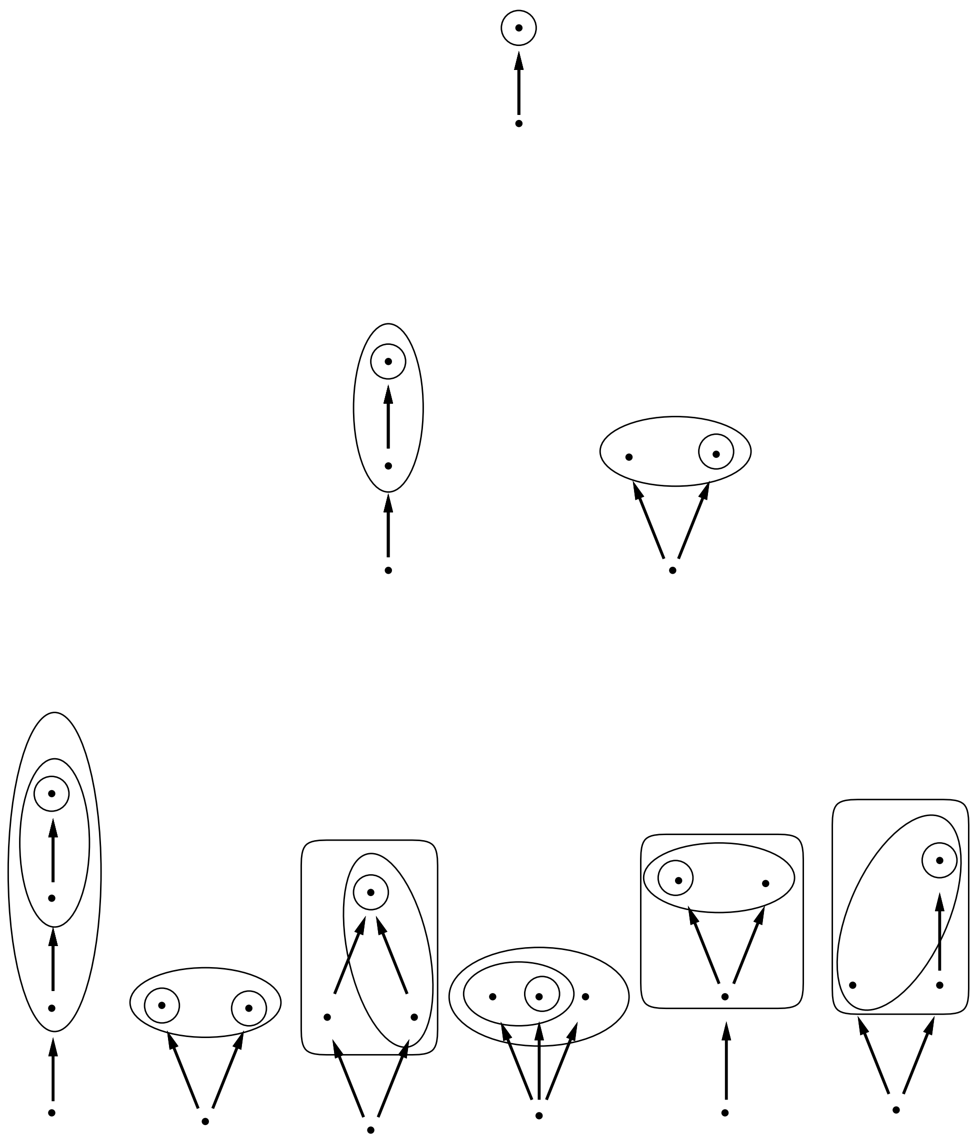


Figure 5.2: A part of **BPL**-consequence including $\vdash_{\mathbf{FPL}}$

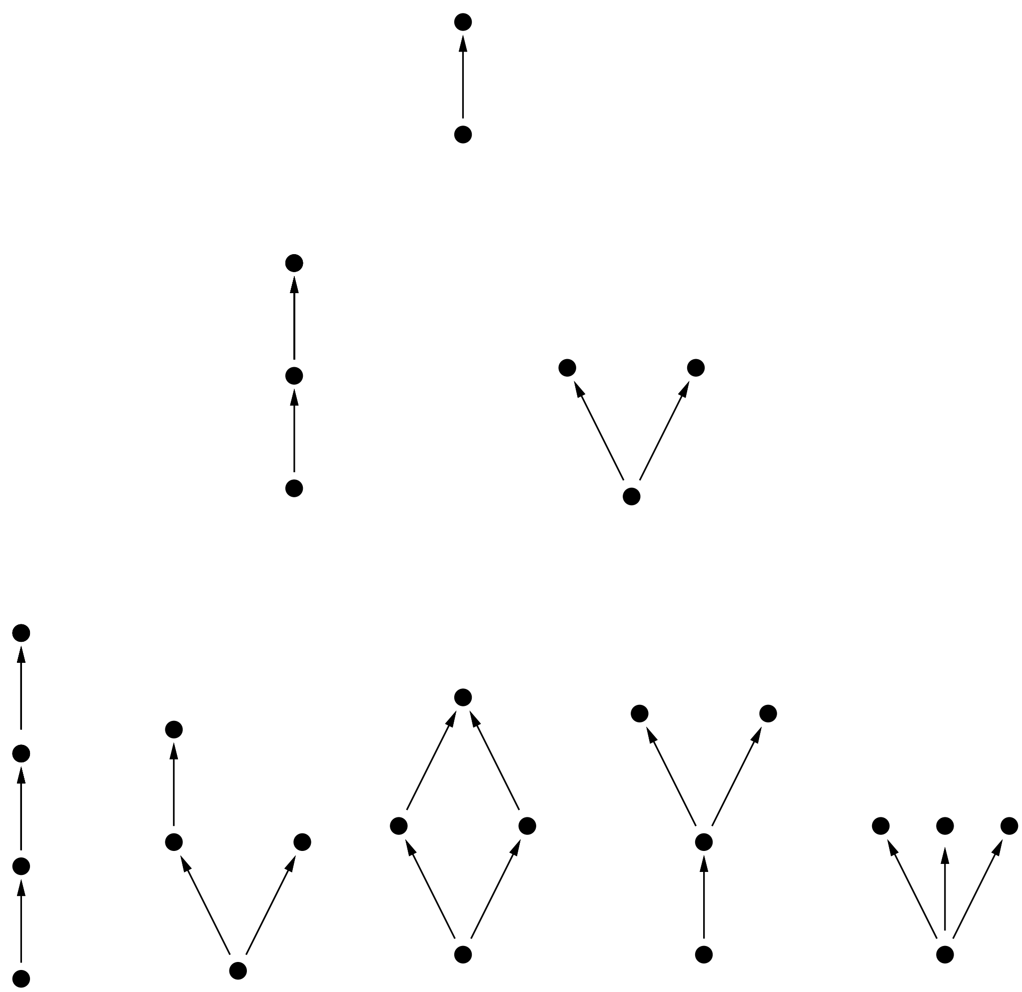


Figure 5.3: A part of NExt**GL** represented by Kripke frames.

Chapter 6

Adding a new implication to the logic BPL

The cause of peculiarities for extensions of **BPL** will come from the fact that intuitionistic implication "$\rightarrow$" expresses the truth of formulas at successors of the present possible world with respect to a relation R in a frame. Any relation R in a frame for intermediate logics is quasi-order (and therefore is reflexive), so the "present" possible world is also a successor of the present world with respect to R. However, relations in frames for extensions of **BPL** may not talk about the truth value of formulas at the present possible world, since we assume that it is transitive.

To removing differences of behaviors of the implication between transitive frames and quasi-ordered frames, it is natural to add a new implication "$\hookrightarrow$" which talks about not only successors of R but also the present possible world. We will denote this new *biarrow language* $\mathcal{L}_2$. In this chapter, we will show that transitive frames for $\mathcal{L}_2$ recover many properties of the intuitionistic logic which are lost in extensions of **BPL**.

It is easily seen that this language $\mathcal{L}_2$ corresponds to some special *intuitionistic modal language* $\mathcal{ML}_{\hookrightarrow}$. This fact will be clarified in the first section. The consequence relation $\vdash_{\mathbf{BiPL}}$ for a logic **BiPL** on the class of transitive frames for $\mathcal{L}_2$ is also introduced.

In the next section, a Hilbert-style system for **BiPL** in $\mathcal{L}_2$ is introduced, and then the Blok-Esakia theorem for transitive frames is proved.

In the last section, it is shown that the both local and global expressive powers between $\mathcal{L}_2$ and $\mathcal{ML}$ are recovered by introducing the implication "$\hookrightarrow$".

6.1 The logic BiPL

In this section, non-modal propositional logic with two kinds of implication in the *biarrow language* $\mathcal{L}_2$ is introduced. The language $\mathcal{L}_2$ is obtained from $\mathcal{L}$ by adding a new implication "$\hookrightarrow$". The set of formulas on $\mathcal{L}_2$ is denoted by For$\mathcal{L}_2$, and any element of For$\mathcal{L}_2$ is called a $\mathcal{L}_2$*-formula.*

Suppose that an intuitionistic transitive frame $\mathfrak{F} = \langle W, R, P\rangle$ is given. For any $X, Y \in P$, an operator $\succeq$ on P is defined as follows:

$$X \succeq Y = \{x : \forall y((x = y) \vee xRy) \wedge y \in X \Rightarrow y \in Y\}.$$

A **BiPL**-*frame* $\mathfrak{F} = \langle W, R, P\rangle$ is defined as follows:
(i) $\mathfrak{F}$ is an intuitionistic transitive frame,
(ii) P is closed under $\succeq$.
A valuation function of a **BiPL**-frame $\mathfrak{F} = \langle W, R, P\rangle$ is a function from Prop to P. For a given valuation $\mathfrak{V}$, the interpretation of $\mathcal{L}_2$-formulas in $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is defined as before except "$\hookrightarrow$" are same to **BPL**. As for "$\hookrightarrow$", for any $x \in W$ and any $\mathcal{L}_2$-formulas φ and ψ,

$$(\mathfrak{M}, x) \models \varphi \hookrightarrow \psi \quad \text{iff} \quad \forall y \in W((x = y) \text{ or } xRy) \text{ and } (\mathfrak{M}, y) \models \varphi \text{ imply } (\mathfrak{M}, y) \models \psi, \tag{6.1}$$

From the above definition, it is easy to understand that "$\hookrightarrow$" in **BiPL**-frames works as the intuitionistic implication "$\rightarrow$" in quasi-ordered frames.

The formulas of the biarrow language $\mathcal{L}_2$ can be translated into formulas of some propositional modal language by interpretations in **BiPL**-frames. It is easily shown as follows: Let $\mathcal{ML}_{\hookrightarrow}$ be the modal language with "$\hookrightarrow$" instead of "$\rightarrow$". Formulas of $\mathcal{ML}_{\hookrightarrow}$ called $\mathcal{ML}_{\hookrightarrow}$-formulas, are defined in the usual way. The set of all $\mathcal{ML}_{\hookrightarrow}$-formulas is denoted by For$\mathcal{ML}_{\hookrightarrow}$. For any model $\mathfrak{M}$ based on a **BiPL**-frame and any $x \in W$

$$(\mathfrak{M}, x) \models \varphi \rightarrow \psi \quad \text{iff} \quad (\mathfrak{M}, x) \models \Box(\varphi \hookrightarrow \psi).$$

It is easily shown that the converse direction holds also. That is,

$$(\mathfrak{M}, x) \models \Box\varphi \quad \text{iff} \quad (\mathfrak{M}, x) \models \top \rightarrow \varphi,$$

where $\top = \bot \rightarrow \bot$. By the above fact, we can translate some properties described on $\mathcal{L}_2$ into properties on $\mathcal{ML}_{\hookrightarrow}$.

The logic **BiPL** is defined as the set of $\mathcal{L}_2$-formulas that are valid in all **BiPL**-frames. Let us define the consequence relation $\vdash_{\mathbf{BiPL}}$ by taking, for any model $\mathfrak{M}$ based on **BiPL**-frames and any element x,

$$, \vdash_{\mathbf{BiPL}} \varphi \text{ if and only if } \forall\mathfrak{M}\forall x((\mathfrak{M}, x) \models , \Rightarrow (\mathfrak{M}, x) \models \varphi).$$

In other words, if $\mathfrak{C}$ be the class of all **BiPL**-frames, $\vdash_{\mathbf{BiPL}} = \models_{\mathfrak{C}}$, in terms of semantic consequence relation treated in the previous chapter. The deduction theorem holds for **BiPL**, which makes a contrast with the case of **BPL**. This can be proved by translating it into modal logics in $\mathcal{ML}_{\hookrightarrow}$.

Theorem 6.1 (deduction theorem) *For any set , of $\mathcal{L}_2$ formulas, any $\mathcal{L}_2$-formulas φ and ψ, , , $\varphi \vdash_{\mathbf{BiPL}} \psi$ holds if and only if , $\vdash_{\mathbf{BiPL}} \varphi \hookrightarrow \psi$ holds.*

6.2 The calculus for BiPL

In the previous section, the consequece relation $\vdash_{\mathbf{BiPL}}$ is defined in terms of frame semantics. In this section, a formal system for **BiPL** is introduced.

As we saw, $\mathcal{L}_2$ corresponds to $\mathcal{ML}_{\hookrightarrow}$. In any **BiPL**-frame, "$\hookrightarrow$" behaves like the implication "$\rightarrow$" in a quasi-ordered frame. Thus, "$\hookrightarrow$" in **BiPL**-frames can be seen as the intuitionistic implication in a quasi-ordered frames, and $\mathcal{ML}_{\hookrightarrow}$ is regarded as a modal propositional language with this intuitionistic implication. Thus, **BiPL** will be formalized as an *intuitionistic modal logic*.

A formal system **U** of **BiPL** is introduced as an extension of the intuitionistic modal logic **IntK**. So, we will give a precise defintion of **IntK** first and show some of its properties. Next, we will show that $\vdash_{\mathbf{U}}$ is complete with respect to the class of all **BiPL**-frames.

6.2.1 Intuitionistic modal logic IntK

We will follows [WZ97a] for the definition of **IntK**. In Section 6.1, we showed that $\hookrightarrow$ works as an implication of **Int**. Thus, we use "$\hookrightarrow$" instead of "$\rightarrow$" in our definition of **IntK**. Axioms of **IntK** includes all axioms of **Int**. (Of course, every "$\rightarrow$" must be replaced by "$\hookrightarrow$". Thus, for instance, (A1) becomes "$\varphi \hookrightarrow (\psi \hookrightarrow \varphi)$".) Formally, **IntK** is a formal system in the language $\mathcal{ML}_{\hookrightarrow}$ as follows:

Axiom schemes:

(Int) all axiom schemes of **Int**,

(dist) $\Box(\varphi \hookrightarrow \psi) \hookrightarrow (\Box\varphi \hookrightarrow \Box\psi)$.

Inference rules:

- MP: from $\mathcal{ML}_{\hookrightarrow}$-formulas φ and $\varphi \hookrightarrow \psi$, infer φ,
- RN: from $\mathcal{ML}_{\hookrightarrow}$-formula φ, infer $\Box\varphi$.

Any subset of $\mathcal{ML}_{\hookrightarrow}$-formulas which contains all the axioms of **IntK** and which is closed under MP and RN is called an *intuitionistic modal logic* (an IntM-logic, for short). For a subset Γ of $\mathcal{ML}_{\hookrightarrow}$-formulas and an IntM-logic L, the smallest IntM-logic containing both Γ and L is denoted by $L \oplus \Gamma$. They are called *normal extensions* of L. The class of all consistent normal extensions of L is denoted by NExt$L$.

An IntM-*frame* is a structure $\langle W, R_{\hookrightarrow}, R, P\rangle$, where W is a non-empty set, $R_{\hookrightarrow}$ a partial order and R a binary relation on W such that

$$R_{\hookrightarrow} \circ R = R \circ R_{\hookrightarrow} = R.$$

Moreover, P is a set of $R_{\hookrightarrow}$-cones which contains $\emptyset$ and is closed under set-intersection, set-union, $\supset$ and $\Box$. Here, operations $\supset$ and $\Box$ are defined by

$$\begin{aligned} X \supset Y &= \{x \in W : \forall y \in W(xR_{\hookrightarrow}y \text{ and } y \in X \text{ imply } y \in Y)\}, \\ \Box X &= \{x \in W : \forall y \in W(xRy \text{ implies } y \in X)\}. \end{aligned}$$

The following equivalence is easily showed:

$$R_{\hookrightarrow} \circ R \circ R_{\hookrightarrow} = R \quad \text{if and only if} \quad R_{\hookrightarrow} \circ R = R \circ R_{\hookrightarrow} = R. \qquad (6.2)$$

A *valuation function* $\mathfrak{V}$ on an IntM-frame $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P\rangle$ is any function from Prop to P, and the pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ of an IntM-frame $\mathfrak{F}$ and a valuation $\mathfrak{V}$ on $\mathfrak{F}$ is called an IntM-*model*. A *satisfaction relation* $\models$ determined by $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is defined as follows:

$$\begin{array}{rll} (\mathfrak{M}, x) \not\models \bot, & & \\ (\mathfrak{M}, x) \models p & \text{iff} & x \in \mathfrak{V}(p), \\ (\mathfrak{M}, x) \models \varphi \wedge \psi & \text{iff} & (\mathfrak{M}, x) \models \varphi \text{ and } (\mathfrak{M}, x) \models \psi, \\ (\mathfrak{M}, x) \models \varphi \vee \psi & \text{iff} & (\mathfrak{M}, x) \models \varphi \text{ or } (\mathfrak{M}, x) \models \psi, \\ (\mathfrak{M}, x) \models \varphi \hookrightarrow \psi & \text{iff} & \forall y \in W(xR_{\hookrightarrow}y \text{ and } (\mathfrak{M}, y) \models \varphi \text{ imply } (\mathfrak{M}, y) \models \psi), \\ (\mathfrak{M}, x) \models \Box\varphi & \text{iff} & \forall y(xRy \text{ implies } (\mathfrak{M}, y) \models \varphi). \end{array}$$

In the usual way, we extend the domain of a valuation $\mathfrak{V}$ to For$\mathcal{ML}_{\hookrightarrow}$, and this extended valuation holds the correspondence like $\mathfrak{V}(\varphi \hookrightarrow \psi) = \mathfrak{V}(\varphi) \supset \mathfrak{V}(\psi)$, $\mathfrak{V}(\Box\varphi) = \Box\mathfrak{V}(\varphi)$ and etc. It is clear that for an IntM-model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$

$$(\mathfrak{M}, x) \models \varphi \text{ if and only if } x \in \mathfrak{V}(\varphi).$$

The validity of $\mathcal{ML}_{\hookrightarrow}$-formulas is defined in the usual way.

The following soundness theorem for **IntK** with respect to the class of all IntM-frames holds.

Proposition 6.2 *Let $\mathfrak{C}$ be the class of all* IntM-*frames. Then* $\mathfrak{C} \models$ **IntK**.

An IntM-*algebra* is a structure $\mathfrak{A} = \langle A, \wedge, \vee, \hookrightarrow, \Box, 0, 1 \rangle$ such that $\langle A, \wedge, \vee, \hookrightarrow, 0, 1 \rangle$ is a Heyting algebra and $\Box$ satisfies both $\Box 1 = 1$ and $\Box(a \hookrightarrow b) \hookrightarrow (\Box a \hookrightarrow \Box b) = 1$ for every $a, b \in A$.

Similarly to definitions in Chapter 4, the *dual structure* $\mathfrak{F}^+$ (and $\mathfrak{A}_+$) is defined for each IntM-frame $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P \rangle$ (and each IntM-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \hookrightarrow, \Box, 0, 1 \rangle$). The dual $\mathfrak{F}^+$ of $\mathfrak{F}$ is the structure $\langle P, \cap, \cup, \supset, \Box, \emptyset, W \rangle$. It is easy to show that $\mathfrak{F}^+$ is an IntM-algebra. On the other hand, the dual $\mathfrak{A}_+$ of $\mathfrak{A}$ is the structure $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}}^{\hookrightarrow}, R_{\mathfrak{A}}^{\Box}, P_{\mathfrak{A}} \rangle$, where $W_{\mathfrak{A}}$ is the set of prime filters and for any prime filters $\nabla, \nabla' \in W_{\mathfrak{A}}$, $R_{\mathfrak{A}}^{\hookrightarrow}$, $R_{\mathfrak{A}}^{\Box}$ and $P_{\mathfrak{A}}$ are defined as follows:

$$\begin{array}{rcl} \nabla R_{\mathfrak{A}}^{\hookrightarrow} \nabla' & \text{iff} & \nabla \subseteq \nabla', \\ \nabla R_{\mathfrak{A}}^{\Box} \nabla' & \text{iff} & \forall a \in A(\Box a \in \nabla \Rightarrow a \in \nabla'), \\ P_{\mathfrak{A}} & = & \{\mathbf{p}(a) : a \in A\}, \\ & & \text{where } \mathbf{p}(a) = \{\nabla : a \in \nabla\}. \end{array}$$

Clearly, $\mathfrak{A}_+$ is an IntM-frame. By the similar argument to that of Chapter 4, we can show that $\mathfrak{A}$ is isomorphic to its bidual $(\mathfrak{A}_+)^+$ for every IntM-algebra $\mathfrak{A}$. An IntM-frame $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P \rangle$ is *descriptive* if $\langle W, R_{\hookrightarrow}, P \rangle$ is an intuitionistic descriptive partially ordered frame and moreover,

$$xRy \text{ if and only if } \forall X \in P(x \in \Box X \Rightarrow y \in X)$$

holds. Then,

Proposition 6.3 (Proposition 1 of [WZ97a]) *For every* IntM-*frame $\mathfrak{F}$, $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$ if and only if $\mathfrak{F}$ is descriptive.*

6.2.2 The calculus for BiPL

In this section, a Hilbert-style system for **BiPL** on $\mathcal{L}_2$ is introduced. This system denoted by **U**. At the end of this section, it will be proved that $\vdash_{\mathbf{U}}$ is strongly charaterized by the class of **BiPL**-frames.

The *Hilbert style system* **U** of **BiPL** consists of the following axiom schemes and inference rules:

Axiom schemes:

(Int) all axiom schemes of **Int**,

(dist) $\Box(\varphi \hookrightarrow \psi) \hookrightarrow (\Box\varphi \hookrightarrow \Box\psi)$,

(rn) $\varphi \hookrightarrow \Box\varphi$,

(U) $\Box\varphi \hookrightarrow (\psi \vee (\psi \hookrightarrow \varphi))$;

Inference rules:

- MP.

Some remarks will be necessary here on the above formalization of **U**. The formal system **U** looks as if it were formalized in $\mathcal{ML}_{\hookrightarrow}$. But, those expressions including $\Box$ should be regareded as abbreviations of expressions in $\mathcal{L}_2$. This is the reason why that **U** contains the axiom scheme (rn), but not the inference rule (RN). It is easy to show that the inference rule (RN) is derivable rule in **U**. Thus the consequence relation of **U** is defined similary to $\vdash_{\mathbf{Int}}$.

Removing (rn) from **U** and adding both (RN) and axiom scheme (4) to **U**, we obtain corresponding system as a normal extension of **IntK4**. Here, we call obtained system $\mathbf{U}_{\mathbf{IntK4}}$. In fact, it is easily proved that the consequence relation of **U** and the consequence relation of $\mathbf{U}_{\mathbf{IntK4}}$ are same.

By the above fact that **U** and $\mathbf{U}_{\mathbf{IntK4}}$ are the same system, we will prove the strong completeness of $\vdash_{\mathbf{U}}$ with respect to the class of **BiPL**-frames via a class of **IntM**-frames, as follows: Let $\mathfrak{C}$ be a class of IntM-frames which satisfies some simple conditions. By these conditions, it is easy to show that $\mathfrak{C}$ is transformed into the class of **BiPL**-frames. We will prove that any IntM-frame $\mathfrak{F}$ which validates all theorems of **U** is an element of $\mathfrak{C}$. The other direction is not explained in detail.

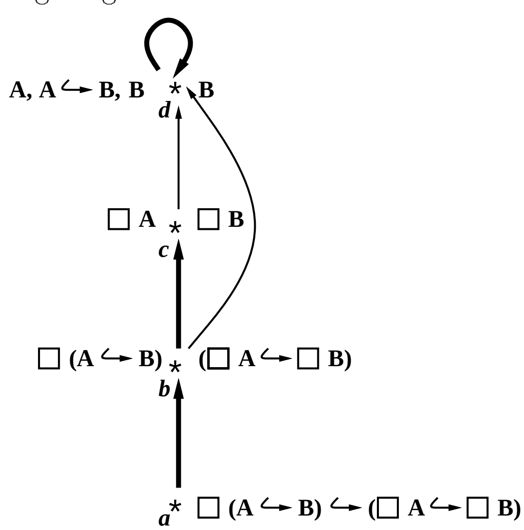
Theorem 6.4 *For any subset* , *of* $\mathcal{ML}_{\hookrightarrow}$ *and any* $\mathcal{ML}_{\hookrightarrow}$*-formula* φ*,*

$$, \vdash_{\mathbf{U}} \varphi \textit{ if and only if } , \vdash_{\mathbf{BiPL}} \varphi.$$

Proof Let $\mathfrak{C}$ be the class of IntM-frames $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P\rangle$ satisfying the following conditions;
i) $R \subseteq R_{\hookrightarrow}$,
ii) R is transitive,
iii) $\mathfrak{F}$ is descriptive,
iv) $R^r = R_{\hookrightarrow}$ where R^r is the reflexive closure of R.
Let $\mathfrak{C}'$ and $\mathfrak{G}$ be the class of all **BiPL**-frames and the class of all descriptive **BiPL**-frames, respectively. It is easy to show that the class $\mathfrak{C}$ can be translated into $\mathfrak{G}$, and $\models_{\mathfrak{G}} = \models_{\mathfrak{C}'}$.

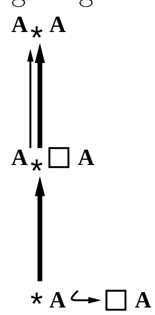
Put the calculus $\mathbf{U}^-$ which is the caluclus deleting the axiom scheme (U) from **U**. Let $\mathfrak{C}^-$ be the class of all IntM-frames satisfying Condition (i) and (ii). First, we will show that $\mathfrak{F} \in \mathfrak{C}^-$ if and only if$\mathfrak{F} \models \mathbf{U}^-$. Suppose $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P\rangle$ and $\mathfrak{F} \not\models \mathbf{U}^-$. By using the induction on the length of derivation, we will show that $\mathfrak{F} \notin \mathfrak{C}^-$. By our assumption $\mathfrak{F} \not\models \mathbf{U}^-$, there exists a theorem φ of $\mathbf{U}^-$ such that $\mathfrak{F} \not\models \varphi$. It is easily showed that $\mathfrak{F} \models \psi$ for any theorem ψ of **Int**, so φ is not a theorem of **Int**. As for MP, it is proved in usual way. Thus, the remains to check axiom schemes (dist) and (rn). We will show this by diagram. In the following diagrams, $R_{\hookrightarrow}$, R and possible world is depicted by thick arrow $\longrightarrow$, thin arrow $\longrightarrow$ and *.

- φ is (dist) : $\Box(A \hookrightarrow B) \hookrightarrow (\Box A \hookrightarrow \Box B)$
  The following diagram is essential.



Suppose φ is false at a. By the definition of $\hookrightarrow$, there exists a $R_{\hookrightarrow}$-successor b of a, where $\Box(A \hookrightarrow B)$ is true but $\Box A \hookrightarrow \Box B$ is false. Since $\Box A \hookrightarrow \Box B$ is false at b, there exists a $R_{\hookrightarrow}$-successor c of b, where $\Box A$ is true but $\Box B$ is false. At c, $\Box A$ is true but $\Box B$ is not false. So there exists a R-successor d of c, where B is false. Since $R_{\hookrightarrow}$ is a partial order on W, reflexivity with respect to $R_{\hookrightarrow}$ holds at all possible worlds of W. Especially, $d\ R_{\hookrightarrow}\ d$ holds. By (6.2), $b\ R_{\hookrightarrow}\ c$, $c\ R\ d$ and $d\ R_{\hookrightarrow}\ d$ deduce $b\ R\ d$. Thus, $A \hookrightarrow B$ is true at d, since $\Box(A \hookrightarrow B)$ is true at b. Then B is true at d. This is contradiction.

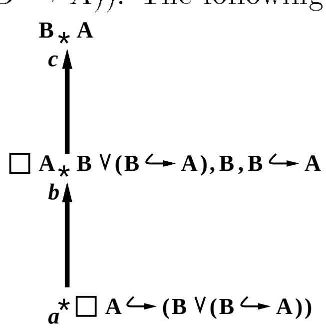
- φ is (rn) : $A \hookrightarrow \Box A$
  The argument is similar to the above case. This time, this is derived from the following diagram.



It is easy to show the converse direction.

Now, we will prove that $\mathfrak{F} \models$ (U) holds if and only if $\mathfrak{F} \in \mathfrak{C}$. In the following, we suppose that a given IntM-frame $\mathfrak{F}$ satisfies Condition (i) and (ii). Condition (iii) does not have any essential role in the proof which is $\mathfrak{F} \in \mathfrak{C}$ implies $\mathfrak{F} \models$ (U).

We will show that $\mathfrak{F}$ satisfies (iv) implies $\mathfrak{F} \models$ (U). Suppose that $\mathfrak{F} \not\models \Box A \hookrightarrow (B \vee (B \hookrightarrow A))$. The following diagram is essential.



The above diagram denotes that there exist possible worlds a, b and c such that $a\ R_{\hookrightarrow}\ b$ and $b\ R_{\hookrightarrow}\ c$ hold. Assume $R^r = R_{\hookrightarrow}$. This assumption deduces that $b\ R\ c$ or $b = c$ holds. If $b\ R\ c$ then A is true at c since $\Box A$ is true at b. If $b = c$ then B is true and false. Both cases derive contradiction.

To show the converse, we need both conditions iii and iv to show that $\mathfrak{F} \models (\mathrm{U})$ implies $\mathfrak{F} \in \mathfrak{C}$. Suppose $\mathfrak{F} \models \Box p \hookrightarrow (q \vee (p \hookrightarrow q))$, $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P\rangle$ is descriptive and $R^r \neq R_{\hookrightarrow}$. Since $\mathfrak{F}$ satisfies Condition (i), $R^r \subseteq R_{\hookrightarrow}$ and $R \neq R_{\hookrightarrow}$ hold. The latter implies that there exist possible worlds x and y such that $xR_{\hookrightarrow}y$ holds but xR^ry does not hold. For these elements x and y, the followings hold since $\mathfrak{F}$ is descriptive: Since xR^ry does not hold, $x \neq y$ holds, and moreover, there exists such an element $Y \in P$ that either $x \in Y$ and $y \notin Y$ hold, or, $x \notin Y$ and $y \in Y$ hold in the bidual structure of $\mathfrak{F}$. That is, there exists an element $X \in P$ such that $x \in \Box X$ and $y \in X$, since xRy does not hold. Put a valuation $\mathfrak{V}$ taking $\mathfrak{V}(p) = X$ and $\mathfrak{V}(q) = Y$. Then, both

1) $x \in \mathfrak{V}(\Box p)$ and $y \notin \mathfrak{V}(p)$, and
2) $x \notin \mathfrak{V}(q)$ and $y \in \mathfrak{V}(q)$,

hold. Under this valuation $\mathfrak{V}$, $(\langle\mathfrak{F}, \mathfrak{V}\rangle, x) \not\models \Box p \hookrightarrow (q \vee (q \hookrightarrow p))$ holds. This is a contradiction. In deed, $\vdash_{\mathbf{BiPL}}$ is introduced by $\models_{\mathfrak{C}}$ for $\mathfrak{C}$ is the class of all **BiPL**-frames. That is, $\mathfrak{F} \models (\mathrm{U})$ if and only $\mathfrak{F}$ satisfies (iii) and (iv). ❑

Theorem 6.5 *Every logic in* NExt**BiPL** *is characterized by a class of descriptive* **BiPL***-frames.*

Theorem 6.6 *Any class of* **BiPL***-frames determines a logic in* NExt**BiPL**.

6.3 Embedding

In this section, a relation between **BiPL** and **K4** is discussed. F. Wolter and M. Zakharyaschev investigate an embedding of IntM-logics into *classical bimodal logics* (CBiM-logics, for short) in [WZ97b]. Modal logic **K4** can be seen a CBiM-logic by putting $\varphi \wedge \Box\varphi$ as $\Box_I\varphi$ for new modal operator $\Box_I$. Our **BiPL** is identical to a IntM-logic as we saw in the previous section. Thus, we will first explain results in [WZ97b] as they are related to our present research, and then, they are applied to our case. As an application, we can get the Blok-Esakia theorem on the class of transitive frames. Thus, our goal in this section is to give an isomorphism between NExt**BiPL** and NExt**Grz**$'$ where $(\mathrm{grz}) = \Box(\Box(\varphi \rightarrow \Box\varphi) \rightarrow \varphi) \rightarrow \varphi$ and $\mathbf{Grz}' = \mathbf{K4} + (\mathrm{grz})$.

6.3.1 Embedding into classical bimodal logics

In [WZ97b], a propositional language with many kinds of modal operators is discussed. Such a propositional language is called *multi-modal propositional language*. In particular, propositional language with two kinds of modal operator is called *bimodal propositional language*. Here, we will take a bimodal propositional language $\mathcal{ML}_2$ with modal operators $\Box_I$ and $\Box$. The set of $\mathcal{ML}_2$-formulas is defined in usaual way, which is denoted by For$\mathcal{ML}_2$. Any logic L on $\mathcal{ML}_2$ is called a *bimodal logic*.

Let L_1 and L_2 be arbitrary monomodal logics in the propositional modal language with $\Box_I$ and with $\Box$, respectively. We define the *fusion* $L_1 \otimes L_2$ of L_1 and L_2 to be the smallest bimodal logics containing both L_1 and L_2. Similarly to the other modal logics, for a subset , of $\mathcal{ML}_2$-formulas and a CBiM-logic L, a *normal extension* $L \oplus$, of L means the smallest CBiM-logic containing both , and L.

In [WZ97b], $\mathcal{ML}_2$ is interpreted using frame-like structures. Let a set W of possible worlds and quasi-order R_I on W be given. Suppose that $\bigcirc_I$ and $\bigcirc$ are unary operators on 2^W. In particular, $\bigcirc_I$ is defined as follows:

$$\bigcirc_I X = \{x \in W : \forall y \in W(xR_I y \Rightarrow y \in X)\}.$$

A *quasi*-CBiM-*frame* is structure $\langle W, R_I, \bigcirc, P\rangle$ where P contains $\emptyset$ and is closed under set-union, set-intersection, set-difference, $\bigcirc_I$ and $\bigcirc$. The validity of $\mathcal{ML}_2$-formulas on quasi-CBiM-frames is determined by its dual structure. Here, the *dual* $\mathfrak{F}^+$ of quasi-CBiM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ is an bimodal algebra $\langle P, \cap, \Leftrightarrow, W, \bigcirc_I, \bigcirc\rangle$. Any valuation $\mathfrak{V}$ on $\mathfrak{F}^+$ is defined as follows:

$$\mathfrak{V}(\bot) = \emptyset, \quad \mathfrak{V}(\varphi \wedge \psi) = \mathfrak{V}(\varphi) \cap \mathfrak{V}(\psi), \quad \mathfrak{V}(\varphi \vee \psi) = \mathfrak{V}(\varphi) \cup \mathfrak{V}(\psi),$$
$$\mathfrak{V}(\neg\varphi) = W \Leftrightarrow \mathfrak{V}(\varphi), \quad \mathfrak{V}(\Box_I\varphi) = \bigcirc_I \mathfrak{V}(\varphi), \quad \mathfrak{V}(\Box\varphi) = \bigcirc\mathfrak{V}(\varphi).$$

The validity of $\mathcal{ML}_2$-formulas on quasi-CBiM-frame is defined by the validity on its dual $\mathfrak{F}^+$, that is, $\mathfrak{F} \models \varphi$ if $\mathfrak{F}^+ \models \varphi$ (in other words, $\mathfrak{V}(\varphi) = W$ holds for any valuation $\mathfrak{V}$). It is obivous that any quasi-CBiM-frame $\mathfrak{F}$ validiates all **S4**-axioms with the modal operator $\Box_I$, since R_I is quasi-order on W.

Let Mix be a formula of the form $\Box_I\Box\Box_I\varphi \leftrightarrow \Box\varphi$ for any formula φ. For a given quasi-CBiM-frame $\mathfrak{F}$ which validiates Mix, a special quasi-CBiM-frame $\rho\mathfrak{F}$ is defined as follows: Suppose quasi-CBiM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ is given. Then, the followings are defined:

- $[x]$: For any possible world x, the cluster of x with respect to R_I is denoted by $[x]$. That is, $[x] = \{y \in W : xR_I y \text{ and } yR_I x\}$.

- $[X]$: For any subset X of W, a set of clusters $[X]$ is defined by $[X] = \{[x] : y \in [x] \text{ for some } y \in X\}$.
  In paticular, $[W]$ is the set of all clusters of W.

- $[R_I]$: A relation $[R_I]$ on $[W]$ inherits R_I, that is, for any $x, y \in W$, $[x][R_I][y]$ if xR_Iy.
  Clearly, $[R_I]$ is well-defined.

- $[P]$: $[P]$ is defined as follows: $[P] = \{[X] : \bigcup[X] \in P\}$.

- $[\bigcirc]$: An operator $[\bigcirc]$ on $[P]$ is defined using $\bigcirc$ as follows: $[\bigcirc][X] = \{[x] : x \in \bigcirc(\bigcup[X])\}$.

- $[\bigcirc_I]$: An operator $[\bigcirc_I]$ on $[P]$ is defined using $[R_I]$ as follows: $[\bigcirc_I][X] = \{[x] : \text{ for any } [y], [x][R_I][y] \text{ implies } [y] \in [X]\}$

- $\rho[P]$: For any $[P]$, the set $\rho[P]$ is defined as follows: $\rho[P] = \{[\bigcirc_I][X] : [X] \in [P]\}$.

The *skelton* $[\mathfrak{F}]$ of $\mathfrak{F}$ is the structure $\langle [W], [R_I], [\bigcirc], [P] \rangle$, and $\rho\mathfrak{F}$ is the structure $\langle [W], [R_I], [\bigcirc], \rho[P] \rangle$. In [WZ97b] it is showed how to construct an IntM-frame from $\rho\mathfrak{F}$, uniquely, and vice versa.

Now, it is ready for discussing an embedding from For$\mathcal{ML}_{\hookrightarrow}$ into For$\mathcal{ML}_2$. Let T'' be the function from For$\mathcal{ML}_{\hookrightarrow}$ into For$\mathcal{ML}_2$ which prefixes $\Box_I$ to all subformulas and replaces $\hookrightarrow$ with $\to$, that is,

$$\begin{array}{rl} \mathsf{T}''(\bot) = \Box_I\bot, & \mathsf{T}''(p) = \Box_I p \text{ for every } p \in \text{Prop}, \\ \mathsf{T}''(\Box\varphi) = \Box_I\Box\mathsf{T}''(\varphi), & \mathsf{T}''(\varphi \wedge \psi) = \Box_I(\mathsf{T}''(\varphi) \wedge \mathsf{T}''(\psi)), \\ \mathsf{T}''(\varphi \vee \psi) = \Box_I(\mathsf{T}''(\varphi) \vee \mathsf{T}''(\psi)), & \mathsf{T}''(\varphi \hookrightarrow \psi) = \Box_I(\mathsf{T}''(\varphi) \to \mathsf{T}''(\psi)). \end{array}$$

Let ρM be the set $\{\varphi \in \text{For}\mathcal{ML}_{\hookrightarrow} : \mathsf{T}''(\varphi) \in M\}$ for a given bimodal logic M, and, $\sigma(\mathbf{IntK} \oplus \Gamma)$ be $(\mathbf{Grz} \otimes \mathbf{K}) \oplus \text{Mix} \oplus \mathsf{T}''(\Gamma)$ for a given $\mathbf{IntK} \oplus \Gamma$, where $\mathsf{T}''(\Gamma) = \{\mathsf{T}''(\varphi) : \varphi \in \Gamma\}$. The following results are shown in [WZ97b].

Proposition 6.7 (Proposition 21 of [WZ97b]) *If a quasi-*CBiM*-logic M is characterized by a class $\mathfrak{C}$ of quasi-*CBiM*-frames, then ρM is characterized by the class $\rho\mathfrak{C} = \{\rho\mathfrak{F} : \mathfrak{F} \in \mathfrak{C}\}$.*

Then the following three propositions denotes relationships between IntM-logic and CBiM-logic.

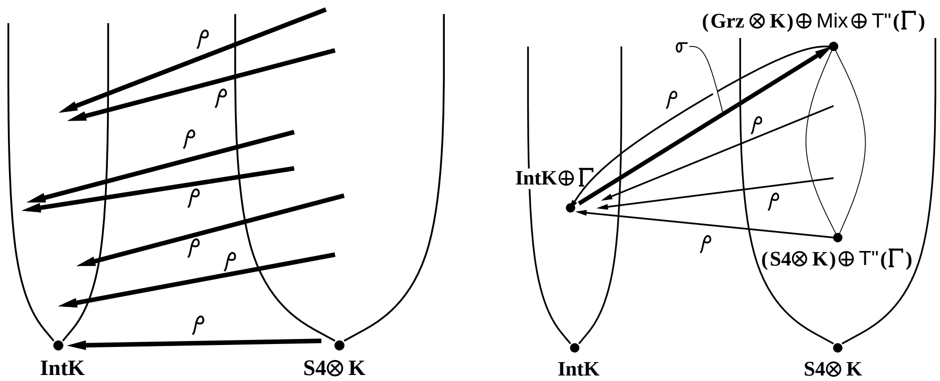


Table 6.1: Propositions 6.8, 6.9 and 6.10

Proposition 6.8 (Theorem 30 of [WZ97b]) *The map* ρ *is a lattice homomorphism from* NExt($\mathbf{S4} \otimes \mathbf{K}$) *onto* NExt**IntK** *(preserving the finite model property and decidability).*

Proposition 6.9 (Theorem 22 of [WZ97b]) *Every logic* $\mathbf{IntK}\oplus,$ *is embedded by* T'' *into any logic* M *in the interval*

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \mathsf{T}''(,\ ) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \mathrm{Mix} \oplus \mathsf{T}''(,\ ).$$

By Proposition 6.9, we have the following: for any bimodal logic M which satisfies $(\mathbf{S4} \otimes \mathbf{K}) \oplus \mathsf{T}''(,\ ) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \mathrm{Mix} \oplus \mathsf{T}''(,\ )$,

$$\vdash_{\mathbf{IntK}\oplus\Gamma} \varphi \text{ iff } \vdash_M \mathsf{T}''(\varphi),$$

that is,

$$\rho M = \mathbf{IntK} \oplus ,\ .$$

Proposition 6.10 (Corollary 28 of [WZ97b]) *The map* σ *is a lattice isomorphism from* NExt**IntK** *onto* NExt($\mathbf{Grz} \otimes \mathbf{K}$) $\oplus$ Mix.

By Proposition 6.9 and 6.10, for any normal extension L of **IntK**,

$$\rho \cdot \sigma(L) = L.$$

These relationships in the above propositions are showed in Table 6.1.

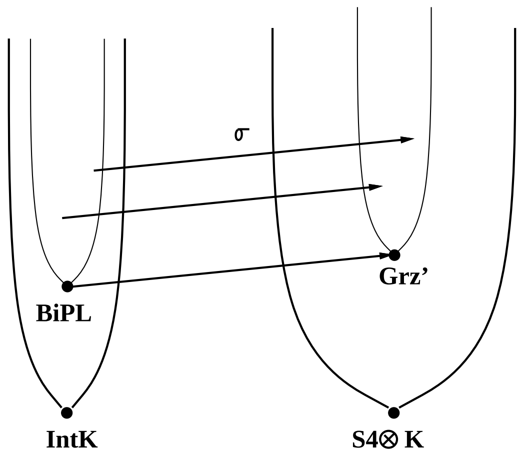


Figure 6.1: Theorem 6.11

6.3.2 Blok-Esakia Theorem for the transitive frames

By Propositions 6.7, 6.8 and 6.10, we will show in this section that Blok-Esakia theorem for transitive frames version holds. The Blok-Esakia theorem can be proved by frame semantics (see [CZ97]). However, by Theorem 5.20, it is proved that there are too many intuitionistic transitive frames, for which the Blok-Esakia type theorem does not hold on the class of intuitionistic transitive frames. So, in this section, we will pay our attention only to transitive frames whose P is closed under $\succeq$. Using the restriction on frames, we can show that an extension of Blok-Esakia Theorem on the transitive frames.

Let $\Box$ be the modal operator of **K4**. For any $\mathcal{ML}$-formula φ, assume that $\Box_I\varphi$ is an abbreviation of $\Box\varphi \wedge \varphi$. Clearly, this new modal operator $\Box_I$ has the properties of **S4**. That is, **K4** which has two modal operators $\Box$ and $\Box_I$ can be regarded as a normal extension of $\mathbf{S4} \otimes \mathbf{K4}$. Thus, any monomodal normal extension of **K4** is a CBiM-logic also. The modal logic $\mathbf{Grz}'$ is defined as $\mathbf{K4} \oplus (\mathrm{grz})$, where (grz) is $\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$. It is known that $\mathbf{Grz}'$ is determined by the class $\mathfrak{C}_{\mathbf{Grz}'}$ of $\mathfrak{C}$ of finite Kripke transitive frames without proper clusters.

Theorem 6.11 *The map σ is an isomorphism from* NExt**BiPL** *onto* NExt$\mathbf{Grz}'$.

Proof In this proof, any normal extension of **K4** is regarded as a bimodal logic by taking $\Box_I\varphi = \varphi\wedge\Box\varphi$. By Proposition 6.7 and **BiPL** is characterized by the class of all **BiPL**-frames, $\rho\mathbf{K4} = \mathbf{BiPL}$ holds. By Proposition 6.8 and the fact that **K4** has the finite model property, **BiPL** has also the finite model property. Hence, **BiPL** is characterized by $\mathfrak{C}_{\mathbf{Grz}'}$, and moreover, $\rho\mathbf{Grz}' = \mathbf{BiPL}$ holds. It is trivial that Mix $\in$ **K4** and **Grz**$'$ is a normal extension of $\mathbf{Grz}\otimes\mathbf{K4}$. That is, when we put $\mathbf{BiPL} = \mathbf{IntK4}\oplus ,$ for some set , , then as we saw by Proposition 6.9, **BiPL** is embedded by T'' into any logic M in the interval

$$(\mathbf{S4}\otimes\mathbf{K4})\oplus\mathsf{T}''(,\,) \subseteq M \subseteq (\mathbf{Grz}\otimes\mathbf{K4})\oplus\mathsf{T}''(,\,).$$

The above embedding holds on every normal extensions of **BiPL**. Applying Proposition 6.10, the lattice isomorphism σ is obtained. ❑

6.4 Expressive powers

It is proved that the deduction theorem holds for $\vdash_{\mathbf{BiPL}}$ in the end of Section 6.1, and that the Blok-Esakia theorem holds for the class of transitive frames in the last of previous section. In this section, like the case of partially ordered frames, it is proved that the expressive powers for transitive frames are recovered in this language.

6.4.1 Local expressive power

Firstly, the local expressive power is discussed. In the previous section, we mentioned that $\mathcal{L}_2$ and $\mathcal{ML}_{\hookrightarrow}$ can be translated into each other on any **BiPL**-frames $\mathfrak{F}$. That is, $\{\varphi_{\mathfrak{F}} : \varphi\in\mathcal{L}_2\} = \{\varphi_{\mathfrak{F}} : \varphi\in\mathcal{ML}_{\hookrightarrow}\}$ holds. However, since the local expressive power was discussed in Kripke frames, we must check the above euqation goes through on Kripke **BiPL**-frames, that is, as follows:

Theorem 6.12 *Let $\mathfrak{F}$ be a transitive Kripke frame. Then,*

$$\{\varphi_{\mathfrak{F}} : \varphi\in\mathcal{ML}_{\hookrightarrow}\} = \{\varphi^{\triangle}_{\mathfrak{F}} : \varphi\in\mathcal{ML}\}.$$

Proof Similar to the proof of Theorem 5.1 using T''. ❑

6.4.2 Global expressive power

To prove that the global expressive power between $\mathcal{L}_2$ and $\mathcal{ML}$ is the same, we will use the terminology of [Zak92] and [CZ97].

Let $\mathfrak{F} = \langle W, R\rangle$ be a finite rooted transitive Kripke frame without proper clusters, W be $\{a_0, \ldots, a_n\}$, a_0 be the root of $\mathfrak{F}$, and $\mathfrak{D}$ be a set of antichains in $\mathfrak{F}$. The following formula $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ is associated with $\mathfrak{F}$ and $\mathfrak{D}$:

$$\gamma(\mathfrak{F}, \mathfrak{D}, \bot) = (\bigwedge_{(i,j)=(0,0)}^{(n,n)} \gamma_{ij} \wedge \bigwedge_{\delta\in\mathfrak{D}} \gamma_\delta \wedge \gamma_\bot) \hookrightarrow p_0,$$

where

$$\begin{array}{rcl}
\gamma_{ij} & = & \left\{\begin{array}{ll} \Box p_0 & \text{if } \neg a_0 R a_0, \\ \Box p_i \hookrightarrow p_i & \text{if } a_i R a_i, \\ (\wedge ,_j \hookrightarrow p_j) \hookrightarrow p_i & \text{if } a_i R a_j \text{ and } a_i \neq a_j, \\ \top & \text{otherwise,} \end{array}\right. \\
\gamma_\delta & = & \bigwedge_{a_j \in W - \delta\underline{\uparrow}} (\wedge ,_j \hookrightarrow p_j) \hookrightarrow \bigvee_{a_i\in\delta} p_i \qquad \text{if } \delta \in \mathfrak{D}, \\
\gamma_\bot & = & \bigwedge_{j=0}^{n} (\wedge ,_j \hookrightarrow p_j) \hookrightarrow \bot, \\
,_j & = & \left\{\begin{array}{ll} \{p_k : p_k \notin a_j\underline{\uparrow}\} & \text{if } a_j R a_j, \\ \{\Box p_j, p_k : p_k \notin a_j\underline{\uparrow}\} & \text{if } \neg a_j R a_j, \end{array}\right. \\
X\underline{\uparrow} & = & X \cup X\uparrow, \\
X\downarrow & = & \{y \in W : \exists x \in X\ yRx\}, \\
X\overline{\downarrow} & = & X \cup X\downarrow.
\end{array}$$

Suppose a Kripke frame $\mathfrak{G} = \langle V, S\rangle$ is given. Then, a partial map f from V onto W is called a *subreduction* of $\mathfrak{G}$ to $\mathfrak{F}$, if, for all $x, y \in \mathrm{dom} f$,
(R1) xSy implies $f(x)Rf(y)$,
(R2) $f(x)Ra_j$ implies $f(y) = a_j$ for some $y \in x\uparrow$.
A subreduction f is *cofinal* if $\mathrm{dom}f\uparrow \subseteq \mathrm{dom}f\overline{\downarrow}$.

For a set $\mathfrak{D}$ of antichain in $\mathfrak{F}$ and a subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$, the following conidtion is defined and called it *closed domain condition*:
(CDC) $\neg\exists x \in \mathrm{dom}f\uparrow \Leftrightarrow \mathrm{dom}f\ \ \exists\delta \in \mathfrak{D} f(x\uparrow) = \delta\underline{\uparrow}$.

Between cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ and canonical formula $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$, the following relationship holds:

Theorem 6.13 *For any intuitionistic transitive Kripke frame* $\mathfrak{G} = \langle V, S\rangle$, $\mathfrak{G} \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ *holds if and only if there is a confinial subreduction of* $\mathfrak{G}$ *to* $\mathfrak{F}$ *satisfying* (CDC).

Proof ($\Rightarrow$) Suppose $\mathfrak{G}$ refutes $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ under some valuation $\mathfrak{U}$. Let $\mathfrak{N}$ be the model $\langle \mathfrak{G}, \mathfrak{U}\rangle$ and π the premise of $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$. Define a partial map from V to W by taking, for $x \in V$,

$$f(x) = \begin{cases} a_i & \text{if } (\mathfrak{N}, x) \not\models p_i, (\mathfrak{N}, x) \models\, ,_i, (\mathfrak{N}, x) \models \pi \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We will show that this map is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC).

[f is a partial function] Let $f(x) = a_i$, $f(x) = a_j$ and $a_i \neq a_j$. Since $\mathfrak{F}$ does not include proper clusters, $a_i \neq a_j$ deduces either $\neg a_i R a_j$ or $\neg a_j R a_i$. For the former case, $p_j \in\, ,_i$ holds, and in the later $p_i \in\, ,_j$. Both cases follows contradiction.

[f satisfies (R1)] Let xSy, $f(x) = a_i$ and $f(y) = a_j$. $(\mathfrak{N}, x) \not\models p_i$ and $(\mathfrak{N}, y) \not\models p_j$ holds because $f(x) = a_i$ and $f(y) = a_j$, respectively. Since the valuation $\mathfrak{U}$ is upward closed and xSy, $(\mathfrak{N}, x) \not\models p_j$. $(\mathfrak{N}, x) \models\, ,_i$ derives $p_j \notin\, ,_i$, namely,

$$a_j \in a_i\underline{\uparrow} \Leftrightarrow a_i R a_j \vee a_i = a_j.$$

If $a_i = a_j$ and $\neg a_i R a_j$, then $(\mathfrak{N}, x) \models\, ,_j$ and $\Box p_j \in\, ,_j$. That is, $(\mathfrak{N}, x) \models \Box p_j$, i.e., $(\mathfrak{N}, y) \models p_j$. This is contradiction.

[f satisfies (R2)] Suppose $f(x) = a_i$ and $a_i R a_j$. $f(x) = a_i$ implies $(\mathfrak{N}, x) \not\models p_i$. If $a_i \neq a_j$, then $(\mathfrak{N}, x) \models\, ,_{ij} = (\wedge ,_j \hookrightarrow p_j) \hookrightarrow p_i$. Since $(\mathfrak{N}, x) \not\models p_i$, $(\mathfrak{N}, x) \not\models \wedge ,_j \hookrightarrow p_j$, namely, there exists the element y of $x\uparrow$ where $(\mathfrak{N}, y) \models \wedge ,_j$ and $(\mathfrak{N}, y) \not\models p_j$. Then, $f(y) = a_j$, and xSy. If $a_i = a_j$, then $(\mathfrak{N}, x) \models \Box p_i \hookrightarrow p_i$. $(\mathfrak{N}, x) \not\models \Box p_i$ since $(\mathfrak{N}, x) \not\models p_i$. It follows that $\exists y \in x\uparrow\ (\mathfrak{N}, y) \not\models p_i$. Then, $f(y) = a_j$, and xSy.

[f is surjective] Since by the definition, $f(x) = a_0$ whenever $(\mathfrak{N}, x) \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$, the map f is a surjection by combining the proof of (R2).

[f is cofinal] Suppose $f(x) = a_0$. Since $(\mathfrak{N}, x) \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ and $(\mathfrak{N}, x) \not\models p_0$, $(\mathfrak{N}, x) \models \pi$. It follows that $x \models \gamma_\bot$. That is, $(\mathfrak{N}, x) \not\models \bigwedge_{j=0}^{n}(\wedge ,_j \hookrightarrow p_j)$ and, for every $y \in x\uparrow$, $(\mathfrak{N}, y) \not\models \bigwedge_{j=0}^{n}(\wedge ,_j \hookrightarrow p_j)$. We consider the case of $(\mathfrak{N}, x) \not\models \bigwedge_{j=0}^{n}(\wedge ,_j \hookrightarrow p_j)$, because the rest case is proved by the similar argument. Assume there exists the element y of successor of x and y is not in dom$f$. Since $\mathfrak{U}$ is an intuitionistic valuation, $(\mathfrak{N}, y) \models\, ,_0$ and $(\mathfrak{N}, y) \models \pi$, and $(\mathfrak{N}, y) \models p_0$. By $(\mathfrak{N}, y) \models \pi$, y satisfies $\gamma_\perp$. That is, there exists the successor z of y such that $(\mathfrak{N}, z) \not\models p_k$, $(\mathfrak{N}, z) \models \pi$ and $(\mathfrak{N}, z) \models \Gamma_k$. Namely, $z \in \text{dom} f$.

[f satisfies (CDC)] Suppose not. Then, there exists the element x_1 which is in $\text{dom} f \uparrow$ but not in $\text{dom} f$, and $f(x_1 \uparrow) = \delta \underline{\uparrow}$ for some $\delta \in \mathfrak{D}$. We can easily deduce that $(\mathfrak{N}, x_1) \not\models \gamma_\delta$ and xSx_1 for some x where $f(x) = a_0$. Since the valuation of $\mathfrak{N}$ is intuitionistic and $(\mathfrak{N}, x) \models \gamma_\delta$, we have $(\mathfrak{N}, x_1) \models \gamma_\delta$. But this is contradiction.

($\Leftarrow$) Let f be a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC). Define a valuation in $\mathfrak{G}$ by taking

$$x \in \mathfrak{U}(p_i) \text{ iff } x \notin f^{-1}(a_i)\overline{\downarrow}.$$

Clearly, this valuation is upward closed. Then, we can easily verify that under this valuation $x \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \perp)$ for every $x \in f^{-1}(a_0)$. ❑

In [Zak92], it is also proved that the similar relation between cofinal subreduction f and canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \perp)$ holds, which is refered in Section 5.1.2:

Proposition 6.14 (Theorem 1 (i) of [Zak92]) *For any transitive Kripke frame $\mathfrak{G}$, $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \perp)$ holds if and only if there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying* (CDC).

Proof This can be proved similarly to Theorem 6.13 .

($\Rightarrow$) A map $f : V \to W$ is defined as follows:

$$f(x) = \begin{cases} a_i & \text{if } (\mathfrak{N}, x) \models \varphi \text{ but } (\mathfrak{N}, x) \not\models p_i, \\ undefined & \text{otherwise,} \end{cases}$$

where φ is the premiss of $\alpha(\mathfrak{F}, \mathfrak{D}, \perp)$.

($\Leftarrow$) A valuation $\mathfrak{U}$ on $\mathfrak{G}$ is defined by $x \in \mathfrak{U}(p_i)$ if $x \notin f^{-1}(a_i)$ holds for any $p_i \in \text{Prop}$. ❑

By Theorem 6.13 and Proposition 6.14, the following results holds.

Corollary 6.15 *For every Kripke transitive frame $\mathfrak{G}$, every finite rooted frame $\mathfrak{F}$ without proper clusters and every set $\mathfrak{D}$ of antichains in $\mathfrak{F}$,*

$$\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}, \perp) \textit{ if and only if } \mathfrak{G} \models \gamma(\mathfrak{F}, \mathfrak{D}, \perp).$$

Now, it is ready to prove that $\mathcal{L}_2$ has the same global expressive power to $\mathcal{ML}$. As like Proposition 5.3, the following holds.

Theorem 6.16 *A skeleton-closed class $\mathcal{C}$ of transitive frames is $\mathcal{ML}_{\hookrightarrow}$-axiomatic if and only if it is $\mathcal{ML}$-axiomatic.*

Chapter 7

Concluding remarks

In this thesis, we have discussed different properties between propositional languages on quasi-oredered frame semantics and that on transitive frame semantics, and showed the way that removes these different points. In the present chapter, we will make a survey of results in this thesis, and then discuss related researches of **BPL** and our further works. As we saw in the preceding chapters, there are several proof systems of **BPL**. However, different definition of semantic consequence relation on Kripke frame is adopted in other researches, although the general frame semantics was introduced here. These logics are investigated by many proof systems. We give these proof systems as a table. In the last section, further works are described.

7.1 Conclusions of this thesis

Our first question which was raised in Chapter 1 is what differences occur between propositional logics for transitive frame semantics and that for quasi-oredered frame semantics, and the second is what method we showed select to dissolve these differences. Our basic non-modal propositional logic with transitive frame semantics is *basic propositional logic* (**BPL**).

On this system, we discussed that many properties of semantic structures (for instance, duality theorem, homomorphisms, p-morphisms and etc) as same as quasi-ordered case. We introuduced a Hilbert style calculus of **BPL**. Also we obtained, the following results which came from the lack of the reflexivity of frames:

- Modus ponens with assmuption ($\rightarrow$-E rule) does not hold in general on

BPL;

- The (local) expressive power of non-modal propositional language is weaker than that of modal propositional language;
- The class of quasi-ordered frames cannot be axiomatizable by a set of non-modal propositional formulas. That is, global expressive power of non-modal propositional language is weaker than that of modal propositional language.

By the above last two results, it becomes necessary to define each extension of **BPL** as a consequence relation. Then we showed that the Blok-Esakia type theorem does not hold on transitive frame semantics.

The answer to the second question is to introduce a new implication $\hookrightarrow$ to non-modal propositional language. This new implication is able to express the truth value not only at each successor of the present point but also at the present point. The Hilbert style calculus is also introduced on this new biarrow language, and we showed that this system has same properties of non-modal propositional language on quasi-ordered frame semantics.

7.2 Classification of proof systems

On research of **BPL**, it is an interesting problem to ask that what provability is expressed by a given system of extension on **BPL**. Visser investigated properties of provability in terms of the formalism via **FPL** and **BPLL** ([Vis81]). In [Vis81], Visser introduced $\mathbf{FPL}^{\mathbf{Cl}}$ as a classical fragment of **FPL** by $\vdash_{\mathbf{FPL}} +(p \to q) \vee ((p \to q) \to p)$. Ruitenburg and Ardeshir are interested in a constructive mathematics that adopts Ruitenburg's interpretation which is more strict than BHK interpretation. They studied **BPL** and its extensions as basis for the constructive mathematics ([AR95]). The standard BHK interpretation and Ruitenburg's are same for logical connectives $\wedge$, $\vee$ and $\perp$. However, a difference comes out in the interpretation of implication. In the standard BHK interpretation,

> a proof of $\varphi \to \psi$ is a construction that converts proofs of φ into proofs of ψ.

On the other hand, Ruitenburg interprets

	BPL	**BPLL**, **FPL**	$\mathbf{FPL}^{\mathbf{Cl}}$	**BQL**, $\mathbf{BQL}^{=}$
nd	[Vis81]	[Vis81]	[Vis81]	
Hs	[SO97], [Sas98], [SWZ98]	[SWZ98]		
sc	[AR95], [Ard95]	[AR95]	[AR95]	[Ard95], [AR96]

Table 7.1: Researches related to **BPL** in terms of syntax

> a proof of $\varphi \to \psi$ is a construction that uses the assumption φ to produce a proof of ψ.

To give a formal meaning of logical connectives in mathematics, predicate logic is necessary. Since subject of Ruitenburg and Ardeshir's research is mathematics, it seems natural that Ruitenburg and Ardeshir's research advances to a predicate calculus (**BQL**) of **BPL**. They call **BPL** and **BQL** *basic logics*. Their researches of **BQL** are, for instance, introduced in [Ard95] and [AR96]. Meanwhile, we showed in this thesis that removing reflexitivity from quasi-ordered frame with the intuitionistic valuation implies that 1) the modus ponens with assumptions and the modus ponens without assumptions are completely different, and that 2) expressive powers among the non-modal propositional language and the modal propositional language are not same. Our main results are deduced in terms of semantics, however these results will be interesting when we consider their syntactical meaning of these results. For instance, Theorem 5.20 which asserts Blok-Esakia type theorem does not hold. We think this theorem denotes that the provability of Peano arithmetic which is different from the provability of ZF-set theory cannot be interpreted into an intuitionistic propositional language in some sense. A relationship between our result and interpretations is following: any axiom in ZF-set theory is used without any proofs, however it is important that the consistency of any proposition in provability theory is *guaranteed* or *declared*. That is, it denotes that any axiom in provability theory is treated as assumption in terms of formalism.

In Table 7.1, we classified papers into logic and type of system. In the above table, *nd*, *Hs* and *sc* denote natural deduction, Hilbert style and sequent calculus, respectively. Sequent calculus (Gentzen style proof system) **GBPL** of **BPL** have beed already discussed in Chapter 3. The logic $\mathbf{BQL}^{=}$ is **BQL** with eqality as special predicate symbol. As for $\mathbf{FPL}^{\mathbf{Cl}}$, the following completeness holds:

Proposition 7.1 (4.10 in [Vis81]) *Let , be a finte set of $\mathcal{L}$-formulas, φ an arbitrary formula and $\mathfrak{C}$ the class of finite, irreflexive, linear frames. Then,*

$$, \vdash_{\mathbf{FPL}^{\mathrm{Cl}}} \varphi \textit{ iff } , \models_{\mathfrak{C}} \varphi.$$

It is an interesting problem what systems will be introduced at blanks in the table.

7.3 Semantic consequence relations in other researches

Visser, Ardeshir and Ruitenburg adopt not only general frame semantics but also Kripke frame semantics. There exists a different point that Visser introduced it for formulas, on the other hand Ardershir and Ruitenburg introduced it for sequents. Firstly, we will explain what semantics they adopt, and then explain a difference between their semantics and ours.

We will discuss Kripke frame semantics. Let $\mathfrak{F}$ be a transitive Kripke frame and $\mathfrak{V}$ a valuation in $\mathfrak{F}$. For any model $\mathfrak{M}$ based on $\mathfrak{F}$ with $\mathfrak{V}$ and any formula, a satisfiable relation $\models$ is defined similarly to that from (2.2) to (2.6). The consequence relation for a class of frames is introduced in Chapter 5, on the other hand, Visser, Ruitenburg and Ardeshir semantics consequence can be introduced as follows: for a given model $\mathfrak{M}$, any set , of formulas and any formula φ,

$$, \models_{\mathfrak{M}} \varphi \text{ iff } (\mathfrak{M}, x) \models , \text{ implies } (\mathfrak{M}, x) \models \varphi \text{ for any possible world } x \text{ of } \mathfrak{M}.$$

In Chapter 3, we said that Visser proved the completeness theorem of **BPL**, however he did not mention the completeness result in the form of Proposition 3.5 . He treated more syntactical way. Let R be a rule in the following form:

$$\frac{\varphi_1 \cdots \varphi_n}{\psi} \quad .$$

Suppose $\vdash_R$ is a consequence relation adding R to **NBPL**. A Kripke model $\mathfrak{M}$ is said R-closed if $\{\varphi_1, \ldots, \varphi_n\} \models_{\mathfrak{M}} \psi$ holds. We denote $, \models_R \varphi$ if $, \models_{\mathfrak{M}} \varphi$ for any R-closed Kripke model $\mathfrak{M}$. Visser showed the following completeness theorem:

Proposition 7.2 (1.10 in [Vis81]) $, \vdash_R \varphi$ *if and only if* $, \models_R \varphi$.

Visser mentioned that the class of models which satisfies the above proposition becomes the class of all transitive Kripke models when any R is not added to **NBPL**. That is, Proposition 3.5 holds.

It is a problem how to treat the arrow "$\Rightarrow$" of sequents semantically. Let $, \Rightarrow \Delta$ be a sequent of formulas, and Ξ a set of sequents. A satisfiable relation is defined as follows by Ruitenburg and Ardeshir:

$$
\begin{aligned}
(\mathfrak{M}, x) \models , \Rightarrow \Delta &\Leftrightarrow \begin{cases} (\mathfrak{M}, x) \models , \text{ implies } (\mathfrak{M}, y) \models \Delta \text{ and} \\ (\mathfrak{M}, y) \models , \text{ implies } (\mathfrak{M}, y) \models \Delta \\ \qquad\qquad \text{for any successor } y \text{ of } x, \end{cases} \\
\mathfrak{M} \models , \Rightarrow \Delta &\Leftrightarrow (\mathfrak{M}, x) \models , \Rightarrow \Delta \text{ for all } x \text{ of } \mathfrak{M}, \\
\mathfrak{M} \models \Xi &\Leftrightarrow \mathfrak{M} \models , \Rightarrow \Delta \text{ for any sequent } , \Rightarrow \Delta \text{ of } \Xi, \\
\Xi \models , \Rightarrow \Delta &\Leftrightarrow \mathfrak{M} \models \Xi \text{ implies } \mathfrak{M} \models , \Rightarrow \Delta \\
&\qquad\qquad \text{for any Kripke model } \mathfrak{M}.
\end{aligned}
$$

By the above definition, it is clear that the arrow "$\Rightarrow$" of sequent behaves as like our new implication "$\hookrightarrow$" of **BiPL** discussed in Chapter 6.

There exists a difference between our consequence relation and consequence relation on their sequent calculus. Let $\mathfrak{C}$ be the class of transitive frames. In our definition, $\Xi \models_{\mathfrak{C}} \varphi$ means that $(\mathfrak{M}, x) \not\models \Xi$ holds or $(\mathfrak{M}, x) \models \varphi$ holds for any model $\mathfrak{M}$ and any x. However, $\Xi \models_{\Rightarrow} \varphi$ denotes that $(\mathfrak{M}, x) \models \Xi$ for any $\mathfrak{M}$ and x, implies $(\mathfrak{M}, x) \models \varphi$ for any $\mathfrak{M}$ and x.

7.4 Further works

We think that the following studies in **BPL** will be worth while to consider:

- Study about relation between **GBPC** and **BiPL**:

  As we mention in the previous section, interpretations of the sequent arrow "$\Rightarrow$" and our new implication "$\hookrightarrow$" are same. Our **BiPL** was introduced by adding some axioms to **Int**, so we think that considering about these arrows denotes another relation between **Int** and **BPL**.

- Study about properties of algebraic semantics for **BPL**:

  Many properties which hold on a class of logics (for instance, disujunction property on a class of logics, interpolation property on a class of logics and etc) can be studied in terms of semantics. As we saw in

Chapter 4, our general frame semantics is quite powerful since it is easy to translate a given frame into a **BPL**-algebra. On the other hand, algebraic semantics has not been studied enough. Thus, it is important to investigate algebraic semantics in detail.

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